

Chapter 2 THE FOUR-FOLD WAY

How to Perceive Complex Mathematical Models and Well-Posed Problems

2.1 PROLOGUE: THE MANAGER AND ANALYST DISCUSS THE ORIGINS OF MULTIDIMENSIONAL MODELS AND WELL-POSEDNESS

“Since complexity has grown so enormously in modern times,” the manager commented, “I presume that the motivations to develop techniques to manage it are relatively recent.”

“On the contrary,” replied the analyst, “many of the concepts and examples of problem recognition are quite old – ancient even.”

Consider the old Indian story of the blind men trying to “understand” an elephant. Depending on what is touched – the leg, ear, tail, trunk, or tusk – the unknown object takes on the attributes of a tree, a leaf, a rope, a snake or a spear. Thus, touching an aspect of a complex object is far removed from understanding the total integrated concept of “elephant.”

A more recent story – but still almost 2000 years old – comes from the Talmud [5]. According to a commentary on the book of Genesis, on the day that the Lord created Man, He took truth and hurled it to the ground, smashing it into thousands of jagged pieces. From then on, truth was dispersed, splintered into fragments like a jigsaw puzzle. While a person might find a piece, it held little meaning until he joined with others who had painstakingly gained different pieces of the puzzle. Only then, slowly and

deliberately, could they try to fit their pieces of Truth together – to make some sense of things.

Mankind's yearning to understand the world over the eons has been aided by the development of mathematical models. Groups of researchers, sometimes spanning centuries contribute their little fragments of data or understanding and eventually a general theory emerges. In many cases, the consequences of the new theory are unexpected by the original contributors, but such is the trust given to mathematics, the unexpected, nonintuitive results are accepted given they are mathematically sound. Examples:

- In the 16th century, Tycho Brahe organized and extended the astronomical observations of Copernicus and others into the world's finest set of data on stellar and planetary objects. Johann Kepler took this data and formulated his famous three laws of planetary motion. Despite his disappointment that planetary orbits were elliptical – rather than the circles the Greeks maintained were necessary for “celestial perfection” – he convinced himself and the scientific world that the ellipse was the correct mathematical form for all the orbits in Tycho's data base.
- Decades later, Isaac Newton, with his greater mathematical understanding, was able to generalize Kepler's laws into his law of universal gravitation – a gigantic intellectual feat which unified the laws of the heavens and earth.
- Centuries later, Albert Einstein provided a refinement of Newton's theory of universal gravitation with his general theory of relativity. Alexander Friedmann solved Einstein's equations and concluded that the universe began in a monstrous big bang. This was so against Einstein's instincts that he added a cosmological constant to his equations of relativity to remove the possibility of an expanding universe or the big bang. However, the rationality of mathematics, as well as new data by Hubble and others have established Friedmann correct and Einstein has referred to the cosmological constant as his greatest blunder.

So in Man's quest to understand, mathematical modeling has taken an increasingly central role in building theories, and indeed in the scientific method itself. The jagged shards of data, incomplete observations and subdimensional theories are pieced together rationally – often resulting in unexpected conclusions and a deeper view of the world. With the advent of modern computer technology, this central importance promises to increase far more.

“You certainly won't get arguments from most practitioners of science and technology about the importance of computers,” remarked the manager,

attempting to be agreeable. “What you have said would be obvious to most observers.”

“What is not obvious is that there are many barriers to the future efficient use of computers in the modeling of complex system,” rebutted the analyst.

“I knew you’d say that,” said the manager, remembering the example of Chapter 1. “What are these barriers?”

“First of all,” began the analyst, “with all the increased capability and flexibility that the digital computer offers over the analog, there comes a subtle but pervasive disadvantage: the model and the computational requests placed on it are inextricably intertwined. In almost all cases, the model is programmed to execute a specific computational flow, and when asked to alter the computation or switch input and output variables on the same model, the programmers tend to tell the managers, “can’t be done” or “too much trouble, or “can’t you make do with all that I’ve given you?”

“Amen,” agreed the manager, “I’ve been told that many a time. The programmers love to overwhelm you with data to show off their powerful computation. Their love of being responsive to your deep needs to understand what the model is teaching us is unfortunately much less.”

Second, until early this century, the general concept of a relation has been quite fuzzy and philosophical. Then in 1913, Norbert Wiener [6], before he became the father of cybernetics, suggested that the definition of a relation be imbedded within set theory – one of the foundations of all mathematics. This served to add needed clarity and rigor to the concept of “relation.”

Third, there was a general expectation that once a model was developed, there were no limitations on what computations could be asked of it. Which questions are “well posed” and which are not? In 1942, Claude Shannon [7], before he became the father of information theory, studied these issues on the recently developed mechanical differential analyzer – the most powerful computer of its time, analog or digital. He discovered that some of the variables desired to be dependant, or output variables – based on the rotation of a shaft assigned to that variable – were “free running”, providing no useful results. In other cases the entire network of rotating shafts, gear trains and integrators would just “lock up” – again providing no useful results. These instances of “free running” and “lockup” are directly related to the concepts of under constraint and over constraint, which we will discuss later.

Fourth, as was mentioned in Chapter 1, there is a vast dimensionality gap between the cognitive capability of man and machine. Our challenge is to make the best partnerships between these cognitive entities. As George Gamow [8] related in his charming book, “One, Two, Three, Infinity,” it was possible to survive with very limited numerical perceptions during our primitive beginnings, but the advent of mathematics, starting with

arithmetic, enormously enriched our lives and ability to understand and control the world.

“I can see the issues are not as new as I thought,” admitted the manager.

“Speaking of all the fathers, you mentioned Gamow – wasn’t he also a student of Alexander Friedmann, the father of the big bang theory as well as the father of George Friedman, this book’s author?”

“Almost correct! You continue to amaze me, I should develop more respect for you,” beamed the analyst. “Yes, George Friedman’s father was Alexander Friedmann, but he was Friedmann the tailor, not Friedmann the cosmologist. But let’s proceed to some substance.”

“OK,” challenged the manager. “I’m ready to enter the mathematical world you tell me that is necessary to bring order to this confusion and ambiguity. Let’s see if the work will prove to be a worthwhile expenditure of intellectual energy.”

“Fair enough,” agreed the analyst. “In the remainder of this chapter, I want to introduce you to the very simplest foundations of set theory and graph theory, which will define for us with rigor and clarity the formerly vague concepts of relation, well-posed, consistent, allowable computation, overconstraint and underconstraint. I believe it will be worth your effort.”

We will begin our exploration of the foundations of constraint theory by presenting four interrelated views of the mathematical model: set theoretic, families of submodels, bipartite graph, and constraint matrix. The first and second are complete and contain all the model’s detail. The third and fourth are metamodels and contain only those abstractions which illuminate the model’s structure as it relates to consistency and computability.

2.2 THE FIRST VIEW: SET THEORETIC

Definition 1: A *set* is a collection of elements. A *subset* is a portion of this collection. The number of elements may be finite, such as the planets of the solar system, or infinite, such as the points on a line. A set with no elements at all is the null set. (Figure 2-1)

Definition 2: A *variable* is an abstraction of one of the model’s characteristics which the analyst considers essential. Associated with each variable is an *allowable set of values*. (Figure 2-2)

The set of variables which define the model can have enormous flexibility. The variables can be continuous and quantitative, such as force, length, or temperature; they can be discrete, such as the variables in Boolean Algebra, or the solutions of Diophantine equations; they can be qualitative, such as hot, rich, salty or sick; or combinations of these.

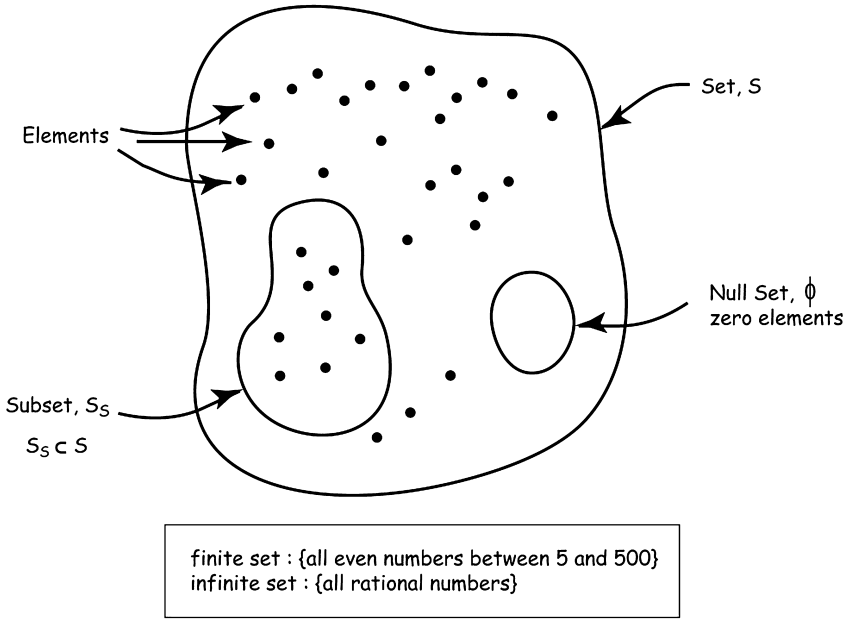


Figure 2-1. It all begins with the simple concept of sets, subsets, and the null set.

VARIABLE TYPE	ALLOWABLE VALUES
Money Length, Mass... Gear Ratio Probability Binary Boolean Ternary Boolean Permutations Taste H ₂ O State Binary Relations	Any real value Positive Real Numbers Rational Numbers $0 < \text{real number} < 1$ 0 or 1 0, 1 or 2 Positive integers Sweet, Sour, Salty, Bitter Ice, Water, Steam Larger than, Older than...

Figure 2-2. The sets of variables and their allowable values have enormous flexibility.

Definition 3: The *model hyperspace* is that multidimensional coordinate system formed by all the variables as axes, each of which is orthogonal to all the others. (Figure 2-3) This is simply a generalization of Descartes’

unification of geometry and algebra. We will frequently refer to these axes as Cartesian coordinates.

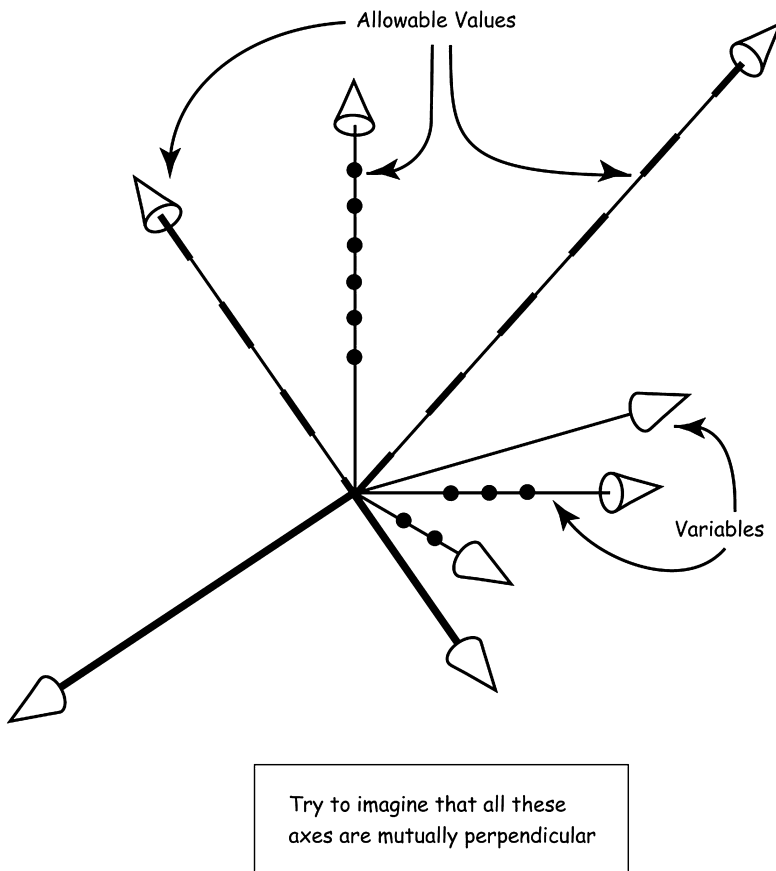


Figure 2-3. The Model Hyperspace formed by the orthogonal axes of the variables is a useful abstraction, although in general it is impossible to be perceived.

Definition 4: The *product set* of a set of variables is the set containing all possible combinations of the allowable values of all the variables. In the case where all the variables are continuous over an infinite range, the product set is merely every point within the hyperspace defined by the set of variables. (Figure 2-4)

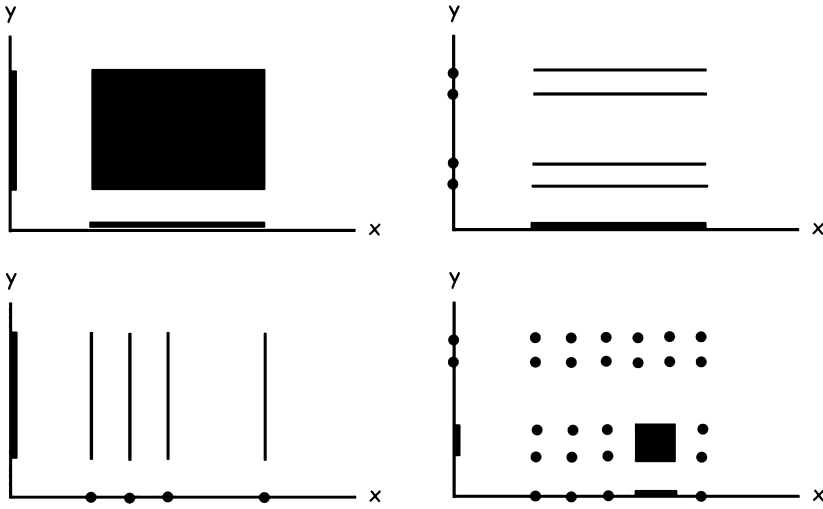


Figure 2-4. The Product Set contains all possible combinations of the allowable sets of the variables' values.

Definition 5: As suggested by Wiener and amplified by Bourbaki [9] and Ashby [10], a *relation* between a set of variables is defined as that subset within the product set of the variables which satisfies that relation. (Figure 2-5) This relation can be between any number of variables and is not restricted to the binary relations of “relation theory.”

The relations can also have enormous flexibility. They can be linear or nonlinear, differential equation, partial differential equations, integral-differential equations, logical equations, binary, ternary, etc., deterministic or probabilistic, inequality relations, or any combination of these. In many cases the relations can be represented by data or “truth tables.”

Definition 6: Since a relation reduces the size of the original product set to a smaller, relation set, the relation can be said to *constrain* or apply a *constraint* to the original product set. (Figure 2-6)

Now that we have embedded the concept of mathematical models within set theory, we will need these four set theoretic operations for further developments: (see definitions 7 and 8; Figure 2-7)

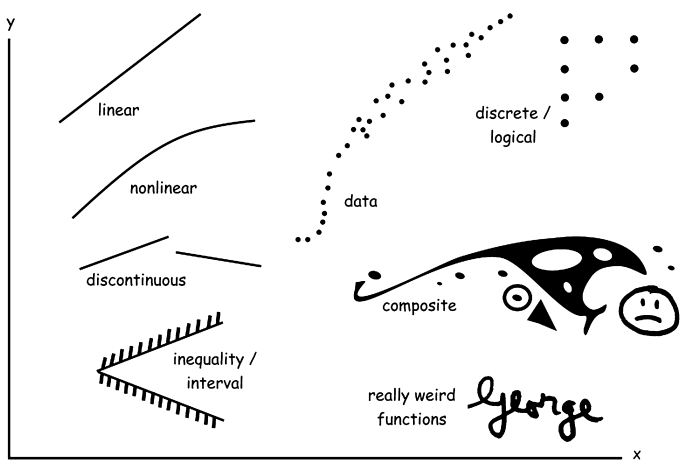


Figure 2-5. Wiener suggested that a “relation” between variables can be defined as the subset within the product set of these variables which satisfies it. This not only provides rigor, but permits a tremendous variety of relation types.

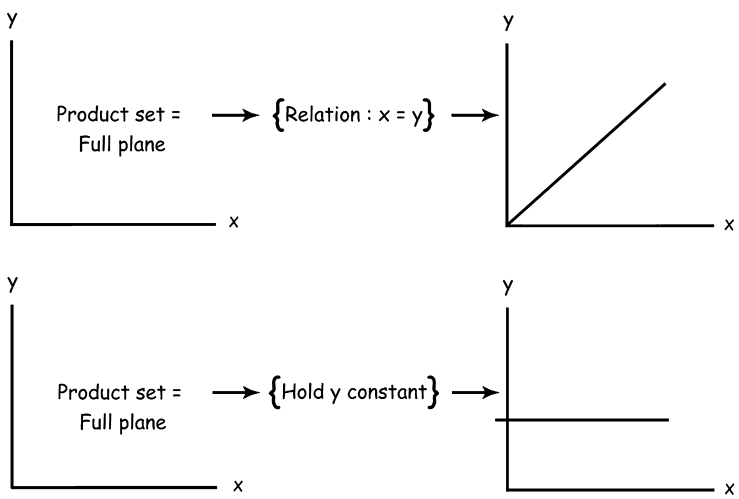
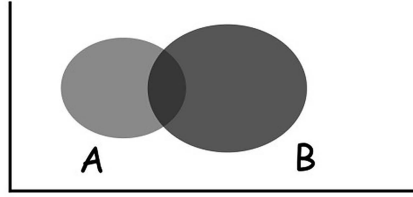


Figure 2-6. Relations, as well as variables held constant, constrain the product set into a much smaller subset.

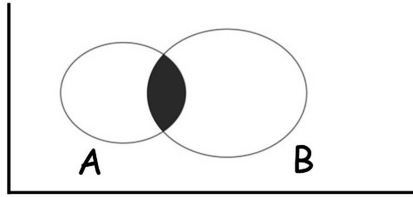
Union:

$\{A\} \cup \{B\}$ = all elements in A or B or both



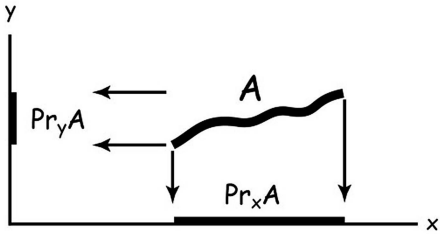
Intersection:

$\{A\} \cap \{B\}$ = all elements that are in both A and B



Projection:

$Pr_y A$ = the shadow of A onto y
 $Pr_x A$ = the shadow of A onto x



Extension:

$Ex A$ = extends each point in A to all points in the new direction

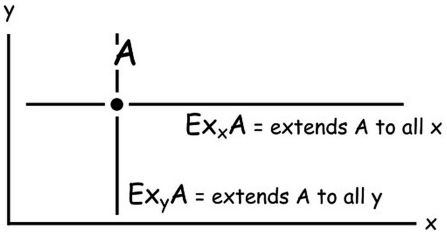


Figure 2-7. Only four operations from set theory are employed for Constraint Theory.

Definitions 7: The *union* of sets A and B is the set of all points which are either in set A or in set B or both. Symbolically:

$$x \subset A \cup B \text{ if: } x \subset A \text{ or } x \subset B$$

The *intersection* of sets A and B is the set of all points which are in both A and B . Symbolically:

$$\mathbf{x} \subset A \cap B \text{ if: } \mathbf{x} \subset A \text{ and } \mathbf{x} \subset B$$

Definitions 8: The *projection* of set A onto dimension x is the set of points within set A with all coordinates except x suppressed. For example, if set A is the point $(2,4)$ in the xy plane, then $\text{Pr}_x(2,4) = 2$ on the x axis only. Projection is a dimension reducing operation. Symbolically:

$$\text{If } A = (2,4) \text{ in } xy\text{-space} \quad (\text{point})$$

$$\text{Pr}_x A = (x=2) \text{ in } x\text{-space} \quad (\text{point})$$

$$\text{Pr}_y A = (y=4) \text{ in } y\text{-space} \quad (\text{point})$$

The *extension* of set A into dimension y is the set of all points within set A plus all possible values of the dimension y . For example, if set A is the point $(2,4)$ in the xy plane, then $\text{Ex}_y(2,4)$ is the line $x=2$ where y varies over all its possible values. Extension is a dimension increasing operation. Symbolically:

$$\text{Ex}_y A = (x=2) \text{ in } xy\text{-space} \quad (\text{line})$$

$$\text{Ex}_z A = (x=2) \cap (y=4) \text{ in } xyz\text{-space} \quad (\text{line})$$

Definition 9: y is a *relevant* variable with respect to relation ϕ in xyz space means that there exist lines in xyz space parallel to the y axis that are neither entirely within nor entirely outside of the relation set. Thus y has an effect on ϕ , or equivalently, the relation ϕ constrains y . Symbolically:

$$\text{If: } \text{Ex}_y(\text{Pr}_{xz} A_\phi) \neq A_\phi, \text{ then } y \text{ is relevant to } \phi$$

Similarly, y is *irrelevant* with respect to relation ϕ if:

$$\text{Ex}_y(\text{Pr}_{xz} A_\phi) = A_\phi \quad (\text{Figure 2-8})$$

2.3 THE SECOND VIEW: FAMILY OF SUBMODELS

The set theoretic definition of relation was chosen to provide the firmest and broadest mathematical foundation for the work to follow. Unfortunately, it cannot also be claimed that this viewpoint is a practical way to *describe* the relation. There are some occasions, such as tabulated or plotted functions, when it is necessary to list every point within the relation subset exhaustively. In these cases, the relation subset is merely the union of all the listed points within the hyperspace of the model. However, in the vast majority of mathematical models, far more efficient means are used to define the usually infinite number of points comprising the relation subset.

These efficient means almost invariably involve the concept of describing the total model as the intersection or union (or both) of a set of submodels or algorithms. (Figure 2-9) This is necessary for at least two reasons: First, and more obvious, a practical way of specifying infinite sets is required. Second, and deeper, model builders cannot conceive of the entire model with their limited perceptual dimensionality and thus attempt to construct higher dimensional models by aggregating in some fashion a series of lower dimensional submodels. The rules of aggregation employ the union, intersection, projection and extension operators defined previously.

Frequently, a function is specified in a piecewise fashion; for example:

$$\begin{aligned}x &= 0 \text{ when } t < 0 \\x &= t^2 \text{ when } t > 0\end{aligned}$$

In cases like this, the meaning is that the contribution of these two sets to the total model is the *union* of the sets.

More frequently, a collection of “simultaneous equations” attempt to define the model; for example:

$$\begin{aligned}x + y + z &= 13 \\x - y &= 8\end{aligned}$$

In cases like this, the meaning is that the contribution of these two sets to the total model is the *intersection* of the sets.

In general, the dimensionality of the total model is far greater than any of the contributing submodels. Thus, the contributing submodels specify only a subset of the total model and, in order for them to be able to intersect in the total dimensional space, they must be *extended* into all the unspecified directions. For example, let the total model space be xyz and let the relation subset for $f_1(x,y)=0$ be A_1 and the relation subset for $f_2(x,z)=0$ be A_2 . Thus,

before these two relations intersect, A_1 must be extended in the missing z direction, and A_2 must be extended in the missing y direction. Defining A_Σ as the total model relation, then:

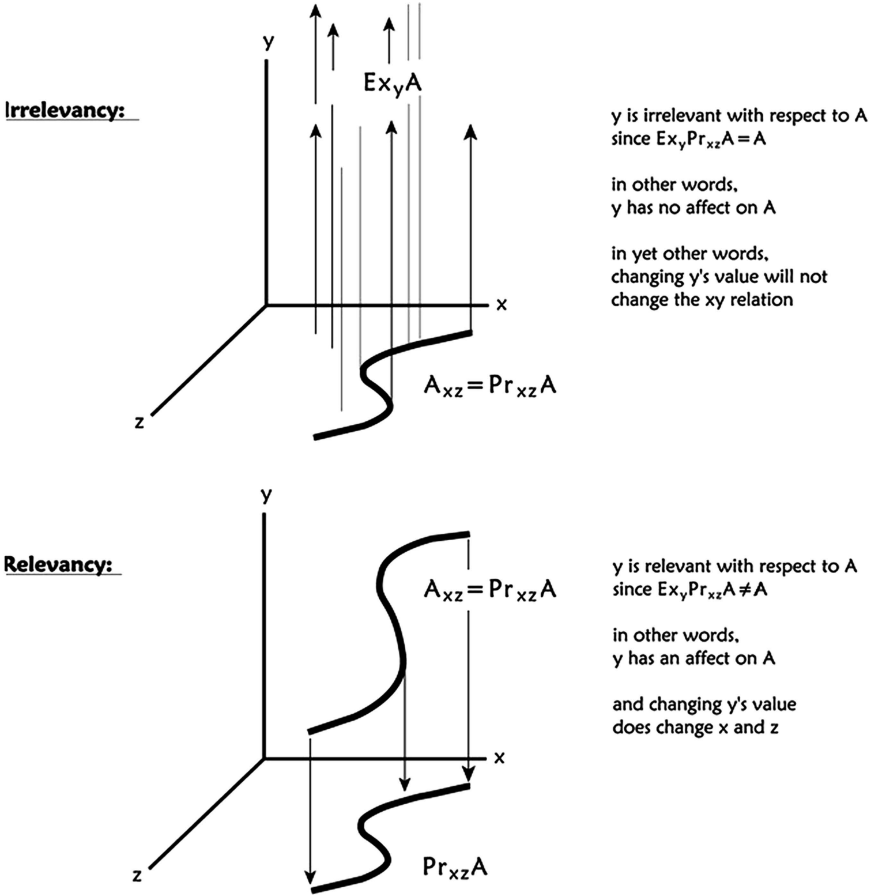


Figure 2-8. Relevancy of a variable with respect to a relation can be defined in terms of the projection and extension operations.

$$A_\Sigma = Ex_z(A_1) \cap Ex_y(A_2)$$

Now, once the model is constructed in the above fashion, an analyst wishes to have a subdimensional “view” – or computational request – of this multidimensional relation. In order for him to view the relation – as A_V – with respect to the xy plane, he must ask for a projection of A_Σ onto the xy plane. (Figure 2-10). Symbolically:

$$A_V = Pr_{xy}(A_\Sigma)$$

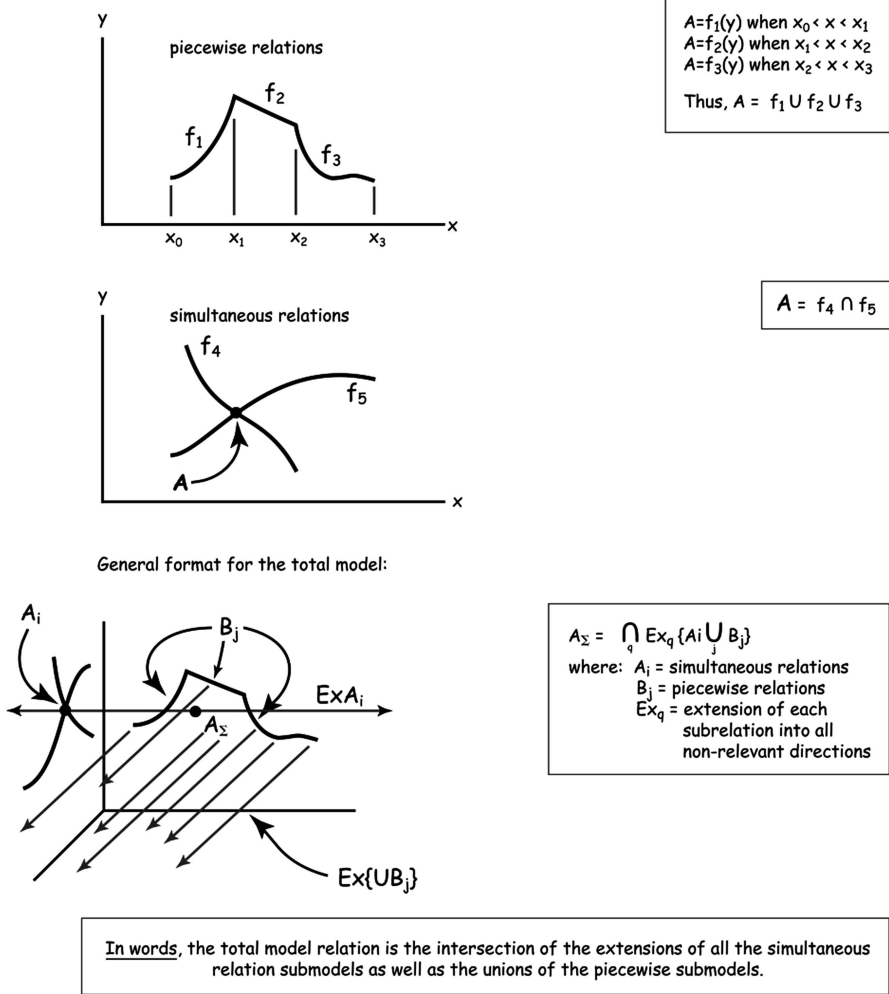


Figure 2-9. Total model relations are generated by families of submodels which are combined by the union, intersection, extension and projection operations.

If the analyst wishes to impose additional restrictions on his view, or computation, prior to the projection, he may hold any number of variables at a constant value. In these cases, the relations corresponding to these variables held constant intersect the total model relation prior to the application of the projection operation.

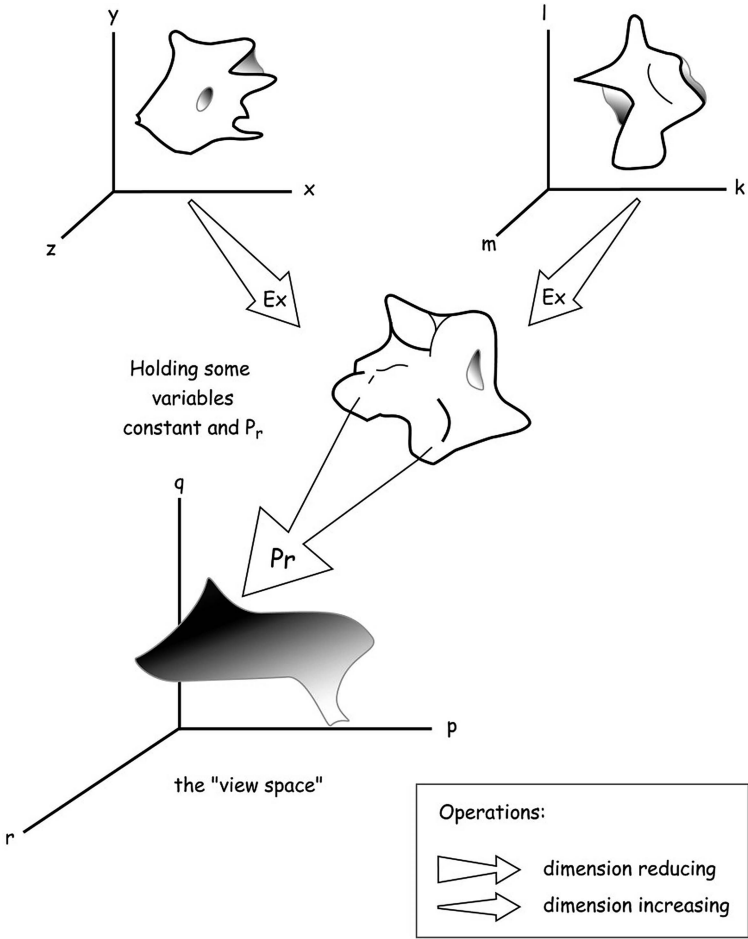


Figure 2-10. After each generating relation is developed, it is *extended* into the full hyperdimensional space of the total model, forming the relation A_Σ , which in turn is *projected* onto the subspace of the computational request, where it can be “viewed” by managers, analysts and others interested in learning from the model.

2.4 THE THIRD VIEW: THE BIPARTITE GRAPH

Although there exist strong implications of topological structure in mathematical models and their computations, neither of the two views described above provides topological insight. In order to provide this additional insight – as well as allow a right-brain perspective to aid the dominantly left-brain views already presented – graph theory will be applied.

Definitions 10: A *graph* is a topological network of points, called *junctions*, or *vertexes*, and lines connecting some of them, called *arcs*, or *edges*. A *bipartite graph* is a special graph having two disjoint sets of vertexes, $\{K\}$ and $\{N\}$, such that any edge is only allowed to connect a vertex in $\{K\}$ to a vertex in $\{N\}$.

Definitions 11: A *model graph*, is a bipartite graph with one set of vertexes, called *nodes*, corresponding to the model's relations and the other set of vertexes, called *knots*, corresponding to the model's variables. A knot will be connected by an edge to a node only if the corresponding variable is relevant to the corresponding relation. As an additional visual aid, nodes will be shown as squares and knots will be shown as circles. (Figure 2-11)

A model graph can be thought of merely as the circuit diagram of a computer hookup of the mathematical model: the nodes are function generators and the knots are wired connections that permit the values of the variables to pass from one function generator to another. Thus, when the edges indicate no direction, the bipartite graph represents a model. When the edges indicate specific directions, then the bipartite graph represents a computation on that model, tracking the flow of computation or constraint across the topological structure.

2.5 THE FOURTH VIEW: THE CONSTRAINT MATRIX

The fourth and final viewpoint of the mathematical model is introduced primarily to provide a format amenable to computer processing. As will be seen later, however, it also furnishes yet another mathematical perspective from which the proof of certain theorems can most easily be made.

Definitions 12: A *constraint matrix* is a rectangular array of elements that presents exactly the information inherent in a bipartite model graph, but is a form that can be easily stored and operated upon by a computer. The columns correspond to variables and the rows correspond to relations. An element in the i th column and the σ th row will be filled if the variable i is relevant to the relation σ , and empty if it is not. (Figure 2.12) Compactly stated, the rows, columns and elements of the constraint matrix are homomorphic to the nodes, knots and edges of the bipartite graph. In order to indicate the direction of computational flow, the elements of the constraint matrix can take on the values: +1 or -1.

To further emphasize the essential similarity between the bipartite graph and the constraint matrix, Figure 2-13 shows a logical evolutionary transition between the two representations. As was stated earlier, both the bipartite graph and constraint matrix are "metamodels" and do not contain the full model information inherent in the set theoretic and family of submodels

versions. Rather, they emphasize the structure and topology important to model consistency and computational allowability.

2.6 MODEL CONSISTENCY AND COMPUTATIONAL ALLOWABILITY

We are now prepared to present rigorous definitions in the area of the “well posed problem.” Saying a problem is well posed means that the mathematical model is consistent and the computation is allowable.

Definition 13: A mathematical model is *consistent* means that its multidimensional relation set contains at least one point. Symbolically,

$$A_{\Sigma} \neq \text{the null set}$$

Definition 14: A computational request made on a model is *allowable* means that the projection of A_{Σ} onto the view space of the computation contains at least one point and in addition, each variable involved in the computation must be relevant to this projection in the sense of definition 9.

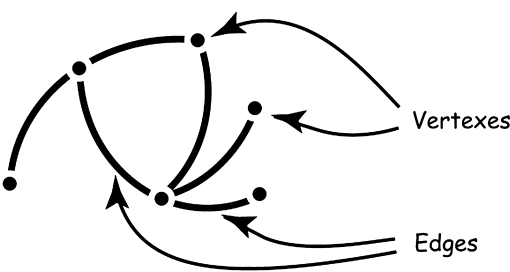
Thus, if the projection onto the desired subspace that the analyst wants to view has been nulled out to no points at all, then the computation, the application of variables held constant or even the total model relation has been *overconstrained*. On the other hand, if the projection has variables that are not relevant, these variables take on all their possible values, and are therefore *underconstrained*. (Figure 2-14)

2.7 THE MANAGER AND ANALYST CONTINUE THEIR DIALOGUE

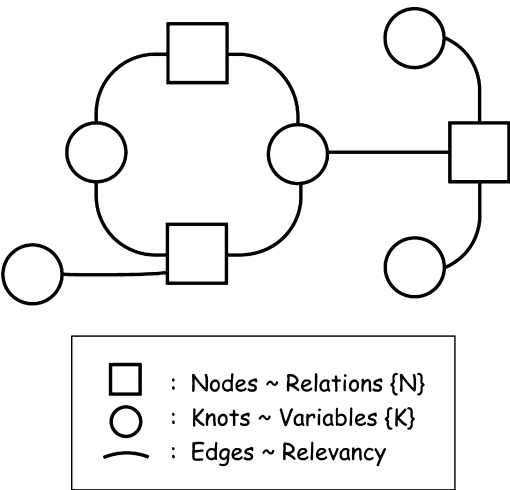
“You started off simply enough,” commented the manager. “What can be easier than the definition of sets and their operations? Also, your extension of Cartesian coordinates into hyperspace can be grasped by extrapolating from what we know of one, two and three dimensional spaces.

As a teenager, I was inspired by Abbot’s *Flatland* [11] and Burger’s *Sphereland* [12]. These extensions are interesting philosophically, but do they really represent the real world and should they be the basis for applied mathematics? I’ve heard professors argue that Descartes himself only was thinking of our familiar three dimensional space, not even the four dimensions for relativity theory, the eleven dimensions for string theory, and certainly not the hundreds of dimensions we need for a modern mathematical model.”

Graph:



**Undirected
Bipartite Graph:**
models topological
structure



**Directed
Bipartite Graph:**
describes possible
computational flow

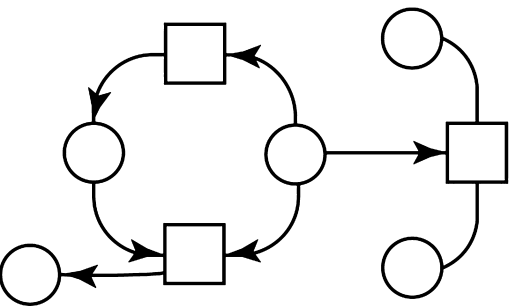
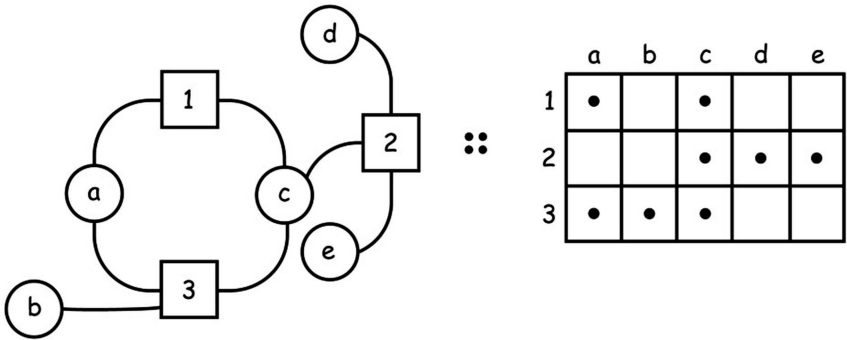


Figure 2-11. The bipartite graph is a metamodel of the full model which illuminates the model's structural and computational properties.

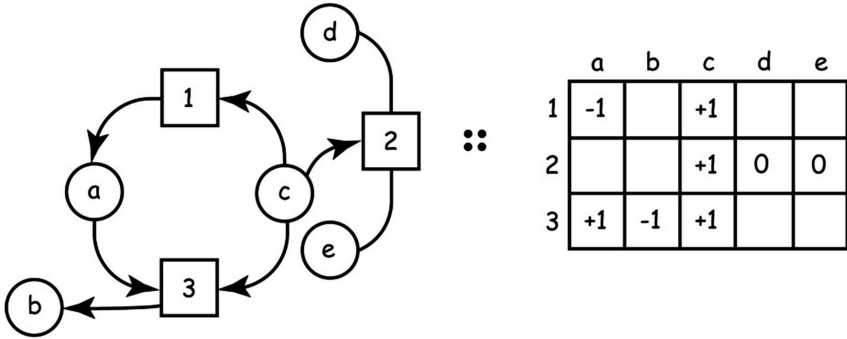
Bipartite Graph {node, knot, edge} :: Constraint Matrix {row, column, element}

The Constraint Matrix displays structure



• denotes that the variable in the column is relevant to the relation in the row

The Constraint Matrix displays computation (or constraint flow)



+1 denotes constraint flow from knot to node
-1 denotes constraint flow from node to knot
0 denotes no constraint flow from, although relevancy exists

Figure 2-12. The Constraint Matrix is the companion to the bipartite graph and displays exactly the same information. It is also a metamodel which contains only that information relating to the model's structural and computational properties.

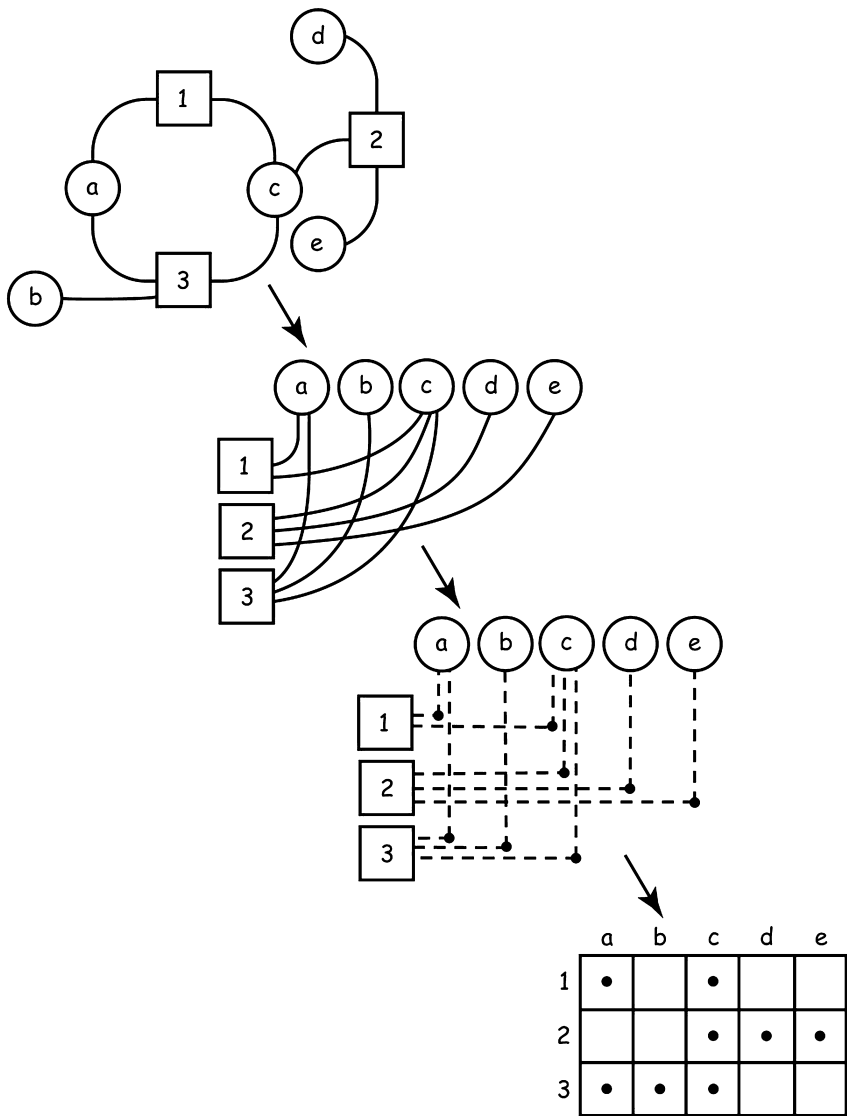


Figure 2-13. Evolutionary transition from the bipartite graph to the constraint matrix.

“What do we really know about ‘reality,’” asked the analyst, launching into a minor tirade. “Irrational numbers were once thought to be irrational, and still bear the label. Negative numbers were once thought to be imaginary, but now play an essential role in every walk of science and finance. Zero was originally thought to be completely unworthy of serious consideration, but we would be crippled if we stuck to the awkward Roman

numerals. Imaginary numbers were originally considered as only interesting abstractions, but today form the basis of several very practical integral transforms and are essential to almost every walk of electrical engineering.

Then there are loops (which we will consider in Chapter Four). Early in control theory, feedback loops were considered impossible, in logic, self-referential statements were considered illogical, and in decision theory, intransitive preferences are still considered irrational."

"So, in light of the above, we face this philosophical question: 'Does hyperdimensional space have to correspond to the space-time continuum of our universe in order to be useful for the understanding of mathematical models?' I claim the answer is 'no' and most mathematicians use as many dimensions as they need. Descartes' crucial intellectual leap was to enrich the algebraic relations with geometric concepts; the extrapolation to any number of dimensions should be trusted as a straightforward extension."

"OK," agreed the manager, feeling a little over-answered. "Your use of the projection and extension operators was less familiar to me and I never really thought how families of submodels contributed to the total model. The concept that a computational request is really a projection of the total model onto a subspace was really beyond my experience. But now I can see the value of this construct. The projection operator provides the dimensionally-limited human an understandable perspective of an inconceivable multi-dimensional relation."

"Or to further extend the lingo that managers like to use," added the analyst, "if you choose the right subdimensional viewspace, you get the 'best angle' on a complex problem – something you guys are always trying to do."

"Figure 2-14 is a good summary of most of the previous ideas. Referring to the ancient stories at the beginning of this chapter, the blind men each observing the elephant from different aspects form pitifully incomplete shards of truth, generating relations which are combined into the 'total truth.' However, since we are hopelessly subdimensional, we cannot perceive or understand the total truth, *A*. Instead, we turn it every which way and attempt to observe it from many different angles, some of which may help us to understand 'more deeply.' Disappointingly, for most of the directions we attempt to look at the relation, we will get no more information. That is the agony of asking questions which are not well-posed."

"*Bottom line*, to use more management jargon," summarized the analyst, "these four views were deemed necessary by the author to understand the underlying foundations of models, computations and well-posedness, rather than rely on the rather opaque, algorithmic crankturning he had been taught in all his courses in mathematics."

Generating Relations

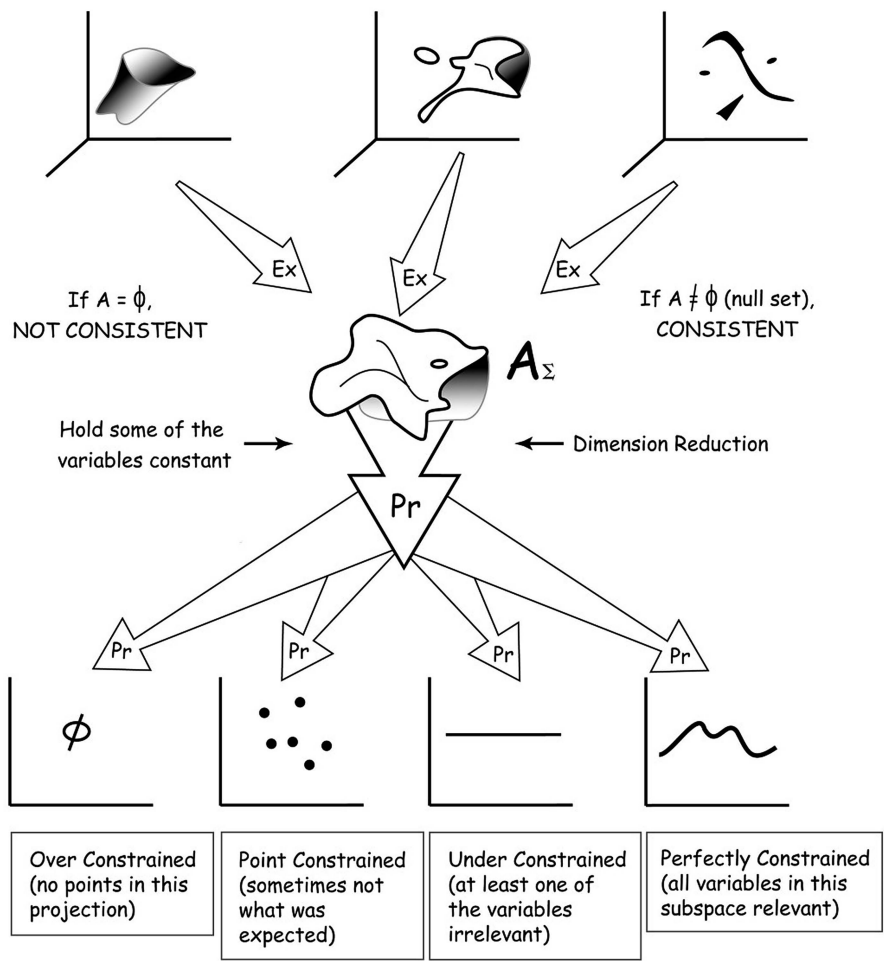


Figure 2-14. In order for a problem to be well-posed, the mathematical model must be *consistent* and the computation must be *allowable*. *Consistency* requires that the hyper-dimensional relation not be the null set. *Allowability* requires that all the variables of the requested computation be relevant to the projection of the total relation onto the computational subspace.

2.8 CHAPTER SUMMARY

- Four interrelated views of a mathematical model and its computations have been presented: set theoretic, family of submodels, bipartite graph and constraint matrix. (Figure 2-15). The first two are full models, containing all the detail necessary for final construction and computation; the latter two are metamodels, and are abstractions of the first two which concentrate on the topological and computational features. In a strong sense, the metamodels can be considered to provide an overarching management perspective on consistency and computability issues. Without this perspective, those who attempt to build models and make computations on them will blunder into difficulties due to either inconsistencies in the model or unallowabilities in the computations – in short, the traditional well posed problems. In any case, once the “well-posedness” of the models and computations have been analyzed and managed by the metamodels, the full models must then be employed for the actual computations.
- The concept of “set” has been used as frequently in this chapter as the word “system” has been used in a book on systems engineering. This was done deliberately because – despite the apparent simplicity of the concept – it is far more precise a concept than “system” and its applicability is wide ranging.
- The *set* was used to define a *relation* between variables. The concept of *set* was also used to identify:
 - the allowable values of a variable
 - the possible values of a product set
 - collections of variables – which can represent computational requests
 - collections of relations – which can represent submodels
 - subsets of bipartite graph vertexes called knots
 - subsets of bipartite graph vertexes called nodes
 - collection of edges connecting subsets of the knots and nodes
 - constraint matrix columns; homomorphic to knots and variables
 - constraint matrix rows; homomorphic to nodes and relations
 - constraint matrix elements; homomorphic to edges and relevancies


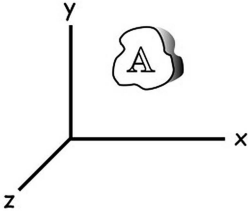
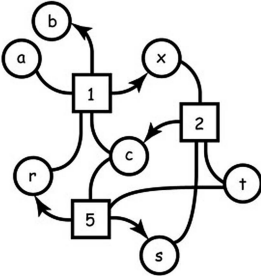
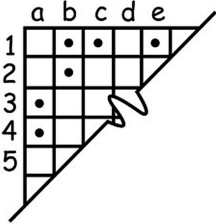
REPRESENTATION	APPEARANCE	PRIMARY PURPOSE
<div> IN K-SPACE</div>		Most fundamental definition for "Relation"
<div>FAMILY OF SUBMODELS</div>	<div>1 : $m > n + k$ 2 : $x = z + 3p$ 3 : $r + \sum a_i y^k$ 4 : $t = \sin^4 s$ 5 : $b = c \int_d^f g(h) dh$ 6 : $v = \bar{w}r + w\bar{r}$</div>	Most practical format for describing models and algorithms
<div>BIPARTITE GRAPH</div>		Best viewpoint for local constraint and computational flow
<div>CONSTRAINT MATRIX</div>		Best format for automatic processing

Figure 2-15. The four representations of a mathematical model. The first two are full models and the latter two are metamodels.

- power set of the knots: all definable computational requests
- power set of the nodes: all possible submodels
- Perhaps Friedman's greatest contribution was the recognition that very useful metamodels of the mathematical model's variables, relations and relevancies are the bipartite graph's knots, nodes and edges and the companion matrix's rows, columns and elements.
- This chapter provided the foundation for a building; its construction and use will continue in subsequent chapters.

2.9 PROBLEMS FOR THE INTERESTED STUDENT

1. Provide a real-world example of a three-dimensional product set where one dimension is continuous, another is discrete and a third is defined in intervals.
2. Employing detailed algebraic equations in three-dimensional space, show an example of y being relevant to the model and another example of y being irrelevant.
3. For a three-dimensional model, show how \cap , \cup , Pr and Ex can be used to combine a family of three algebraic equations into the total model.
4. Draw the constraint matrix for Figures 1, 5, 6 and 7 of Chapter One.
5. Draw the constraint matrix for Figure 3 of Chapter One. Can you suggest how the term, "Basic Nodal Square" was developed?
6. Regarding the mathematical model depicted in Figure 1-3, which of the following computational requests are allowable and which are not allowable?

For the allowable requests, draw the directed bipartite graph which depicts the computational flow. For the unallowable requests, discuss the reason(s) for the unallowability.

Computational Requests:

$E=f(T,M)$, $A=f(T,E)$, $A=f(P,M)$

Constraint Theory

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