Chapter 2
Simply Connected Spaces and Groups

2.1 The Czechist Viewpoint

Let[1] Γ be a discrete group acting properly and freely on a locally compact space X, so that the quotient $B = \Gamma\backslash X$ is separable (and locally compact). As seen on p. 19, in this case, the canonical map p from X onto B satisfies the property that, for any sufficiently small open subset U of B, the inverse image $p^{-1}(U)$ is the disjoint union of open sets mapped homeomorphically onto U under p. In practice, the space B is *locally connected*, i.e. every point of B admits a fundamental system of connected neighbourhoods. In the above, we can then restrict ourselves to open subsets U that are also connected, and the "components" of $p^{-1}(U)$ that are homeomorphically mapped onto U under p must then be the *connected components* of the open set $p^{-1}(U)$.

The ensuing definition of a *covering* of a locally connected space B follows from this example: it is a pair (X, p) consisting of a locally connected space X and of a continuous map $p : X \to B$ such that every point of B has an open connected neighbourhood U satisfying the following condition: p induces a homeomorphism from each connected component of the open set $p^{-1}(U)$ onto U itself.

Note that neither B nor X is assumed to be connected, but this degree of generality is deceptive (albeit practical). First of all, for every open subset U of B (and even for every locally connected subspace U of B), the pair consisting of $p^{-1}(U)$ and of the map induced by p is a covering of U, which will be called the *restriction* to U of the given covering of B. In particular, every covering of B defines a covering of each connected component of B and proving the converse is not difficult...

Besides, the map p from X onto B is clearly open since each $x \in X$ has a neighbourhood mapped homeomorphically by p onto a neighbourhood of the point $p(x)$. The image under p of a connected component Y of X is thus open in B. It is also closed. Indeed, let b be an adherent point of $p(Y)$ and U a connected open neighbourhood of b in B such that every connected component of $p^{-1}(U)$ is

[1]The material in this chapter will not really be needed before Chap. 6. See MA X, no 3.

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homeomorphically mapped onto U under p. There exists a $b' \in p(Y) \cap U$ and if $y \in Y$ is chosen so that $b' = p(y)$, then the connected component of y in $p^{-1}(U)$ meets Y, and so is fully in Y, whence $b \in p(Y)$ as claimed. It obviously follows that if B is connected, then each connected component of X is also a connected covering space of B.

If (X, p) and (X', p') are coverings of B, a *homomorphism* from (X, p) to (X', p') is a continuous map $f : X \to X'$ such that $p' \circ f = p$. If f is also a homeomorphism, it is said to be an *isomorphism* of covering spaces. We will show how to classify the coverings of a given space B up to isomorphism, subject to some restrictions on the local nature of B.

A covering of B is said to be *trivial* or *decomposable* if it is isomorphic to a covering (X, p) obtained by choosing a discrete space F (i.e. a set equipped with the discrete topology) and setting $X = B \times F$ and $p = pr_1$. An arbitrary covering (X, p) of B is not all that different from a trivial covering. Indeed, consider a sufficiently small connected open subset U of X so that each connected component of $p^{-1}(U)$ is homeomorphically mapped onto U under p. Let F be the set of connected components, and to each $x \in p^{-1}(U)$, associate the pair (b, f), where $b = p(x)$ and $f \in F$ is the connected component of x in $p^{-1}(U)$. This is obviously a bijection from $p^{-1}(U)$ onto $U \times F$ and in fact an isomorphism of $p^{-1}(U)$, regarded as a covering space of U, onto the trivial covering space $U \times F$. In other words, *the restriction of* (X, p) *to every sufficiently small open connected*[2] *subset of* B *is trivial.*

Moreover, note that if the restrictions of (X, p) to two open subsets U and V are trivial and if U and V meet, then there is a set F such that $p^{-1}(U)$ and $p^{-1}(V)$ are isomorphic to $U \times F$ and $V \times F$. To see this, it suffices to observe that, for any $c \in U \cap V$, the set $p^{-1}(c)$ meets every connected component of $p^{-1}(U)$ or of $p^{-1}(V)$ at exactly one point. This gives a (non-canonical if $U \cap V$ is not connected) method of constructing a bijection between the set of connected components of $p^{-1}(U)$ and the analogous set with respect to V.

We show that if B is connected, which by the way has been assumed from the beginning, then the above implies the existence of a set F such that, for every sufficiently small connected open subset U of X, the restriction $p^{-1}(U)$ of (X, p) to U is isomorphic to $U \times F$. This is equivalent to showing that *the cardinality of the set* $p^{-1}(b)$ *is independent of the point* $b \in B$. But let us choose a cover of B consisting of non-empty connected open sets U_i, $i \in I$, such that $p^{-1}(U_i)$ is trivial for all i, and hence isomorphic to $U_i \times F_i$, where F_i is the set of connected components of $p^{-1}(U_i)$. We saw above that F_i and F_j are equipotent if U_i and U_j meet, hence more generally if there is a finite sequence of indices $i = k_1, \ldots, k_n = j$ in I such that U_{k_p} and $U_{k_{p+1}}$ meet for $1 \leq p < n$. But the existence of such a sequence defines an obvious equivalence relation on I, whence a partition of I into classes I_λ and a corresponding *partition* of the space B into non-empty open sets

[2]This assumption is not really necessary since every “sufficiently small” open subset is contained in a connected open subset over which (X, p) is trivial.

$$\bigcup_{i \in I_\lambda} U_i.$$

As B is connected, there is only one class I_λ, from which our assertion readily follows.

If we carry on with this Czechist viewpoint (from the Czech mathematician, Eduard Čech, who was the first to systematically use open covering spaces and their intersection properties to define homology groups of a space), then we must choose an isomorphism ϕ_i from $p^{-1}(U_i)$ onto $U_i \times F$ for all i, and, for every pair of indices i and j, compare ϕ_i and ϕ_j over

$$U_{ij} = U_i \cap U_j. \tag{2.1.1}$$

We obviously get two isomorphisms from $p^{-1}(U_{ij})$ onto $U_{ij} \times F$, which therefore can only differ by an automorphism of the trivial covering $U_{ij} \times F$, i.e. by an automorphism of the topological space $U_{ij} \times F$ compatible with pr_1, hence of the form

$$(u, f) \mapsto (u, \theta_{ij}(u, f)), \tag{2.1.2}$$

where θ_{ij} is a map from $U_{ij} \times F$ to F. For (2.1.2) to be continuous, this must also be the case for θ_{ij}. As F is discrete this means that, for all f, the map $u \mapsto \theta_{ij}(u, f)$ must be locally constant, and hence *constant on every connected component of* U_{ij} (such a component is open since B is assumed to be locally connected). Moreover, (2.1.2) must be bijective, which means that for all $u \in U_{ij}$, the map

$$\theta_{ij}(u) : f \mapsto \theta_{ij}(u, f) \tag{2.1.3}$$

from F to F must be bijective. Conversely, for each $u \in U_{ij}$ and permutation $\theta_{ij}(u)$ of the set F constant on each connected component of U_{ij}, formula (2.1.2) clearly defines an automorphism of the trivial covering $U_{ij} \times F$.

Since, over U_{ij}, (2.1.2) transforms the isomorphism $\phi_i : p^{-1}(U_i) \to U_i \times F$ into the isomorphism $\phi_j : p^{-1}(U_j) \to U_j \times F$, the family of maps θ_{ij} from the various intersections U_{ij} to the group Γ of permutations of the set F has the following properties:

(i) $\theta_{ii}(u) = e$ for all i and $u \in U_i$;
(ii) $\theta_{ij}(u)\theta_{ji}(u) = e$ for all i, j and $u \in U_{ij} = U_{ji}$;
(iii) if i, j and k are three indices such that $U_{ijk} = U_i \cap U_j \cap U_k \neq \emptyset$, then

$$\theta_{ij}(u) = \theta_{ik}(u)\theta_{kj}(u) \quad \text{for all} \quad u \in U_{ijk}. \tag{2.1.4}$$

Conversely, let us take an open cover $(U_i)_{i \in I}$ of B, a set F, and, for every pair (i, j) such that $U_i \# U_j$, a *locally constant* map θ_{ij} from U_{ij} to the group Γ of permutations of F satisfying the above conditions (i), (ii) and (iii). We use them to construct a covering (X, p) of B and, for each i, an isomorphism ϕ_i from $p^{-1}(U_i)$ onto $U_i \times F$. We proceed as follows.

First consider the topological space

$$S = \coprod_{i \in I} U_i \times F = \bigcup_{i \in I} \{i\} \times U_i \times F \tag{2.1.5}$$

defined as follows: the elements of S are the pairs consisting of an index $i \in I$ and of an element of $U_i \times F$, in other words of the triples (i, u, f) with $i \in I$, $u \in U_i$ and $f \in F$, so that S is the "disjoint sum" of the sets $U_i \times F$ (*no element* of $U_i \times F$ *being allowed* to be identified with an element of $U_j \times F$ for $i \neq j$, contrary to expectations...). Finally, define an open subset of S as a subset whose intersection with $U_i \times F$ is open in $U_i \times F$ for all i. This gives a topology on S. The reader will naturally note that the use of the term "intersection" is a misnomer since $U_i \times F$ is not strictly speaking a subset of S; but there is a canonical injection $(u, f) \mapsto (i, u, f)$ from $U_i \times F$ to S, which justifies the terminology used.

X will be constructed through an identification process in S by means of an equivalence relation R, defined as follows: two elements (i, u, f) and (j, v, g) of S are in the same class if and only if

$$u = v \quad \text{and} \quad f = \theta_{ij}(u)g. \tag{2.1.6}$$

The condition is empty unless $U_i \# U_j$. As is easily seen, this is an equivalence relation in (2.1.5) if and only if the above conditions (i), (ii) and (iii) hold. It is *open*.[3] Since, by construction, every open subset of S is the union of sets of the form $W \times \{f\} \subset U_i \times F$, where W is an open subset of U_i and f a fixed element of F, it suffices to show that, for all j, the elements $(j, v, g) \in U_j \times F$ whose classes meet $W \times \{f\}$ form an open subset of $U_j \times F$. This is clear since these are elements for which $v \in W \cap U_j$ and $g = \theta_{ij}(v)f$, with $\theta_{ji}(v)$ locally constant. The closure of the graph of relation R follows from similar arguments. Then set $X = S/R$ and define $p : X \mapsto B$ by observing that the map from S onto B which transforms (i, u, f) into u is constant on the classes mod R.

Finally, (X, p) needs to be shown to be a covering of B. Let q be the canonical map from S onto X. By definition (2.1.6) of R, q clearly induces a *bijection* from every subset $\{i\} \times U_i \times F$ of S onto the subset $X_i = p^{-1}(U_i)$ of X. As $U_i \times F$ is open in S and as relation R is open, this bijection from $U_i \times F$ onto X_i is a homeomorphism onto an open subset of X. As the projection $S \to B$ is continuous, so is p. Next, fix a point $a \in B$ and choose i such that $a \in U_i$. Let U be a connected open neighbourhood of a such that $U \subset U_i$. Then consider the homeomorphism from $U_i \times F$ onto $p^{-1}(U_i)$ induced by q. Its restriction to the open subset $U \times F$ of $U_i \times F$ is a homeomorphism onto $p^{-1}(U)$. Therefore, the connected components

[3] An equivalence relation R on a space X is said to be open if the R-saturation of every open subset is open. If, moreover, the graph of R is closed and if X is separable, the same holds for X/R.

of $p^{-1}(U)$ are the images under q of the sets $U \times \{f\}$ with $f \in F$. Consequently, (X, p) is a covering of B, as claimed.

We leave it to the reader to check that if, for all i, ϕ_i denotes the inverse isomorphism of q from $p^{-1}(U_i) = X_i$ onto $U_i \times F_r$, then over $U_i \cap U_j$, ϕ_i is transformed into ϕ_j by using the given θ_{ij}.

It is useful to find the conditions under which the above covering is trivial. Let θ be an isomorphism from the trivial covering $B \times F$ onto X. For each i, the composition of θ and the isomorphism $\phi_i : p^{-1}(U_i) \to U_i \times F$ defined above gives an automorphism $\theta_i = \phi_i \circ \theta$ of the trivial covering $U_i \times F$ of U_i, hence of type $(u, f) \mapsto (u, \psi_i(u) f)$, where ψ_i is a locally constant map from U_i to the group Γ of permutations of F. But for $X \in p^{-1}(U_{ij})$, the automorphism $(u, f) \mapsto (u, \theta_{ij}(u) f)$ of $U_{ij} \times F$ transforms $\phi_i(x)$ into $\phi_j(x)$. Hence

$$\psi_i(u) = \theta_{ij}(u) \circ \psi_j(u) \quad \text{in} \quad U_{ij},$$

or else computations in the group Γ imply

$$\theta_{ij}(u) = \psi_i(u)\psi_j(u)^{-1} \text{ in } U_{ij}. \tag{2.1.7}$$

The converse is readily verified; namely, every family of *locally constant* maps $(\psi_i)_{i \in I}$ from the open subsets U_i to the discrete group Γ satisfying (2.1.7) leads to an isomorphism from the trivial covering $B \times F$ onto (X, p). The existence of such maps is therefore a necessary and sufficient condition for the covering defined by the θ_{ij} to be trivial.

Note that if the sets U_i are connected, which is the most common case, the functions ψ_i are necessarily constant, and hence so are the θ_{ij} (although the sets U_{ij} may not be connected). This condition—that the θ_{ij} are constant on each U_{ij}—although necessary for the triviality of the covering (X, p) considered, is not sufficient in general. There is, however, a very particular case where the situation is much simplified, namely the case where the cover $(U_i)_{i \in I}$ consists of only *two* sets U_1 and U_2. It is then a matter of showing that if there is a *constant* map θ_{12} from U_1 to U_2 in Γ, then there are *constant* maps ψ_1 and ψ_2 from U_1 and U_2 to Γ such that $\theta_{12}(u) = \psi_1(u)\psi_2(u)^{-1}$ in $U_1 \cap U_2$—a trivial problem! As θ_{ij} is necessarily constant whenever U_{ij} is connected, it notably follows that *if $B = U \cup V$ with U and V open and $U \cap V$ connected, then every trivial covering of B over U and V is globally trivial*, a result that can be directly proved from the definitions (take x such that $p(x) = c \in U \cap V$, consider the components U' and V' of $p^{-1}(U)$ and $p^{-1}(V)$ containing x, check that $U' \cap V'$ is the component of $p^{-1}(U \cap V)$ containing x, and deduce that p bijectively maps $U' \cup V'$ onto B).

More specifically, let us call a space B *simply connected* if it is connected, locally connected and if it does not admit any non-trivial connected covering (or equivalently if every covering of B is trivial). Then, *if $B = U \cup V$, where U and V are simply connected and open and if $U \cap V$ is connected, then B is necessarily simply connected.* Applying this argument to a sphere of dimension at least two, we will later show that such a sphere is simply connected. To finish the proof, it is best to use the "homotopic" processes set out in the next section. (The Earth can be covered by (i) everything located above latitude 10° south (ii) everything south of latitude 10° north. It remains to check that a space homeomorphic to a ball is simply connected.) Note that the argument no longer holds in dimension one (nor in fact the result: $\mathbb{R} \to \mathbb{T}$)...

2.2 Extensions of Local Homomorphisms of a Simply Connected Group

Since these notes are supposed to address the theory of Lie groups, in this section we give an example to motivate the presentation of these theories in this context. Let G be a connected and locally connected group, U a connected open neighbourhood of e in G and φ a map from U to an "abstract" group F such that

$$\varphi(xy) = \varphi(x)\varphi(y) \text{ whenever } x, y \text{ and } xy \text{ are in } U. \tag{2.2.1}$$

We may ask ourselves whether φ can be extended to a homomorphism from G to F. This is not always the case (were it always possible, the canonical map from $\mathbb{R}$ to $\mathbb{T}$ would have an inverse homomorphism from $\mathbb{T}$ to $\mathbb{R}$). However, the non-existence of such an extension can be expressed in terms of the non-triviality of a covering. In particular, the following result can be proved:

Theorem 1 *Let G be a simply connected topological group. Let U be an open connected neighbourhood of e in G and φ a map from U to a group F such that*

$$\varphi(xy) = \varphi(x)\varphi(y) \textit{ whenever } x, y \textit{ and } xy \textit{ are in } U.$$

Then, φ can be extended in a unique way to a homomorphism from G to F.

The proof rests on constructions that hold in all cases, regardless of whether G is simply connected or not.

Consider the set I of pairs $i = (V, \theta)$ consisting of a sufficiently small connected open set $V \subset G$ so that $V^{-1}V \subset U$ and of a map $\theta : V \to F$ such that

$$\theta(x)^{-1}\theta(y) = \varphi(x^{-1}y) \quad \text{for all } x, y \in V. \tag{2.2.2}$$

A first remark: (2.2.2) has solutions for every open subset V such that $V^{-1}V \subset U$: some of them can be constructed by choosing an element $a \in V$ and setting $\theta(x) = \varphi(a^{-1}x)$. For $x, y \in V$, $a^{-1}y = a^{-1}x.x^{-1}y$, and assumption (2.2.1) then shows that $\theta(y) = \theta(x)\varphi(x^{-1}y)$ as desired. Next, for $i = (V, \theta) \in I$, let us set $V = U_i$ and $\theta = \theta_i$, and consider indices i and j such that $U_i \# U_j$. Clearly,

$$\theta_i(x)^{-1}\theta_i(y) = \theta_j(x)^{-1}\theta_j(y) \quad \text{for all } x, y \in U_{ij} = U_i \cap U_j. \tag{2.2.3}$$

As a consequence there is a *constant* map θ_{ij} from U_{ij} to F such that

$$\theta_i(x) = \theta_{ij}(x)\theta_j(x) \quad \text{for all } x \in U_{ij}. \tag{2.2.4}$$

If the group F is identified (under left translations) with a subset of the group Γ of permutations of the set F, the family of maps θ_{ij} from U_{ij} to Γ satisfy conditions (i), (ii) and (iii) given on page 35. *Hence specifying U and φ enables us to canonically construct a covering of the topological space G.*

Suppose that this covering is trivial. Then, for each $i \in I$, there is a locally constant map ψ_i from U_i to the group Γ of permutations of the set F such that, for all $x \in U_{ij}$, the permutation $\psi_j(x)$ is the composition of the permutation $\psi_j(x)$ and the left translation by $\theta_{ij}(x)$. Denoting by $s_i(x) \in F$ the image of the identity of F under $\psi_i(x)$, we thus get locally constant maps (hence *constant* since the U_i are connected) s_i from the subsets U_i to F such that $s_i(x) = \theta_{ij}(x)s_j(x)$ (product in the group F) in U_{ij} whenever U_i and U_j meet. Taking into account (2.2.4), we get

$$s_i(x)^{-1}\theta_i(x) = s_j(x)^{-1}\theta_j(x) \quad \text{in } U_{ij} \tag{2.2.5}$$

whenever U_i and U_j meet. As the U_i cover G, there is one and only one map Φ from G to F such that $s_i(x)^{-1}\theta_i(x) = \Phi(x)$ in U_i for all i. But as (2.2.2) can be rewritten as

$$\theta_i(x)^{-1}\theta_i(y) = \varphi(x^{-1}y) \quad \text{for all } x, y \in U_i, \tag{2.2.6}$$

a trivial calculation shows that $\varphi(x^{-1}y) = \Phi(x)^{-1}\Phi(y)$ for all $i \in I$ and $x, y \in U_i$.

We then choose a connected open neighbourhood V of e such that $V = V^{-1}$, $V.V \subset U$. For all $a \in G$, there exists an i such that $aV = U_i$ since aV is connected and satisfies $(aV)^{-1}aV \subset U$. Then $\varphi(x^{-1}y) = \Phi(x)^{-1}\Phi(y)$ for all $a \in G$ and $x, y \in aV$, whence

$$\Phi(xv) = \Phi(x)\varphi(v) \quad \text{for all } x \in G \text{ and } v \in V \tag{2.2.7}$$

and more generally,

$$\Phi(v_1 \dots v_n) = \Phi(e)\varphi(v_1)\dots\varphi(v_n) \tag{2.2.8}$$

for all n and points $v_1, \dots, v_n \in V$. The connected set G is the union of the sets V^n since $V = V^{-1}$. So (2.2.8) defines Φ on the whole of G and shows that if $\Phi(x)$ is replaced by $\Phi(e)^{-1}\Phi(x)$, the map Φ is a *homomorphism* from G to F coinciding with φ on V.

We still need to show that $\Phi(x) = \varphi(x)$ for all $x \in U$. But for $x \in U$, the elements $v \in V$ such that $xv \in U$ form a neighbourhood of e. For such a v,

$$\Phi(xv)\varphi(xv)^{-1} = \Phi(x)\varphi(v)[\varphi(x)\varphi(v)]^{-1} = \Phi(x)\varphi(x)^{-1} \tag{2.2.9}$$

clearly holds by (2.2.1) and (2.2.7). The function $\Phi(x)\varphi(x)^{-1}$ is therefore locally constant on U. As U is connected, it is constant, and hence everywhere equal to e since this is the case at the origin. As a result, Φ and φ are indeed equal on U as claimed. This completes the proof of Theorem 1 since, if G is simply connected, then the covering associated to φ by the preceding construction is necessarily trivial.

In the general case, if φ extends to a homomorphism from G to F, the associated covering is trivial. Indeed, let φ also denote such an extension. Then (2.2.2) or, if preferred (2.2.6), shows that for each i there exists an $a_i \in F$ such that $\theta_i(x) = a_i\varphi(x)$ in U_i. By (2.2.4), it follows that $\theta_{ij}(x) = a_i a_j^{-1}$ for all i and j and $x \in U_{ij}$. For every $x \in U_i$, denoting by $\psi_i(x)$ the left translation on F defined by a_i obviously gives us solutions of (2.1.7), and, as seen above, the covering is trivial. We leave the rest for Sect. 2.6.

Remark. Instead of using the cover $(U_i)_{i\in I}$ defined at the beginning of the proof, to construct the covering associated to φ it would be simpler, although not canonical, to use the cover $(aV)_{a\in G}$, where V is chosen as above. Then, in each *non-empty* intersection $aV \cap bV$, there is a constant "transition function" $\theta_{ab}(x) = \varphi(a^{-1}b)$ and the family of functions θ_{ab} defines the desired covering just as well. The proof can be easily adapted to this viewpoint.

2.3 Covering Spaces and the Fundamental Group

Applications of the theory of covering spaces nearly always concern *path*-connected spaces. Let us remind the reader of their definition. A *path* in a space B is a continuous map γ from the interval $I = [0, 1]$ to B. The points $\gamma(0)$ and $\gamma(1)$ are respectively the *origin* and *extremity* (or the endpoints, in the plural) of γ. If $\gamma(0) = \gamma(1) = b$, γ is said to be a *loop based at the point* b. A space B is then said to be *path-connected* if any two arbitrary points of B can be connected by a path, and *locally path-connected* when each of its points has a fundamental system of path-connected open neighbourhoods. A covering (X, p) of a locally path-connected space B is clearly of the same kind. Given a path γ in B, a path γ' in X such that $\gamma = p \circ \gamma'$ is called a *lifting* of γ to X.

Lemma 1 *Let (X, p) be a covering of B and γ a path in B. For any $x \in X$ such that $p(x) = \gamma(0)$, there exists a unique lifting γ' of γ such that $\gamma'(0) = x$.*

Existence is readily proved. Covering $\gamma(I)$ with a finite number of open sets over which (X, p) is trivial, well-known arguments give us a sequence of numbers $0 = t_0 < t_1 < \cdots < t_n = 1$ such that, for each k, the covering (X, p) is trivial over some open subset U_k containing $\gamma(I_k)$, where $I_k = [t_k, t_{k+1}]$. Consider first the connected component in $p^{-1}(U_0)$ containing the point x given in the statement. There is evidently a unique continuous map γ' from I_0 to it such that $p \circ \gamma' = \gamma$ on I_0. Once this is done, consider the connected component in $p^{-1}(U_1)$ of the point $\gamma'(t_1)$, which enables us to construct a map, still denoted by γ', from I_1 to this component such that $p \circ \gamma' = \gamma$ on I_1, and so on from index to index, whence the existence of γ'.

To prove the uniqueness of γ', it is preferable to show that more generally *if* f_0 *and* f_1 *are two continuous maps from a space* Y *to* X, *such that* $p \circ f_0 = p \circ f_1$, *then the set of* $y \in Y$ *such that* $f_0(y) = f_1(y)$ *is both open and closed in* Y, and hence is the whole of Y if Y is connected and if the set in question is not empty. Now, the closure of the set in question follows from the continuity of f_0 and f_1 (remember that in these Notes, we only consider separable spaces, except spaces that may need to be derived from others, by taking the quotient for example...). To show that it is open, let us suppose that $f_0(b) = f_1(b)$ for some $b \in Y$ and let U be a neighbourhood of $f_0(b)$ homeomorphically and hence injectively mapped to B by p. There is a neighbourhood V of b in Y mapped to U by f_0 and f_1. For any $y \in V$, the points $f_0(y)$ and $f_1(y)$ of U have the same projection in B, and hence coincide since p is injective on U; as a result, $f_0(y) = f_1(y)$ in V, qed.

To go further, the notion of *homotopy* between two paths γ_0 and γ_1 in a space B is needed: a homotopy is a continuous map f from the square $I \times I$ to B such that $f(0, t) = \gamma_0(t)$, $f(1, t) = \gamma_1(t)$, $f(s, 0) = \gamma_0(0) = \gamma_1(0)$, and finally, $f(s, 1) = \gamma_0(1) = \gamma_1(1)$ for all s and t. So the definition assumes that the two paths considered have the same endpoints. When such a homotopy exists, γ_0 and γ_1 are said to be *homotopic* and this is often written $\gamma_0 \sim \gamma_1$.

Lemma 2 *Let* (X, p) *be a covering of* B *and* γ_0, γ_1 *two paths in* X. *Suppose that* $\gamma_0(0) = \gamma_1(0)$ *and that the paths* $p \circ \gamma_0$ *and* $p \circ \gamma_1$ *are homotopic in* B. *Let* $f : I \times I \to B$ *be a homotopy between* $p \circ \gamma_0$ *and* $p \circ \gamma_1$. *Then* γ_0 *and* γ_1 *are homotopic, and there exists a unique homotopy* $g : I \times I \to X$ *between* γ_0 *and* γ_1 *such that* $f = p \circ g$.

The proof is suggested by the similarity between this statement and the previous one. We start by constructing sequences of points $0 = s_0 < s_1 < \cdots < s_m = 1$ and $0 = t_0 < t_1 < \cdots < t_n = 1$ such that for all k and h, the covering (X, p) is trivial over an open subset U_{kh} of B containing $f(I_k \times J_h)$, where $I_k = [s_k, s_{k+1}]$ and $J_h = [t_h, t_{h+1}]$. Considering the connected component of $p^{-1}(U_{00})$ containing the point $\gamma_0(0) = \gamma_1(0)$, the construction of a continuous map g from $I_0 \times J_0$ to this connected component which "lifts" the map f from $I_0 \times J_0$ to U_{00} is immediate. This being done, there is a connected component of $p^{-1}(U_{10})$ containing $g(s_1, t)$ for all $t \in J_0$. This gives a (continuous) map g from $I_1 \times J_0$ which coincides with the previous one on the common border and which "lifts" f over $I_1 \times J_0$. Continuing from index to index, we find a continuous map $g : I \times I \to X$ which lifts f. It is a homotopy since $s \mapsto g(s, 0)$ and $s \mapsto g(s, 1)$ are continuous and hence *constant* maps from I to discrete subsets of X. As $g(0, 0) = \gamma_0(0) = \gamma_1(0)$, the uniqueness part of Lemma 1 shows that g is a homotopy from γ_0 to γ_1. Finally, uniqueness of g follows as in Lemma 1. See also MA X, no 3, (iii).

Note that if we start from a loop γ based at some point of B, the liftings of γ to X *are not* generally loops in X (counterexample: take $B = \mathbb{T}$ and $X = \mathbb{R}$ with the map p given by $p(\theta) = e^{2\pi i\theta}$, and try to lift the loop in $\mathbb{T}$ given by $\gamma(t) = e^{2\pi it}$ for all $t \in I$ to a loop in $\mathbb{R}$...). However, Lemma 2 tells us that the liftings to X of a loop *homotopic to a point* are themselves loops of the same kind.

To make this type of observation systematic and to reach a classification of the coverings of a given space B, the *fundamental group* based at a point of a path-connected space needs to be introduced. It is defined as follows.

Let E be a path-connected space and a a given point of E. The product of two loops γ_0 and γ_1 based at a in E is defined to be the loop given by

$$\gamma_0\gamma_1(t) = \begin{cases} \gamma_0(2t) & \text{if } 0 \le t \le \frac{1}{2}, \\ \gamma_1(2t-1) & \text{if } \frac{1}{2} \le t \le 1. \end{cases} \tag{2.3.1}$$

(Note that the definition only assumes that the extremity of γ_0 and the origin of γ_1 coincide, and in a more general framework makes it possible to define the product of two paths satisfying this condition; it will be needed later.) Consider the equivalence relation defined by the homotopy. It trivially follows that the homotopy class of the product of two loops based at a only depends on the class of the two factors. Hence there is a composition law in the set $\pi_1(E, a)$ of loops at a. This turns $\pi_1(E, a)$ into a *group*, which is shown as follows. First, if there are three loops, γ_0, γ_1 and γ_2 based at a, we get a homotopy between the loops $\gamma_0(\gamma_1\gamma_2)$ and $(\gamma_0\gamma_1)\gamma_2$ by considering the map $f : I \times I \to E$ given by

$$f(s,t) = \begin{cases} \gamma_0(\frac{4t}{1+s}) & \text{for } 0 \le t \le \frac{s+1}{4}, \\ \gamma_1(4t-1-s) & \text{for } \frac{s+1}{4} \le t \le \frac{s+2}{4}, \\ \gamma_2(1-4\frac{1-t}{2-s}) & \text{for } \frac{s+2}{4} \le t \le 1. \end{cases} \tag{2.3.2}$$

This implies the associativity of the multiplication in $\pi_1(E, a)$. The existence of an identity is clear: it is the class of the loop $t \mapsto a$ reduced to a. Finally, every class admits an inverse which can be seen by associating to each loop γ the loop γ' given by $\gamma'(t) = \gamma(1-t)$, and whose geometric interpretation is obvious.

The *fundamental group* $\pi_1(E, a)$ of E based at a seemingly depends on a, but if a is replaced by another point b of E, fixing a path γ connecting a to b gives an isomorphism from $\pi_1(E, a)$ onto $\pi_1(E, b)$ by associating to a loop σ based at a the loop based at b which consists in going from b to a by following the inverse of γ, then in going along σ, and finally in returning to b via γ (we leave it to the reader to transform this colourful description into formulas). In this sense, we can talk of a fundamental group $\pi_1(E)$ of E; it is defined up to isomorphism.

Now a remark: *every path-connected and locally path-connected space B such that $\pi_1(B) = \{e\}$ is simply connected*. Indeed, let (X, p) be a connected covering of B and consider the points x' and x'' of X such that $p(x') = p(x'') = b$. They can be connected by a path γ', projecting onto a loop $\gamma = p \circ \gamma'$ based at b. It is homotopic to a point so that, as seen after the proof of Lemma 2, the lifting γ' of γ is a loop based at x'. Hence $x' = x''$ and p is bijective, proving the result. The converse will be shown later.

This result obviously shows that every *contractible* space B (i.e. for which there exists a continuous map from $I \times B$ to B which "starts" with the identity map from B to B and "ends" with a map from B onto a point) is simply connected. This is for instance the case of a ball in the space $\mathbb{R}^n$ (as remarked on p. 37, this implies that a sphere is also simply connected in dimension ≥ 2).

Let E and F be two path-connected spaces and f a continuous map from E to F. For all $a \in E$, with $b = f(a)$, f clearly defines a homomorphism from $\pi_1(E, a)$ to $\pi_1(F, b)$ obtained by associating to each loop γ based at a the loop $f \circ \gamma$ based at b. The map (sic) $(E, a) \mapsto \pi_1(E, a)$ is therefore a *covariant functor* defined on the category of pointed topological spaces, and taking values in the category of groups, a pointed topological space being a pair (E, a) consisting of a topological space E and of a point a of E, a morphism from a pointed space (E, a) to a pointed space (F, b) being a continuous map from E to F mapping a onto b, the composition of these maps being defined as usual, etc., etc. Envy the children of the XXIth century who, after a hundred and thirty two educational and syllabus reforms, each more final than the preceding one, will learn about functors as soon as they enter middle school. (False: in 2003, they are learning to calculate 6×7 with the help of a computer).

In particular, consider a connected covering (X, p) of a space B, so that for all $x \in X$, the map p defines a homomorphism from $\pi_1(X, x)$ to $\pi_1(B, p(x))$. By Lemma 2, which enables us to "lift" homotopies to B, this homomorphism is *injective*. If, moreover, x is replaced by another point x' located "over" the same point of B as x, then the images of $\pi_1(X, x)$ and of $\pi_1(X, x')$ in $\pi_1(B, p(x))$ are its *conjugate* subgroups, as is immediately seen by choosing a path in X connecting x to x' and using it as above to transform every loop based at x into a loop based at x'. Besides, as every loop over B based at $p(x)$ can be lifted to a path in X having x as origin, by making x' vary, we get all the conjugates in $\pi_1(B, p(x))$ of the image of $\pi_1(X, x)$ under p. We conclude that, *for each point b of B, the class of conjugate subgroups in $\pi_1(B, b)$ is fully determined by a covering (X, p)*. Furthermore, for any $b, b' \in B$, the classes of subgroups defined by (X, p) in $\pi_1(B, b)$ and $\pi(B, b')$ correspond under any isomorphism from the first group onto the second one obtained by connecting b to b' by a path. The classification of the coverings of B rests on these observations since, as will be seen later, the connected coverings of B are in *bijective* correspondence (up to isomorphism) with the classes of conjugate subgroups in $\pi_1(B)$, at least under some simple local assumptions on B.

2.4 The Simply Connected Covering of a Space

The classification of covering spaces is immediate when B is *simply path-connected*, i.e. when $\pi_1(B, b) = \{e\}$ for some point (and hence for all points) of B, since, as seen above, the space B is then simply connected and only has trivial coverings. In the general case, the classification of covering spaces is particularly easy if B is assumed to be *locally simply path-connected*, i.e. if every $b \in B$ has an open path-connected and simply path-connected neighbourhood. This is the case if every $b \in B$ admits a neighbourhood homeomorphic to a ball in a space $\mathbb{R}^n$. The first stage in the classification of covering spaces consists in proving the following result, which is anyhow fundamental:

Theorem 2 *Let B be a path-connected and a locally simply path-connected space. Then B admits a path-connected and a locally simply path-connected covering, and the latter is unique up to isomorphism.*

The proof of this result is somewhat lengthy, but is not very difficult and we trust the reader will fill in the details.

(a) Suppose the problem has been solved and let (X, p) be a connected and simply connected covering of B (*path-connected—this indication will be omitted from now on*). Fix some $b \in B$ and $a \in X$ such that $p(a) = b$. As X is simply connected, two paths in X having a as origin are homotopic if and only if they have the same extremity x. On the other hand, by Lemma 2, two such paths are homotopic in X if and only if their projections in B are homotopic. Since every path in B with b as origin can be lifted to a path in X with a as origin, there must be a *bijection* between the elements of X and the classes of paths in B with origin b. This leads to a construction of (X, p) starting from B and the point b.

(b) To see this, start with a connected and locally simply connected space B and fix a point b of B. Let X denote the set of *classes of paths in B originating at b* and p the map from X onto B which associates to each class of paths originating at b, its extremity (which only depends on the class considered). All that is required is to endow X with a topology turning (X, p) into a simply connected covering of B.

For this purpose, let U be a *connected and simply connected* open subset of B. Then $p^{-1}(U) \subset X$ is partitioned into classes of paths originating at b and ending in U. Let us define (for fixed U) two paths γ_0 and γ_1 originating at b and ending at points u_0 and u_1 of U as being equivalent if γ_1 is homotopic to the product of γ_0 and a path in U connecting u_0 to u_1. This gives an equivalence relation in the set of homotopy classes of paths originating at b and ending in U. In other words, this defines an equivalence relation on the set $p^{-1}(U)$, which is thus partitioned into classes. If V is one of these classes, p induces a *bijection* from V onto U. Indeed, fix a point x of V, defined by the path γ connecting b to $u = p(x) \in U$. The points of V then correspond to paths homotopic to $\gamma\gamma'$, where γ' is an arbitrary path in U having origin u. It is already the case that p maps V *onto* U since U is path-connected. Moreover, U being simply connected, two paths in U having origin u are homotopic whenever they have the same extremity. Taking their product with γ shows that the map p from V onto U is also injective.

Let the equivalence classes defined above be called the components of $p^{-1}(U)$. They are mutually disjoint and their union is $p^{-1}(U)$. Letting U vary within the stated conditions, the components of the inverse images of $p^{-1}(U)$ are clearly seen to form a covering of X. We are now going to endow X with the following topology: *a subset W of X is open if and only if, for every open connected and simply connected set $U \subset B$ and every component V of $p^{-1}(U)$, the set $p(W \cap V)$ is open in U* (i.e. in B). The axioms of a topology hold for trivial set theoretic reasons.

We are now going to show that, *if V is as before a component of $p^{-1}(U)$ with U connected and simply connected, then V is open in X and p is a homeomorphism from V onto U*. We show that (X, p) is a covering of B.

To do so, let us take a subset W of V such that $p(W)$ is open. As p is a bijection from V onto U, it amounts to showing that W is open in X. For this it suffices to check that if U' is an open connected and simply connected subset of B and if V' is a component of $p^{-1}(U')$, then $p(W \cap V')$ is open in B. But let us consider a point $x \in W \cap V' \subset V \cap V'$, its projection $u = p(x) \in U \cap U'$, and a connected neighbourhood U'' of u contained in the open set $p(W) \cap U'$. The point x is the homotopy class of a path γ in B connecting b to u. As U'' is connected, u can be connected to every $v \in U''$ by a path in U''. Taking its product with γ, we get an element $y \in X$ such that $p(y) = v$, and as the path in U'' connecting u to v is also a path in U or in U', x and y clearly belong to the same component of $p^{-1}(U)$ or $p^{-1}(U')$. As $x \in V \cap V'$, it follows that $y \in V \cap V'$ as well. However, as p maps V bijectively onto U, $v = p(y) \in p(W)$ implies that $y \in W \cap V'$. As a consequence, $p(W \cap V')$ contains every point v of U'', and so is open in U as claimed.

To finish this part of the proof, the *connectedness and simple connectedness* of the *covering* (X, p) of B constructed above remains to be checked. First note that if γ is a path in B originating at b, then for every $s \in I$, let γ_s be the "partial" path defined by $\gamma_s(t) = \gamma(st)$. It defines a point x_s of X. For s' sufficiently near s, $\gamma_{s'}$ is obtained (up to equivalence) by the product of γ_s with a path connecting $\gamma_s(1)$ to $\gamma_{s'}(1)$ in a connected and simply connected neighbourhood of $\gamma_s(1)$. Hence the map $s \mapsto x_s$ from I to X is continuous, and so defines a lifting of the path γ to X. It is obviously *the* lifting of γ having as "origin" the base point of X representing the class of the path consisting solely of b in B. Hence this particular point of X can be connected to any other point by a path. This is why X is path-connected like B.

Denoting by a this "base" point of X, let λ be a loop based at a over X and let us consider its projection $\gamma = p \circ \lambda$. Defining partial paths γ_s as above and denoting the class of γ_s by x_s, the map $s \mapsto x_s$ is a lifting of γ to X originating at a. Since such a lifting is unique, it is necessarily λ. In other words, $x_s = \lambda(s)$ for all $s \in I$, and in particular $x_1 =$ class of $\gamma = \lambda(1) = a$, which means that the path γ and the path reduced to b are homotopic. Equivalently, the image of the homomorphism from $\pi_1(X, a)$ to $\pi_1(B, b)$ induced by p is the identity subgroup. As this homomorphism is injective, it follows that X is simply connected.

So far, we have proved the existence of a simply connected covering of the space B (the one constructed in the proof is called the *universal covering* of B at the base point b; the terminology will become clear in the next section). In the following section, we prove its uniqueness and at the same time show how to classify the other coverings of B.

2.5 Classification of Covering Spaces

We continue to assume that B is connected and locally connected. In this case, as seen above, if (X, p) is a connected covering of B, then, for all $b \in B$, (X, p) canonically defines a class of mutually conjugate subgroups in $\pi_1(B, b)$. We are now going to show that the correspondence thus defined between connected coverings of B (defined up to isomorphism) and classes of subgroups of $\pi_1(B, b)$ is *bijective*. Uniqueness up to isomorphism of the simply connected covering (which corresponds to the trivial subgroup) will notably follow.

(i) Let (X, p) be the universal covering of B at the point b constructed in the previous section, and $\Gamma = \pi_1(B, b)$ be the fundamental group of B based at b. We first show that Γ can be made to act on X in such a way that B is identified with the quotient $\Gamma\backslash X$. For this, it suffices to observe that for $\gamma \in \Gamma$ and $x \in X$, γx admits a natural definition: choose a loop based at b in the homotopy class γ, a path with origin b in the homotopy class x, and take their product. This gives a new path with origin b whose class only depends on classes γ and x; thus γx is well-defined. Similarly, the constructions outlined above to check the group axioms in $\pi_1(B, b)$ show that Γ *acts* on the set X. Obviously, $p(\gamma x) = p(x)$ by construction, and moreover Γ acts *transitively* on each "fibre" $p^{-1}(u)$ in X. Equivalently, if there are two paths in B with origin b and extremity u, the latter is homotopic to the product of the former with a loop based at b, which is obvious. If, in particular, a denotes as above the base point of X, in other words the class of the trivial loop based at b, then γa must be the extremity of the lifting with origin a of any loop based at b belonging to the class γ. Finally, Γ *acts freely on* X. This means that if λ is a path having origin b and γ a loop based at b, and if the path $\gamma\lambda$ is homotopic to the path λ, then the loop γ is homotopic to a point. This follows immediately by considering the "reverse" path λ' of λ. Paths $(\lambda'\gamma)\lambda \simeq \gamma'(\gamma\lambda)$ and $\lambda'\lambda$ are then homotopic. However the latter is a loop homotopic to a point at $b' = \lambda(1)$. Hence so is the former. Moreover, as the map $\gamma \mapsto (\lambda'\gamma)\lambda$ specifically defines (see p. 42) an isomorphism from $\pi_1(B, b)$ onto $\pi_1(B, b')$, the loop γ is homotopic to a point, as claimed. [Note that we have used path composition in a more general setting than that of loops].

These remarks provide a canonical bijection between $\Gamma\backslash X$ and B. It remains to show that the action of Γ on X is compatible with the topology on X. For this, consider a loop σ based at b, a point x of X represented by a path λ connecting b to the point $u = p(x)$, and fix a connected and simply connected neighbourhood U of u in B. Let V and V' be the components of $p^{-1}(U)$ respectively containing x and γx, where $\gamma \in \Gamma$ is the class of σ so that V and V' are open neighbourhoods of x and $x' = \gamma x$ homeomorphically mapped onto U by p. By the construction of components, V is the set of classes of paths consisting of λ followed by a path in U connecting u to any point of U. As x' is the class of the path $\sigma\lambda$, similarly V' is the set of classes of paths obtained by adjoining to $\sigma\lambda$ a path in U originating at u. Since path composition is associative up to homotopy, $V' = \gamma V$. As a result, first of all Γ acts continuously on X, and secondly every point in X has a neighbourhood whose images under the elements of Γ are mutually disjoint. More generally, if x and y are points in X belonging to distinct orbits of Γ, i.e. such that $p(x) \neq p(y)$, then there clearly exist neighbourhoods V and W of x and y in X such that γV and $\gamma' W$ are disjoint for all $\gamma, \gamma' \in \Gamma$.

Therefore, if B is locally compact, in which case so is X, Γ acts properly and freely on X as claimed (Chap. 1, Sect. 1.4), the quotient space $\Gamma\backslash X$ being canonically identified with B in all cases.

(ii) Let us next show how to *construct a covering of B corresponding to a given class of subgroups of* $\pi_1(B, b) = \Gamma$. For this, fix a subgroup Γ' in the given class and make it act on the simply connected covering (X, p) as in (i) above. Replace X by the quotient space $X' = \Gamma' \backslash X$ and p by the map p' from X' onto B obtained by observing that two points of X corresponding under the action of Γ' have the same image in B. It is necessary to check that (X', p') is also a covering of B and that Γ' belongs to the class of subgroups of $\pi_1(B, b)$ defined by (X', p').

First note that because of the way Γ acts on X (see end of (i) above), *the canonical map π from X onto X' is locally a homeomorphism*. In particular, if U is a connected and simply connected open subset of B, in which case $p^{-1}(U)$ is the disjoint union of (connected...) components simply transitively permuted by Γ, then the restriction of π to any one of these components, say V, is a homeomorphism from V onto an open subset of X'. Besides if V and W are two components of $p^{-1}(U)$, the images $\pi(V)$ and $\pi(W)$ are clearly either identical or disjoint depending on whether V and W correspond or not under the action of Γ'. It follows that the inverse image $p'^{-1}(U) = \Gamma' \backslash p^{-1}(U)$ of U in X' is the disjoint union of open sets mapped homeomorphically onto U under p'. The pair (X', p') is therefore a covering of B as expected. A similar argument shows that the pair (X, π) is a simply connected covering of X', and with good reason.

The image in $\pi_1(B, b)$ of the fundamental group of X' based at a point over b remains to be found. For example, the point $a' = \pi(a)$, may be chosen, where a is the base point of X (i.e. the class of the identity loop in B). We show that the image of $\pi_1(X', a')$ in Γ is precisely the subgroup Γ', thereby completing the construction. To do so, let us start with an element γ of Γ' and let σ be a loop based at b representing the homotopy class γ. It can be lifted to a path λ in X with origin a and whose extremity, as seen in part (i) of the proof, is the point γa of X. As $\pi(a) = \pi(\gamma a)$ since $\gamma \in \Gamma'$, $\pi \circ \lambda$ is clearly a *loop* based at a' in X', which is mapped onto $p' \circ \pi \circ \lambda = \sigma$ under p'. This argument shows that the image of $\pi_1(X', a')$ in Γ contains Γ'.

To prove the inverse inclusion, we start with a loop λ' based at a' in X'. Since (X, a) is a covering of X', the loop λ' can be lifted to a path λ in X with origin a. Then the map $p' : X' \to B$ transforms the homotopy class of λ' into that of the loop $\sigma = p' \circ \lambda' = p' \circ \pi \circ \lambda = p \circ \lambda$. Denote by γ the element of $\Gamma = \pi_1(B, b)$ thus obtained. Since λ is a path in X with origin a, the extremity $\lambda(1)$ of λ must be the point γa of X. But as $\pi \circ \lambda$ is a loop, $\pi(\gamma(a)) = \pi(\lambda(1)) = \pi(\lambda(0)) = \pi(a)$, and as π is the canonical map from X onto $\Gamma' \backslash X$, it follows that $\gamma \in \Gamma'$, giving the desired result.

(iii) To finish the proof, it remains to show that the covering of B corresponding to a given class of subgroups of Γ is unique up to isomorphism.

For this, let (X', p') be a connected covering of B and choose an element $a' \in X'$ such that $p'(a') = b$, and let Γ' be the image in Γ of the fundamental group of X' at a'. We need to show that (X', p') can be identified with the quotient of the simply connected quotient of (X, p) by Γ', constructed above.

We first construct a map π from X onto X'. So, let x be a point of X, and σ a path in B with origin b representing x, so that the unique lifting λ of σ to X having origin a connects the base point a of X to the given point x. As (X', p') is a covering of B, the path σ can be lifted in a unique way to a path λ' in X' having origin a'. The path σ is determined up to homotopy by x, and so, by Lemma 2, so is λ'. This shows that the extremity $x' = \lambda'(1)$ of λ' only depends on x and not on the choice of σ. We set $x' = \pi(x)$. Surjectivity of π obviously follows from the path-connectedness of X', and clearly $p = p' \circ \pi$.

As (X', p') is a covering of B, any sufficiently small neighbourhood V' in X' of x' has the following properties: p' induces a homeomorphism from V' onto a neighbourhood U of the point $u = p'(x')$ in B, and moreover (X', p') and (X, p) are trivial over U. Suppose that V' (and hence also U) is connected so that V' is the connected component of x' in $p'^{-1}(U)$. If $x \in X$ is chosen so that $\pi(x) = x'$ and if V denotes the connected component of x in $p^{-1}(U)$, then the map π induces a map from V to V' since $\pi(V)$ is connected, contains x' and is contained in $p'^{-1}(U)$. Moreover, the composition of this map from V to V' and the homeomorphism p' from V' onto U gives the homeomorphism p from V onto U. As a result, π induces a homeomorphism from V onto V', which shows that the pair (X, π) is a covering of X', trivial over V'.

We still need to check that if x and y are two given points of X, then $\pi(x) = \pi(y)$ if and only if there exists a $\gamma \in \Gamma'$ such that $y = \gamma x$. Let σ and τ be paths in B with origin b belonging to the homotopy classes x and y and let us consider their liftings λ and μ to X having origin a, as well as their liftings λ' and μ' to X' having origin a'. By definition, $\pi(x) = \lambda'(1)$ and $\pi(y) = \mu'(1)$. Equality $\pi(x) = \pi(y)$ therefore tells us that $\lambda'(1) = \mu'(1)$, namely that the loop based at b obtained by adjoining to the path σ the reverse of the path τ is the projection on B of a loop in X' based at a', or equivalently that the loop based at b defined previously belongs, up to homotopy, to the image subgroup Γ' of $\pi_1(X', a')$ in $\pi_1(B, b)$. This means that $y \in \Gamma' x$, completing the proof.

Note that in part (ii) of the construction of the covering $X' = \Gamma' \backslash X$ corresponding to the given subgroup Γ' of Γ, it is in general impossible to make Γ act on the quotient X'. This is nonetheless the case when the subgroup Γ' is *normal* in Γ. Trivial considerations show that Γ then acts on the left on X', or more precisely that the quotient group $\Gamma \backslash \Gamma'$ acts *freely* on X' similarly to the action of Γ on X. Then, (X', p'), endowed with these group operations, is said to be a *Galois covering* of B.

2.6 The Simply Connected Covering of a Topological Group

Let G be a topological group satisfying the assumptions of Theorem 2, i.e. path-connected and locally simply path-connected (this will be the case for connected Lie groups since each point of such a group has neighbourhoods homeomorphic to $\mathbb{R}^n$ for some appropriate n). Let us choose e as the base point of G and apply to (G, e) the constructions used for (B, b) in Sect. 2.4. This gives a simply connected canonical covering of G, which will be denoted by $(\widetilde{G}, p)$ and which is therefore defined as follows: the paths of $\widetilde{G}$ are the paths in G with origin e, projection p associates to the class of a path γ its extremity $\gamma(1)$, and the topology on $\widetilde{G}$ is defined as in Sect. 2.4.

But as G is a topological group, if λ and μ are paths in G with origin e, then the map

$$\lambda * \mu : t \mapsto \lambda(t)\mu(t) \tag{2.6.1}$$

from I to G defines another path with origin e. The homotopy class of $\lambda * \mu$ depends solely on those of λ and μ (as for paths, we get a "composition" or a "convolution" of homotopies $I \times I \to G$ using the given composition law of G). Hence, this gives a composition law $(x, y) \mapsto x * y$ (which from now will be written simply as xy) in $\widetilde{G}$. Its associativity follows trivially. It admits an identity element—the path reduced to e which will be written $\widetilde{e}$. And every element of $\widetilde{G}$ has an inverse, which can be seen by associating to a path λ the path $t \mapsto \lambda(t)^{-1}$. In other words, $\widetilde{G}$, endowed with this composition law, is a *group*, the map $p : \widetilde{G} \to G$ clearly becoming a homomorphism.

The group law and the topology of $\widetilde{G}$ are easily seen to be compatible. As an example, let us show the continuity of the map $(x, y) \mapsto x * y^{-1}$ at the origin. For this, we choose a connected and simply connected open neighbourhood U of e, and set $\widetilde{U}$ to be the component of $p^{-1}(U)$ containing e, so that p induces a homeomorphism from $\widetilde{U}$ onto U. Let V be a sufficiently small connected open neighbourhood of e so that $V \cdot V^{-1} \subset U$, and let $\widetilde{V}$ be the component of $p^{-1}(V)$ containing e, i.e. the inverse image of V in $\widetilde{U}$. The elements of $\widetilde{U}$ (resp. $\widetilde{V}$) are clearly the classes of the paths in U (resp. V) with origin e. But if $x, y \in \widetilde{V}$ are represented by the paths σ and τ in V with origin e, then $x * y^{-1}$ is clearly represented by the path $t \mapsto \sigma(t)\tau(t)^{-1}$ in U. As a consequence, $\widetilde{V} \cdot \widetilde{V}^{-1} \subset \widetilde{U}$ and so, identifying $\widetilde{U}$ and $\widetilde{V}$ with U and V using p, the map $(x, y) \mapsto x * y^{-1}$ from $\widetilde{V} \times \widetilde{V}$ to $\widetilde{U}$ is identified with the similar map from $V \times V$ to U, thereby implying the desired continuity property. The other properties are obtained likewise.

Hence $\widetilde{G}$ can be considered a connected and simply connected topological group and the map p a homomorphism of topological groups. Then G can be clearly identified (as a topological group) with the quotient of $\widetilde{G}$ by $\ker(p)$. This kernel, which equals $p^{-1}(e)$ and which, as a *set* for now, can be canonically identified with $\pi_1(G, e) = \pi_1(G)$, is a discrete subgroup of $\widetilde{G}$ since p is locally a homeomorphism. This discrete subgroup is abelian and is even contained in the centre of $\widetilde{G}$ (*all normal discrete subgroups of a connected group are central*, because the set of conjugates of an element x of such a subgroup is a connected subset, which therefore necessarily reduces to the point x itself). Let us show that it can be identified *as a group* with $\pi_1(G)$. It suffices to show that if σ and τ are loops based at e in G, then the loop $\sigma * \tau$ is homotopic to the product $\sigma\tau$ of the loops σ and τ (product defined for every space even in the absence of a composition law). For this let ε denote the path reduced to the point e and let us replace σ and τ by the loops $\sigma' = \sigma\varepsilon$ and $\tau' = \varepsilon\tau$ given by

$$\sigma'(t) = \begin{cases} \sigma(2t) & \text{if } t \leq \frac{1}{2}, \\ e & \text{if } t \geq \frac{1}{2}, \end{cases} \quad \tau'(t) = \begin{cases} e & \text{if } t \leq \frac{1}{2}, \\ \tau(2t-1) & \text{if } t \geq \frac{1}{2}. \end{cases} \tag{2.6.2}$$

Clearly, σ' and τ' are homotopic to σ and τ and hence $\sigma * \tau$ is homotopic to $\sigma' * \tau'$. However, from the definition it is obvious that $\sigma' * \tau' = \sigma\tau$, and so the desired result is immediate.

The previous proof remains valid if σ is a loop based at e and τ an arbitrary path with origin e. As the composition law $(\sigma, \tau) \mapsto \sigma\tau$ defines the action of $\pi_1(G)$ on $\widetilde{G}$ in the sense used in the preceding section, *in the case of a topological group G, the general theory prompts us to endow $\widetilde{G}$ with the structure of a topological group, to identify $\pi_1(G)$ with the kernel of the canonical homomorphism p from $\widetilde{G}$ onto G, and to then make $\pi_1(G)$ act on $\widetilde{G}$ by left translations*. $\widetilde{G}$ is called the *universal covering* of G. Since, according to the previous section, any other connected covering of G is the quotient of the space $\widetilde{G}$ by a subgroup Γ' of $\pi_1(G)$, i.e. by a normal (since central) subgroup of G, *every other connected covering of G is the quotient group of $\widetilde{G}$ by the corresponding subgroup of* $\pi_1(G)$ (as an aside, note that since $\pi_1(G)$ is abelian, the coverings of G correspond to the subgroups, and not just to the classes of subgroups, of $\pi_1(G)$).

Theorem 1 on extensions of local homomorphisms can now be completed:

Theorem 3 *Let G be a connected and locally connected group, U a connected open neighbourhood of the identity e in G and ϕ a local homomorphism from U to a group F. Let $(\widetilde{G}, p)$ be the universal covering of G and $\widetilde{U}$ the connected component of the identity in $p^{-1}(U)$. Then there is a uniquely defined homomorphism $\widetilde{\phi}$ from $\widetilde{G}$ to F such that*

$$\widetilde{\phi}(x) = \phi(p(x)) \quad \textit{for all} \quad x \in \widetilde{U}. \tag{2.6.3}$$

It suffices to apply Theorem 1 to the local homomorphism $x \mapsto \phi(p(x))$ of the *connected* open subgroup $\widetilde{U}$ of the simply connected group $\widetilde{G}$.

Note that the above statement does not intend to say more than it does. For example, it does not say that p is a bijection from $\widetilde{U}$ onto U (this would be the case if U was assumed to be simply connected—but we do not make this assumption). Neither does it say that equality $\widetilde{\phi}(x) = \phi(p(x))$ holds throughout $p^{-1}(U)$—if that was the case, the homomorphism $\widetilde{\phi}$ would be trivial on $p^{-1}(e)$, and as G is the quotient of $\widetilde{G}$ by this discrete subgroup, it would in fact follow that $\widetilde{\phi}$ is the composition of p and a homomorphism from G to F extending ϕ, a conclusion that would destroy the entire theory of coverings! In fact, $\widetilde{\phi}$ may well be constant on a non-trivial subgroup D of $p^{-1}(e)$, in which case $\widetilde{\phi}$ "passes to the quotient" and defines a homomorphism to F from the "intermediate" covering $\widetilde{G}/D$ of G, but in general, this does not reduce to G.

Exercise. Describe in terms of $\widetilde{G}$ and $\widetilde{\phi}$ the covering of G associated to ϕ by the proof of Theorem 1.

2.7 The Universal Covering of the Group $SL_2(\mathbb{R})$

We now illustrate the construction of the universal covering with an example using the group $SL_2(\mathbb{R})$. Its fundamental group will be shortly shown to be $\mathbb{Z}$. Let us begin with a completely different case, that of the group $SL_2(\mathbb{C})$ consisting of complex matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with determinant $ad - bc = 1$.

$SL_2(\mathbb{C})$ is *simply connected*. To see this, consider the subgroup K of unitary matrices in $G = SL_2(\mathbb{C})$, in other words of matrices with $c = -\bar{b}$ and $d = \bar{a}$. So K can be topologically identified with the subset of $\mathbb{C}^2$ defined by the equality $a\bar{a} + b\bar{b} = 1$, i.e. with the unit sphere in $\mathbb{R}^4$. As a consequence, K is simply connected. Consider the subgroup B_+ of triangular matrices of G, namely $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ with $ad = 1$ and $a > 0$. It is easy to check that the map $(k, b) \mapsto kb$ is a homeomorphism from $K \times B_+$ onto G. As B_+, homeomorphic to $\mathbb{R}^*{}_+ \times \mathbb{C}$, is simply connected, so is G (exercise: compute the fundamental group of a Cartesian product).

There are (MA XII, Chap. 4) similar subgroups in $G = SL_2(\mathbb{R})$, namely the subgroup K of rotations

$$k = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \tag{2.7.1}$$

and the subgroup B_+ of matrices

$$b = \begin{pmatrix} 1/t & u \\ 0 & t \end{pmatrix} \quad \text{with } u \in \mathbb{R} \text{ and } t > 0. \tag{2.7.2}$$

Once again G is homeomorphic to $K \times B_+$ and B_+, homeomorphic to $\mathbb{R}^*{}_+ \times \mathbb{R}$, is simply connected. Hence,

$$\pi_1(G) = \pi_1(K) = \pi_1(\mathbb{T}) = \mathbb{Z}, \tag{2.7.3}$$

as claimed above.

The universal covering $\widetilde{G}$ of $SL_2(\mathbb{R})$ will therefore be a simply connected group with a central subgroup D isomorphic to $\mathbb{Z}$ and such that the quotient $\widetilde{G}/D$ is isomorphic to G. Searching for a "concrete" description of $\widetilde{G}$ using a group of matrices satisfying appropriate conditions is pointless; indeed, it can be proved that every continuous homomorphism from $\widetilde{G}$ to a group $GL_n(\mathbb{R})$ is trivial on the kernel of $p : \widetilde{G} \to G$, in other words is reduced to a linear representation of the group G itself. The description of $\widetilde{G}$ given below follows by taking into account the argument of the function $\phi : G \to \mathbb{T}$ given by

$$\phi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \mathrm{Am}(ci + d), \tag{2.7.4}$$

where $\mathrm{Am}(z) = z/|z|$ for every non-trivial complex number. More precisely, we consider the set $\widetilde{G}$ of pairs (g, θ) in $G \times \mathbb{R}$ for which

$$\phi(g) = e^{i\theta} \tag{2.7.5}$$

and turn it into a simply connected covering of G. The construction will then be complete once the composition law of $\widetilde{G}$ is described. Hence this construction consists of several parts.

(a) *The covering of a space B defined by a continuous map ϕ from B to $\mathbb{T}$*. Let B be a topological space and ϕ a continuous map from B to $\mathbb{T}$. Consider the set X of pairs (b, θ) in $B \times \mathbb{R}$ such that $\phi(b) = e^{i\theta}$. It is a closed subset of $B \times \mathbb{R}$ and the map p from X onto B with the obvious definition is continuous. We show that the pair (X, p) is a (not necessarily connected) *covering* of B, at least if B is locally connected, since we will restrict ourselves to this case in these Notes.

Let U be an open subset of B. A *continuous* map α from U to $\mathbb{R}$ satisfying

$$\phi(u) = e^{i\alpha(u)} \quad \text{for all } u \in U \tag{2.7.6}$$

will be said to be a *uniform branch of the argument of ϕ in U*. As will be seen, the choice of such a function makes it possible to "trivialize" (X, p) over U and, more precisely, to construct a homeomorphism from $p^{-1}(U)$ onto $U \times \mathbb{Z}$ which transforms p into the projection $U \times \mathbb{Z} \to U$. For this, observe that if $x = (u, \theta)$ is in $p^{-1}(U)$, i.e. if $u \in U$, then there exists an $n \in \mathbb{Z}$ such that $\theta = \alpha(u) + 2\pi n$ and conversely. Since $2\pi n = \theta - \alpha(u) = \theta - \alpha(p(x))$, the map $x \mapsto (p(x), n)$ from $p^{-1}(U)$ to $U \times \mathbb{Z}$ is continuous and bijective; the inverse map, given by $(u, n) \mapsto (u, \alpha(u) + 2\pi n)$, is also continuous, giving the desired homeomorphism.

The pair (X, p) will therefore be a covering of B if for every sufficiently small open subset U of B, a uniform branch of the argument of ϕ is shown to exist in U. To this end, take any $b \in B$ and consider the set $U = U(b)$ of $u \in B$ such that

$$|\phi(u) - \phi(b)| < 2. \tag{2.7.7}$$

If $\mathbb{T}(b)$ denotes the open subset of $\mathbb{T}$ obtained by omitting the point $-\phi(b)$, then $U(b) = \phi^{-1}(\mathbb{T}(b))$, and so is open in B. Let us show that there is a uniform branch of the argument of ϕ in $U(b)$. To do so, set $\mathbb{T}^* = \mathbb{T} - \{-1\}$ and for $z \in \mathbb{T}^*$, let $\arg(z)$ denote the value of the argument of z strictly contained between $-\pi$ and $+\pi$ ("principal value"); it is a continuous map from $\mathbb{T}^*$ to $\mathbb{R}$ such that

$$z = e^{i \cdot \arg(z)} \text{ for all } z \in \mathbb{T}^* = \mathbb{T} - \{-1\}. \tag{2.7.8}$$

We then choose an arbitrary value θ for the argument of the complex number $\phi(b)$. For $u \in U(b)$, $\phi(u)/\phi(b) \in \mathbb{T}^*$ and so we can define a continuous map α from $U(b)$ to $\mathbb{R}$ by setting

$$\alpha(u) = \theta + \arg(\phi(u)/\phi(b)). \tag{2.7.9}$$

This gives the desired uniform branch.

Hence, in short, *every continuous map ϕ from B to $\mathbb{T}$ canonically defines a covering of B* acted on freely by the discrete group $\mathbb{Z}$ in an obvious way—a good example of a Galois covering in the sense of Sect. 2.5, which the reader will check as an exercise. Besides, the previous arguments show that this covering is (globally) trivial if and only if there exists a uniform branch of the argument of ϕ defined throughout B. As a result, *if ϕ is a continuous map from a simply connected space B to $\mathbb{T}$, then there exists a continuous map α from B to $\mathbb{R}$ such that*

$$\phi(b) = e^{i\cdot\alpha(b)} \text{ for all } b \in B. \tag{2.7.10}$$

This result is frequently used in the classical theory of analytic functions [construction of uniform branches of $\log f(z)$, where f is a holomorphic function without any zeros in a simply connected domain B of $\mathbb{C}$; the choice of a uniform branch α of the argument of the function $z \mapsto f(z)/|f(z)|$ enables us to construct a uniform branch of $\log f(z)$, namely the function $\log|f(z)| + i\cdot\alpha(z)$, where $\log|f(z)| \in \mathbb{R}$; as is well known, the uniform branch $\log|f(z)| + i\cdot\alpha(z)$ is then not only continuous but also holomorphic in B].

Exercise. Let (R, q) be a covering of a space T (the choice of the letters R and T is meant to suggest analogies...) and let ϕ be a continuous map from a space B to T. Consider the set $X \subset B \times R$ of pairs (b, θ) consisting of $b \in B$ and $\theta \in R$ such that $q(\theta) = \phi(b)$. Let p be the projection of X onto B. Show that (X, p) is a covering of B (the "inverse image" of a covering by a continuous map). In what way is this exercise related to the content of the present section? Suppose B is simply connected. Show that there is a continuous map $\psi : B \to R$ such that $\phi = q \circ \psi$. What about the case when B, R, T are groups and ϕ and q homomorphisms?

(b) *Construction of the universal covering of $G = SL_2(\mathbb{R})$*. To obtain a universal covering of $SL_2(\mathbb{R})$, we apply the preceding construction to the map from G to $\mathbb{T}$ given by (2.7.4). The pair (X, p) of (a) is therefore transformed into the set of $(g, \theta) \in G \times \mathbb{R}$ such that $\phi(g) = e^{i\theta}$. With the benefit of hindsight regarding the success of this operation, we will denote this set by $\widetilde{G}$. The map p is then the first projection. We thereby get a closed subset, hence locally compact, of $G \times \mathbb{R}$ and we know that $(\widetilde{G}, p)$ is indeed a covering of the space G. Let us show that it is connected and simply connected. For this, we use the decomposition $G = B_+ \cdot K$, where B_+ and K are the subgroups of G described earlier.

Since $t > 0$, for

$$g = bk = \begin{pmatrix} 1/t & u \\ 0 & t \end{pmatrix} \begin{pmatrix} * & * \\ \sin\omega & \cos\omega \end{pmatrix} = \begin{pmatrix} * & * \\ t\cdot\sin\omega & t\cdot\cos\omega \end{pmatrix}, \tag{2.7.11}$$

we get

$$\phi(bk) = e^{i\omega} = \phi(k). \tag{2.7.12}$$

The map $(b, k) \mapsto bk$ being a homeomorphism, $\widetilde{G}$ is identified (as a set and topologically) with the subset of $B_+ \times K \times \mathbb{R} = B_+ \times \mathbb{T} \times \mathbb{R}$ consisting of triples (b, z, θ) with $b \in B_+$, $z \in \mathbb{T}$, $\theta \in \mathbb{R}$ and $z = e^{i\theta}$. As $\theta \mapsto (e^{i\theta}, \theta)$ is a homeomorphism from $\mathbb{R}$ to the set of pairs (z, θ) considered, $\widetilde{G}$ is finally seen to be homeomorphic to $B_+ \times \mathbb{R}$. The space $\widetilde{G}$ is therefore indeed a universal covering of G up to isomorphism. Its general construction was given in Sect. 2.6. It has a group structure which we now need to explicitly describe in the context of the above model. We already know that the map p should be a homomorphism from the *group* $\widetilde{G}$ to G, but this information is naturally insufficient to compute the composition law of $\widetilde{G}$. To do so, we first present another realization of G.

(c) *The covering* $\widetilde{G}$ *as the set of uniform branches of the arguments of functions* $\mathrm{Am}(cz + d)$. Let $P : \mathrm{Im}(z) > 0$ be the upper half-plane. On $G \times P$ consider the function

$$J(g, z) = cz + d \quad \text{if} \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{2.7.13}$$

It is continuous, without any zeros, and an easy computation shows that[4]

$$J(gg'', z) = J(g', g''(z))J(g'', z), \tag{2.7.14}$$

where G acts on P in the usual way:

$$g(z) = \frac{az + b}{cz + d}. \tag{2.7.15}$$

A formula analogous to (2.7.14) follows if the ($\mathbb{C}^*$-valued) function J is replaced by the continuous map A from $G \times P$ to $\mathbb{T}$ given by

$$A(g, z) = J(g, z)/|J(g, z)| = \mathrm{Am}(cz + d). \tag{2.7.16}$$

As $G \times P$ is not simply connected, a uniform branch of the argument of the function A cannot be globally defined, but this becomes feasible if G is replaced by $\widetilde{G}$ since $\widetilde{G} \times P$ is simply connected. As $A(e, i) = 1$, *there exists a unique continuous map* ω *from* $\widetilde{G} \times P$ *to* $\mathbb{R}$ *satisfying the conditions*

$$A(p(\gamma), z) = e^{i \cdot \omega(\gamma, z)} \quad \text{for all } \gamma \in \widetilde{G} \text{ and } z \in P, \tag{2.7.17}$$

$$\omega(\widetilde{e}, i) = 0 \tag{2.7.18}$$

where $\widetilde{e}$ is the "base" point of $\widetilde{G}$ defined by

$$\widetilde{e} = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, 0 \right). \tag{2.7.19}$$

[4]The similarity with equality $d(g'g''z)/dz = d(g'g''z)/d(g''z) \times d(g''z)/dz$ is not entirely a coincidence.

Note that a more general equality

$$\omega(\gamma, i) = \theta \text{ if } \gamma = (g, \theta) \tag{2.7.20}$$

holds because the maps $\gamma \mapsto \omega(\gamma, i)$ and $\gamma = (g, \theta) \mapsto \theta$ from $\widetilde{G}$ to $\mathbb{R}$ are obviously two uniform branches of the argument of the function $(g, \theta) \mapsto A(g, i)$, and as they coincide for $\gamma = \widetilde{e}$, they are equal everywhere.

Formula (2.7.17) also shows that, for a given γ, the map $z \mapsto \omega(\gamma, z)$ from P to $\mathbb{R}$ is a uniform branch of the argument of the map $z \mapsto \mathrm{Am}(cz + d)$ from P to $\mathbb{T}$, where $p(\gamma) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Hence the pair (g, α) consisting of the matrix $g = p(\gamma)$ and the function $\alpha(z) = \omega(\gamma, z)$ can be associated to γ. *We thereby get a bijection from $\widetilde{G}$ onto the set of pairs (g, α) consisting of $g \in G$ and of a uniform branch $z \mapsto \alpha(z)$ of the argument of $z \mapsto A(g, z)$*. Indeed, consider such a pair (g, α); if it is the image of an element γ of $\widetilde{G}$, it is necessarily of the form $\gamma = (g, \theta)$, where θ is a solution of $e^{i\theta} = \phi(g) = \mathrm{Am}(ci + d) = w(g, i)$. Hence, it is a matter of showing that there is a unique solution for which $\omega(\gamma, z) = \alpha(z)$ for all z. But the other two factors, being uniform branches of the argument of the same function—namely $z \mapsto \mathrm{Am}(cz + d)$, only differ by a constant multiple of 2π, so that to check that they are equal for all z, it suffices to do so for $z = i$. Equality (2.7.20) then shows that we only need to choose $\theta = \alpha(i)$, which indeed determines the pair (g, θ) unambiguously.

Let γ' and γ'' then be elements of $\widetilde{G}$, and consider the pairs (g', α') and (g'', α'') corresponding to them in the previous construction. The equalities

$$A(g', z) = e^{i \cdot \alpha'(z)}, \quad A(g'', z) = e^{i \cdot \alpha''(z)} \tag{2.7.21}$$

and the equality analogous to (2.7.13) for the function A then imply that

$$A(g'g'', z) = e^{i \cdot \alpha(z),} \text{ where } \alpha(z) = \alpha'(g''(z)) + \alpha''(z). \tag{2.7.22}$$

The new function α is obviously a uniform branch of the argument of the function $z \mapsto A(g'g'', z)$. As a consequence, the pair $(g'g'', \alpha)$ corresponds to an element γ of $\widetilde{G}$. Hence if $\widetilde{G}$ is identified with the set of pairs (g, α), it is possible to define a composition law on $\widetilde{G}$ by setting

$$(g', \alpha')(g'', \alpha'') = (g'g'', \alpha) \quad \text{where} \quad \alpha(z) = \alpha'(g''(z)) + \alpha''(z). \tag{2.7.23}$$

The last part of the construction consists in showing that this *composition law is indeed the one following from the general constructions of Sect.* 2.6, a result that, on the face of it, is not at all obvious since it is defined using *paths in G* that have not yet been mentioned.

Exercise. A group G acts on a set P. Let F be the set of maps from P to a (not necessarily abelian) group R. Define a composition law on $G \times F$ by setting

$(g', \alpha')(g'', \alpha'') = (g'g'', \alpha)$, where $\alpha(z) = \alpha'(g''(z)) \cdot \alpha''(z)$ for all $z \in P$. Show that $G \times F$ is a group.

(d) *Compatibility with path multiplication*. Formula (2.7.23) enabled us to transform the set $\widetilde{G}$ into the set whose identity element is obviously $\widetilde{e}$. Defining the composition law (2.7.23) by involving the function ω considered above is both useful and easy. Indeed, if (g', α') and (g'', α'') correspond to elements γ' and γ'' of $\widetilde{G}$, then, as seen, by construction,

$$g' = p(\gamma'), \quad g'' = p(\gamma''), \quad \alpha'(z) = \omega(\gamma', z), \quad \alpha''(z) = \omega(\gamma'', z). \tag{2.7.24}$$

Denote the product given in (2.7.23) by $\gamma'\gamma'' = \gamma$, so that γ corresponds to the pair $(g'g'', \alpha)$ given by the formula in question. Therefore

$$\omega(\gamma, z) = \alpha(z) = \omega(\gamma', g''(z)) + \omega(\gamma'', z). \tag{2.7.25}$$

Hence if $\widetilde{G}$ is made to act on P by setting

$$\gamma(z) = g(z) \quad \text{where } g = p(\gamma), \tag{2.7.26}$$

multiplication (2.7.23) in $\widetilde{G}$ is such that the function ω satisfies the identity

$$\omega(\gamma'\gamma'', z) = \omega(\gamma', \gamma''(z)) + \omega(\gamma'', z), \tag{2.7.27}$$

whose likeness with (2.7.14) is clear. Besides, (2.7.27) fully determines $\gamma'\gamma''$, given that, in addition,

$$p(\gamma'\gamma'') = p(\gamma')p(\gamma''). \tag{2.7.28}$$

If indeed we return to the definition of $\widetilde{G}$ as a subset of $G \times \mathbb{R}$, and if we set $\gamma' = (g', \theta')$ and $\gamma'' = (g'', \theta'')$, so that $\theta' = \omega(\gamma', i)$ and $\theta'' = \omega(\gamma'', i)$, conditions (2.7.27) and (2.7.28) show that $\gamma'\gamma'' = (g'g'', \theta)$, with

$$\theta = \omega(\gamma'\gamma'', i) = \omega(\gamma', g''(i)) + \omega(\gamma'', i) = \omega(\gamma', g''(i)) + \theta''. \tag{2.7.29}$$

Note that $\omega(\gamma', g''(i))$ is the value at $z'' = g''(i)$ of the uniform branch of the argument of the function $z \mapsto \text{Am}(c'z + d')$, where c' and d' are the obvious entries of the matrix g', which takes value θ' at $z = i$.

A comparison with the composition law following from Sect. 2.6 requires the latter to be explicitly formulated. Let us temporarily denote it by $(\gamma', \gamma'') \mapsto \gamma' * \gamma''$ so as to avoid confusion with (2.7.23). It is obtained in the following manner, by setting $\widetilde{e}$ to be its identity.

Let γ' and γ'' be two elements of $\widetilde{G}$. Connect them to $\widetilde{e}$ by the paths

$$\gamma'(t) = (g'(t), \theta'(t)) \quad \text{and} \quad \gamma''(t) = (g''(t), \theta''(t)), \tag{2.7.30}$$

so that $t \mapsto g'(t)$ and $t \mapsto g''(t)$ are their projections in G. Then there is a unique path $t \mapsto \gamma(t)$ in $\widetilde{G}$ with origin $\widetilde{e}$ whose projection follows the path $t \mapsto g'(t)g''(t)$ (product in the group G) and the extremity $\gamma(1)$ of this lifting is the desired element $\gamma' * \gamma''$ of $\widetilde{G}$.

As (2.7.28) and (2.7.29) determine $\gamma'\gamma''$ and as

$$p(\gamma' * \gamma'') = p(\gamma')p(\gamma''), \tag{2.7.31}$$

to prove $\gamma'\gamma'' = \gamma' * \gamma''$, all that needs to be shown is that

$$\omega(\gamma' * \gamma'', i) = \omega(\gamma', g''(i)) + \omega(\gamma', i). \tag{2.7.32}$$

But let $\gamma(t) = (g'(t)g''(t), \theta(t))$ be the lifting with origin e of the path $g'(t)g''(t)$, so that $\gamma(1) = \gamma' * \gamma''$. More generally, $\gamma(t) = \gamma'(t) * \gamma''(t)$ for all $t \in I$, and hence, instead of proving (2.7.32), we need to show that

$$\omega(\gamma(t), i) = \omega(\gamma'(t), g''(t)(i)) + \omega(\gamma''(t), i). \tag{2.7.33}$$

By (2.7.20) this can be rewritten as

$$\theta(t) = \omega(\gamma'(t), g''(t)(i)) + \theta''(t). \tag{2.7.34}$$

Now, the right-hand side is a *continuous* map from I to $\mathbb{R}$ since every function appearing in it—ω, $t \mapsto \gamma'(t)$, $g \mapsto g(i)$, etc...—are continuous and this map vanishes at $t = 0$ since $g'(0) = g''(0) = e$ and $\theta''(0) = 0$. If

$$(g'(t)g''(t), \omega(\gamma'(t), g''(t)(i)) + \theta''(t)) \in \widetilde{G} \tag{2.7.35}$$

is shown to hold for all $t \in I$, then the continuous map

$$t \mapsto (g'(t)g''(t), \omega(\gamma'(t), g''(t)(i)) + \theta''(t)) \tag{2.7.36}$$

from I to $\widetilde{G}$ will be a path with origin $\widetilde{e}$ lifting the path $t \mapsto g'(t)g''(t)$. As its lifting with origin $\widetilde{e}$ is unique, (2.7.36) will then be the lifting $(g'(t)g''(t), \theta(t))$, and Equality (2.7.34) will then follow. Hence, all that is needed is to check (2.7.35). For this, according to the definition of $\widetilde{G}$ as a subset of $G \times \mathbb{R}$ given in subsection (b), it suffices to show that

$$e^{i[\omega(\gamma'(t), g''(t)(i)) + \theta''(t)]} = \phi[g'(t)g''(t)]. \tag{2.7.37}$$

Now, generally speaking $\phi(g) = \mathrm{Am}(ci + d) = A(g, i)$, where we have inserted function (2.7.16). Then, taking into account identity (2.7.14) for A, it follows that

$$\phi[g'(t)g''(t)] = A[g'(t)g''(t), i] = A[g'(t), g''(t)(i)] \cdot A[g''(t), i]. \tag{2.7.38}$$

So it suffices to separately prove the equalities

$$\begin{aligned} e^{i\cdot\omega(\gamma'(t),g''(t)(i))} &= A[g'(t), g''(t)(i)], \\ e^{i\cdot\theta''(t)} &= A[g''(t), i]. \end{aligned} \tag{2.7.39}$$

But these follow readily from the formula

$$A(p(\gamma), z) = e^{i\cdot\omega(\gamma,z)} \tag{2.7.40}$$

applied to (2.7.39), with $\gamma = \gamma'(t)$ and $z = g''(t)(i)$ and to (2.7.40), with $\gamma = \gamma''(t)$ and $z = i$, completing the proof.

(e) *Conclusions*. As these constructions are not very obvious,[5] it may be useful to summarize the essential points obtained along the way:

(i) As a *set*, $\widetilde{G}$ consists of pairs (g, α), where $g \in G$, and $\alpha : P \mapsto \mathbb{R}$ is a uniform branch of the argument of the function $z \mapsto cz + d$, $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$; the map $p : \widetilde{G} \to G$ is given by $p(g, \alpha) = g$;

(ii) as a *group*, $\widetilde{G}$ is endowed with composition law (2.7.22):

$$(g', \alpha')(g'', \alpha'') = (g'g'', \alpha), \quad \text{where} \quad \alpha(z) = \alpha'(g''(z)) + \alpha''(z);$$

(iii) as a *topological space*, $\widetilde{G}$ is identified with a closed subspace of $G \times \mathbb{R}$ under the map

$$(g, \alpha) \mapsto (g, \alpha(i));$$

the image of $\widetilde{G}$ under this map consists of all pairs (g, θ) such that

$$ci + d = |ci + d|e^{i\theta} \quad \text{if} \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

These data fully define $\widetilde{G}$ and its structures, and the rest can be forgotten...

[5] They do not appear to be in the published literature. The construction of $\widetilde{G}$ as the set of pairs (g, α) with composition law (2.7.23) has been known to experts for a long time, but for experts, of which I am supposed to be one, this construction obviously leads to a genuine *covering* of the group G. It assumes that a topology is established on the set of these pairs (g, α), that its compatibility with the composition law (2.7.23) holds, and finally that so does the "local triviality" condition of the coverings for the map p. Hence, these "trivial verifications" are usually omitted as it is reckoned that, if a central extension of G by $\mathbb{Z}$ is (abstractly) constructed, Providence will provide the topology that will turn it into a universal covering of G. This expectation is of course fully justified in hindsight, which explains why the detailed construction is little more than a tedious exercise for "beginners".

Exercise. Show that the inverse image $\widetilde{K}$ of K in $\widetilde{G}$ is isomorphic to $\mathbb{R}$, that the subgroup B_+ is isomorphically embedded in $\widetilde{G}$ by $b \mapsto (b, 0)$, where 0 is the map $z \mapsto 0$ from P to $\mathbb{R}$, and that the obvious map from $\widetilde{K} \times B_+$ to $\widetilde{G}$ is a homeomorphism.[6]
Exercise. Let n be a positive integer. Consider the set $\widetilde{G}_n$ of pairs (g, ρ) consisting of $g \in G$ and of a continuous (hence holomorphic) map $\rho : P \mapsto \mathbb{C}^*$ such that

$$\rho(z)^n = cz + d \quad \text{if} \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{2.7.41}$$

(we know that a holomorphic function with no zeros in a simply connected domain has nth roots for all n). Turn $\widetilde{G}_n$ into a group using the exercise of pages 55–56 and consider the map $\widetilde{G}$ to $\widetilde{G}_n$ (justify this!) which associates to the pair (g, α) the pair (g, ρ) given by

$$\rho(z) = |cz + d|^{1/n} \cdot e^{i\cdot\alpha(z)/n}, \tag{2.7.42}$$

where $|cz + d|^{1/n}$ denotes the *positive* nth root of $|cz + d|$. Show that this map is a surjective homomorphism and that $\widetilde{G}_n$ is identified with the quotient of G by the subgroup $D = \mathbb{Z}$ consisting of the multiples of n (where D is the kernel of $\widetilde{G} \mapsto G$). Deduce that $\widetilde{G}_n$ can be endowed with a topology with respect to which $\widetilde{G}_n$ is a covering of G of order n (which is in fact Galois—we have $G = \widetilde{G}_n/D_n$ where D_n is a cyclic central subgroup of order n).

Exercise. Let r be a real number and Γ a discrete subgroup of G. A (for example) continuous (but not necessarily holomorphic) map f from P to $\mathbb{C}$ is said to be of weight r with respect to Γ if, for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, there is a holomorphic function $M_\gamma(z)$ on P satisfying $|M_\gamma(z)| = |cz + d|^r$ and such that

$$f(\gamma(z)) = M_\gamma(z) f(z) \quad \text{for all } z \in P. \tag{2.7.43}$$

The equality

$$M_{\gamma'\gamma''}(z) = M_{\gamma'}(\gamma''(z)) M_{\gamma''}(z) \tag{2.7.44}$$

must obviously be assumed if we want functions f that do not vanish everywhere. As $M_\gamma(z)$ is holomorphic,

$$M_\gamma(z) = |cz + d|^r e^{ir\cdot\lambda(z)}, \tag{2.7.45}$$

where λ is a uniform branch on P of the argument of $cz + d$. Note that this formula determines (2.7.45) up to multiplication by a factor of type $e^{2\pi irn}$, $n \in \mathbb{Z}$.

[6]This result could have been proved beforehand since B_+ is simply connected and it would seem that it could have then easily provided a direct construction of $\widetilde{G}$. Unfortunately, we would have needed to express the composition law of $\widetilde{G}$ in terms of the decomposition considered...

Associate to f a function ϕ on $\widetilde{G}$, considered as the set of pairs (g, α), by setting

$$\phi(g, \alpha) = |ci + d|^{-r} e^{-ir\cdot\alpha(i)} \cdot f(g(i)) \quad \text{for all } (g, \alpha) \in \widetilde{G}, \tag{2.7.46}$$

where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Show that the function ϕ is multiplied by a constant with absolute value 1 when it undergoes left translation by an element of the discrete subgroup $p^{-1}(\Gamma)$ of $\widetilde{G}$.

Suppose that $r = p/q$ with p, q integers, $q > 0$. Show that, by passing to the quotient, ϕ can be defined on the group $\widetilde{G}_q$ of the previous exercise. In particular, take $r = 1/2$, and for Γ the subgroup of $SL_2(\mathbb{Z})$ generated by $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and for f the Jacobi function

$$\theta(z) = \sum e^{\pi i n^2 z}, \tag{2.7.47}$$

for which

$$\theta(z + 2) = \theta(z), \quad \theta(-1/z) = \left(\frac{z}{i}\right)^{\frac{1}{2}} \theta(i) \tag{2.7.48}$$

is known to hold, $(z/i)^{1/2}$ being the uniform branch of the square root of the function z/i on P which has value 1 at $z = i$. Define directly the function corresponding to θ on the "double" covering $\widetilde{G}_2$ of G, and show that when it undergoes a left translation by an element of the inverse image of Γ in $\widetilde{G}_2$, the function obtained is multiplied by a constant equal to $+1$ or -1 (and whose full computation is one of the hardest exercises in the classical theory of modular functions).

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