# Preface

The non-commutative geometry program for aperiodic condensed matter systems in the presence of magnetic fields was initiated by Jean Bellissard in the mid-1970s. Its foundations stand on a natural generalization of the Brillouin torus in terms of a suitable $C^*$-algebra and a set of non-commutative rules of calculus [J. Bellissard, *K-theory of $C^*$-algebras in solid state physics*, Lecture Notes in Physics **257**, 99–156, (1986)]. Together, they form a non-commutative differential manifold, dubbed the non-commutative Brillouin torus (see Chap. 3), on which one can write closed-form expressions for the intrinsic thermodynamic coefficients of condensed matter systems and for their response functions to external fields. Examples are the non-commutative formulas for the integrated density of states, linear and nonlinear transport coefficients, polarization, and magnetization. These and other formulas will be discussed in Sect. 3.8. Furthermore, extremely difficult issues in condensed matter physics can be naturally formalized and attacked with mathematical rigor. One such example is the question of existence and characterization of the thermodynamic limit for the intensive thermodynamic variables and response functions of homogeneous aperiodic condensed matter systems, which is discussed in Chap. 1. Another example is understanding the quantization and stability against various perturbations of certain response functions, such as the Hall conductance, in the regime of strong disorder. The pairing of Bellissard's program in condensed matter and Alain Conne's program in non-commutative geometry produced an extremely elegant and natural explanation of the integer quantum Hall effect [J. Bellissard et al., *The non-commutative geometry of the quantum Hall effect*, J. Math. Phys. **35**, 5373–5451 (1994)], which came in the form of an index theorem that is stable even in conditions where the Fermi energy is located in the essential, though Anderson localized, spectrum. This result is regarded as one of the most important applications of non-commutative geometry in physics. Using precisely the same methods, this index theorem has been recently generalized to higher dimensions and to condensed matter systems with chiral symmetry. These aspects are discussed in Chap. 2. Worth mentioning are also the magnetic derivations introduced by Bellissard and discussed in Sect. 3.7, essential for the derivation of closed-form expressions for the magnetoelectric transport coefficients.

Since the works of Mathew Hastings and Terry Loring around 2009, it became gradually clear that the framework of operator algebras and non-commutative geometry is quite natural for developing computer-assisted calculations for aperiodic solids. Our first calculation of this type was reported around the same time and contained the first numerical evaluation of the non-commutative formula for the Hall conductance. An extended discussion of these and other calculations, together with the empirical principles behind them, can be found in the review article *Disordered topological insulators: A non-commutative geometry perspective*, J. Phys. A: Math. Theor. 44, 113001 (2011). Somewhat ironically, the non-commutative Brillouin torus is defined in the strict thermodynamic limit and there is no exact analog of it at finite volumes. For this reason, the non-commutative formulas become ill-defined at finite volumes. However, in a 2013 study, we realized that the non-commutative Brillouin torus accepts a canonical finite-volume approximation [E. Prodan, *Quantum transport in disordered systems under magnetic fields: A study based on operator algebras*, Appl. Math. Res. Express **2013**, 176–255 (2013)]. This observation enabled us not only to better understand our previous calculations (see Chap. 5) but also to improve on them to a point where the quantization of various topological invariants was reproduced with machine precision, in the regime of strong disorder where the random fluctuations in Hamiltonian coefficients far exceed their average (see Table 9.1). Furthermore, it was found that the setting of $C^*$-algebras, which is at the core of the non-commutative torus, provides just the right framework for the analysis of the numerical errors. These aspects will be revisited in Chaps. 6 and 8. The resulting numerical algorithms have been applied to a whole range of problems, spanning from the critical behavior of the transport coefficients at the plateau–insulator transition of the integer quantum Hall effect to computations of phase diagrams for various disordered models of topological insulators. These applications will be revisited in Chaps. 7 and 9.

The present notes build on almost a decade of practice in the field of disordered topological insulators. At this point, we are definitely at a crossroad, trying to look beyond the class of disordered crystals. In many respects, the disordered crystals are the simplest and best understood among the homogeneous condensed matter systems and we are quite eager to see whether the amorphous, quasicrystalline, fractal and network-like condensed matter systems can be analyzed by similar techniques. We felt, however, that, in order to move forward, we need to understand the fundamental principles that make the current algorithms work so well. This is precisely the main scope of the present notes, which are intended to serve the following purposes:

1. We tried to fit the finite-volume approximations in a more general context, which should be quite familiar to the practitioners of operator algebras. For example, we show that the process of taking the thermodynamic limit of these approximations fits into a projective tower of algebras (see Chap. 5). It is our hope that this observation can serve as a guiding principle for a generic computational non-commutative program with a much broader range of

applicability. Some of the algebras which we think can be managed this way are listed at the beginning of Sect. 3.4.

2. The analysis of the numerical errors is greatly improved compared to our 2013 study (see Chap. 6). This enabled us to treat arbitrary Hamiltonians with rapidly decaying coefficients while keeping the complexity of the calculations within quite reasonable bounds. The improvements stem from a major shift toward purely algebraic arguments, notably, the use of the smooth algebra introduced in Sect. 3.5 and the exploitation of its invariance w.r.t. the smooth functional calculus. We have also corrected (see Sect. 4.4) a shortcoming of the 2013 study, where the link between the algebra of physical observables and its finite-volume approximations was given by an approximate homomorphism.
3. The present analysis covers an important aspect that was entirely omitted in the 2013 study, namely, the error estimates for correlations involving functional calculus with piecewise smooth function (see Chap. 8). Here, we make use of the Sobolev spaces and their associated Frechét algebra introduced in Sect. 3.6. The latter is shown to be invariant against the functional calculus with piecewise smooth functions, as long as the singularities occur in the Anderson localized spectrum. This together with the Aizenman-Molchanov bound enabled us to confirm that the canonical finite-volume algorithms continue to display the same rapid convergence to the thermodynamic limit.
4. As we already mentioned, the notes also showcase several numerical applications and numerical tests of the algorithms, which are mostly reproduced from the published literature. They are still unique in many respects. For example, the critical regime at the plateau–insulator transition in the integer quantum Hall effect and Chern insulators was resolved computationally for the first time using these algorithms (see Chap. 7). Also, to our knowledge, these algorithms are still the only existing ones that can reproduce the quantization of the topological invariants with machine precision in the regime of strong disorder (see Chap. 9).

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