Chapter 2
Wormhole Basics

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2.1 Static and Spherically Symmetric Traversable Wormholes

2.1.1 Spacetime Metric

Throughout this book, unless stated otherwise, we will consider the following spherically symmetric and static wormhole solution [1]

$$ds^2 = -e^{2\Phi(r)}\, dt^2 + \frac{dr^2}{1 - b(r)/r} + r^2\,(d\theta^2 + \sin^2\theta\, d\phi^2)\,. \tag{2.1}$$

The metric functions $\Phi(r)$ and $b(r)$ are arbitrary functions of the radial coordinate r. As $\Phi(r)$ is related to the gravitational redshift, it has been denoted the redshift function, and $b(r)$ is called the shape function, as it determines the shape of the wormhole [1–3], which will be shown below using embedding diagrams. The radial coordinate r is non-monotonic in that it decreases from $+\infty$ to a minimum value r_0, representing the location of the throat of the wormhole, where $b(r_0) = r_0$, and then increases from r_0 to $+\infty$. Although the metric coefficient g_{rr} becomes divergent at the throat, which is signalled by the coordinate singularity, the proper radial distance $l(r) = \pm \int_{r_0}^{r} [1 - b(r)/r]^{-1/2}\, dr$ is required to be finite everywhere. The proper distance decreases from $l = +\infty$, in the upper universe, to $l = 0$ at the throat, and then from zero to $-\infty$ in the lower universe. One must verify the absence of horizons,

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in order for the wormhole to be traversable. This condition must imply that $g_{tt} = -e^{2\Phi(r)} \neq 0$, so that $\Phi(r)$ must be finite everywhere.[1]

Another interesting feature of the redshift function is that its derivative with respect to the radial coordinate also determines the "attractive" or "repulsive" nature of the geometry. In order to verify this, consider the four-velocity of a static observer given by $U^\mu = dx^\mu/d\tau = (e^{-\Phi(r)}, 0, 0, 0)$. The observer's four-acceleration is $a^\mu = U^\mu{}_{;\nu}\, U^\nu$, which has the following components:

$$a^t = 0\,, \qquad a^r = \Phi' \left(1 - \frac{b}{r}\right)\,, \tag{2.2}$$

where the prime denotes a derivative with respect to the radial coordinate r. Now, note that from the geodesic equation, a radially moving test particle which starts from rest initially has the equation of motion

$$\frac{d^2 r}{d\tau^2} = -\Gamma^r_{tt} \left(\frac{dt}{d\tau}\right)^2 = -a^r\,. \tag{2.3}$$

Here, a^r is the radial component of proper acceleration that an observer must maintain in order to remain at rest at constant $r,\ \theta,\ \phi$, so that from Eq. (2.2), a static observer at the throat for generic $\Phi(r)$ is a geodesic observer. In particular, for a constant redshift function, $\Phi'(r) = 0$, static observers are also geodesic. Thus, a wormhole is "attractive" if $a^r > 0$, i.e. observers must maintain an outward-directed radial acceleration to keep from being pulled into the wormhole. If $a^r < 0$, the geometry is "repulsive", i.e. observers must maintain an inward-directed radial acceleration to avoid being pushed away from the wormhole. Indeed, this distinction depends on the sign of Φ', as is transparent from Eq. (2.2).

2.1.2 The Mathematics of Embedding

We can use embedding diagrams to represent a wormhole and extract some useful information for the choice of the shape function, $b(r)$. Due to the spherically symmetric nature of the problem, one may consider an equatorial slice, $\theta = \pi/2$, without loss of generality. The respective line element, considering a fixed moment of time, $t =$ const, is given by

$$ds^2 = \frac{dr^2}{1 - b(r)/r} + r^2\, d\phi^2\,. \tag{2.4}$$

[1] This follows from a result originally due to C.V. Vishveshwara stated as follows: In any asymptotically flat spacetime with a Killing vector ξ ($\xi = \mathbf{e_0}$ for the metric (2.1)) which (i) is the ordinary time-translation Killing vector at spatial infinity and (ii) is orthogonal to a family of three-dimensional surfaces, the 3-surface $\xi \cdot \xi = 0$, i.e. $\mathbf{e_0} \cdot \mathbf{e_0} = g_{tt} = 0$, is a null surface that cannot be crossed by any outgoing, future-directed timelike curves, i.e. a horizon.

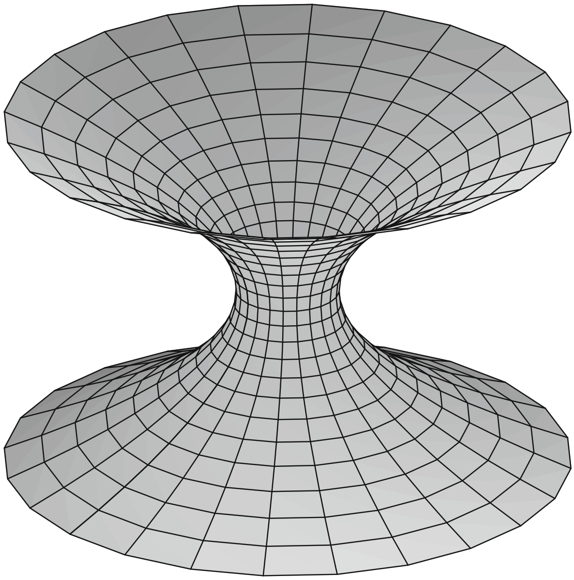

Fig. 2.1 The embedding diagram of a two-dimensional section along the equatorial plane ($t = \text{const}, \theta = \pi/2$) of a traversable wormhole. For a full visualization of the surface sweep through a 2π rotation around the $z-$axis, as can be seen from the graphic on the right

To visualize this slice, one embeds this metric into three-dimensional Euclidean space, in which the metric can be written in cylindrical coordinates, (r, ϕ, z), as

$$ds^2 = dz^2 + dr^2 + r^2\, d\phi^2\,. \tag{2.5}$$

In the three-dimensional Euclidean space the embedded surface has equation $z = z(r)$, so that the metric of the surface can be written as

$$ds^2 = \left[1 + \left(\frac{dz}{dr}\right)^2\right] dr^2 + r^2\, d\phi^2\,. \tag{2.6}$$

Comparing Eq. (2.4) with (2.6), one deduces the equation for the embedding surface, which is given by

$$\frac{dz}{dr} = \pm\left(\frac{r}{b(r)} - 1\right)^{-1/2}\,. \tag{2.7}$$

To be a solution of a wormhole, the geometry has a minimum radius, $r = b(r) = r_0$, denoted as the throat, at which the embedded surface is vertical, i.e. $dz/dr \to \infty$. Far from the throat, one may consider that space is asymptotically flat, $dz/dr \to 0$ as $r \to \infty$.

To be a solution of a wormhole, one also needs to impose that the throat flares out (see Fig. 2.1 for details). This flaring-out condition entails that the inverse of the embedding function $r(z)$ must satisfy $d^2r/dz^2 > 0$ at or near the throat r_0. Differentiating $dr/dz = \pm(r/b(r) - 1)^{1/2}$ with respect to z, we have

$$\frac{d^2 r}{dz^2} = \frac{b - b'r}{2b^2} > 0 \,. \tag{2.8}$$

This "flaring-out" condition is a fundamental ingredient of wormhole physics, and plays a fundamental role in the analysis of the violation of the energy conditions. At the throat we verify that the form function satisfies the condition $b'(r_0) < 1$. Note, however, that this treatment has the drawback of being coordinate dependent, and we refer the reader to Refs. [4, 5] for a covariant treatment.

2.1.3 Equations of Structure for the Wormhole

From the metric expressed in the form $ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu$, one may determine the Christoffel symbols (connection coefficients), $\Gamma^\mu{}_{\alpha\beta}$, defined as

$$\Gamma^\mu{}_{\alpha\beta} = \frac{1}{2}\, g^{\mu\nu} \, \left(g_{\nu\alpha,\beta} + g_{\nu\beta,\alpha} - g_{\alpha\beta,\nu} \right) \,, \tag{2.9}$$

which for the metric (2.1) have the following nonzero components:

$$\begin{aligned}
&\Gamma^t{}_{rt} = \Phi' \,, \qquad \Gamma^r{}_{tt} = \left(1 - \frac{b}{r}\right) \Phi'\, e^{2\Phi} \,, \qquad \Gamma^r{}_{rr} = \frac{b'r - b}{2r(r-b)} \,, \\
&\qquad\qquad \Gamma^r{}_{\theta\theta} = -r + b \,, \qquad \Gamma^r{}_{\phi\phi} = -(r-b)\, \sin^2\theta \,, \\
&\Gamma^\theta{}_{r\theta} = \Gamma^\phi{}_{r\phi} = \frac{1}{r} \,, \qquad \Gamma^\theta{}_{\phi\phi} = -\sin\theta\, \cos\theta \,, \qquad \Gamma^\phi{}_{\theta\phi} = \tan\theta \,.
\end{aligned} \tag{2.10}$$

The Riemann tensor is defined as

$$R^\alpha{}_{\beta\gamma\delta} = \Gamma^\alpha{}_{\beta\delta,\gamma} - \Gamma^\alpha{}_{\beta\gamma,\delta} + \Gamma^\alpha{}_{\lambda\gamma}\Gamma^\lambda{}_{\beta\delta} - \Gamma^\alpha{}_{\lambda\delta}\Gamma^\lambda{}_{\beta\gamma} \,. \tag{2.11}$$

However, the mathematical analysis and the physical interpretation is simplified using a set of orthonormal basis vectors. These may be interpreted as the proper reference frame of a set of observers who remain at rest in the coordinate system (t, r, θ, ϕ), with (r, θ, ϕ) fixed. Denote the basis vectors in the coordinate system as $(\mathbf{e}_t, \mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi)$. Thus, the orthonormal basis vectors are given by

$$\begin{cases}
\mathbf{e}_{\hat{t}} = e^{-\Phi}\; \mathbf{e}_t \\
\mathbf{e}_{\hat{r}} = (1 - b/r)^{1/2}\; \mathbf{e}_r \\
\mathbf{e}_{\hat{\theta}} = r^{-1}\; \mathbf{e}_\theta \\
\mathbf{e}_{\hat{\phi}} = (r \sin\theta)^{-1}\; \mathbf{e}_\phi
\end{cases} \,. \tag{2.12}$$

The nontrivial Riemann tensor components, given in the orthonormal reference frame, take the following form:

$$R^{\hat{t}}{}_{\hat{r}\hat{t}\hat{r}} = -R^{\hat{t}}{}_{\hat{r}\hat{r}\hat{t}} = R^{\hat{r}}{}_{\hat{t}\hat{t}\hat{r}} = -R^{\hat{r}}{}_{\hat{t}\hat{r}\hat{t}} = \left(1-\frac{b}{r}\right)\left[-\Phi'' - (\Phi')^2 + \frac{b'r-b}{2r(r-b)}\Phi'\right], \tag{2.13}$$

$$R^{\hat{t}}{}_{\hat{\theta}\hat{t}\hat{\theta}} = -R^{\hat{t}}{}_{\hat{\theta}\hat{\theta}\hat{t}} = R^{\hat{\theta}}{}_{\hat{t}\hat{t}\hat{\theta}} = -R^{\hat{\theta}}{}_{\hat{t}\hat{\theta}\hat{t}} = -\left(1-\frac{b}{r}\right)\frac{\Phi'}{r}, \tag{2.14}$$

$$R^{\hat{t}}{}_{\hat{\phi}\hat{t}\hat{\phi}} = -R^{\hat{t}}{}_{\hat{\phi}\hat{\phi}\hat{t}} = R^{\hat{\phi}}{}_{\hat{t}\hat{t}\hat{\phi}} = -R^{\hat{\phi}}{}_{\hat{t}\hat{\phi}\hat{t}} = -\left(1-\frac{b}{r}\right)\frac{\Phi'}{r}, \tag{2.15}$$

$$R^{\hat{r}}{}_{\hat{\theta}\hat{r}\hat{\theta}} = -R^{\hat{r}}{}_{\hat{\theta}\hat{\theta}\hat{r}} = R^{\hat{\theta}}{}_{\hat{r}\hat{\theta}\hat{r}} = -R^{\hat{\theta}}{}_{\hat{r}\hat{r}\hat{\theta}} = \frac{b'r-b}{2r^3}, \tag{2.16}$$

$$R^{\hat{r}}{}_{\hat{\phi}\hat{r}\hat{\phi}} = -R^{\hat{r}}{}_{\hat{\phi}\hat{\phi}\hat{r}} = R^{\hat{\phi}}{}_{\hat{r}\hat{\phi}\hat{r}} = -R^{\hat{\phi}}{}_{\hat{r}\hat{r}\hat{\phi}} = \frac{b'r-b}{2r^3}, \tag{2.17}$$

$$R^{\hat{\theta}}{}_{\hat{\phi}\hat{\theta}\hat{\phi}} = -R^{\hat{\theta}}{}_{\hat{\phi}\hat{\phi}\hat{\theta}} = R^{\hat{\phi}}{}_{\hat{\theta}\hat{\phi}\hat{\theta}} = -R^{\hat{\phi}}{}_{\hat{\theta}\hat{\theta}\hat{\phi}} = \frac{b}{r^3}, \tag{2.18}$$

where, as before, a prime denotes a derivative with respect to the radial coordinate r.

The Ricci tensor, $R_{\hat{\mu}\hat{\nu}}$, is given by the contraction $R_{\hat{\mu}\hat{\nu}} = R^{\hat{\alpha}}{}_{\hat{\mu}\hat{\alpha}\hat{\nu}}$, and the nonzero components are the following:

$$R_{\hat{t}\hat{t}} = \left(1-\frac{b}{r}\right)\left[\Phi'' + (\Phi')^2 - \frac{b'r-3b+4r}{2r(r-b)}\Phi'\right], \tag{2.19}$$

$$R_{\hat{r}\hat{r}} = -\left(1-\frac{b}{r}\right)\left[\Phi'' + (\Phi')^2 + \frac{b-b'r}{2r(r-b)}\Phi' + \frac{b-b'r}{r^2(r-b)}\right], \tag{2.20}$$

$$R_{\hat{\theta}\hat{\theta}} = R_{\hat{\phi}\hat{\phi}} = \left(1-\frac{b}{r}\right)\left[\frac{b'r+b}{2r^2(r-b)} - \frac{\Phi'}{r}\right]. \tag{2.21}$$

The curvature scalar or Ricci scalar, defined by $R = g^{\hat{\mu}\hat{\nu}}\,R_{\hat{\mu}\hat{\nu}}$, is given by

$$R = -2\left(1-\frac{b}{r}\right)\left[\Phi'' + (\Phi')^2 - \frac{b'}{r(r-b)} - \frac{b'r+3b-4r}{2r(r-b)}\Phi'\right]. \tag{2.22}$$

Thus, the Einstein tensor, given in the orthonormal reference frame by $G_{\hat{\mu}\hat{\nu}} = R_{\hat{\mu}\hat{\nu}} - \frac{1}{2}R\,g_{\hat{\mu}\hat{\nu}}$, yields for the metric (2.1), the following nonzero components:

$$G_{\hat{t}\hat{t}} = \frac{b'}{r^2}, \tag{2.23}$$

$$G_{\hat{r}\hat{r}} = -\frac{b}{r^3} + 2\left(1-\frac{b}{r}\right)\frac{\Phi'}{r}, \tag{2.24}$$

$$G_{\hat{\theta}\hat{\theta}} = \left(1-\frac{b}{r}\right)\left[\Phi'' + (\Phi')^2 - \frac{b'r-b}{2r(r-b)}\Phi' - \frac{b'r-b}{2r^2(r-b)} + \frac{\Phi'}{r}\right], \tag{2.25}$$

$$G_{\hat{\phi}\hat{\phi}} = G_{\hat{\theta}\hat{\theta}}, \tag{2.26}$$

respectively.

2.1.4 Stress–Energy Tensor

Through the Einstein field equation, $G_{\hat{\mu}\hat{\nu}} = 8\pi T_{\hat{\mu}\hat{\nu}}$, one verifies that the stress–energy tensor $T_{\hat{\mu}\hat{\nu}}$ has the same algebraic structure as $G_{\hat{\mu}\hat{\nu}}$, Eqs. (2.23)–(2.26), and the only nonzero components are precisely the diagonal terms $T_{\hat{t}\hat{t}}$, $T_{\hat{r}\hat{r}}$, $T_{\hat{\theta}\hat{\theta}}$ and $T_{\hat{\phi}\hat{\phi}}$. Using the orthonormal basis, these components carry a simple physical interpretation, i.e.

$$T_{\hat{t}\hat{t}} = \rho(r)\,, \qquad T_{\hat{r}\hat{r}} = -\tau(r)\,, \qquad T_{\hat{\theta}\hat{\theta}} = T_{\hat{\phi}\hat{\phi}} = p(r)\,, \tag{2.27}$$

where $\rho(r)$ is the energy density, $\tau(r)$ is the radial tension, with $\tau(r) = -p_r(r)$, i.e. it is the negative of the radial pressure, $p(r)$ is the pressure measured in the tangential directions, orthogonal to the radial direction.

Thus, the Einstein field equation provides the following stress–energy scenario:

$$\rho(r) = \frac{1}{8\pi}\,\frac{b'}{r^2}\,, \tag{2.28}$$

$$\tau(r) = \frac{1}{8\pi}\left[\frac{b}{r^3} - 2\left(1-\frac{b}{r}\right)\frac{\Phi'}{r}\right], \tag{2.29}$$

$$\begin{aligned} p(r) = \frac{1}{8\pi}\left(1-\frac{b}{r}\right)\Bigg[&\Phi'' + (\Phi')^2 - \frac{b'r-b}{2r^2(1-b/r)}\Phi' \\ &-\frac{b'r-b}{2r^3(1-b/r)} + \frac{\Phi'}{r}\Bigg]. \end{aligned} \tag{2.30}$$

Note that one now has three equations with five unknown functions of the radial coordinate. Several strategies to solve these equations are available, for instance, one can impose an equation of state [6–10] and consider a specific choice of the shape function or of the redshift function.

Note that the sign of the energy density depends on the sign of $b'(r)$. One often comes across the misleading statement, in the literature, that wormholes should necessarily be threaded by negative energy densities, or negative matter; however, this is not necessarily the case. Note, however, that due to the flaring-out condition, observers traversing the wormhole with sufficiently high velocities, $v \to 1$, will measure a negative energy density. This will be shown below. Furthermore, one should perhaps correctly state that it is the radial pressure that is necessarily negative at the throat, which is transparent for the radial tension at the throat, which is given by $p_r(r) = -\tau(r_0) = -(8\pi r_0^2)^{-1}$.

By taking the derivative with respect to the radial coordinate r, of Eq. (2.29), and eliminating b' and Φ'', given in Eqs. (2.28) and (2.30), respectively, we obtain the following equation:

$$\tau' = (\rho - \tau)\Phi' - \frac{2}{r}(p+\tau)\,. \tag{2.31}$$

Equation (2.31) is the relativistic Euler equation, or the hydrostatic equation for equilibrium for the material threading the wormhole, and can also be obtained using the conservation of the stress–energy tensor, $T^{\hat{\mu}\hat{\nu}}{}_{;\hat{\nu}} = 0$, inserting $\hat{\mu} = r$.

The effective mass, $m(r) = b(r)/2$ contained in the interior of a sphere of radius r, can be obtained by integrating Eq. (2.28), which yields

$$m(r) = \frac{r_0}{2} + \int_{r_0}^{r} 4\pi\, \rho(r')\, r'^2\, dr' \,. \tag{2.32}$$

Therefore, the form function has an interpretation which depends on the mass distribution of the wormhole.

2.1.5 Exotic Matter and Modified Gravity

2.1.5.1 Exoticity Function

To gain some insight into the matter threading the wormhole, Morris and Thorne defined the dimensionless function $\xi = (\tau - \rho)/|\rho|$ [1], which taking into account Eqs. (2.28) and (2.29) yields

$$\xi = \frac{\tau - \rho}{|\rho|} = \frac{b/r - b' - 2r(1 - b/r)\Phi'}{|b'|} \,. \tag{2.33}$$

Combining the flaring-out condition, given by Eq. (2.8), with Eq. (2.33), the exoticity function takes the form

$$\xi = \frac{2b^2}{r|b'|}\,\frac{d^2r}{dz^2} - 2r\left(1 - \frac{b}{r}\right)\frac{\Phi'}{|b'|} \,. \tag{2.34}$$

Now, taking into account the finite character of ρ, and consequently of b', and the fact that $(1 - b/r)\Phi' \to 0$ at the throat, we have the following relationship:

$$\xi(r_0) = \frac{\tau_0 - \rho_0}{|\rho_0|} > 0 \,. \tag{2.35}$$

The restriction $\tau_0 > \rho_0$ is a somewhat troublesome condition, depending on one's point of view, as it states that the radial tension at the throat should exceed the energy density. Thus, Morris and Thorne coined matter constrained by this condition "exotic matter" [1]. We shall verify below that this is defined as matter that violates the null energy condition (in fact, it violates all the energy conditions) [1, 2].

Exotic matter is particularly troublesome for measurements made by observers traversing through the throat with a radial velocity close to the speed of light. Consider a Lorentz transformation, $x^{\hat{\mu}'} = \Lambda^{\hat{\mu}'}{}_{\hat{\nu}}\, x^{\hat{\nu}}$, with $\Lambda^{\hat{\mu}}{}_{\hat{\alpha}'}\, \Lambda^{\hat{\alpha}'}{}_{\hat{\nu}} = \delta^{\hat{\mu}}{}_{\hat{\nu}}$ and $\Lambda^{\hat{\mu}}{}_{\hat{\nu}'}$ defined as

$$(\Lambda^{\hat{\mu}}{}_{\hat{\nu}'}) = \begin{bmatrix} \gamma & 0 & 0 & \gamma v \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma v & 0 & 0 & \gamma \end{bmatrix} . \tag{2.36}$$

The energy density measured by these observers is given by $T_{\hat{0}'\hat{0}'} = \Lambda^{\hat{\mu}}{}_{\hat{0}'}\, \Lambda^{\hat{\nu}}{}_{\hat{0}'}\, T_{\hat{\mu}\hat{\nu}}$, i.e.

$$T_{\hat{0}'\hat{0}'} = \gamma^2 \left(\rho_0 - v^2 \tau_0\right) , \tag{2.37}$$

with $\gamma = (1 - v^2)^{-1/2}$. For sufficiently high velocities, $v \to 1$, the observer will measure a negative energy density, $T_{\hat{0}'\hat{0}'} < 0$.

This feature also holds for any traversable, nonspherical and nonstatic wormhole. To see this, one verifies that a bundle of null geodesics that enters the wormhole at one mouth and emerges from the other must have a cross-sectional area that initially increases, and then decreases. This conversion of decreasing to increasing is due to the gravitational repulsion of matter through which the bundle of null geodesics traverses.

2.1.5.2 The Violation of the Energy Conditions

The exoticity function (2.33) is closely related to the null energy condition (NEC), which asserts that for any null vector k^μ, we have $T_{\mu\nu}k^\mu k^\nu \geq 0$. For a diagonal stress–energy tensor, this implies $\rho - \tau \geq 0$ and $\rho + p \geq 0$. Using the Einstein field equations (2.28) and (2.29), evaluated at the throat r_0, and taking into account the finite character of the redshift function so that $(1 - b/r)\Phi'|_{r_0} \to 0$, we verify the condition $(\rho - \tau)|_{r_0} < 0$. This violates the NEC. In fact, it implies the violation of all the pointwise energy condition. Although classical forms of matter are believed to obey the energy conditions, it is a well-known fact that they are violated by certain quantum fields, amongst which we may refer to the Casimir effect. Thus, the flaring-out condition (2.8) entails the violation of the NEC, at the throat. Note that negative energy densities are not essential, but negative pressures are necessary to sustain the wormhole throat.

It is interesting to note that the violations of the pointwise energy conditions led to the averaging of the energy conditions over timelike or null geodesics [11]. The averaged energy conditions permit localized violations of the energy conditions, as long on average the energy conditions hold when integrated along timelike or null geodesics. Now, as the averaged energy conditions involve averaging over a line integral, with dimensions (mass)/(area), not a volume integral, they do not provide useful information regarding the "total amount" of energy condition violating matter. In order to overcome this shortcoming, the "volume integral quantifier" was proposed [12]. Thus, the amount of energy condition violations is then the extent that these integrals become negative.

2.1.5.3 Wormholes in Modified Theories of Gravity

Generally, the NEC arises when one refers back to the Raychaudhuri equation, which is a purely geometric statement, without the need to refer to any gravitational field equations. Now, in order for gravity to be attractive, the positivity condition $R_{\mu\nu}k^{\mu}k^{\nu} \geq 0$ is imposed in the Raychaudhuri equation. In general relativity, contracting both sides of the Einstein field equation $G_{\mu\nu} = \kappa^2 T_{\mu\nu}$ (where $\kappa^2 = 8\pi$) with any null vector k^{μ}, one can write the above condition in terms of the stress–energy tensor given by $T_{\mu\nu}k^{\mu}k^{\nu} \geq 0$, which is the statement of the NEC.

In modified theories of gravity the gravitational field equations can be rewritten as an effective Einstein equation, given by $G_{\mu\nu} = \kappa^2 T^{\rm eff}_{\mu\nu}$, where $T^{\rm eff}_{\mu\nu}$ is an effective stress–energy tensor containing the matter stress–energy tensor $T_{\mu\nu}$ and the curvature quantities, arising from the specific modified theory of gravity considered [13]. Now, the positivity condition $R_{\mu\nu}k^{\mu}k^{\nu} \geq 0$ in the Raychaudhuri equation provides the *generalized* NEC, $T^{\rm eff}_{\mu\nu}k^{\mu}k^{\nu} \geq 0$, through the modified gravitational field equation.

Therefore, the necessary condition to have a wormhole geometry is the violation of the generalized NEC, i.e. $T^{\rm eff}_{\mu\nu}k^{\mu}k^{\nu} < 0$. In classical general relativity this simply reduces to the violation of the usual NEC, i.e. $T_{\mu\nu}k^{\mu}k^{\nu} < 0$. However, in modified theories of gravity, one may in principle impose that the matter stress–energy tensor satisfies the standard NEC, $T_{\mu\nu}k^{\mu}k^{\nu} \geq 0$, while the respective generalized NEC is necessarily violated, $T^{\rm eff}_{\mu\nu}k^{\mu}k^{\nu} < 0$, in order to ensure the flaring-out condition.

More specifically, consider the generalized gravitational field equations for a large class of modified theories of gravity, given by the following field equation: [13]

$$g_1(\Psi^i)(G_{\mu\nu} + H_{\mu\nu}) - g_2(\Psi^j)\, T_{\mu\nu} = \kappa^2\, T_{\mu\nu}\,, \tag{2.38}$$

where $H_{\mu\nu}$ is an additional geometric term that includes the geometrical modifications inherent in the modified gravitational theory under consideration; $g_i(\Psi^j)$ $(i = 1, 2)$ are multiplicative factors that modify the geometrical sector of the field equations, and Ψ^j denote generically curvature invariants or gravitational fields such as scalar fields; the term $g_2(\Psi^i)$ covers the coupling of the curvature invariants or the scalar fields with the matter stress–energy tensor, $T_{\mu\nu}$.

It is useful to rewrite this field equation as an effective Einstein field equation, as mentioned above, with the effective stress–energy tensor, $T^{\rm eff}_{\mu\nu}$, given by

$$T^{\rm eff}_{\mu\nu} \equiv \frac{1 + \bar{g}_2(\Psi^j)}{g_1(\Psi^i)}\, T_{\mu\nu} - \bar{H}_{\mu\nu}\,, \tag{2.39}$$

where $\bar{g}_2(\Psi^j) = g_2(\Psi^j)/\kappa^2$ and $\bar{H}_{\mu\nu} = H_{\mu\nu}/\kappa^2$ are defined for notational convenience.

In modified gravity, the violation of the generalized NEC, $T^{\rm eff}_{\mu\nu}k^{\mu}k^{\nu} < 0$, implies the following restriction:

$$\frac{1+\bar{g}_2(\Psi^j)}{g_1(\Psi^i)}\, T_{\mu\nu}k^\mu k^\nu \,<\, \bar{H}_{\mu\nu}k^\mu k^\nu \,. \tag{2.40}$$

For general relativity, with $g_1(\Psi^j) = 1$, $g_2(\Psi^j) = 0$, and $H_{\mu\nu} = 0$, we recover the standard violation of the NEC for the matter threading the wormhole, i.e. $T_{\mu\nu}k^\mu k^\nu \,<\, 0$.

If the additional condition $[1+\bar{g}_2(\Psi^j)]/g_1(\Psi^i) > 0$ is met, then one obtains a general bound for the normal matter threading the wormhole, in the context of modified theories of gravity, given by

$$0 \le T_{\mu\nu}k^\mu k^\nu < \frac{g_1(\Psi^i)}{1+\bar{g}_2(\Psi^j)}\, \bar{H}_{\mu\nu}k^\mu k^\nu \,. \tag{2.41}$$

2.1.6 Traversability Conditions

In constructing traversable wormhole geometries, we will be interested in specific solutions by imposing specific traversability conditions. Assume that a traveller of an absurdly advanced civilization begins the trip in a space station in the lower universe, at proper distance $l = -l_1$, and ends up in the upper universe, at $l = l_2$. Furthermore, consider that the traveller has a radial velocity $v(r)$, as measured by a static observer positioned at r. One may relate the proper distance travelled dl, radius travelled dr, coordinate time lapse dt, and proper time lapse as measured by the observer $d\tau$, by the following relations:

$$v = e^{-\Phi}\,\frac{dl}{dt} = \mp\, e^{-\Phi}\,\left(1-\frac{b}{r}\right)^{-1/2}\,\frac{dr}{dt}\,, \tag{2.42}$$

$$v\gamma = \frac{dl}{d\tau} = \mp\,\left(1-\frac{b}{r}\right)^{-1/2}\,\frac{dr}{d\tau}\,. \tag{2.43}$$

It is also important to impose certain conditions at the space stations [1]. First, consider that space is asymptotically flat at the stations, i.e. $b/r \ll 1$. Second, the gravitational redshift of signals sent from the stations to infinity should be small, i.e. $\Delta\lambda/\lambda = e^{-\Phi} - 1 \approx -\Phi$, so that $|\Phi| \ll 1$. The condition $|\Phi| \ll 1$ imposes that the proper time at the station equals the coordinate time. Third, the gravitational acceleration measured at the stations, given by $g = -(1-b/r)^{-1/2}\,\Phi' \simeq -\Phi'$, should be less than or equal to the Earth's gravitational acceleration, $g \le g_\oplus$, so that the condition $|\Phi'| \le g_\oplus$ is met.

For a convenient trip through the wormhole, certain conditions should also be imposed [1]. First, the entire journey should be done in a relatively short time as measured both by the traveller and by observers who remain at rest at the stations. Second, the acceleration felt by the traveller should not exceed the Earth's gravitational acceleration, $g_\oplus$. Finally, the tidal accelerations between different parts of the traveller's body should not exceed, once again, Earth's gravity.

2.1.6.1 Total Time in a Traversal

The trip should take a relatively short time, for instance, Morris and Thorne considered 1 year, as measured by the traveller and for observers that stay at rest at the space stations, $l = -l_1$ and $l = l_2$, i.e.

$$\Delta\tau_{\text{traveller}} = \int_{-l_1}^{+l_2} \frac{dl}{v\gamma} \leq 1 \text{ year}, \tag{2.44}$$

$$\Delta t_{\text{space station}} = \int_{-l_1}^{+l_2} \frac{dl}{v e^{\Phi}} \leq 1 \text{ year}, \tag{2.45}$$

respectively.

2.1.6.2 Acceleration Felt by a Traveller

An important traversability condition required is that the acceleration felt by the traveller should not exceed Earth's gravity [1]. Consider an orthonormal basis of the traveller's proper reference frame, $(\mathbf{e}_{\hat{0}'}, \mathbf{e}_{\hat{1}'}, \mathbf{e}_{\hat{2}'}, \mathbf{e}_{\hat{3}'})$, given in terms of the orthonormal basis vectors of Eq. (2.12) of the static observers, by a Lorentz transformation, i.e.

$$\mathbf{e}_{\hat{0}'} = \gamma\, \mathbf{e}_{\hat{t}} \mp \gamma\, v\, \mathbf{e}_{\hat{r}}\ , \qquad \mathbf{e}_{\hat{1}'} = \mp \gamma\, \mathbf{e}_{\hat{r}} + \gamma\, v\, \mathbf{e}_{\hat{t}}\ , \qquad \mathbf{e}_{\hat{2}'} = \mathbf{e}_{\hat{\theta}}\ , \qquad \mathbf{e}_{\hat{3}'} = \mathbf{e}_{\hat{\phi}}\ , \tag{2.46}$$

where $\gamma = (1 - v^2)^{-1/2}$, and $v(r)$ being the velocity of the traveller as he passes r, as measured by a static observer positioned there. Thus, the traveller's four-acceleration expressed in his proper reference frame, $a^{\hat{\mu}'} = U^{\hat{\nu}'} U^{\hat{\mu}'}{}_{;\hat{\nu}'}$, yields the following restriction:

$$|\mathbf{a}| = \left| \left(1 - \frac{b}{r}\right)^{1/2} e^{-\Phi} \, \left(\gamma\, e^{\Phi}\right)' \right| \leq g_\oplus\,. \tag{2.47}$$

2.1.6.3 Tidal Acceleration Felt by a Traveller

It is also convenient that an observer traversing through the wormhole should not be ripped apart by enormous tidal forces. Thus, another of the traversability conditions required is that the tidal accelerations felt by the traveller should not exceed, for instance, the Earth's gravitational acceleration [1]. The tidal acceleration felt by the traveller is given by

$$\Delta a^{\hat{\mu}'} = -R^{\hat{\mu}'}{}_{\hat{\nu}'\hat{\alpha}'\hat{\beta}'}\, U^{\hat{\nu}'} \eta^{\hat{\alpha}'} U^{\hat{\beta}'}, \tag{2.48}$$

where $U^{\hat{\mu}'} = \delta^{\hat{\mu}'}{}_{\hat{0}'}$ is the traveller's four-velocity and $\eta^{\hat{\alpha}'}$ is the separation between two arbitrary parts of his body. Note that $\eta^{\hat{\alpha}'}$ is purely spatial in the traveller's reference frame, as $U^{\hat{\mu}'}\eta_{\hat{\mu}'} = 0$, so that $\eta^{\hat{0}'} = 0$. For simplicity, assume that $|\eta^{\hat{i}'}| \approx 2\,\mathrm{m}$ along any spatial direction in the traveller's reference frame. Taking into account the antisymmetric nature of $R^{\hat{\mu}'}{}_{\hat{\nu}'\hat{\alpha}'\hat{\beta}'}$ in its first two indices, we verify that $\Delta a^{\hat{\mu}'}$ is purely spatial with the components

$$\Delta a^{\hat{i}'} = -R^{\hat{i}'}{}_{\hat{0}'\hat{j}'\hat{0}'}\, \eta^{\hat{j}'} = -R_{\hat{i}'\hat{0}'\hat{j}'\hat{0}'}\, \eta^{\hat{j}'}. \tag{2.49}$$

Using a Lorentz transformation of the Riemann tensor components in the static observer's frame, $(\mathbf{e}_{\hat{t}}, \mathbf{e}_{\hat{r}}, \mathbf{e}_{\hat{\theta}}, \mathbf{e}_{\hat{\phi}})$, to the traveller's frame, $(\mathbf{e}_{\hat{0}'}, \mathbf{e}_{\hat{1}'}, \mathbf{e}_{\hat{2}'}, \mathbf{e}_{\hat{3}'})$, the nonzero components of $R_{\hat{i}'\hat{0}'\hat{j}'\hat{0}'}$ are given by

$$\begin{aligned} R_{\hat{1}'\hat{0}'\hat{1}'\hat{0}'} &= R_{\hat{r}\hat{t}\hat{r}\hat{t}} \\ &= -\left(1-\frac{b}{r}\right)\left[-\Phi'' - (\Phi')^2 + \frac{b'r-b}{2r(r-b)}\Phi'\right], \end{aligned} \tag{2.50}$$

$$\begin{aligned} R_{\hat{2}'\hat{0}'\hat{2}'\hat{0}'} &= R_{\hat{3}'\hat{0}'\hat{3}'\hat{0}'} = \gamma^2\, R_{\hat{\theta}\hat{t}\hat{\theta}\hat{t}} + \gamma^2\, v^2\, R_{\hat{\theta}\hat{r}\hat{\theta}\hat{r}} \\ &= \frac{\gamma^2}{2r^2}\left[v^2\left(b' - \frac{b}{r}\right) + 2(r-b)\Phi'\right]. \end{aligned} \tag{2.51}$$

Thus, Eq. (2.49) takes the form

$$\Delta a^{\hat{1}'} = -R_{\hat{1}'\hat{0}'\hat{1}'\hat{0}'}\, \eta^{\hat{1}'}, \qquad \Delta a^{\hat{2}'} = -R_{\hat{2}'\hat{0}'\hat{2}'\hat{0}'}\, \eta^{\hat{2}'}, \qquad \Delta a^{\hat{3}'} = -R_{\hat{3}'\hat{0}'\hat{3}'\hat{0}'}\, \eta^{\hat{3}'}. \tag{2.52}$$

The constraint $|\Delta a^{\hat{\mu}'}| \leq g_{\oplus}$ provides the tidal acceleration restrictions as measured by a traveller moving radially through the wormhole, given by the following inequalities:

$$\left|\left(1-\frac{b}{r}\right)\left[\Phi'' + (\Phi')^2 - \frac{b'r-b}{2r(r-b)}\Phi'\right]\right| \, |\eta^{\hat{1}'}| \leq g_{\oplus}, \tag{2.53}$$

$$\left|\frac{\gamma^2}{2r^2}\left[v^2\left(b' - \frac{b}{r}\right) + 2(r-b)\Phi'\right]\right| \, |\eta^{\hat{2}'}| \leq g_{\oplus}. \tag{2.54}$$

The radial tidal constraint, Eq. (2.53), constrains the redshift function, and the lateral tidal constraint, Eq. (2.54), constrains the velocity with which observers traverse the wormhole. These inequalities are particularly simple at the throat, r_0,

$$|\Phi'(r_0)| \leq \frac{2g_{\oplus}\, r_0}{(1-b')\,|\eta^{\hat{1}'}|}\,, \tag{2.55}$$

$$\gamma^2 v^2 \leq \frac{2g_{\oplus}\, r_0^2}{(1-b')\,|\eta^{\hat{2}'}|}\,, \tag{2.56}$$

For the particular case of a constant redshift function, $\Phi' = 0$, the radial tidal acceleration is zero, and Eq. (2.54) reduces to

$$\frac{\gamma^2 v^2}{2r^2}\left|\left(b' - \frac{b}{r}\right)\right|\,\left|\eta^{\hat{2}'}\right| \leq g_{\oplus}\,. \tag{2.57}$$

For this specific case one verifies that stationary observers with $v = 0$ measure null tidal forces.

2.2 Dynamic Spherically Symmetric Thin-Shell Traversable Wormholes

An interesting and efficient manner to minimize the violation of the null energy condition is to construct thin-shell wormholes using the thin-shell formalism [2, 14] and the cut-and-paste procedure as described in [2, 15–18]. Motivated in minimizing the usage of exotic matter, the thin-shell construction was generalized to nonspherically symmetric cases [2, 15], and in particular, it was found that a traveller may traverse through such a wormhole without encountering regions of exotic matter. In the context of a linearized stability analysis [16], two Schwarzschild spacetimes were surgically grafted together in such a way that no event horizon is permitted to form. This surgery concentrates a nonzero stress energy on the boundary layer between the two asymptotically flat regions and a dynamical stability analysis (with respect to spherically symmetric perturbations) was explored. In the latter stability analysis, constraints were found on the equation of state of the exotic matter that comprises the throat of the wormhole. Indeed, the stability of the latter thin-shell wormholes was considered for certain specially chosen equations of state [2, 16], where the analysis addressed the issue of stability in the sense of proving bounded motion for the wormhole throat.

This dynamical analysis was generalized to the stability of spherically symmetric thin-shell wormholes by considering linearized radial perturbations around some assumed static solution of the Einstein field equations, without the need to specify an equation of state [18]. This linearized stability analysis around a static solution was soon generalized to the presence of charge [19], and of a cosmological constant [20], and was subsequently extended to a plethora of individual scenarios (see [21] and references therein). The key point of the present section is to develop an extremely general, flexible, and robust framework that can quickly be adapted to general spherically symmetric traversable wormholes in $3+1$ dimensions see [21].

We shall consider standard general relativity, with traversable wormholes that are spherically symmetric, with all of the exotic material confined to a thin shell.

2.2.1 Generic Static Spherically Symmetric Spacetimes

To set the stage, consider two distinct spacetime manifolds, $\mathscr{M}_+$ and $\mathscr{M}_-$, with metrics given by $g^+_{\mu\nu}(x^\mu_+)$ and $g^-_{\mu\nu}(x^\mu_-)$, in terms of independently defined coordinate systems x^μ_+ and x^μ_-. A single manifold $\mathscr{M}$ is obtained by gluing together the two distinct manifolds, $\mathscr{M}_+$ and $\mathscr{M}_-$, i.e. $\mathscr{M} = \mathscr{M}_+ \cup \mathscr{M}_-$, at their boundaries. The latter are given by Σ_+ and Σ_-, respectively, with the natural identification of the boundaries $\Sigma = \Sigma_+ = \Sigma_-$.

Consider two generic static spherically symmetric spacetimes given by the following line elements:

$$ds^2 = -e^{2\Phi_\pm(r_\pm)}\left[1 - \frac{b_\pm(r_\pm)}{r_\pm}\right] dt^2_\pm + \left[1 - \frac{b_\pm(r_\pm)}{r_\pm}\right]^{-1} dr^2_\pm + r^2_\pm d\Omega^2_\pm, \quad (2.58)$$

on $\mathscr{M}_\pm$, respectively. Using the Einstein field equation, $G_{\mu\nu} = 8\pi\, T_{\mu\nu}$ (with $c = G = 1$), the (orthonormal) stress–energy tensor components are given by

$$\rho(r) = \frac{1}{8\pi r^2} b', \quad (2.59)$$

$$\bar{\tau}(r) = \frac{1}{8\pi r^2}\left[2\Phi'(b-r) + b'\right], \quad (2.60)$$

$$p_t(r) = -\frac{1}{16\pi r^2}\big[(-b + 3rb' - 2r)\Phi' + 2r(b-r)(\Phi')^2 + 2r(b-r)\Phi'' + b''r\big], \quad (2.61)$$

where we have denoted the quantity $\bar{\tau}(r)$ here as the radial tension (the variable τ in this section denotes the proper time, as measured by a comoving observer on the thin shell). The $\pm$ subscripts were (temporarily) dropped so as not to overload the notation.

2.2.2 Extrinsic Curvature

The manifolds are bounded by hypersurfaces Σ_+ and Σ_-, respectively, with induced metrics g^+_{ij} and g^-_{ij}. The hypersurfaces are isometric, i.e. $g^+_{ij}(\xi) = g^-_{ij}(\xi) = g_{ij}(\xi)$, in terms of the intrinsic coordinates, invariant under the isometry. As mentioned above, a single manifold $\mathscr{M}$ is obtained by gluing together $\mathscr{M}_+$ and $\mathscr{M}_-$ at their boundaries, i.e. $\mathscr{M} = \mathscr{M}_+ \cup \mathscr{M}_-$, with the natural identification of the boundaries

$\Sigma = \Sigma_+ = \Sigma_-$. The three holonomic basis vectors $\mathbf{e}_{(i)} = \partial/\partial \xi^i$ tangent to Σ have the following components $e^\mu_{(i)}|_\pm = \partial x^\mu_\pm/\partial \xi^i$, which provide the induced metric on the junction surface by the following scalar product $g_{ij} = \mathbf{e}_{(i)} \cdot \mathbf{e}_{(j)} = g_{\mu\nu} e^\mu_{(i)} e^\nu_{(j)}|_\pm$. The intrinsic metric to Σ is thus provided by

$$ds^2_\Sigma = -d\tau^2 + a^2(\tau)\,(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{2.62}$$

where τ is the proper time of an observer comoving with the junction surface, as mentioned above.

Thus, for the static and spherically symmetric spacetime considered in this section, the single manifold, $\mathscr{M}$, is obtained by gluing $\mathscr{M}_+$ and $\mathscr{M}_-$ at Σ, i.e. at $f(r,\tau) = r - a(\tau) = 0$. The position of the junction surface is given by

$$x^\mu(\tau,\theta,\phi) = (t(\tau), a(\tau), \theta, \phi)\,, \tag{2.63}$$

and the respective 4-velocities (as measured in the static coordinate systems on the two sides of the junction) are

$$U^\mu_\pm = \left(\frac{e^{-\Phi_\pm(a)}\sqrt{1 - \frac{b_\pm(a)}{a} + \dot{a}^2}}{1 - \frac{b_\pm(a)}{a}},\ \dot{a}, 0, 0 \right), \tag{2.64}$$

where the overdot denotes a derivative with respect to τ.

We shall consider a timelike junction surface Σ, defined by the parametric equation of the form $f(x^\mu(\xi^i)) = 0$. The unit normal $4-$vector, n^μ, to Σ is defined as

$$n_\mu = \pm \left| g^{\alpha\beta} \frac{\partial f}{\partial x^\alpha} \frac{\partial f}{\partial x^\beta} \right|^{-1/2} \frac{\partial f}{\partial x^\mu}\,, \tag{2.65}$$

with $n_\mu\, n^\mu = +1$ and $n_\mu e^\mu_{(i)} = 0$. The Israel formalism requires that the normals point from $\mathscr{M}_-$ to $\mathscr{M}_+$ [14]. Thus, the unit normals to the junction surface, determined by Eq. (2.65), are given by

$$n^\mu_\pm = \pm \left(\frac{e^{-\Phi_\pm(a)}}{1 - \frac{b_\pm(a)}{a}}\,\dot{a}, \sqrt{1 - \frac{b_\pm(a)}{a} + \dot{a}^2}, 0, 0 \right). \tag{2.66}$$

Note that the above expressions can also be deduced from the contractions $U^\mu n_\mu = 0$ and $n^\mu n_\mu = +1$. The extrinsic curvature, or the second fundamental form, is defined as $K_{ij} = n_{\mu;\nu} e^\mu_{(i)} e^\nu_{(j)}$. Taking into account the differentiation of $n_\mu e^\mu_{(i)} = 0$ with respect to ξ^j, the extrinsic curvature is given by

$$K_{ij}^{\pm} = -n_{\mu}\left(\frac{\partial^2 x^{\mu}}{\partial\xi^i\,\partial\xi^j} + \Gamma^{\mu\pm}_{\alpha\beta}\,\frac{\partial x^{\alpha}}{\partial\xi^i}\,\frac{\partial x^{\beta}}{\partial\xi^j}\right) . \tag{2.67}$$

Note that for the case of a thin shell K_{ij} is not continuous across Σ, so that for notational convenience, the discontinuity in the second fundamental form is defined as $\kappa_{ij} = K_{ij}^{+} - K_{ij}^{-}$.

Thus, using Eq. (2.67), the nontrivial components of the extrinsic curvature can easily be computed to be

$$K^{\theta\ \pm}_{\ \theta} = \pm\frac{1}{a}\sqrt{1 - \frac{b_{\pm}(a)}{a} + \dot{a}^2}\ , \tag{2.68}$$

$$K^{\tau\ \pm}_{\ \tau} = \pm\left\{\frac{\ddot{a} + \frac{b_{\pm}(a) - b_{\pm}'(a)a}{2a^2}}{\sqrt{1 - \frac{b_{\pm}(a)}{a} + \dot{a}^2}} + \Phi_{\pm}'(a)\sqrt{1 - \frac{b_{\pm}(a)}{a} + \dot{a}^2}\right\} , \tag{2.69}$$

where the prime now denotes a derivative with respect to the coordinate a.

2.2.3 Lanczos Equations: Surface Stress–Energy

The Lanczos equations follow from the Einstein equations applied to the hypersurface joining the four-dimensional spacetimes, and are given by

$$S^{i}_{\ j} = -\frac{1}{8\pi}\left(\kappa^{i}_{\ j} - \delta^{i}_{\ j}\kappa^{k}_{\ k}\right) , \tag{2.70}$$

where $S^{i}_{\ j}$ is the surface stress–energy tensor on Σ. In particular, because of spherical symmetry considerable simplifications occur, namely $\kappa^{i}_{\ j} = \mathrm{diag}\left(\kappa^{\tau}_{\ \tau}, \kappa^{\theta}_{\ \theta}, \kappa^{\theta}_{\ \theta}\right)$. The surface stress–energy tensor may be written in terms of the surface energy density, σ, and the surface pressure, $\mathscr{P}$, as $S^{i}_{\ j} = \mathrm{diag}(-\sigma, \mathscr{P}, \mathscr{P})$. The Lanczos equations then reduce to

$$\sigma = -\frac{1}{4\pi}\,\kappa^{\theta}_{\ \theta}\,, \tag{2.71}$$

$$\mathscr{P} = \frac{1}{8\pi}(\kappa^{\tau}_{\ \tau} + \kappa^{\theta}_{\ \theta})\,. \tag{2.72}$$

Taking into account the computed extrinsic curvatures, Eqs. (2.68) and (2.69), we see that Eqs. (2.71) and (2.72) provide us with the following expressions for the surface stresses:

$$\sigma = -\frac{1}{4\pi a}\left[\sqrt{1-\frac{b_+(a)}{a}+\dot{a}^2}+\sqrt{1-\frac{b_-(a)}{a}+\dot{a}^2}\right], \tag{2.73}$$

$$\begin{aligned}\mathscr{P} = \frac{1}{8\pi a}&\left[\frac{1+\dot{a}^2+a\ddot{a}-\frac{b_+(a)+ab'_+(a)}{2a}}{\sqrt{1-\frac{b_+(a)}{a}+\dot{a}^2}}+\sqrt{1-\frac{b_+(a)}{a}+\dot{a}^2}\; a\Phi'_+(a)\right.\\ &\left.+\frac{1+\dot{a}^2+a\ddot{a}-\frac{b_-(a)+ab'_-(a)}{2a}}{\sqrt{1-\frac{b_-(a)}{a}+\dot{a}^2}}+\sqrt{1-\frac{b_-(a)}{a}+\dot{a}^2}\; a\Phi'_-(a)\right].\end{aligned} \tag{2.74}$$

Note that the surface energy density σ is always negative, consequently violating the energy conditions. The surface mass of the thin shell is given by $m_s = 4\pi a^2\sigma$, which will be used below.

2.2.4 Conservation Identity

The first contracted Gauss–Codazzi equation is given by

$$G_{\mu\nu}\, n^\mu\, n^\nu = \frac{1}{2}\,(K^2 - K_{ij}K^{ij} - {}^3R)\,, \tag{2.75}$$

which combined with the Einstein equations provides the evolution identity

$$S^{ij}\;\overline{K}_{ij} = -\left[T_{\mu\nu}n^\mu n^\nu\right]^+_- . \tag{2.76}$$

The convention $[X]^+_- \equiv X^+|_\Sigma - X^-|_\Sigma$ and $\overline{X} \equiv \frac{1}{2}(X^+|_\Sigma + X^-|_\Sigma)$ is used.

The second contracted Gauss–Codazzi equation is

$$G_{\mu\nu}e^\mu_{(i)}n^\nu = K^j_{i|j} - K_{,i}\;, \tag{2.77}$$

which together with the Lanczos equations provides the conservation identity

$$S^i_{\;j|i} = \left[T_{\mu\nu}\; e^\mu_{\;(j)}n^\nu\right]^+_- . \tag{2.78}$$

When interpreting the conservation identity Eq. (2.78), consider the momentum flux defined by

$$\left[T_{\mu\nu}\; e^\mu_{\;(\tau)}\, n^\nu\right]^+_- = \left[T_{\mu\nu}\; U^\mu\, n^\nu\right]^+_- = \left[\pm\left(T_{\hat{t}\hat{t}}+T_{\hat{r}\hat{r}}\right)\,\frac{\dot{a}\sqrt{1-\frac{b(a)}{a}+\dot{a}^2}}{1-\frac{b(a)}{a}}\right]^+_- , \tag{2.79}$$

where $T_{\hat{t}\hat{t}}$ and $T_{\hat{r}\hat{r}}$ are the four-dimensional stress–energy tensor components given in an orthonormal basis. This flux term corresponds to the net discontinuity in the momentum flux $F_\mu = T_{\mu\nu} U^\nu$ which impinges on the shell. Applying the Einstein equations, we have

$$\left[T_{\mu\nu}\, e^{\mu}_{(\tau)}\, n^\nu\right]^+_- = \frac{\dot{a}}{4\pi a}\left[\Phi'_+(a)\sqrt{1-\frac{b_+(a)}{a}+\dot{a}^2}+\Phi'_-(a)\sqrt{1-\frac{b_-(a)}{a}+\dot{a}^2}\right]. \tag{2.80}$$

It is useful to define the quantity

$$\Xi = \frac{1}{4\pi a}\left[\Phi'_+(a)\sqrt{1-\frac{b_+(a)}{a}+\dot{a}^2}+\Phi'_-(a)\sqrt{1-\frac{b_-(a)}{a}+\dot{a}^2}\right]. \tag{2.81}$$

and to let $A = 4\pi a^2$ be the surface area of the thin shell. Then in the general case, the conservation identity provides the following relationship:

$$\frac{d(\sigma A)}{d\tau} + \mathscr{P}\,\frac{dA}{d\tau} = \Xi\, A\, \dot{a}\,. \tag{2.82}$$

The first term represents the variation of the internal energy of the shell, the second term is the work done by the shell's internal force, and the third term represents the work done by the external forces.

If we assume that the equations of motion can be integrated to determine the surface energy density as a function of radius a, that is, assuming the existence of a suitable function $\sigma(a)$, then the conservation equation can be written as

$$\sigma' = -\frac{2}{a}\,(\sigma + \mathscr{P}) + \Xi\,, \tag{2.83}$$

where $\sigma' = d\sigma/da$. Note that the flux term Ξ is zero whenever $\Phi_\pm = 0$, which is actually a quite common occurrence, for instance in either Schwarzschild or Reissner–Nordström geometries, or more generally whenever $\rho + p_r = 0$. In particular, for a vanishing flux $\Xi = 0$ one obtains the so-called "transparency condition", $\left[G_{\mu\nu}\, U^\mu\, n^\nu\right]^+_- = 0$ [22]. The conservation identity, Eq. (2.78), then reduces to the simple relationship $\dot{\sigma} = -2\,(\sigma + \mathscr{P})\dot{a}/a$, which is extensively used in the literature.

2.2.5 Equation of Motion

To qualitatively analyse the stability of the wormhole, it is useful to rearrange Eq. (2.73) into the thin-shell equation of motion given by

$$\frac{1}{2}\dot{a}^2 + V(a) = 0\,, \tag{2.84}$$

where the potential $V(a)$ is given by

$$V(a) = \frac{1}{2}\left\{1 - \frac{\bar{b}(a)}{a} - \left[\frac{m_s(a)}{2a}\right]^2 - \left[\frac{\Delta(a)}{m_s(a)}\right]^2\right\}. \tag{2.85}$$

Here $m_s(a) = 4\pi a^2\,\sigma(a)$ is the mass of the thin shell. The quantities $\bar{b}(a)$ and $\Delta(a)$ are defined, for simplicity, as

$$\bar{b}(a) = \frac{b_+(a) + b_-(a)}{2}, \qquad \Delta(a) = \frac{b_+(a) - b_-(a)}{2}, \tag{2.86}$$

respectively. This gives the potential $V(a)$ as a function of the surface mass $m_s(a)$. By differentiating with respect to a, we see that the equation of motion implies $\ddot{a} = -V'(a)$.

It is useful to reverse the logic flow and determine the surface mass as a function of the potential. Following the techniques used in [23], suitably modified for the present wormhole context, we have

$$m_s^2(a) = 2a^2\left[1 - \frac{\bar{b}(a)}{a} - 2V(a) + \sqrt{1 - \frac{b_+(a)}{a} - 2V(a)}\sqrt{1 - \frac{b_-(a)}{a} - 2V(a)}\right], \tag{2.87}$$

and in fact

$$m_s(a) = -a\left[\sqrt{1 - \frac{b_+(a)}{a} - 2V(a)} + \sqrt{1 - \frac{b_-(a)}{a} - 2V(a)}\right], \tag{2.88}$$

with the negative root now being necessary for compatibility with the Lanczos equations. Note the novel approach used here, namely, by specifying $V(a)$ dictates the amount of surface mass that needs to be inserted on the wormhole throat. This implicitly makes demands on the equation of state of the exotic matter residing on the wormhole throat.

In a completely analogous manner, after imposing the equation of motion for the shell one has

$$\sigma(a) = -\frac{1}{4\pi a}\left[\sqrt{1 - \frac{b_+(a)}{a} - 2V(a)} + \sqrt{1 - \frac{b_-(a)}{a} - 2V(a)}\right], \tag{2.89}$$

$$\begin{aligned}\mathscr{P}(a) = \frac{1}{8\pi a}&\left[\frac{1 - 2V(a) - aV'(a) - \frac{b_+(a)+ab_+'(a)}{2a}}{\sqrt{1 - \frac{b_+(a)}{a} - 2V(a)}} + \sqrt{1 - \frac{b_+(a)}{a} - 2V(a)}\; a\Phi_+'(a)\right.\\ &\left. + \frac{1 - 2V(a) - aV'(a) - \frac{b_-(a)+ab_-'(a)}{2a}}{\sqrt{1 - \frac{b_-(a)}{a} - 2V(a)}} + \sqrt{1 - \frac{b_-(a)}{a} - 2V(a)}\; a\Phi_-'(a)\right].\end{aligned} \tag{2.90}$$

and the flux term is given by

$$\Xi(a) = \frac{1}{4\pi a}\left[\Phi'_+(a)\sqrt{1-\frac{b_+(a)}{a}-2V(a)} + \Phi'_-(a)\sqrt{1-\frac{b_-(a)}{a}-2V(a)}\right]. \tag{2.91}$$

The three quantities $\{\sigma(a),\ \mathscr{P}(a),\ \Xi(a)\}$ are related by the differential conservation law, so at most two of them are independent.

2.2.6 Linearized Equation of Motion

Consider a linearization around an assumed static solution a_0 to the equation of motion $\frac{1}{2}\dot{a}^2 + V(a) = 0$, and so also a solution of $\ddot{a} = -V'(a)$. A Taylor expansion of $V(a)$ around a_0 to second order yields

$$V(a) = V(a_0) + V'(a_0)(a-a_0) + \frac{1}{2}V''(a_0)(a-a_0)^2 + O[(a-a_0)^3], \tag{2.92}$$

and since we are expanding around a static solution, $\dot{a}_0 = \ddot{a}_0 = 0$, we automatically have $V(a_0) = V'(a_0) = 0$, which reduces Eq. (2.92) to

$$V(a) = \frac{1}{2}V''(a_0)(a-a_0)^2 + O[(a-a_0)^3]. \tag{2.93}$$

The assumed static solution at a_0 is stable if and only if $V(a)$ has a local minimum at a_0, which requires $V''(a_0) > 0$, which is the primary criterion for wormhole stability.

For instance, it is extremely useful to express $m'_s(a)$ and $m''_s(a)$ by the following expressions:

$$m'_s(a) = +\frac{m_s(a)}{a} + \frac{a}{2}\left\{\frac{(b_+(a)/a)' + 2V'(a)}{\sqrt{1-b_+(a)/a-2V(a)}} + \frac{(b_-(a)/a)' + 2V'(a)}{\sqrt{1-b_-(a)/a-2V(a)}}\right\}, \tag{2.94}$$

and

$$\begin{aligned} m''_s(a) = &\left\{\frac{(b_+(a)/a)' + 2V'(a)}{\sqrt{1-b_+(a)/a-2V(a)}} + \frac{(b_-(a)/a)' + 2V'(a)}{\sqrt{1-b_-(a)/a-2V(a)}}\right\} \\ &+\frac{a}{4}\left\{\frac{[(b_+(a)/a)' + 2V'(a)]^2}{[1-b_+(a)/a-2V(a)]^{3/2}} + \frac{[(b_-(a)/a)' + 2V'(a)]^2}{[1-b_-(a)/a-2V(a)]^{3/2}}\right\} \\ &+\frac{a}{2}\left\{\frac{(b_+(a)/a)'' + 2V''(a)}{\sqrt{1-b_+(a)/a-2V(a)}} + \frac{(b_-(a)/a)'' + 2V''(a)}{\sqrt{1-b_-(a)/a-2V(a)}}\right\}. \end{aligned} \tag{2.95}$$

Doing so allows us to easily study linearized stability, and to develop a simple inequality on $m''_s(a_0)$ using the constraint $V''(a_0) > 0$. Similar formulae hold for

$\sigma'(a)$, $\sigma''(a)$, for $\mathscr{P}'(a)$, $\mathscr{P}''(a)$, and for $\Xi'(a)$, $\Xi''(a)$. In view of the redundancies coming from the relations $m_s(a) = 4\pi\sigma(a)a^2$ and the differential conservation law, the only interesting quantities are $\Xi'(a)$, $\Xi''(a)$.

For practical calculations, it is extremely useful to consider the dimensionless quantity $m_s(a)/a$ and then to express $[m_s(a)/a]'$ and $[m_s(a)/a]''$. It is similarly useful to consider $4\pi\,\Xi(a)a$, and then evaluate $[4\pi\,\Xi(a)\,a]'$ and $[4\pi\,\Xi(a)\,a]''$ (see Ref. [21] for more details). We shall evaluate these quantities at the assumed stable solution a_0.

2.2.7 The Master Equations

For practical calculations it is more useful to work with $m_s(a)/a$, so that in view of the above, in order to have a stable static solution at a_0 we must have

$$m_s(a_0)/a_0 = -\left\{\sqrt{1-\frac{b_+(a_0)}{a_0}} + \sqrt{1-\frac{b_-(a_0)}{a_0}}\right\}, \tag{2.96}$$

while

$$[m_s(a)/a]'\big|_{a_0} = +\frac{1}{2}\left\{\frac{(b_+(a)/a)'}{\sqrt{1-b_+(a)/a}} + \frac{(b_-(a)/a)'}{\sqrt{1-b_-(a)/a}}\right\}\Bigg|_{a_0}, \tag{2.97}$$

and the stability condition $V''(a_0) \geq 0$ is translated into

$$\begin{aligned}[m_s(a)/a]''\big|_{a_0} \geq &+\frac{1}{4}\left\{\frac{[(b_+(a)/a)']^2}{[1-b_+(a)/a]^{3/2}} + \frac{[(b_-(a)/a)']^2}{[1-b_-(a)/a]^{3/2}}\right\}\Bigg|_{a_0} \\ &+\frac{1}{2}\left\{\frac{(b_+(a)/a)''}{\sqrt{1-b_+(a)/a}} + \frac{(b_-(a)/a)''}{\sqrt{1-b_-(a)/a)}}\right\}\Bigg|_{a_0}.\end{aligned} \tag{2.98}$$

In the absence of external forces this inequality (or the equivalent one for $m_s''(a_0)$ above) is the only stability constraint one requires. However, once one has external forces ($\Phi_\pm \neq 0$), there is additional information:

$$\begin{aligned}[4\pi\,\Xi(a)\,a]'\big|_{a_0} = &+\left\{\Phi_+''(a)\sqrt{1-b_+(a)/a} + \Phi_-''(a)\sqrt{1-b_-(a)/a}\right\}\Big|_{a_0} \\ &-\frac{1}{2}\left\{\Phi_+'(a)\frac{(b_+(a)/a)'}{\sqrt{1-b_+(a)/a}} + \Phi_-'(a)\frac{(b_-(a)/a)'}{\sqrt{1-b_-(a)/a}}\right\}\Bigg|_{a_0},\end{aligned} \tag{2.99}$$

and (provided $\Phi_\pm'(a_0) \geq 0$)

$$
\begin{aligned}
[4\pi\, \Xi(a)\, a]''\big|_{a_0} \leq\; & \left\{\Phi_+'''(a)\sqrt{1-b_+(a)/a} + \Phi_-'''(a)\sqrt{1-b_-(a)/a}\right\}\Big|_{a_0} \\
& - \left\{\Phi_+''(a)\frac{(b_+(a)/a)'}{\sqrt{1-b_+(a)/a}} + \Phi_-''(a)\frac{(b_-(a)/a)'}{\sqrt{1-b_-(a)/a}}\right\}\Bigg|_{a_0} \\
& - \frac{1}{4}\left\{\Phi_+'(a)\frac{[(b_+(a)/a)']^2}{[1-b_+(a)/a]^{3/2}} + \Phi_-'(a)\frac{[(b_-(a)/a)']^2}{[1-b_-(a)/a]^{3/2}}\right\}\Bigg|_{a_0} \\
& - \frac{1}{2}\left\{\Phi_+'(a)\frac{(b_+(a)/a)''}{\sqrt{1-b_+(a)/a}} + \Phi_-'(a)\frac{(b_-(a)/a)''}{\sqrt{1-b_-(a)/a}}\right\}\Bigg|_{a_0} . \qquad (2.100)
\end{aligned}
$$

If $\Phi_\pm'(a_0) \leq 0$, we simply have

$$
\begin{aligned}
[4\pi\, \Xi(a)\, a]''\big|_{a_0} \geq\; & \left\{\Phi_+'''(a)\sqrt{1-b_+(a)/a} + \Phi_-'''(a)\sqrt{1-b_-(a)/a}\right\}\Big|_{a_0} \\
& - \left\{\Phi_+''(a)\frac{(b_+(a)/a)'}{\sqrt{1-b_+(a)/a}} + \Phi_-''(a)\frac{(b_-(a)/a)'}{\sqrt{1-b_-(a)/a}}\right\}\Bigg|_{a_0} \\
& - \frac{1}{4}\left\{\Phi_+'(a)\frac{[(b_+(a)/a)']^2}{[1-b_+(a)/a]^{3/2}} + \Phi_-'(a)\frac{[(b_-(a)/a)']^2}{[1-b_-(a)/a]^{3/2}}\right\}\Bigg|_{a_0} \\
& - \frac{1}{2}\left\{\Phi_+'(a)\frac{(b_+(a)/a)''}{\sqrt{1-b_+(a)/a}} + \Phi_-'(a)\frac{(b_-(a)/a)''}{\sqrt{1-b_-(a)/a}}\right\}\Bigg|_{a_0} . \qquad (2.101)
\end{aligned}
$$

Note that these last three equations are entirely vacuous in the absence of external forces, which is why they have not appeared in the literature until now.

In discussing specific examples one now merely needs to apply the general formalism described above. Several examples are particularly important, some to emphasize the features specific to possible asymmetry between the two universes used in traversable wormhole construction, some to emphasize the importance of NEC non-violation in the bulk, and some to assess the simplifications due to symmetry between the two asymptotic regions [21, 24].

2.2.8 Discussion

These linearized stability conditions reflect an extremely general, flexible and robust framework, which is well-adapted to general spherically symmetric thin-shell traversable wormholes and, in this context, the construction confines the exotic material to the thin shell. The latter, while constrained by spherical symmetry, is allowed to move freely within the four-dimensional spacetimes, which permits a fully dynamic analysis. Note that to this effect, the presence of a flux term has been, although widely ignored in the literature, considered in great detail. This flux term corresponds to the net discontinuity in the conservation law of the surface stresses of the bulk momentum flux, and is physically interpreted as the work done by external forces on the thin shell.

Relative to the linearized stability analysis, we have reversed the logic flow typically considered in the literature, and introduced a novel approach to the analysis. We recall that the standard procedure extensively used in the literature is to define a parametrization of the stability of equilibrium, so as not to specify an equation of state on the boundary surface [18–20]. More specifically, the parameter $\eta(\sigma) = d\mathscr{P}/d\sigma$ is usually defined, and the standard physical interpretation of η is that of the speed of sound. In this section, rather than adopting the latter approach, we considered that the stability of the wormhole is fundamentally linked to the behaviour of the surface mass $m_s(a)$ of the thin shell of exotic matter, residing on the wormhole throat, via a pair of stability inequalities.

More specifically, we have considered the surface mass as a function of the potential. This novel procedure implicitly makes demands on the equation of state of the matter residing on the transition layer, and demonstrates in full generality that the stability of thin-shell wormholes is equivalent to choosing suitable properties for the material residing on the thin shell. Furthermore, specific applications were explored and we refer the reader to Ref. [21] for more details.

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