Chapter 1
Introduction

1.1 What Do We Mean by (Mathematical) Modeling?

With *(mathematical) modeling* we denote the translation of a specific problem from the natural sciences (experimental physics, chemistry, biology, geosciences) or the social sciences, or from technology, into a well-defined mathematical problem. The mathematical problem may range in complexity from a single equation to a system of several equations, to an ordinary or partial differential equation or a system of such equations, to an optimization problem, where the state is described by one of the aforementioned equations. In more complicated cases we can also have a combination of the problems mentioned. A mathematical problem is *well-posed*, if it has a unique solution and if the solution of the problem depends continuously on its data, where continuity has to be measured in such a way that the results are meaningful for the application problem in mind. In general the phenomena to be described are very complex and it is not possible or sensible to take all its aspects into account in the process of modeling, because for example

- not all the necessary data are known,
- the model thus achieved cannot be solved anymore, meaning that its (numerical) solution is expensive and time consuming, or it is not possible to show the well-posedness of the model.

Therefore nearly every model is based on *simplifications* and *modeling assumptions*. Typically the *influence of unknown data* are neglected, or only taken into account in an approximative fashion. Usually *complex effects* with only *minor influences* on the solution are neglected or strongly simplified. For example if the task consists of the computation of the ballistic trajectory of a soccer ball then it is sensible to use classical Newtonian mechanics without taking into account relativity theory. In principle using the latter one would be more precise, but the difference in results for a typical velocity of a soccer ball is negligible. In particular this holds true if one takes into account that there are errors in the data, for example slight variations in the size, the weight, and the kickoff velocity of the soccer ball. Typically available data

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are measured and therefore afflicted with measurement errors. Furthermore in this example certainly the gravitational force of the Earth has to be taken into account, but its dependence on the flight altitude can be neglected. In a similar way the influence of the rotation of the Earth can be neglected. On the other hand the influence of air resistance cannot be neglected. The negligible effects are exactly those which make the model equations more complex and require additional data, but do not improve the accuracy of the results significantly.

In deriving a model one should make oneself clear what is the question to be answered and which effects are of importance and have to be taken into account in any case and which effects are possibly negligible. The aim of the modeling therefore plays a decisive role. For example the model assumptions mentioned above are sensible for the flight trajectory of a soccer ball, but certainly not for the flight trajectory of a rocket in an orbit around the Earth. Another aspect shows the following example from weather forecasting: An exact model to compute the future weather for the next seven days from the data of today cannot serve for the purposes of weather forecast if the numerical solution of this model would need nine days of computing time of the strongest available supercomputer. Therefore often a balance between the accuracy required for the predictions of a model and the costs to achieve a solution is necessary. The costs can be measured for example by the time which is necessary to achieve a solution of the model and for numerical solutions also by the necessary computer capacities. Thus at least in industrial applications costs often mean *financial* costs. Because of these reasons there can be no clear separation between *correct* or *false* models, a given model can be sensible for certain applications and aims but not sensible for others.

An important question in the construction of models is: Does the *mathematical structure* of a model change by neglecting certain terms? For example in the initial value problem

$$\varepsilon\, y'(x) + y(x) = 0\,, \quad y(0) = 1$$

with the small parameter ε one could think about omitting the term $\varepsilon y'$. However, this would lead to an obviously unsolvable algebraic system of equations

$$y(x) = 0\,, \quad y(0) = 1\,.$$

The term neglected is decisive for the mathematical structure of the problem independent of the smallness of parameter ε. Therefore sometimes terms which are identified as small, cannot be neglected. Hence, constructing a good mathematical model also means to take aspects of analysis (well-posedness) and numerics (costs) of the model into account.

The essential ingredients of a *mathematical model* are

- an *application problem* to be described,
- a number of *model assumptions*,

- a mathematical problem formulation, for example in the form of a mathematical *relation*, specifically an equation, an inequality, or differential equation, or several coupled relations, or an optimization problem.

The knowledge of the model assumptions is of importance to estimate the scope of applications and the accuracy of the predictions of a model. The aim of a good model is, starting from known but probably only estimated data and accepted laws of nature to give an answer as good as possible for a given question in an application field. A sensible model should only need data which are known or for which at least plausible approximations can be used. Therefore the task consists in extracting as much as possible information from known data.

1.2 Aspects of Mathematical Modeling: Example of Population Dynamics

To illustrate some important aspects of modeling in this section we consider a very simple example: A farmer has a herd of 200 cattle and he wants to increase this herd to 500 cattle, but only by natural growth, i.e., without buying additional animals. After a year the cattle herd has grown to 230 animals. He wants to estimate how long it lasts till he has reached his goal.

A sensible modeling assumption is the statement that the growth of the population depends on the size of the population, as a population of the double size should also have twice as much offspring. The data available are

- the initial number $x(t_0) = 200$ of animals at the initial time t_0,
- the increment in time $\Delta t = 1$ year,
- a growth factor of $r = 230/200 = 1.15$ per animal and per time increment Δt.

If one sets $t_n = t_0 + n\,\Delta t$ and if $x(t)$ denotes the number of animals at time t, then knowing the growth factor leads to the recursion formula

$$x(t_{n+1}) = r\,x(t_n)\,. \tag{1.1}$$

From this recursion formula one gets

$$x(t_n) = r^n x(t_0)\,.$$

Therefore the question can be formulated as:

Find a number n such that $x(t_n) = 500$.

The solution is

$$n\ln(r) = \ln\left(\frac{x(t_n)}{x(t_0)}\right), \quad \text{or } n = \frac{\ln\left(\dfrac{500}{200}\right)}{\ln(1.15)} \approx 6.6.$$

Hence, the farmer has to wait for 6.6 years.

This is a simple *population model* which in principle can also be applied to other problems from biology, for example the growth of other animal population, of plants or bacteria. But it can also be used in apparently totally different fields of applications, for example the computation of interests or the cooling of bodies (see Exercises 1.1 and 1.2). Without possibly noticing, in deriving the above model we have used several important modeling assumptions, which are fulfilled sometimes, but which are not fulfilled in a lot of cases. In particular the influence of the following effects has been neglected:

- the *spatial distribution* of the population,
- limited *resources*, for example limited nutrients,
- a loss of population by natural enemies.

Further details which also have been neglected, are for example the age distribution in the population, which has influence on the death rate and the birth rate, and the subdivision in female and male animals. Additionally the model leads to non-integer population quantities, which strictly speaking is not correct. The simplifications and deficiencies do not render the model worthless but they have to be recognized and taken into account to assess the result correctly. In particular the specific result of 6.6 years should not be taken too seriously, and an appropriate interpretation rather is that the farmer presumably will reach his goal in the 7$^{\text{th}}$ year.

An aspect, which is not optimal for intrinsic mathematical reasons, is the time increment of one year, because it is chosen arbitrarily. For the application under consideration it has a sensible meaning, nevertheless also an increment of three months or of two years could have been chosen. Furthermore we need two data, the increment in time and the growth rate. Both data depend on each other, meaning that the growth characteristics possibly can only be described by one number. As a first approach one can conjecture that the growth rate depends *linearly* on the time increment, i.e.,

$$r = 1 + \Delta t\, p$$

with a factor p still unknown. From $r = 1.15$ for $\Delta t = 1$ year we conclude that $p = 0.15/\text{year}$. Taking this for granted then for $\Delta t = 2$ years one has $r = 1.3$. Therefore after 6 years, (6 = 3 times 2) the farmer has

$$200 \cdot 1.3^3 = 439.4$$

cattle. But in the "old" model with $\Delta t = 1$ year he has

$$200 \cdot 1.15^6 \approx 462.61$$

animals. Therefore the assumption of a linear relation between r and Δt is wrong.

A better approach can be gained by the *limiting process* $\Delta t \to 0$:

$$x(t + \varDelta t) \approx (1 + \varDelta t\, p)\, x(t) \text{ for "small" } \varDelta t\,,$$

or more precisely

$$\lim_{\varDelta t \to 0} \frac{x(t + \varDelta t) - x(t)}{\varDelta t} = p\, x(t)\,,$$

i.e.,

$$x'(t) = p\, x(t)\,. \tag{1.2}$$

This is a *continuous* model in the form of an *ordinary differential equation*, which does not contain an arbitrarily chosen time increment anymore. It possesses the exact solution

$$x(t) = x(t_0)\, e^{p(t-t_0)}\,.$$

If the data are as above, i.e., a time increment of $\varDelta t = 1$ year and a growth rate $r = 1.15$, this means

$$e^{p \cdot 1\,\text{year}} = 1.15$$

and therefore

$$p = \ln(1.15)/\text{year} \approx 0.1398/\text{year}\,.$$

This is a *continuous exponent of growth.*

The discrete model (1.1) can be perceived as a special *numerical discretization* of the continuous model. An application of the explicit Euler method with time step $\varDelta t$ to (1.2) leads to

$$x(t_{i+1}) = x(t_i) + \varDelta t\, p\, x(t_i)\,, \quad \text{or} \quad x(t_{i+1}) = (1 + \varDelta t\, p)\, x(t_i)\,,$$

this is (1.1) with $r = 1 + \varDelta t\, p$. In the case $p < 0$ a time increment of the size $\varDelta t < (-p)^{-1}$ has to be chosen to achieve a sensible sequence of numbers. On the other hand using the implicit Euler method one gets

$$x(t_{i+1}) = x(t_i) + \varDelta t\, p\, x(t_{i+1})\,, \quad \text{or } x(t_{i+1}) = (1 - \varDelta t\, p)^{-1} x(t_i)\,,$$

i.e., (1.1) with $r = (1 - \varDelta t\, p)^{-1}$. Here for $p > 0$ the time step has to be chosen such that $\varDelta t < p^{-1}$. By Taylor series expansion one can see that the different growth factors coincide for *small* $\varDelta t$ "up to an error of the order $O\big((\varDelta t)^2\big)$":

$$(1 - \varDelta t\, p)^{-1} = 1 + \varDelta t\, p + O\big((\varDelta t\, p)^2\big)\,.$$

The connection between the continuous and the discrete model therefore can be established by an analysis of the *convergence properties* of the numerical method. For the (explicit or implicit) Euler method one gets for example

$$|x(t_i) - x_i| \le C(t_e)\, \varDelta t\,,$$

where $x(t_i)$ is the exact solution of (1.2) at time t_i and x_i is the approximate solution of the numerical method, assuming that $t_i \le t_e$, where t_e is the given final time for the model. For details about the analysis of numerical methods for ordinary differential equations we refer to the textbooks of Stoer and Bulirsch [123] and Deuflhard and Bornemann [28].

Both models, the discrete and the continuous, have the seeming disadvantage that they also allow *non-integer* solutions, which obviously are not realistic for the considered example. The model describes — as it is true for every other model — not the total reality but only leads to an idealized picture. For *small* populations the model is not very precise, as in general population growth also depends heavily on stochastic effects and therefore cannot be computed precisely in a deterministic way. In addition for small populations the model assumptions are questionable, in particular one neglects the age and the sex of the animals. In the extreme case of a herd of two animals obviously the growth will depend heavily on the fact whether there is a male and a female animal, or not. For large populations on the other hand one can assume with a certain qualification that it possesses a characteristic uniform distribution in age and in sex, such that the assumption of a growth proportional to population size make sense.

The substitution of integer values by real numbers reflects the inaccuracy of the model. Therefore it is not sensible to change the model such that integer values in the solutions are enforced. This would only lead to an unrealistic perception of high accuracy of the model. For a small population a *stochastic* model, which then "only" provides statements about the *probability distribution* of the population size, makes sense instead of deterministic models.

Nondimensionalization

The quantities in a mathematical model generally have a *physical dimension*. In the population model (1.2) we have the units *number* and *time*. We denote the physical dimension of a quantity f with $[f]$ and abbreviate the units number of entities by A and time by T. Therefore we have

$$\begin{aligned}
[t] &= T\,,\\
[x(t)] &= A\,,\\
[x'(t)] &= \frac{A}{T}\,,\\
[p] &= \frac{1}{T}\,.
\end{aligned}$$

The specification of a physical dimension is not yet a decision about the physical unit of measurements. As a unit of measurement for time one can use seconds, minutes, hours, days, weeks, or years, for example. If we measure time in years, then t is indicated in years, $x(t)$ by a number, $x'(t)$ in number/years and p in number/years.

To get models as simple as possible and furthermore in order to determine characteristic quantities in a model, one can *nondimensionalize* the model equations.

For this aim one defines a characteristic value for every appearing dimension and correspondingly a unit of measurement. Here it is not necessary to choose one of the common units as for example seconds or hours but it is more appropriate to choose a unit *adapted to the problem*. For the population model there are two dimensions, therefore two characteristic values are needed, the characteristic number $\overline{x}$ and the characteristic time $\overline{t}$. These are chosen in such a way that the *initial data* t_0 and $x_0 = x(t_0)$ are as simple as possible. Therefore a convenient unit of measurement for time is given by

$$\tau = \frac{t - t_0}{\overline{t}},$$

where $\overline{t}$ denotes a unit of time which still has to be specified, and as a unit for number we choose

$$\overline{x} = x_0 .$$

Setting

$$y = \frac{x}{\overline{x}}$$

and expressing y as a function of τ,

$$y(\tau) = \frac{x(\overline{t}\tau + t_0)}{\overline{x}},$$

one obtains

$$y'(\tau) = \frac{\overline{t}}{\overline{x}}\, x'(t)$$

and therefore the model becomes

$$\frac{\overline{x}}{\overline{t}}\, y'(\tau) = p\,\overline{x}\, y(\tau) .$$

This model gets its most simple form for the choice

$$\overline{t} = \frac{1}{p} . \tag{1.3}$$

The model thus derived is the initial value problem

$$\begin{aligned} y'(\tau) &= y(\tau) , \\ y(0) &= 1 . \end{aligned} \tag{1.4}$$

This model has the solution

$$y(\tau) = e^{\tau} .$$

From this solution all solutions of the original model (1.2) can be achieved by using a transformation:

$$x(t) = \bar{x}\, y(\tau) = x_0\, y(p(t - t_0)) = x_0\, e^{p(t-t_0)} .$$

The advantage of the nondimensionalization therefore is the reduction of the solution of all population models of a given type by the choice of units to *one single problem*. Note that this holds true *independent of the sign of* p, although the behavior of the solutions for $p > 0$ and $p < 0$ is different. For $p < 0$ the solution of (1.2) is given by the solution (1.4) for the range $\tau < 0$.

The scaling condition (1.3) also can be obtained by means of *dimensional analysis*. In this procedure the characteristic time $\bar{t}$ to be determined is expressed as a product of the other characteristic parameters in the model,

$$\bar{t} = p^n x_0^m \quad \text{with } n, m \in \mathbb{Z} .$$

By computing the dimension one obtains

$$[\bar{t}] = [p]^n [x_0]^m \quad \text{and therefore} \quad T = \left(\frac{1}{T}\right)^n A^m .$$

The only possible solution of this equation is given by $n = -1, m = 0$, if the number of animals is interpreted as a dimension of its own. Thus we get exactly (1.3).

In more complex models typically the model cannot be reduced to a single problem by nondimensionalization but the *number of relevant parameters* can be strongly reduced and the characteristic parameters can be identified. This also relates to the corresponding experiments: For instance, from the nondimensionalization of the equations for airflows one can conclude how the circulation around an airplane can be experimentally measured by using a (physical) model for the airplane much smaller in scale. We will explain dimensional analysis in one of the following sections using a more meaningful example.

1.3 Population Models with Restricted Resources

For large populations in nature a constant growth rate is not realistic anymore. A restriction of the habitat, or the available nutrients, or other mechanisms impose limitations on the growth. To construct a model it is feasible for such situations to assume that there is a certain capacity $x_M > 0$ for which the resources of the habitat are still sufficient. For population quantities x smaller than x_M the population still can grow, but for values larger than x_M the population decreases. This means that the growth rate p now depends on the population x, $p = p(x)$, and that

$$
\begin{aligned}
p(x) &> 0 \;\text{ for } 0 < p < x_M\,, \\
p(x) &< 0 \;\text{ for } p > x_M
\end{aligned}
$$

have to hold true. The most simple functional form satisfying these conditions is given by a *linear* ansatz for p, i.e.,

$$p(x) = q(x_M - x) \;\text{ for all }\; x \in \mathbb{R}$$

with a parameter $q > 0$. With this ansatz we obtain the differential equation

$$x'(t) = q\, x_M\, x(t) - q\, x(t)^2 \tag{1.5}$$

as a model. The additional term $-q\, x(t)^2$ is proportional to the probability for the *number of encounters* of two specimens of the population per unit of time. The term represents the more competitive situation if the population size increases, the so-called "social friction". The Eq. (1.5) has been proposed by the Dutch biomathematician Verhulst and is called *logistic differential equation* or *equation of limited growth*.

Equation (1.5) also can be solved in closed form (compare Exercise 1.3). From

$$\frac{x'}{x(x_M - x)} = q$$

we conclude using the partial fraction decomposition

$$\frac{1}{x(x_M - x)} = \frac{1}{x_M}\left(\frac{1}{x} + \frac{1}{x_M - x}\right)$$

and by integration

$$\ln(x(t)) - \ln|x_M - x(t)| = x_M q t + c_1, \quad c_1 \in \mathbb{R}\,.$$

After the choice of an appropriate constant $c_2 \in \mathbb{R}$ we obtain

$$\frac{x(t)}{x_M - x(t)} = c_2 e^{x_M q t}\,,$$

and

$$x(t) = \frac{c_2 x_M e^{x_M q t}}{1 + c_2 e^{x_M q t}} = \frac{x_M}{1 + c_3 e^{-x_M q t}}\,.$$

Incorporating the initial condition $x(t_0) = x_0$ we obtain

$$x(t) = \frac{x_M x_0}{x_0 + (x_M - x_0)e^{-x_M q (t - t_0)}}\,. \tag{1.6}$$

From this exact solution the following properties can be easily derived:

- If x_0 is positive, the solution always stays positive.
- If x_0 is positive, then for $t \to +\infty$ the solution converges to the equilibrium point $x_\infty = x_M$.

The graph of x can be sketched also without knowing the exact solution. From (1.5) first we conclude

$$\begin{aligned} x' &> 0\,, \text{ if } x < x_M\,, \\ x' &< 0\,, \text{ if } x > x_M\,. \end{aligned}$$

Furthermore we have

$$\begin{aligned} x'' = (x')' = (q\,(x_M - x)\,x)' &= q(x_M - x)\,x' - q\,x\,x' \\ &= q(x_M - 2x)x' = q^2(x_M - 2x)(x_M - x)\,x\,. \end{aligned}$$

From these results we conclude

$$\begin{aligned} x'' &> 0\,, \text{ if } x \in (0, x_M/2) \cup (x_M, \infty)\,, \\ x'' &< 0\,, \text{ if } x_M/2 < x < x_M\,. \end{aligned}$$

Thus the solution curves have an inflection point at $x_M/2$ and the curves are concave in the interval between $x_M/2$ and x_M, and convex otherwise. Solutions of the logistic differential equation are depicted in Fig. 1.1.

Stationary Solutions

For more complex time-depending models a closed form solution often cannot be found. Then it is useful to identify *time independent* solutions. Such solutions can be computed using the time dependent model by just setting all time derivatives to zero. For our model with restricted growth one gets

$$0 = q x_M x - q x^2\,.$$

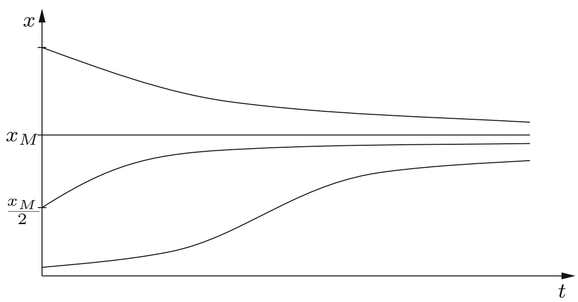

Fig. 1.1 Solutions of the logistic differential equation

This equation has the two solutions

$$x_0 = 0 \;\text{ and }\; x_1 = x_M\,.$$

These are the solutions of the original model for specific initial data. Often time independent solutions appear as so-called *stationary limits* of arbitrary solutions for large times, meaning that they are solutions constant in time towards which time dependent solutions converge for large times. Typically this only appears if the stationary solution is *stable* in the following sense: If the initial data is only changed slightly then also the solution changes only slightly. Using the exact solution (1.6) the question of stability can be easily answered for the logistic differential equation: The solution for the initial value

$$x(t_0) = \varepsilon$$

with a small $\varepsilon > 0$ is given by

$$x_\varepsilon(t) = \frac{x_M\varepsilon}{\varepsilon + (x_M - \varepsilon)e^{-x_M q(t-t_0)}}\,,$$

it converges for $t \to +\infty$ towards x_M, therefore the stationary solution $x_0 = 0$ is *not stable*. For

$$x(t_0) = x_M + \varepsilon$$

with a small $\varepsilon \neq 0$ the solution is given by

$$x_\varepsilon(t) = \frac{x_M(x_M + \varepsilon)}{(x_M + \varepsilon) - \varepsilon\, e^{-x_M q(t-t_0)}}\,,$$

it converges for $t \to +\infty$ towards x_M. From

$$x'_\varepsilon(t) = q\, x_\varepsilon(t)(x_M - x_\varepsilon(t))$$

one can conclude also without knowing the exact solution that the distance to x_M can only decrease for increasing time as from $x_\varepsilon(t) > x_M$ it follows $x'_\varepsilon(t) < 0$ and from $x_\varepsilon(t) < x_M$ it follows $x'_\varepsilon(t) > 0$. Therefore the stationary solution x_M is *stable*. Stability is of importance, as in nature in general no *instable* stationary solution can be observed, therefore they are irrelevant for most practical applications. For more complex models sometimes no closed form solution for the time dependent equation can be derived. However, there are techniques of *stability analysis*, with which often the stability properties of stationary solutions can be deduced. Often this is done by means of a *linearization* of the problem at the stationary solution followed by a computation of the *eigenvalues* of the linearized problem. This will be explained in more detail in Chap. 4.

1.4 Dimensional Analysis and Scaling

Now we want to explain dimensional analysis using a slightly more significant example. We consider a body of mass m, which is thrown bottom-up in vertical direction with respect to the gravitational field of a planet (for example the Earth). The motion of the body is described by Newton's law

$$a = \frac{F}{m} \,,$$

where a denotes the acceleration of the body and F the force acting on the body. This force is described by Newton's law of gravitation

$$F = -G \frac{m_E \, m}{(x + R)^2} \,,$$

where $G \approx 6.674 \cdot 10^{-11} \mathrm{N} \cdot \mathrm{m}^2/\mathrm{kg}^2$ denotes the gravitational constant, m_E the mass of the planet, R the radius of the planet and x the height of the body, measured from the surface of the planet. We neglect the air resistance in the atmosphere and consider the planet to be a sphere. If one defines the constant g by

$$g = \frac{G m_E}{R^2} \,,$$

one gets

$$F = -\frac{g R^2 m}{(x + R)^2} \,.$$

For the Earth we have $g = 9.80665\,\mathrm{m/s}^2$, the gravitational acceleration. The motion of the body then is described by the differential equation

$$x''(t) = -\frac{g R^2}{(x(t) + R)^2} \,. \tag{1.7}$$

This has to be completed by two initial conditions,

$$x(0) = 0 \,, \quad x'(0) = v_0 \,,$$

where v_0 denotes the initial velocity.

For trajectories expected in our application typically the term $x(t)$ is very small compared to the radius of the Earth and seems to be negligible in the denominator in the right-hand side in (1.7). We want to investigate the validity of this ansatz in a systematic fashion. To do so first we perform a nondimensionalization. As a specific example we use the data

$$g = 10\,\mathrm{m/s}^2\,, \quad R = 10^7\,\mathrm{m}\,, \quad \text{and } v_0 = 10\,\mathrm{m/s}\,,$$

which have an order of magnitude corresponding to the application in mind.

The dimensions appearing here are L for the length and T for time. The given data are the initial velocity v_0 with dimension $[v_0] = L/T$, the "planet acceleration" g with dimension $[g] = L/T^2$, and the radius R with dimension $[R] = L$. The independent variable is the time t with dimension $[t] = T$, and the quantity to be computed is the height x with dimension $[x] = L$. First we look for all representations of the form

$$\Pi = v_0^a\, g^b R^c\,,$$

which are either dimensionless (case (i)), or have the dimension of a length (case (ii)), or have the dimension of a time (case (iii)). From

$$[\Pi] = \left(\frac{L}{T}\right)^a \left(\frac{L}{T^2}\right)^b L^c = L^{a+b+c} T^{-a-2b}$$

it follows:

Case (i): We have $a + b + c = 0$, $-a - 2b = 0$, therefore $a = -2b$, $c = b$, and finally

$$\Pi = \left(\frac{gR}{v_0^2}\right)^b\,.$$

This leads to the identification of

$$\varepsilon = \frac{v_0^2}{gR} \tag{1.8}$$

as a characteristic dimensionless parameter. In fact all other dimensionless parameters are powers of this specific one.

Case (ii): We have $a + b + c = 1$, $a + 2b = 0$ and therefore $a = -2b$, $c = 1 + b$. As a specific unit for length one obtains

$$\ell = v_0^{-2b} g^b R^{1+b} = R\,\varepsilon^{-b}\,,$$

where b denotes a constant not yet specified.

Case (iii): We have $a + b + c = 0$, $a + 2b = -1$ and therefore $a = -1 - 2b$, $c = b + 1$. Therefore a characteristic unit for time is given by

$$\tau = v_0^{-1-2b} g^b R^{b+1} = \frac{R}{v_0}\varepsilon^{-b}.$$

We will now try to nondimensionalize Eq. (1.7). To this purpose we consider a unit for length $\overline{x}$ and a unit for time $\overline{t}$ and represent $x(t)$ in the form

$$x(t) = \bar{x}\, y(t/\bar{t})\,.$$

From (1.7) we obtain

$$\frac{\bar{x}}{\bar{t}^2}\, y''(\tau) = -\frac{gR^2}{(\bar{x}\, y(\tau) + R)^2}\,,$$

i.e.,

$$\frac{\bar{x}}{\bar{t}^2 g}\, y''(\tau) = -\frac{1}{((\bar{x}/R)\, y(\tau) + 1)^2}\,. \tag{1.9}$$

This equation has to be provided with the initial conditions

$$y(0) = 0 \;\text{ and }\; y'(0) = \frac{\bar{t}}{\bar{x}} v_0\,.$$

Now we want to choose $\bar{x}$ and $\bar{t}$ in such a way that as many of the appearing parameters as possible equal 1. However, here we have more parameters than scaling units, namely the three parameters

$$\frac{\bar{x}}{\bar{t}^2 g}\,, \quad \frac{\bar{x}}{R}, \quad \text{and} \quad \frac{\bar{t}}{\bar{x}} v_0\,.$$

Hence, only two of these parameters can be transformed to one and therefore there are three different possibilities:

(a) $\dfrac{\bar{x}}{\bar{t}^2 g} = 1$ and $\dfrac{\bar{x}}{R} = 1$ are a consequence of $\bar{x} = R$, $\bar{t} = \sqrt{\dfrac{R}{g}}$, then the parameter is given by $\dfrac{\bar{t}}{\bar{x}} v_0 = \dfrac{v_0}{\sqrt{Rg}} = \sqrt{\varepsilon}$ using ε from (1.8). Therefore the model reduces to

$$y''(\tau) = -\frac{1}{(y(\tau) + 1)^2}\,, \quad y(0) = 0\,, \quad y'(0) = \sqrt{\varepsilon}\,. \tag{1.10}$$

(b) $\dfrac{\bar{x}}{R} = 1$ and $\dfrac{\bar{t}}{\bar{x}} v_0 = 1$ can be deduced from $\bar{x} = R$ and $\bar{t} = \dfrac{R}{v_0}$, leading to $\dfrac{\bar{x}}{\bar{t}^2 g} = \dfrac{v_0^2}{Rg} = \varepsilon$ for the third parameter. Then the dimensionless model is given by

$$\varepsilon\, y''(\tau) = -\frac{1}{(y(\tau) + 1)^2}\,, \quad y(0) = 0\,, \quad y'(0) = 1\,.$$

(c) $\dfrac{\bar{x}}{\bar{t}^2 g} = 1$ and $\dfrac{\bar{t}}{\bar{x}} v_0 = 1$ are a consequence of $\bar{t} = \dfrac{v_0}{g}$ and $\bar{x} = \dfrac{v_0^2}{g}$. Then the third parameter is given by $\dfrac{\bar{x}}{R} = \dfrac{v_0^2}{gR} = \varepsilon$. Thus the dimensionless model reads

$$y''(\tau) = -\frac{1}{(\varepsilon\, y(\tau) + 1)^2}\,, \quad y(0) = 0\,, \quad y'(0) = 1\,. \tag{1.11}$$

Let us mention that there is a fourth possibility. Besides (1.9) we may also use the equivalent formulation

$$\frac{\overline{x}^3}{\overline{t}^2 g R^2} y''(\tau) = -\frac{1}{(y(\tau) + R/\overline{x})^2}\,.$$

This leads, by setting $\overline{t} = gR^2/v_0^3$ $\overline{x} = gR^2/v_0^2$, to a fourth possibility (d)

$$y''(\tau) = -\frac{1}{(y(\tau) + \varepsilon)^2}\,, \quad y(0) = 0\,, \quad y'(0) = 1$$

with $\varepsilon = v_0^2/(gR)$.

Now we want to assess and compare the four dimensionless equations for the application example displayed above. For $R = 10^7$ m, $g = 10\,\text{m/s}^2$ and $v_0 = 10\,\text{m/s}$ the parameter ε is very small,

$$\varepsilon = \frac{v_0^2}{Rg} = 10^{-6}\,.$$

This suggests to neglect terms of the order of ε in the equations.

The model (a) is then reduced to

$$y''(\tau) = -\frac{1}{(y(\tau) + 1)^2}\,, \quad y(0) = 0\,, \quad y'(0) = 0\,.$$

Because of $y''(0) < 0$ and $y'(0) = 0$ this model leads to negative solutions and therefore it is extremely inexact and of no use. The reason lies in the scaling within the nondimensionalization: The parameters $\overline{t}$ and $\overline{x}$ here are given by

$$\overline{t} = \sqrt{\frac{R}{g}} = 10^3\,\text{s} \;\text{ and } \overline{x} = 10^7\,\text{m}\,,$$

both scales are much too large for the problem under investigation. The maximal height to be reached and the instance of time for which it is reached are much smaller than the scales $\overline{x}$ for length and $\overline{t}$ for time and therefore are "hardly visible" in the nondimensionalized model.

The model (b) reduces to

$$0 = -\frac{1}{(y(\tau) + 1)^2}\,, \quad y(0) = 0, \; y'(0) = 1\,.$$

This problem is not well posed, as it has no solution. Also here the chosen scales for time and length are much too large,

$$\bar{t} = \frac{R}{v_0} = 10^6\,\mathrm{s} \ \text{ and } \ \bar{x} = R = 10^7\,\mathrm{m}\,.$$

Model (c) reduces to

$$y''(\tau) = -1\,, \quad y(0) = 0\,, \quad y'(0) = 1\,. \tag{1.12}$$

This model has the solution

$$y(\tau) = \tau - \frac{1}{2}\tau^2$$

and thus describes a typical parabola shaped path-time curve for a throw within the gravitational field of the Earth neglecting the air resistance. The back transformation

$$x(t) = \bar{x}\, y(t/\bar{t}) = \frac{v_0^2}{g}\, y(gt/v_0)$$

leads to

$$x(t) = v_0 t - \frac{1}{2} g t^2\,.$$

This corresponds to the solution of (1.7), if the term $x(t)$ in the denominator of the right-hand side of (1.7) is neglected. The scales in the nondimensionalization here have reasonable values,

$$\bar{t} = \frac{v_0}{g} = 1\,\mathrm{s}\,, \quad \bar{x} = \frac{v_0^2}{g} = 10\,\mathrm{m}\,.$$

Model (d) reduces to

$$y''(\tau) = -\frac{1}{y(\tau)^2}\,, \quad y(0) = 0\,, \quad y'(0) = 1\,.$$

This model does not correspond to a constant acceleration force. Also, the initial condition seems to be problematic for this differential equation, and the time and the length scale chosen are much too large.

Hence, for the application considered the nondimensionalization in version (c) is the "correct one". The versions (a), (b), and (d) are equally well mathematically correct, but there the small parameter ε cannot be neglected anymore because its influence is amplified by the (too) large scaling parameters $\bar{t}$ and $\bar{x}$.

1.5 Asymptotic Expansions

Now we will introduce a technique with which the simplified model can be improved. The basic idea is not to neglect the terms of order ε in the exact model (1.11), but rather to do a series expansion of the solution of (1.11) with respect to ε to achieve more precise solutions by keeping some of the terms beyond the zeroth order term. The terms of higher order in ε are determined from equations which we get by substituting a series expansion into (1.11).

We want to discuss this procedure which is called the *method of asymptotic expansion*, first for a simple algebraic example. We consider the equation

$$x^2 + 0.002\,x - 1 = 0\,. \tag{1.13}$$

The second summand has a small factor in front. Setting $\varepsilon = 0.001 \ll 1$, we obtain

$$x^2 + 2\varepsilon x - 1 = 0\,. \tag{1.14}$$

Now we want to approximate solutions x of this equation by a series expansion of the form

$$x_0 + \varepsilon^\alpha x_1 + \varepsilon^{2\alpha} x_2 + \cdots \quad \text{with } \alpha > 0. \tag{1.15}$$

Before doing so we first define in general what we mean by an asymptotic expansion. Let $x : (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}$, $\varepsilon_0 > 0$, be a given function. A sequence $(\phi_n(\varepsilon))_{n\in\mathbb{N}_0}$ is called an asymptotic sequence if and only if

$$\phi_{n+1}(\varepsilon) = o(\phi_n(\varepsilon)) \quad \text{as} \quad \varepsilon \to 0$$

for each $n = 0, 1, 2, 3, \ldots$. An example is the sequence $\phi_n(\varepsilon) = \varepsilon^{n\alpha}$ used above. A series $\sum\limits_{k=0}^{N} \phi_k(\varepsilon)x_k$ is called asymptotic expansion of $x(\varepsilon)$ of the order $N \in \mathbb{N} \cup \{\infty\}$ with respect to the sequence $(\phi_n(\varepsilon))_{n\in\mathbb{N}_0}$, if for $M = 0, 1, 2, 3, \ldots, N$ we have

$$x(\varepsilon) - \sum_{k=0}^{M} \phi_k(\varepsilon)x_k = o(\phi_M(\varepsilon)) \quad \text{as} \quad \varepsilon \to 0.$$

If $\sum_{k=0}^{N} \phi_k(\varepsilon)x_k$ is an asymptotic expansion of $x(\varepsilon)$ we write

$$x \sim \sum_{k=0}^{N} \phi_k(\varepsilon)x_k \quad \text{as} \quad \varepsilon \to 0\,.$$

If $N = \infty$, we write

$$x(\varepsilon) \sim \sum_{k=0}^{\infty} \phi_k(\varepsilon) x_k \quad \text{as} \quad \varepsilon \to 0\,.$$

In the special case $\phi_k(\varepsilon) = \varepsilon^k$, we speak of an asymptotic expansion of $x(\varepsilon)$ in powers of ε. Note that asymptotic expansions of arbitrary order can exist, even if the corresponding infinite series are divergent for every $\varepsilon \neq 0$. In particular an asymptotic expansion can exist, although the Taylor expansion for $x(\varepsilon)$ does not converge for any $\varepsilon \neq 0$.

Now we substitute the asymptotic expansion (1.15) into (1.14) and obtain

$$x_0^2 + 2\varepsilon^\alpha x_0 x_1 + \cdots + 2\varepsilon(x_0 + \varepsilon^\alpha x_1 + \cdots) - 1 = 0\,.$$

If this identity shall hold true, it must hold true in particular for small ε. Therefore all terms which do not contain a factor ε (or ε^α), must add up to zero. Such terms are of order 1. We write $\mathcal{O}(1)$ or $\mathcal{O}(\varepsilon)$, respectively, and collect only those terms, which are exactly of order 1 or ε, respectively. The equation of order 1 then reads

$$\mathcal{O}(1): \quad x_0^2 - 1 = 0\,.$$

Its solutions are given by $x_0 = \pm 1$. In particular we see that the equation of order $\mathcal{O}(1)$ has exactly as many solutions as the original problem. This is a condition necessary in order to speak of a *regularly perturbed problem*. Later on we will see when to speak of regular or of *singular* perturbations.

Now we consider the terms of the next higher order in ε. What the next higher order is depends on whether we have $\alpha < 1$, $\alpha > 1$, or $\alpha = 1$. If $\alpha < 1$, then we conclude from the term of order ε^α that $x_1 = 0$, and from the terms of order $\varepsilon^{j\alpha}$ in a successive fashion $x_j = 0$ for $1 \leq j < 1/\alpha$. If $\alpha = 1/k$ for some $k \in \mathbb{N}$, then it follows from the term of order $k\alpha = 1$ that

$$2x_0 x_k + 2x_0 = 0$$

and therefore $x_k = -1$. Proceeding with the asymptotic expansion one sees that the terms of order j with $\alpha j \notin \mathbb{N}$ always lead to $x_j = 0$. Therefore only the terms x_{kn} for $n \in \mathbb{N}$ remain. For the corresponding powers $\varepsilon^{kn\alpha}$ we have $kn\alpha \in \mathbb{N}$. Therefore the power series ansatz with $\alpha < 1$, $\alpha = 1/k$, for some $k \in \mathbb{N}$, leads to the same result as the ansatz $\alpha = 1$, and therefore it is unnecessarily complicated. If $\alpha \neq 1/k$ for all $k \in \mathbb{N}$, then from the term of order ε we get

$$2x_0 = 0\,,$$

but this is in contradiction to the already computed solutions $x_0 = \pm 1$. Therefore the ansatz $\alpha < 1$ is not sensible. In the case $\alpha > 1$ the term of order ε also leads to $2x_0 = 0$ which is impossible as we have already seen. Therefore $\alpha = 1$ remains as the only sensible choice and we obtain the equation

$$\mathcal{O}(\varepsilon): \quad 2x_0x_1 + 2x_0 = 0$$

for the order ε. Its only solution is given by $x_1 = -1$.

If we also take terms of the next higher order ε^2 into account, we obtain

$$x_0^2 + 2\varepsilon x_0x_1 + \varepsilon^2 x_1^2 + 2\varepsilon^2 x_2x_0 + 2\varepsilon(x_0 + \varepsilon x_1 + \varepsilon^2 x_2) - 1 = 0$$

and the terms of order ε^2 lead to the identity

$$\mathcal{O}(\varepsilon^2): \quad x_1^2 + 2x_2x_0 + 2x_1 = 0\,.$$

Therefore we have

$$x_2 = \frac{1}{2}(x_0)^{-1} = \pm\frac{1}{2}\,.$$

The Eq. (1.13) corresponds to (1.14) for $\varepsilon = 10^{-3}$. Therefore we expect that the numbers

$$x_0,\ x_0 + \varepsilon x_1,\ x_0 + \varepsilon x_1 + \varepsilon^2 x_2$$

are good approximations of the solutions of (1.13) if we set $\varepsilon = 10^{-3}$. In fact we have

x_0	$x_0 + \varepsilon x_1$	$x_0 + \varepsilon x_1 + \varepsilon^2 x_2$	exact solutions
1	0.999	0.9990005	$0.9990005\cdots$
-1	-1.001	-1.0010005	$-1.0010005\cdots$

Therefore in this simple example the series expansion leads to very good approximations taking only a few terms into account.

This procedure becomes more interesting for complex problems without a closed form solution. Now we want to discuss the method of asymptotic expansion for the example (1.11), a throw in a gravitational field of a planet. We apply Taylor expansion at $z = 0$

$$\frac{1}{(1+z)^2} = 1 - 2z + 3z^2 - 4z^3 \pm \cdots$$

to the right-hand side of the differential equation

$$y_\varepsilon''(\tau) = -\frac{1}{(1 + \varepsilon\, y_\varepsilon(\tau))^2} \tag{1.16}$$

and get

$$y_\varepsilon''(\tau) = -1 + 2\varepsilon\, y_\varepsilon(\tau) - 3\varepsilon^2 y_\varepsilon^2(\tau) \pm \cdots\,. \tag{1.17}$$

We assume that the solution y_ε also possesses a series expansion of the form

$$y_\varepsilon(\tau) = y_0(\tau) + \varepsilon^\alpha y_1(\tau) + \varepsilon^{2\alpha} y_2(\tau) + \cdots \tag{1.18}$$

with coefficient functions $y_j(\tau)$ and a parameter α to be specified. This ansatz will be substituted into (1.17) and then the coefficients of the same powers of ε will be grouped together. The aim of this procedure is to determine a reasonable value for a parameter α and to obtain solvable equations for the coefficient functions $y_j(\tau)$, $j = 0, 1, 2, \ldots$. The substitution of (1.18) into (1.17) leads to

$$\begin{aligned} & y_0''(\tau) + \varepsilon^\alpha y_1''(\tau) + \varepsilon^{2\alpha} y_2''(\tau) + \cdots \\ = & - 1 + 2\varepsilon\big(y_0(\tau) + \varepsilon^\alpha y_1(\tau) + \varepsilon^{2\alpha} y_2(\tau) + \cdots\big) \\ & - 3\varepsilon^2\big(y_0(\tau) + \varepsilon^\alpha y_1(\tau) + \varepsilon^{2\alpha} y_2(\tau) + \cdots\big)^2 \pm \cdots . \end{aligned} \tag{1.19}$$

In the same way the series expansion can be substituted into the initial conditions and one obtains

$$\begin{aligned} y_0(0) + \varepsilon^\alpha y_1(0) + \varepsilon^{2\alpha} y_2(0) + \cdots &= 0\,, \\ y_0'(0) + \varepsilon^\alpha y_1'(0) + \varepsilon^{2\alpha} y_2'(0) + \cdots &= 1\,. \end{aligned}$$

By comparing the coefficients of $\varepsilon^{k\alpha}$, $k \in \mathbb{N}$, one immediately gets

$$y_j(0) = 0 \text{ for } j \in \mathbb{N} \cup \{0\},\ y_0'(0) = 1 \text{ and } y_j'(0) = 0 \text{ for } j \in \mathbb{N}\,. \tag{1.20}$$

To compare the coefficients appearing in (1.19) for the same powers of ε on the left and right-hand side is more complicated. The lowest appearing power of ε is $\varepsilon^0 = 1$. The comparison of the coefficients of ε^0 leads to

$$y_0''(\tau) = -1\,.$$

Together with the initial conditions $y_0(0) = 0$ and $y_0'(0) = 1$ we obtain the already known problem (1.12) with its solution

$$y_0(\tau) = \tau - \frac{1}{2}\tau^2\,.$$

The next exponent to be considered depends on the choice of α. For $\alpha < 1$ it is ε^α, comparison of the coefficients leads to

$$y_1''(\tau) = 0\,.$$

Together with the initial conditions $y_1(0) = y_1'(0) = 0$ we have the unique solution $y_1(\tau) = 0$. The term $2\varepsilon y_0$ in (1.19) can only be compensated by a term of the form $\varepsilon^{k\alpha} y_k''$, $k \in \mathbb{N}, k\alpha = 1$. As in the case of y_1 we conclude that $y_j \equiv 0$ for $1 \le j \le k - 1$. Analogously one sees that the terms y_k, where $k\alpha \notin \mathbb{N}$, all have to be zero. Hence, one could have started with the ansatz $\alpha = 1$.

For $\alpha > 1$ the next exponent is given by ε^1, and the comparison of coefficients leads to $y_0(\tau) = 0$. This is a contradiction to the solution computed above. Therefore $\alpha > 1$ is the wrong choice.

In summary, the only reasonable exponent is $\alpha = 1$. Then the coefficients of ε^1 are given by

$$y_1''(\tau) = 2\, y_0(\tau) = 2\tau - \tau^2 .$$

Together with the initial conditions $y_1(0) = y_1'(0) = 0$ one obtains the unique solution

$$y_1(\tau) = \frac{1}{3}\tau^3 - \frac{1}{12}\tau^4.$$

The coefficients of ε^2 lead to the problem

$$y_2''(\tau) = 2\, y_1(\tau) - 3\, y_0^2(\tau) = \frac{2}{3}\tau^3 - \frac{1}{6}\tau^4 - 3\tau^2 + 3\tau^3 - \frac{3}{4}\tau^4$$

together with the initial conditions $y_2(0) = y_2'(0) = 0$. Its solution is given by

$$y_2(\tau) = -\frac{11}{360}\tau^6 + \frac{11}{60}\tau^5 - \frac{1}{4}\tau^4 .$$

Correspondingly further coefficients $y_3(\tau), y_4(\tau), \cdots$ can be computed, but the effort becomes larger and larger with increasing order. In particular the first three terms of the series expansion are

$$y_\varepsilon(\tau) = \tau - \frac{1}{2}\tau^2 + \varepsilon\left(\frac{1}{3}\tau^3 - \frac{1}{12}\tau^4\right) + \varepsilon^2\left(-\frac{1}{4}\tau^4 + \frac{11}{60}\tau^5 - \frac{11}{360}\tau^6\right) + O\big(\varepsilon^3\big) .$$

Figure 1.2 shows the graph of the approximations $y_0(\tau)$ of order 0, $y_0(\tau) + \varepsilon\, y_1(\tau)$ of order 1 and of the exact solution for $\varepsilon = 0.2$. One sees that visually the approximation of order 1 can hardly be distinguished from the exact solution, but the approximation of order 0 still contains a clearly visible error.

Now we want to use the series expansion to obtain a better approximation for the height of the throw. For this purpose we first compute an approximation for the instant of time $\tau = \tau_\varepsilon$, at which this maximal height is reached, using the equation

$$y_\varepsilon'(\tau) = 0 .$$

From $y_\varepsilon(\tau) = y_0(\tau) + \varepsilon\, y_1(\tau) + \varepsilon^2 y_2(\tau) + \cdots$ it follows that

$$y_0'(\tau) + \varepsilon\, y_1'(\tau) + \varepsilon^2 y_2'(\tau) + O\big(\varepsilon^3\big) = 0$$

with y_0, y_1, y_2 as above. Again we solve this equation in an approximative fashion using the series ansatz

$$\tau_\varepsilon = \tau_0 + \varepsilon\, \tau_1 + \varepsilon^2 \tau_2 + \cdots .$$

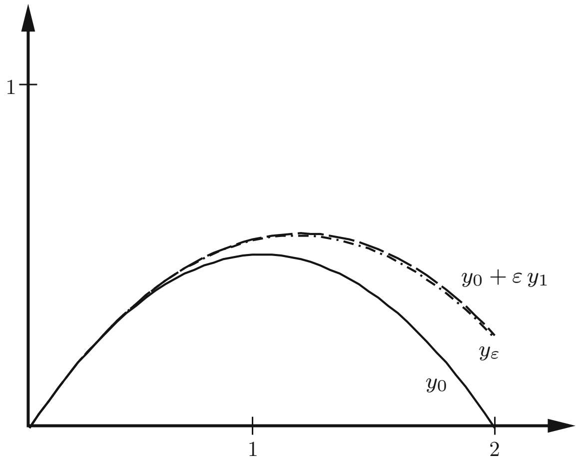

Fig. 1.2 Asymptotic expansion for the vertical throw with $\varepsilon = 0.2$

Comparing the coefficients of ε^0 leads to

$$y_0'(\tau_0) = 1 - \tau_0 = 0\,,$$

and therefore $\tau_0 = 1$. From the coefficients of ε and the expansion $y_i'(\tau_\varepsilon) = y_i'(\tau_0) + \varepsilon\, y_i''(\tau_0)\tau_1 + \cdots$, $i = 0, 1$, one obtains

$$y_0''(\tau_0)\tau_1 + y_1'(\tau_0) = -\tau_1 + \tau_0^2 - \frac{1}{3}\tau_0^3 = 0$$

and therefore $\tau_1 = 2/3$. Thus, the approximation of first order for τ_ε is given by

$$1 + \frac{2}{3}\varepsilon\,.$$

The corresponding height is

$$\begin{aligned} h_\varepsilon = y_\varepsilon(\tau_\varepsilon) &= y_0(\tau_0) + \varepsilon\big(y_0'(\tau_0)\tau_1 + y_1(\tau_0)\big) + O\big(\varepsilon^2\big) \\ &= y_0(\tau_0) + \varepsilon\, y_1(\tau_0) + O\big(\varepsilon^2\big) = \frac{1}{2} + \frac{1}{4}\varepsilon + O\big(\varepsilon^2\big)\,. \end{aligned}$$

If one takes into account that the gravitational force decreases with height, then the maximal height of the throw becomes slightly bigger. For our original example with $\varepsilon = 10^{-6}$ this affects the results in the seventh digit.

A priori it is not clear whether a series expansion of the form (1.18) exists. In order to develop a model in a mathematically rigorous fashion by using such a series expansion makes it necessary to justify the results obtained, for example by the derivation of an error estimate of the form

$$\left| y_\varepsilon(\tau) - \sum_{j=0}^{N} \varepsilon^j y_j(\tau) \right| \le C_N \varepsilon^{N+1} . \tag{1.21}$$

We show such an estimate for $N = 1$, i.e.,

$$|y_\varepsilon(\tau) - y_0(\tau) - \varepsilon\, y_1(\tau)| \le C\varepsilon^2 , \tag{1.22}$$

for $\tau \in (0, T)$ for an appropriate final time T, and ε small enough in the sense that $\varepsilon < \varepsilon_0$ for an appropriate ε_0 to be specified. As a first step we construct a differential equation for the *error*

$$z_\varepsilon(\tau) = y_\varepsilon(\tau) - y_0(\tau) - \varepsilon\, y_1(\tau) .$$

From the differential equations for y_ε, y_0, and y_1 we obtain

$$z_\varepsilon''(\tau) = y_\varepsilon''(\tau) - y_0''(\tau) - \varepsilon\, y_1''(\tau) = -\frac{1}{(1 + \varepsilon\, y_\varepsilon(\tau))^2} + 1 - 2\varepsilon\, y_0(\tau) .$$

A Taylor expansion with a representation of the remainder leads to

$$\frac{1}{(1+y)^2} = 1 - 2y + 3\frac{1}{(1+\vartheta y)^4} y^2,$$

where $\vartheta = \vartheta(y) \in (0, 1)$. Hence, we obtain

$$z_\varepsilon''(\tau) = -1 + 2\varepsilon\, y_\varepsilon(\tau) - 3\varepsilon^2 \frac{1}{(1 + \varepsilon\vartheta\, y_\varepsilon(\tau))^4} y_\varepsilon^2(\tau) + 1 - 2\varepsilon\, y_0(\tau) .$$

Substitution of $y_\varepsilon(\tau) = z_\varepsilon(\tau) + y_0(\tau) + \varepsilon\, y_1(\tau)$ leads to

$$z_\varepsilon''(\tau) = 2\varepsilon\, z_\varepsilon(\tau) + \varepsilon^2 R_\varepsilon(\tau), \tag{1.23}$$

where

$$R_\varepsilon(\tau) = -\frac{3\, y_\varepsilon^2(\tau)}{(1 + \varepsilon\vartheta\, y_\varepsilon(\tau))^4} + 2y_1(\tau) .$$

Additionally the initial conditions

$$z_\varepsilon(0) = 0 \ \text{ and } \ z_\varepsilon'(0) = 0$$

are valid. For an estimation of $R_\varepsilon(\tau)$ we need *lower* and *upper* bounds for $y_\varepsilon(\tau)$. These can be derived from the differential equation for y_ε and the representation

$$
\begin{aligned}
y_\varepsilon(\tau) = y_\varepsilon(0) + \int_0^\tau y'_\varepsilon(t)\,dt &= \int_0^\tau \left(y'_\varepsilon(0) + \int_0^t y''_\varepsilon(s)\,ds \right) dt \\
&= \tau + \int_0^\tau \int_0^t y''_\varepsilon(s)\,ds\,dt\,.
\end{aligned}
$$

We set $t_\varepsilon := \inf \left\{ t \,\middle|\, t > 0,\;\; y_\varepsilon(t) < 0 \right\}$. Obviously it follows from (1.16), that $y''_\varepsilon(\tau) \geq -1$ for $0 < \tau < t_\varepsilon$ and therefore

$$
y_\varepsilon(\tau) \geq \tau - \frac{1}{2}\tau^2\,.
$$

As y_ε is continuous, in particular we have $t_\varepsilon \geq 2$. Because of $y''_\varepsilon(\tau) \leq 0$ for $\tau < t_\varepsilon$ we also have $y_\varepsilon(\tau) \leq t_\varepsilon$ for $\tau < t_\varepsilon$. Now let $T \leq t_\varepsilon$ be chosen and fixed, for example $T = 2$. Then we have

$$
|R_\varepsilon(\tau)| \leq 3|y_\varepsilon(\tau)|^2 + 2|y_1(\tau)| \leq C_1
$$

for $\tau < T$ with a constant $C_1 = C_1(T)$. For a given $C_0 > 0$ we define a further instant of time $\tau_\varepsilon > 0$ by $\tau_\varepsilon := \inf \left\{ t \,\middle|\, t > 0,\;\; |z_\varepsilon(t)| \geq C_0\varepsilon^2 \right\}$. As z_ε is continuous and $z_\varepsilon(0) = 0$, we have $\tau_\varepsilon > 0$. For $\tau < \min(T, \tau_\varepsilon)$ we conclude from (1.23) that

$$
|z_\varepsilon(\tau)| = \left| \int_0^\tau \int_0^t z''_\varepsilon(s)\,ds\,dt \right| \leq \int_0^\tau \int_0^t |z''_\varepsilon(s)|\,ds\,dt \leq \frac{1}{2}T^2(2C_0\varepsilon + C_1)\varepsilon^2\,.
$$

Then for $C_0 > T^2C_1$ there exists a $\varepsilon_0 > 0$, such that $\frac{1}{2}T^2(2C_0\varepsilon_0 + C_1) = \frac{C_0}{2}$, namely

$$
\varepsilon_0 = \frac{1}{2T^2} - \frac{C_1}{2C_0}\,.
$$

For all $\varepsilon \leq \varepsilon_0$ and all $t \leq \min\{T, \tau_\varepsilon\}$ it holds true that

$$
|z_\varepsilon(t)| \leq \frac{C_0}{2}\varepsilon^2\,.
$$

As z_ε is continuous, in particular we have $\tau_\varepsilon \geq T$. Hence, (1.22) has been shown, for $C = C_0/2$.

The procedure to determine an asymptotic expansion can also be formulated more generally and abstractly in *Banach spaces*, i.e., in complete, normed vector spaces. Let B_1, B_2 be Banach spaces and

$$
F : B_1 \times [0, \varepsilon_0) \to B_2
$$

be a smooth mapping, which is sufficiently differentiable for the following considerations. For $\varepsilon \in [0, \varepsilon_0)$ we look for a solution y_ε of the equation

$$F(y, \varepsilon) = 0\,.$$

We make the ansatz

$$y_\varepsilon = \sum_{i=0}^{\infty} \varepsilon^i y_i$$

and expand

$$\begin{aligned}
F(y_\varepsilon, \varepsilon) &= \sum_{i=0}^{\infty} \varepsilon^i F_i(y_\varepsilon) = \sum_{i=0}^{\infty} \varepsilon^i F_i\left(\sum_{j=0}^{\infty} \varepsilon^j y_j\right) \\
&= \sum_{i=0}^{\infty} \varepsilon^i \left(F_i(y_0) + DF_i(y_0)\left(\sum_{j=1}^{\infty} \varepsilon^j y_j\right)\right. \\
&\quad \left. + \tfrac{1}{2} D^2 F_i(y_0)\left(\sum_{j=1}^{\infty} \varepsilon^j y_j, \sum_{j=1}^{\infty} \varepsilon^j y_j\right) + \dots \right) \\
&= F_0(y_0) + \varepsilon(F_1(y_0) + DF_0(y_0)(y_1)) + \\
&\quad + \varepsilon^2\big(F_2(y_0) + DF_1(y_0)(y_1) + DF_0(y_0)(y_2) + \tfrac{1}{2} D^2 F_0(y_0)(y_1, y_1)\big) \\
&\quad + \cdots .
\end{aligned}$$

We try to solve these equations successively, with increasing order and obtain

$$\begin{aligned}
F_0(y_0) &= 0\,, \\
DF_0(y_0)(y_1) &= -F_1(y_0)\,, \\
DF_0(y_0)(y_2) &= -F_2(y_0) - DF_1(y_0)(y_1) - \frac{1}{2} D^2 F_0(y_0)(y_1, y_1)\,, \\
&\vdots \\
DF_0(y_0)(y_k) &= G_k(y_0, \dots, y_{k-1})\,.
\end{aligned}$$

If the linear mapping $DF_0(y_0) : B_1 \to B_2$ possesses an inverse then the values $y_1, y_2, y_3, \dots$ can be computed successively.

Definition 1.1 If the values $y_0, \dots, y_N$ are solutions of the above displayed equations, then the series

$$y_\varepsilon^N := \sum_{i=0}^{N} \varepsilon^i y_i$$

is called asymptotic expansion of order N.

An important question now is: Are the solutions for the problems "perturbed" by a small parameter ε a good approximation for the original problem? A positive answer is encoded in the following definition.

Definition 1.2 *(Consistency)* The equations

$$F(y, \varepsilon) = 0, \quad \varepsilon > 0\,,$$

are called *consistent* with

$$F(y, 0) = 0\,,$$

if for all solutions y_0 of $F(y_0, 0) = 0$ it holds true that

$$\lim_{\varepsilon \to 0} F(y_0, \varepsilon) = 0\,.$$

Remarks

1. In general consistency does *not* imply convergence: also in a consistent situation the solutions y_ε of $F(y, \varepsilon) = 0$ does not need to fulfill

$$y_\varepsilon - y_0 \to 0 \quad \text{in} \quad B_1$$

(see also Exercise 1.11).
2. An important case in asymptotic analysis is characterized by the fact that the small parameter appears as a factor in a term which is decisive for the mathematical structure of the problem. For differential equations this term in general is the highest order derivative of the unknown function appearing in the equation. In this case one speaks of a *singular perturbation*. At the end of Chap. 6 we will investigate corresponding examples.

Examples:

(i) The equation

$$\varepsilon\, x^2 - 1 = 0$$

changes its order for $\varepsilon \to 0$. In particular for $\varepsilon = 0$ the equation becomes insolvable and the solutions $x_\varepsilon^\pm$ of $\varepsilon\, x^2 - 1 = 0$ converge to infinity for $\varepsilon \to 0$.

(ii) The initial value problem

$$\varepsilon\, y_\varepsilon'' = \frac{1}{(y_\varepsilon + 1)^2}\,, \quad y_\varepsilon(0) = 0\,, \quad y_\varepsilon'(0) = 1$$

changes its character if one sets $\varepsilon = 0$. For $\varepsilon > 0$ one has a differential equation and for $\varepsilon = 0$ one obtains an insolvable algebraic equation.

1.6 Applications from Fluid Mechanics

Now we will discuss the introduced notions of dimensional analysis, asymptotic expansion and singular perturbation for a considerably more complex example from fluid mechanics. The models which we use will be derived systematically in the framework of continuum mechanics in Chap. 5.

We consider the following example: a fluid, i.e., a liquid or a gas, flows past a body K (Fig. 1.3). We are interested in the velocity field

$$v = v(t, x) \in \mathbb{R}^3,\ t \in \mathbb{R},\ x \in \mathbb{R}^3\,,$$

of the fluid. We assume that for $|x| \to \infty$ the velocity converges to a constant value, i.e.,

$$v(t, x) \to V \in \mathbb{R}^3 \quad \text{as} \quad |x| \to \infty\,.$$

From conservation principles and using certain constitutive assumptions about the properties of the fluid the *Navier–Stokes equations* can be derived, see Chap. 5. For an incompressible fluid with constant density ϱ_0 neglecting exterior forces we obtain

$$\varrho_0(\partial_t v + (v \cdot \nabla)v) = -\nabla p + \mu \varDelta v\,, \tag{1.24}$$

$$\nabla \cdot v = 0\,, \tag{1.25}$$

where p denotes the pressure and μ the *dynamic viscosity* of the fluid. The viscosity is caused by *internal friction*. It is high for honey and low for gases. Furthermore, expressed in Cartesian coordinates we have

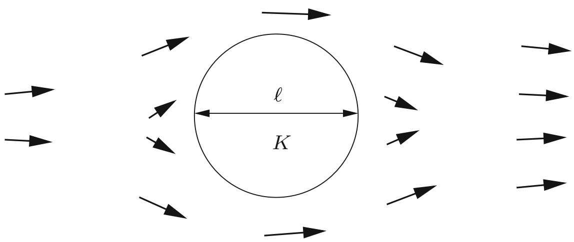

Fig. 1.3 Flow past an obstacle

$$\nabla \cdot v = \sum_{i=1}^{3} \frac{\partial}{\partial x_i} v_i \in \mathbb{R} \qquad \text{for the } \textit{divergence} \text{ of a vector field } v,$$

$$\Delta v = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2} v \in \mathbb{R}^3 \qquad \text{the } \textit{Laplacian operator} \text{ and}$$

$$(v \cdot \nabla) v = \left(\sum_{i=1}^{3} v_i \, \partial_i v_j \right)_{j=1,2,3} \in \mathbb{R}^3 .$$

First we discuss the *dimensions* of the appearing terms.

variables	dimension
v velocity	L/T
ϱ_0 mass density	M/L^3
p pressure = force/area	$(M \cdot L/T^2)/L^2 = M/(LT^2)$

Furthermore we have

$$[\mu] = M/(LT) .$$

As a consequence all terms in (1.24) have the dimension $M/(L^2T^2)$.

As an example for the potential of *dimension analysis* we consider the behavior of a fluid flowing around a large ship. The goal is to perform experiments with a ship whose size is reduced by the factor 100. Under which circumstances the results of such experiments with a ship model can be transferred to the behavior of a large real ship?

For this purpose the equation has to be transferred into a dimensionless form. Here the relevant parameters are the characteristic length $\overline{x}$, for example the length of the ship, the velocity v of the ship, the density ϱ_0, and the viscosity μ. We form dimensionless quantities from combinations of these parameters, for example

$$y = \frac{x}{\overline{x}}, \quad \tau = \frac{t}{\overline{t}} ,$$

where $\overline{t} = \overline{t}(\overline{x}, V, \varrho_0, \mu)$ is a characteristic time still to be determined. Furthermore we set

$$u(\tau, y) = \frac{v}{|V|}$$

and

$$q(\tau, y) = \frac{p}{\overline{p}}, \quad \text{where } \overline{p} \text{ is still to be determined.}$$

Multiplication of the Navier–Stokes equations by $\overline{t}/(\varrho_0|V|)$ and using the transformation rules $\partial_t = \frac{1}{\overline{t}}\partial_\tau$ and $\nabla_x = \frac{1}{\overline{x}}\nabla_y$ leads to the equation

$$\partial_\tau u + \frac{\bar{t}|V|}{\bar{x}}(u \cdot \nabla)u = -\frac{\bar{p}}{\varrho_0}\frac{\bar{t}}{\bar{x}|V|}\nabla q + \frac{\mu}{\varrho_0}\frac{\bar{t}}{(\bar{x})^2}\Delta u\,.$$

We set $\bar{t} = \bar{x}/|V|$, $\bar{p} = |V|^2\varrho_0$ and $\eta = \mu/\varrho_0$ — this is called the *kinematic viscosity* — and obtain

$$\partial_\tau u + (u \cdot \nabla)u = -\nabla q + \frac{1}{\mathrm{Re}}\Delta u\,,$$
$$u(\tau, y) \to V/|V| \quad \text{as} \quad |y| \to \infty\,.$$

Here $\mathrm{Re} := \bar{x}|V|/\eta$ is called the *Reynolds number*. For large $|y|$ the Euclidean norm of the nondimensionalized velocity converges to 1. Furthermore it still holds that

$$\nabla \cdot u = 0\,.$$

This means that flow situations with different $\bar{x}$ lead to the *same* dimensionless form, if the Reynolds number is the same for the different situations. If we reduce the size of the ship by the factor 100, then one possibility is to enlarge the approach velocity by the factor 10 and to reduce the kinematic viscosity by the factor 10 to obtain the same Reynolds number.

With the help of the Reynolds number one can estimate which effects are of importance for a flow and which effects are not. We discuss this for the example of two different models for the flow resistance of a body, which we motivate by means of heuristic considerations. For the case of small Reynolds numbers, i.e., for high viscosity or a small approach velocity, the viscous friction dominates the flow resistance. Then the characteristic quantities are the velocity v of the obstacle relative to the flow, a characteristic quantity $\bar{x}$ for the size of the obstacle, and the dynamic viscosity μ of the fluid. The dimensions are given by $[v] = L/T$, $[\bar{x}] = L$, and $[\mu] = FT/L^2$, where F denotes the dimension of force. Then a combination of these quantities has the dimension

$$\left[v^a \bar{x}^b \mu^c\right] = L^{a+b-2c}T^{-a+c}F^c\,.$$

This is the dimension of force, if

$$a + b - 2c = 0, \quad -a + c = 0, \text{ and } c = 1$$

and therefore $a = b = c = 1$. A law for the friction resistance in a viscous fluid therefore has to have the form

$$F_R = -c_R\,\mu\,\bar{x}\,v \tag{1.26}$$

where F_R is the friction force acting on the body, v the velocity of the body relative to the flow velocity, and c_R is the friction coefficient which is depending on the shape of the body. For a sphere with radius r it can be shown that

$$F_R = -6\pi r \mu v$$

holds true, and this is called *Stokes' law*.

For high Reynolds numbers, however, this part of the flow resistance is dominated by a force which is necessary to accelerate the part of the fluid lying in the direction of the motion of the body. The mass to be accelerated per time interval Δt may fulfill

$$\Delta m \approx \varrho\, A\, |v|\, \Delta t\,,$$

where ϱ is the density of the fluid and A the cross sectional area of the body. Here the term $A\,|v|\,\Delta t$ just describes the volume replaced by the body in the time interval Δt. This fluid volume is accelerated to velocity v. The supplied kinetic energy is

$$\Delta E_{\text{kin}} \approx \tfrac{1}{2}\Delta m\, |v|^2 \approx \tfrac{1}{2}\varrho\, A\, |v|^3\, \Delta t\,.$$

The friction force is related to the kinetic energy by

$$|F_R|\,|v|\,\Delta t \propto \Delta E_{\text{kin}}$$

where $\propto$ indicates that the two sides are proportional to each other. Hence, we obtain

$$|F_R| \propto \tfrac{1}{2}\varrho\, A\, |v|^2\,.$$

The corresponding proportionality constant c_d is called drag coefficient and we obtain

$$F_R = -\tfrac{1}{2}c_d\,\varrho A|v|v\,. \tag{1.27}$$

As this force is proportional to the square of the velocity, for large velocities it dominates the viscous frictional force (1.26), on the other hand for small velocities it can be neglected compared to (1.26). Formula (1.27) can also be justified by a dimensional analysis (see Exercise 1.12). In applications the drag coefficient has to be determined by measurements since for most body shapes a theoretical derivation as for a sphere in the case of Stokes' law does not exist any more. In any case (1.27) is only a relatively coarse approximation to reality, the real dependence of the flow resistance on the velocity is considerably more complex. On the other hand Stokes' law is a relatively good approximation, if only the velocity is sufficiently small.

In order to assess for a given application which of the two laws (1.26) or (1.27) is reasonable, the coefficient of the two frictional forces can be considered:

$$\frac{|F_R^{(1.27)}|}{|F_R^{(1.26)}|} \propto \frac{\varrho \overline{x}|v|}{\mu} = \text{Re}\,.$$

In doing so we choose the scale $\overline{x}$ such that $A = \overline{x}^2$. Therefore Stokes' law (1.26) makes sense for Reynolds numbers $\text{Re} \ll 1$. On the other hand the flow resistance given by (1.27) dominates for $\mathrm{Re} \gg 1$. For $\mathrm{Re} \approx 1$ both effects have the same importance.

The Navier–Stokes equations being a complex model the question arises whether certain terms can be neglected in specific situations. As we have transferred the equation to a nondimensional form it is possible to speak of large or small independent of the choice of units: now the number 1 can be interpreted as a *medium sized* quantity. The only parameter is the Reynolds number and for many problems Re is very large. Then $\varepsilon = 1/\mathrm{Re}$ is a small term which suggests to neglect the term $\varepsilon \Delta u = \frac{1}{\mathrm{Re}} \Delta u$. In this way we obtain the *Euler equations* of fluid mechanics

$$\partial_\tau u + (u \cdot \nabla) u = -\nabla q \,,$$
$$\nabla \cdot u = 0 \,.$$

How good is the description of a real fluid by this reduced model? Later we will see that the Euler equations do not allow for the formation of vortices (see Sect. 6.1.4). Most specifically we have

$$\nabla \times u(t,x) = \begin{pmatrix} \partial_{x_2} u_3 - \partial_{x_3} u_2 \\ \partial_{x_3} u_1 - \partial_{x_1} u_3 \\ \partial_{x_1} u_2 - \partial_{x_2} u_1 \end{pmatrix} = 0$$

for $t > 0$ if $\nabla \times u(0,x) = 0$. Additionally it turns out that

- the term $\varepsilon \Delta u$ *cannot* be small in the vicinity of the boundary ∂K of the body,

and if the Euler equations describe the flow,

- a bounded body would exert no resistance to the flow, therefore no forces would act against the flow. This is called *d'Alembert's paradox* and
- no lift force would act on the body (in 3-D).

In reality we see that a flow past an obstacle induces the formation of vortices, which sometimes separate from K. Additionally so-called *boundary layers* can be observed. These are "thin" regions of the flow in the vicinity of the body in which the flow field changes drastically.

What is the deficiency of the reduced model?

In the Navier–Stokes equations second derivatives with respect to the spatial variables appear, but in the Euler equations only first derivatives. In the theory of partial differential equations different types of equations are distinguished (see for example Evans, [37]). According to this classification the Navier–Stokes equations are *parabolic*, but the Euler equations are *hyperbolic*. The qualitative behavior of hyperbolic and parabolic differential equations differs considerably. For example in the solutions of the Euler equations even for arbitrary smooth data discontinuities may occur, on the other hand for the Navier–Stokes equations smooth solutions are to be expected in this case, even if the rigorous theoretical proof for three dimensions is still outstanding. The small factor ε belongs to the term which is decisive for the behavior of the solution. Therefore one speaks of *singular perturbation*. To obtain approximate solutions of the Navier–Stokes equations the method of asymptotic expansion cannot be used in the form we have discussed so far. It breaks down in boundary layers, where the solution changes strongly. In Chap. 6 we will develop the *singular perturbation theory*, to obtain asymptotic expansions also in boundary layers.

1.7 Literature

An extensive description and analysis of biological growth models can be found in [105]. For further information on the subjects nondimensionalization, scaling, and asymptotic analysis we recommend [89], Chaps. 6 and 7, for scaling and dimensional analysis also [41], Chap. 1, is a good reference and for various aspects of asymptotic analysis we refer to [68]. A presentation of singular perturbation theory with many examples can be found in [77]. Parts of the presentation in this chapter are based on the lecture notes [116].

1.8 Exercises

Exercise 1.1 A bank offers four different variants of a savings account:
variant A with monthly payment of interest and an interest rate of 0.3% per month,
variant B with a quarterly payment of interest and an interest rate of 0.9% per quarter,
variant C with a semiannual payment of interest and an interest rate of 1.8% per half-year,
variant D with an annual payment of interest and an interest rate of 3.6% per year.

(a) Compute and compare the *effective* interest rate which is obtained after a year (reinvesting all paid interest).
(b) How must the interest rates be adjusted, such that they lead to the same yearly interest rate of 3.6%?
(c) Develop an interest model that is continuous in time, which does not need a time increment for the payment of interest.

Exercise 1.2 A police officer wants to determine the time of death of the victim of a homicide. He measures the temperature of the victim at 12.36 p.m. and obtains 80°F. According to Newton's law of cooling the cooling of a body is proportional to the difference between the body's temperature and the ambient temperature. Unfortunately the proportionality constant is unknown to the officer. Therefore he measures the temperature at 1.06 p.m. once more and now he obtains 77°F. The ambient

temperature is 68°F and it is assumed that the body's temperature at the time of death has been 98°F.
At what time the homicide took place?

Exercise 1.3 (*Separation of variables, uniqueness, continuation of solutions*) For given functions f and g consider the ordinary differential equation

$$x'(t) = f(t)\, g(x(t)).$$

We look for solutions passing through the point (t_0, x_0), i.e., $x(t_0) = x_0$ holds true.

(a) Show that in the case $g(x_0) \neq 0$ locally a unique solution through the given point exists.
(b) Assume that $g \neq 0$ in the interval (x_-, x_+), where $g(x_-) = g(x_+) = 0$ and let g be differentiable at x_- and x_+. Show that the solution of the differential equation through the point (t_0, x_0), where $x_0 \in (x_-, x_+)$, exists globally and is unique.
Hint: Is the solution through the point (t_+, x_+) unique?

Exercise 1.4 We consider the model of limited growth of populations

$$x'(t) = q\, x_M\, x(t) - q\, x^2(t)\,, \quad x(0) = x_0\,.$$

(a) Nondimensionalize the model using appropriate units for t and x. Which possibilities exist?
(b) What nondimensionalization is appropriate for $x_0 \ll x_M$ (x_0 "much smaller than" x_M) in the sense that omitting small terms leads to a reasonable model?

Exercise 1.5 (*Nondimensionalization, scale analysis*) A body of mass m is thrown upwards in a vertical direction from the Earth's surface with a velocity v. The air resistance is supposed to be taken into account by Stokes' law $F_R = -cv$ for the flow resistance in viscous fluids, which is reasonable for small velocities. Here c is a coefficient depending on the shape and the size of the body. The motion is supposed to depend on the mass m, the velocity v, the gravitational acceleration g and the friction coefficient c with dimension $[c] = M/T$.

(a) Determine the possible dimensionless parameters and reference values for height and time.
(b) The initial value problem for the height is assumed to take the form

$$mx'' + cx' = -mg\,, \quad x(0) = 0\,, \quad x'(0) = v\,.$$

Nondimensionalize the differential equation. Again different possibilities are available.
(c) Discuss the different possibilities of a reduced model if $\beta := cv/(mg)$ is small.

Exercise 1.6 A model for the vertical throw on the Earth taking into account the air resistance is given by

$$mx''(t) = -mg - c|x'(t)|x'(t)\,, \quad x(t_0) = 0\,, \quad x'(t_0) = v_0\,.$$

In this model the gravitational force is approximated by $F = -mg$, the air resistance for a given velocity v is described by $-c|v|v$ with a proportionality constant c depending on the shape and size of the body and the density of the air. This law is reasonable for high velocities.

(a) Nondimensionalize the model. What possibilities exist?
(b) Compute the maximal height of the throw for the data $m = 0.1\,\mathrm{kg}$, $g = 10\,\mathrm{m/s^2}$, $v_0 = 10\,\mathrm{m/s}$, $c = 0.01\,\mathrm{kg/m}$ and compare the result with the corresponding result for the model without air resistance.

Exercise 1.7 (*Nondimensionalization*) We want to compute the power P, which is necessary to move a body with known shape (for example a ship) in a liquid (for example water). We assume that the power depends on the length ℓ and the velocity v of the ship, the density ϱ and the kinematic viscosity η of the liquid, and the gravitational acceleration g. The dimensions of the data are $[\ell] = L$, $[\varrho] = M/L^3$, $[v] = L/T$, $[\eta] = L^2/T$, $[P] = ML^2/T^3$, and $[g] = L/T^2$, where L denotes the length, M the mass and T the time. Show that under these assumptions the power P is given by

$$\frac{P}{\varrho \ell^2 v^3} = \Phi(\mathrm{Fr}, \mathrm{Re})$$

with a function $\Phi : \mathbb{R}^2 \to \mathbb{R}$ and the dimensionless quantities

$$\mathrm{Re} = \frac{|v|\ell}{\eta} \text{ (Reynolds number) and } \mathrm{Fr} = \frac{|v|}{\sqrt{\ell g}} \text{ (Froude number).}$$

Exercise 1.8 (*Formal asymptotic expansion*)

(a) For the initial value problem

$$x''(t) + \varepsilon\, x'(t) = -1\,, \quad x(0) = 0\,, \quad x'(0) = 1$$

compute the formal asymptotic expansion of the solution $x(t)$ up to the second order in ε.

(b) Compute the formal asymptotic expansion for the instance of time $t^* > 0$, for which $x(t^*) = 0$ holds true, up to first order in ε, by substituting the series expansion $t^* \sim t_0 + \varepsilon\, t_1 + O(\varepsilon^2)$ into the approximation obtained for x leading to a determination of t_0 and t_1.

Exercise 1.9 A model already nondimensionalized for the vertical throw with *small* air resistance is given by

$$x''(t) = -1 - \varepsilon(x'(t))^2\,, \quad x(0) = 0,\; x'(0) = 1\,.$$

The model describes the throw up to the maximal height.

(a) Compute the first two coefficients $x_0(t)$ and $x_1(t)$ in the asymptotic expansion

$$x(t) = x_0(t) + \varepsilon\, x_1(t) + \varepsilon^2 x_2(t) + \cdots$$

for small ε.

(b) Compute the maximal height of the throw up to terms of order ε using asymptotic expansion.

(c) Compare the results from (b) for the data of Exercise 1.6(b) with the exact result and the result neglecting the air resistance.

Exercise 1.10 (*Multiscale approach*) The function $y(t)$ is supposed to solve the initial value problem

$$y''(t) + 2\varepsilon\, y'(t) + (1 + \varepsilon^2)y(t) = 0\,, \quad y(0) = 0\,, \quad y'(0) = 1\,,$$

for $t > 0$ and a small parameter $\varepsilon > 0$.

(a) Compute the approximation of the solution by means of formal asymptotic expansion up to first order in ε.

(b) Compare the function obtained in (a) with the exact solution

$$y(t) = e^{-\varepsilon t} \sin t\,.$$

For which times t the approximation from (a) is good?

(c) To get a better approximation one can try the approach

$$y \sim y_0(t, \tau) + \varepsilon\, y_1(t, \tau) + \varepsilon^2 y_2(t, \tau) + \cdots\,,$$

here $\tau = \varepsilon t$ is a slow time scale.
Substitute this ansatz in the differential equation and compute y_0 such that the approximation becomes better.
Hint: The equation of lowest order does not determine y_0 uniquely and coefficient functions in τ appear. Choose them in a clever way such that y_1 is easily computable.

Exercise 1.11 (*Consistency versus convergence*) For a parameter $\varepsilon \in [0, \varepsilon_0)$ with $\varepsilon_0 > 0$ we consider the family of operators

$$F(\cdot, \varepsilon) : B_1 := C_b^2([0, \infty)) \to B_2 := C_b^0([0, \infty)) \times \mathbb{R}^2\,,$$
$$F(y, \varepsilon) = (y'' + (1 + \varepsilon)y,\, y(0),\, y'(0) - 1)\,.$$

Here $C_b^n([0,\infty))$ denotes the vector space of n-times differentiable functions with bounded derivatives up to order n. The norms of the spaces B_1 and B_2 are given by

$$\|y\|_{B_1} = \sup_{t\in(0,\infty)} \{|y(t)| + |y'(t)| + |y''(t)|\},$$
$$\|(f,a,b)\|_{B_2} = \sup_{t\in(0,\infty)} \{|f(t)|\} + |a| + |b|.$$

(a) For the problem $F(y,\varepsilon) = (0,0,0)$ compute the exact solution y_ε.
(b) Show: $F(\cdot,\varepsilon)$ is consistent with $F(\cdot,0)$, but y_ε does not converge to y_0 in B_1 as $\varepsilon \to 0$.

Exercise 1.12 Derive the friction law for the flow resistance in the case of high Reynolds numbers,

$$F_R = -\tfrac{1}{2} c_W A\, \varrho\, |v| v\,,$$

by means of a dimensional analysis. Use the assumptions that the frictional force depends on the density ϱ of the liquid, a characteristic quantity r of the body, and the velocity v of the flow. As the drag coefficient depends on the shape of the body choosing r such that $A \approx r^2$ is feasible.

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