Chapter 2
Optimality Criteria

Abstract A first optimality criterion, that of the fully stressed design, was already introduced in the previous chapter. The buckling of a circular tube in compression is used to illustrate a second criterion, that of simultaneous buckling modes. In fact, when the tube forms part of a truss structure, this might be seen as a logical extension of the principle of the fully stressed design. This second optimality criterion leads directly to an efficiency formula, expressing the maximum stress that can be achieved in a thin tube or other component in terms of a suitable structural index and the elastic modulus of the material. The concept of the design space, widely used in subsequent chapters, is introduced with the circular tube. A third criterion is developed for the maximum stiffness of a structure, on the basis of a simple truss but taken in principle to apply more widely. It is shown that under certain conditions, a fully stressed design, with maximum strength-to-weight ratio, also has maximum stiffness. A spreadsheet program is presented for the optimization of circular and square tubes in compression, subject to dimensional restrictions and specified maximum allowable stress.

Optimality criteria are conditions that are assumed to be satisfied in an optimum design. When known to be valid, they can be used either directly to find an optimum or otherwise to reduce the size of an optimization problem. The first of these criteria, that of a fully stressed design, was introduced in the previous chapter, along with some necessary conditions to establish its validity. In short, a fully stressed design implies that the maximum allowable stress is reached in all parts of a structure, or in the case of a truss structure in each member. This was discussed without regard to the possibility of other modes of failure, such as buckling of some parts of the structure at a lower stress. A second optimality criterion relates specifically to the design of a structural component when buckling is the principal design condition. This is introduced here through the optimization of a circular tube loaded in compression, subject to both buckling and maximum stress limitations. This might be regarded as one of the members of a truss structure in the previous chapter. At the same time, the circular tube is used to introduce the concept of the 'design space', an invaluable aid in the visualization of a numerical optimization

A. Rothwell, *Optimization Methods in Structural Design*, Solid Mechanics and Its Applications 242, DOI 10.1007/978-3-319-55197-5_2

procedure and sometimes, as now for the circular tube, as a direct means of solving an optimization problem. With this, an efficiency formula can be derived expressing the maximum stress that can be achieved in terms of a structural index. A third optimality criterion concerns the design of a structure for maximum stiffness, and again conditions under which this is valid have to be established. This is introduced later in the present chapter, in the context of a truss structure but taken to apply more generally to other types of structure.

2.1 Circular Tube in Compression

In the previous chapter, the design of a truss structure was explored on the basis of a specified maximum stress for all the members. However, for those members of the truss which are in compression, if they are relatively slender the maximum stress may be limited by buckling instead of by an allowable material stress. The maximum stress will depend then on the actual size and shape of cross section of each member, as well as on its length. For a tubular member, such as the circular tube considered here, buckling may be either in flexural buckling in a long-wave mode, as illustrated in Fig. 2.1, or by local buckling in a short-wave mode.

The critical compressive load P_E for flexural buckling is given by the well-known Euler's formula:

$$P_E = \frac{\pi^2 EI}{L^2},$$

where E is the elastic modulus of the material, I is the (minimum) second moment of area of the cross section, and L is the length of the member (assumed to be pinned at its ends). For a thin circular tube of radius R and thickness t, its cross-sectional area is

$$A = 2\pi Rt$$

and second moment of area:

$$I = \pi R^3 t,$$

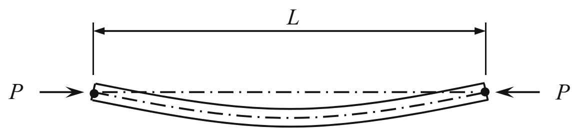


Fig. 2.1 Flexural buckling of a *bar* in compression

giving:

$$P_E = \frac{\pi^3 E}{L^2} R^3 t \tag{2.1}$$

and buckling stress:

$$\sigma_E = \frac{P_E}{A} = \frac{\pi^2 E R^2}{2L^2}.$$

The above formulae are sufficiently accurate if the thickness of the tube is small compared with its radius, and if R is taken to be the *mean* radius of the tube (measured to the mid-thickness). Use of Euler's formula above also implies that the tube is initially perfectly straight. If the tube is clamped at its ends, instead of pinned, an effective length $L/2$ should be used in the formula. Different effective lengths exist for other end conditions (see Young and Budynas [4], and many other texts).

It can be seen that the flexural buckling stress σ_E increases without limit with increasing radius R. If the design of the tube were based just on the stress σ_E, and leaving aside for the present any material strength limitation, this would imply that the strength-to-weight ratio of the tube also increases without limit. However, the restriction on R is through local, or short wavelength, buckling of the tube in which the cross section is deformed out of its initially circular shape into a pattern of small buckles, both around and along the length of the tube. The standard formula for this local mode of buckling is given as follows:

$$\sigma_L = KE\frac{t}{R},$$

where the buckling coefficient $K = 0.605$ for Poisson's ratio $\nu = 0.3$. (Formulae for Euler buckling and for the buckling of a thin, circular tube can be found in any standard textbook on buckling theory, including the classic text by Timoshenko and Gere [3], and more recent texts such as that by Megson [1].) It should be noted, however, that for very thin tubes the coefficient K is sensitive to imperfections in the circular form of the tube, as well as to the end conditions, and even for the thicker tubes assumed here may still have to be reduced. The above formula gives for the buckling load:

$$P_L = KE\frac{t}{R} \cdot A = 2\pi K E t^2, \tag{2.2}$$

independent of R.

We can now represent the design problem of the circular tube in a so-called design space, as shown in Fig. 2.2. The axes of the diagram are the radius R and thickness t of the tube—termed the 'design variables'. The design conditions, or 'constraints', imposed on the design are the Euler and local buckling loads, in

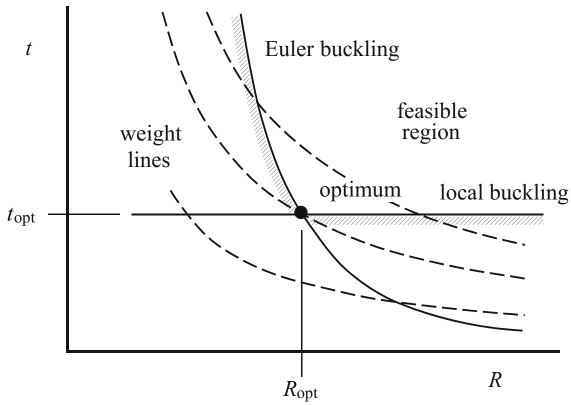


Fig. 2.2 Design space for a *circular* section tube in compression

Eqs. (2.1) and (2.2), plotted as solid lines in the design space. Each of these lines represents critical combinations of R and t to precisely satisfy the particular constraint. The shading on the constraint lines marks the side on which the constraints are not satisfied, in other words the design is unsafe, or infeasible. The unshaded side is therefore the safe, or feasible, side. The two constraints together define the boundary of the feasible region. To locate the optimum, lines of constant cross-sectional area (so-called 'weight lines') are added to the design space, shown as broken lines. The cross-sectional area of the tube reduces towards the origin of the diagram, so it is clear that the optimum is located at the intersection of the two constraint lines, at which we have the smallest cross-sectional area and at which Euler and local buckling occur simultaneously.

This condition—that of simultaneous modes of buckling—is the second optimality criterion referred to earlier. In a sense, this might also be seen more generally as an extension of the principle of the fully stressed design—referring again to a truss structure, simultaneous failure of all members and now simultaneous failure in the different modes of buckling within a member. However, as before for a fully stressed design, the validity of this criterion cannot be guaranteed in every case, and has therefore first to be verified when applied to a different class of problem. For the circular tube, as well as for other shapes of cross section, use of this criterion leads to the definition of efficiency, and a direct solution for the optimum dimensions. However, when we include a material strength limitation we shall see that the condition of simultaneous buckling modes may no longer apply. Neither can it be guaranteed that *all* buckling modes will occur simultaneously. This is evidently so when the number of possible buckling modes exceeds the number of design variables available for optimization. Finally, it should be pointed out that while the discussion here has been about simultaneous buckling in the different modes, this is unlikely to occur in reality. Imperfections of various kinds reduce the buckling stress from its theoretical value, and in practice determine in which mode buckling will actually occur.

2.1.1 Efficiency Formula

In a practical problem, the design space in Fig. 2.2 might have been drawn for specific values of P, L and E. However, with the optimality criterion that we now have, explicit formulae can be derived for the optimum radius and thickness, and for the maximum stress that can be achieved in the tube. For simultaneous Euler and local buckling:

$$P = P_E = P_L,$$

and substituting from Eqs. (2.1) and (2.2)

$$P = \frac{\pi^3 E}{L^2} \cdot R^3 t = 2\pi K E t^2.$$

Here, we have two equations which can be solved for t and R to give:

$$t^* = \left(\frac{P}{2\pi K E}\right)^{1/2} \tag{2.3}$$

and

$$R^* = \left(\frac{2K}{\pi^5} \cdot \frac{PL^4}{E}\right)^{1/6}, \tag{2.4}$$

where the asterisk conventionally denotes optimum values of the given dimensions. The maximum stress is then as follows:

$$\sigma_{\max} = \frac{P}{2\pi R^* t^*} = \left(\frac{\pi K}{4}\right)^{1/3} E^{2/3} \left(\frac{P}{L^2}\right)^{1/3},$$

which can be written as follows:

$$\sigma_{\max} = \eta\, E^{2/3} \left(\frac{P}{L^2}\right)^{1/3}, \tag{2.5}$$

where the ‘efficiency’ η of the tube is

$$\eta = \left(\frac{\pi K}{4}\right)^{1/3}.$$

If the buckling coefficient K is taken to have its maximum theoretical value $K = 0.605$, we obtain an efficiency $\eta = 0.780$.

The above formulae define the maximum stress that can be achieved in an optimized circular tube in compression, together with the optimum radius and thickness, and show how these depend on the modulus E, load P and length L. The term P/L^2 in Eq. (2.5) is referred to as the 'structural index' and represents the non-material parameters of the problem. The different powers in the formula show the sensitivity of the maximum stress $\sigma_{\max}$ to the particular parameters. For example, while the individual buckling loads are directly proportional to the modulus E, the effect of change in E on the maximum stress in an optimized tube depends only on $E^{2/3}$. Similarly, the effect of any reduction in K, to compensate for imperfections in the tube, is felt only as $K^{1/3}$. Whereas the weight of a bar in tension is directly proportional to both the load on it and its length ($W \propto PL$), from the efficiency formula it is deduced that for a circular tube in compression the relation becomes

$$W \propto \left(\frac{P}{L^2}\right)^{-1/3} PL.$$

An effective length L can be used in all the formulae, if it is necessary to allow for different end conditions. Finally, it should be emphasized that the term efficiency, as used here, is simply an efficiency *coefficient*—the larger its value the greater the maximum stress that can be achieved. Its maximum value is *not* 1.0, and it should not be expressed as a percentage.

Similar efficiency formulae can be derived for other shapes of cross section, as shown in Table 2.1. For a solid section, there is, of course, no local buckling mode, so the design condition is simply $P = P_E$ to solve for the required radius or other cross-sectional dimension. For a square section tube, the second moment of area to calculate the Euler buckling load is $I = 2b^3t/3$, where b is the *mean* side of the square and t is the thickness of the tube. For local buckling, each side can be treated individually as a thin plate, for which the local buckling stress is:

$$\sigma_L = KE\left(\frac{t}{b}\right)^2,$$

Table 2.1 Efficiency formulae for circular and square sections

Circular section tube	$\sigma_{\max} = \left(\frac{\pi}{4}\right)^{\frac{1}{3}} K^{\frac{1}{3}} E^{\frac{2}{3}} \left(\frac{P}{L^2}\right)^{\frac{1}{3}} = 0.780 E^{\frac{2}{3}} \left(\frac{P}{L^2}\right)^{\frac{1}{3}}$ with $K = 0.605$
Square section tube	$\sigma_{\max} = \left(\frac{\pi^2}{24}\right)^{\frac{2}{5}} K^{\frac{1}{5}} E^{\frac{3}{5}} \left(\frac{P}{L^2}\right)^{\frac{2}{5}} = 0.907 E^{\frac{3}{5}} \left(\frac{P}{L^2}\right)^{\frac{2}{5}}$ with $K = 3.62$
Solid circular section	$\sigma_{\max} = \left(\frac{\pi}{4}\right)^{\frac{1}{2}} E^{\frac{1}{2}} \left(\frac{P}{L^2}\right)^{\frac{1}{2}} = 0.886 E^{\frac{1}{2}} \left(\frac{P}{L^2}\right)^{\frac{1}{2}}$
Solid square section	$\sigma_{\max} = \left(\frac{\pi^2}{12}\right)^{\frac{1}{2}} E^{\frac{1}{2}} \left(\frac{P}{L^2}\right)^{\frac{1}{2}} = 0.907 E^{\frac{1}{2}} \left(\frac{P}{L^2}\right)^{\frac{1}{2}}$

where buckling coefficient $K = 3.62$ for Poisson's ratio $\nu = 0.3$ (plate buckling is discussed further in Chap. 7). The different form of the local buckling formula above for a square tube results in different powers in the efficiency formula. This means that the efficiency coefficient of a circular tube cannot be compared directly with that of the square tube, or other flat-sided sections, instead of which the maximum stress has to be calculated from the appropriate efficiency formula in each case, at the required structural index, and the stresses compared.

Open sections, such as angle and I-sections, have significantly lower efficiency, as seen in Table 2.2 (with values taken from [2]. The I section in the table when optimized buckles simultaneously in all four modes, that is, flexural buckling about each axis and local buckling of the web and of the flanges. The angle section after optimization buckles simultaneously in local buckling and in flexural buckling about the axis shown. With only two variables, it is not possible for buckling to take place simultaneously in all three modes (local buckling and flexural buckling about both axes). The optimized X-section buckles simultaneously in flexural buckling about any axis and in torsional—local buckling. For many other open sections, such as a channel section, the shear centre does not coincide with the centre of gravity of the section, and these are subject to coupled flexural—torsional buckling with further reduction in efficiency.

In the foregoing text, by 'efficiency' is implied the *maximum* efficiency of a particular shape of section. Of course, for a tube or bar with given limits on dimensions (for example, if some minimum thickness is imposed), the maximum stress that can be reached will be less. This can be expressed as reduced or 'achieved' efficiency. For example, with Eq. (2.5) for a circular tube, if σ is the reduced maximum stress the achieved efficiency becomes:

$$\eta = \frac{\sigma}{E^{2/3}\left(P/L^2\right)^{1/3}}. \tag{2.6}$$

Finally, it is perhaps interesting to observe that, for a tube of any shape, if *all* its cross-sectional dimensions as well as its length are increased by, say, a factor of two, then both flexural and local buckling stresses are unchanged. An already optimized tube, with simultaneous buckling modes, therefore remains an optimum.

Table 2.2 Maximum efficiency η for some thin-walled open sections (values for angle and I sections taken from Rees [2])

$\sigma_{max} = \eta\, E^{\frac{3}{5}} \left(\frac{P}{L^2}\right)^{\frac{2}{5}}$		
	$\eta = 0.705$	Buckles simultaneously about both axes
	$\eta = 0.439$	Equal flange width and thickness, buckles about axis shown
	$\eta = 0.205$	Equal flange width and thickness, buckles simultaneously in flexural buckling and torsional/local buckling

The same applies to a solid bar, but now with only flexural buckling. Since in both cases, the cross-sectional area is increased by a factor of four, the maximum load is also increased by a factor of four (=2 squared), whereas the volume and therefore the weight is increased by a factor of 8 (=2 cubed). This is a demonstration of the 'square-cube law' in structural design.

Example 2.1 Find the optimum diameter and thickness of a circular tube, with simply supported length L = 1000 mm, to carry a compressive load P = 10,000 N. Take the elastic modulus E = 72,000 N/mm^2 for an aluminium alloy material, and local buckling coefficient K = 0.605.

From Eq. (2.3), the optimum thickness is

$$t^* = \left(\frac{P}{2\pi KE}\right)^{1/2} = 0.191\,\text{mm},$$

and from Eq. (2.4) the optimum (mean) radius is

$$R^* = \left(\frac{2K}{\pi^5}\cdot\frac{PL^4}{E}\right)^{1/6} = 28.6\,\text{mm}.$$

The corresponding outer diameter is 57.4 mm.

The thickness found above may in practice be considered too small for a tube of this diameter. Suppose we choose now a minimum thickness $t = 1.0$ mm. From Eq. (2.1), the required Euler buckling load is

$$P_E = \frac{\pi^3 E}{L^2}R^3 t = 10,000\,\text{N}.$$

Solving for R gives R = 16.48 mm, with corresponding outer diameter 34.0 mm.

With reduced radius and increased thickness, the local buckling condition is clearly more than satisfied.

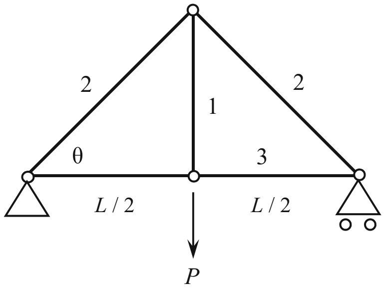
The cross-sectional area of the tube is now

$$A = 2\pi Rt = 103.6\,\text{mm}^2,$$

and the compressive stress

$$\sigma = \frac{P}{A} = 96.6\,\text{N/mm}^2.$$

Fig. 2.3 Five-bar truss



With the formula for 'achieved efficiency' in Eq. (2.6), we find:

$$\eta = \frac{\sigma}{E^{2/3}(P/L^2)^{1/3}} = 0.259.$$

We see that the effect of the chosen minimum thickness is to reduce substantially the efficiency of the tube, from its maximum value $\eta = 0.780$ to its present value $\eta = 0.259$. Note also that, while the compressive stress is comparatively low in this example, no account has yet been taken of a material stress limitation. ■

Example 2.2 Find the optimum angle θ of the truss in Fig. 2.3, with members composed of circular tubes, taking into account the maximum compressive stress due to buckling of members in compression. Take load $P = 1000$ N, span $L = 1000$ mm, modulus $E = 72{,}000$ N/mm^2 and the allowable tensile stress $\sigma_t = 400$ N/mm^2.

An efficiency formula enables the maximum stress that can be achieved in a compression member to be calculated directly, without actually performing the design. The two sloping members (both numbered 2 in the figure) are in compression. The compressive force in each of these is

$$F_2 = \frac{P}{2 \sin \theta}$$

and their length is

$$l_2 = \frac{L}{2 \cos \theta},$$

giving a structural index

$$\frac{F_2}{l_2^2} = \frac{2P}{L^2} \cdot \frac{\cos^2 \theta}{\sin \theta}.$$

Since both members are circular tubes, with efficiency $\eta = 0.780$, the maximum stress in these members is found by substituting the above formula for the structural index into the efficiency formula, Eq. (2.5), to give:

$$\sigma_{\max} = 0.780E^{2/3}\left(\frac{2P}{L^2}\cdot\frac{\cos^2\theta}{\sin\theta}\right)^{1/3}.$$

The forces in the remaining tension members are

$$F_1 = P, \qquad F_3 = \frac{P}{2\tan\theta}.$$

With maximum stress $\sigma_{\max}$ in the compression members and stress σ_t in the tension members, the volume of the truss becomes:

$$V = \left(\frac{1}{\sigma_t}\cdot\tan\theta + \frac{1}{\sigma_{\max}}\cdot\frac{1}{\sin\theta\cos\theta} + \frac{1}{\sigma_t}\cdot\frac{1}{\tan\theta}\right)\frac{PL}{2}.$$

Varying θ to minimize the volume V gives an optimum angle of the truss in this example

$$\theta^* = 35.9^\circ$$

and maximum compressive stress

$$\sigma_{\max} = 176\,\text{N/mm}^2.$$

The optimum angle θ^* is less than the 45° found earlier ($H/L = 0.5$ in Fig. 1.11) when all bars were assumed to have the same allowable stress. The explanation for this is that reducing the angle θ reduces the length of the compression members but increases the load in them. The optimum angle θ^* is therefore a compromise between the two, with reduction in length having the greater effect. Note that both optimality criteria have now been used in this example. All members are fully stressed in the sense that their area is based either on the maximum tensile stress σ_t of the material in the tension members or on the maximum compressive stress $\sigma_{\max}$ in the compression members, the latter based on simultaneous flexural and local buckling of the tubular members. Even in this small problem, this illustrates how optimality criteria can be used to reduce the size of an optimization problem. All constraints—tensile strength and both buckling modes—have been eliminated, or better said directly satisfied, and only one design variable, angle θ, remains. ■

2.1.2 Material Limitation

Up to now, no reference has been made to the allowable compressive stress of the material of a compression member, which may limit the stress predicted by the efficiency formula in Eq. (2.5). This is illustrated in Fig. 2.4, where the maximum compressive stress for four different shapes of cross section is plotted against

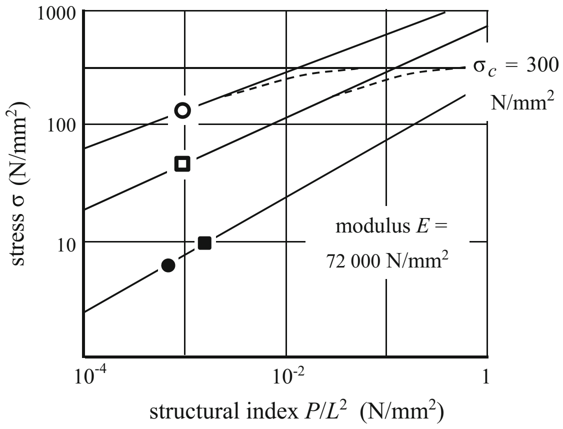


Fig. 2.4 Maximum compressive stress for *circular* and *square* section *tubes* and *bars* with material stress limitation σ_c

structural index, together with an allowable compressive stress $\sigma_c = 300\ \text{N/mm}^2$ for the material. Note the logarithmic scale which, by the nature of the efficiency formula, gives straight lines in this plot but also tends to obscure the difference in performance of the different cross sections (lines for the solid circular and square sections are indistinguishable in the figure). The superiority of tubular sections over solid ones is nevertheless clear in this figure. In reality, there is a loss of modulus as the maximum allowable stress of the material is approached, due to progressive yielding, resulting in blending of the lines as suggested by the dotted lines in the figure.

A material limitation

$$P_M = \sigma_c A = \sigma_c \cdot 2\pi R t$$

has been added in Fig. 2.5 to the previous design space in Fig. 2.2. This new constraint clearly coincides with a line of constant cross-sectional area $A = P/\sigma_c$, meaning that all combinations of R and t between points 1 and 2 represent an optimum design. There is therefore no unique optimum, and unless point 1 or point 2 is chosen, neither buckling mode is critical. However, if a material with a larger allowable stress had been chosen, the curve representing the material limitation on the design space would be lowered and the problem would revert to the original one of simultaneous buckling modes. If, say, a minimum thickness limitation $t_{\min}$ or a maximum radius $R_{\max}$ is imposed (for manufacturing or other practical reasons), then these can also be added to the design space. It is clear that many different versions of the same diagram can exist, producing different design conditions in the now more highly constrained optimum.

The situation in Fig. 2.5 is representative of the majority of optimization problems, albeit here a very simple one. With only two design variables R and t, it is clearly not possible to satisfy all constraints simultaneously, whether they arise from buckling or material stress limitations, dimensional restrictions or perhaps other conditions that might be imposed. The principal task in optimization is

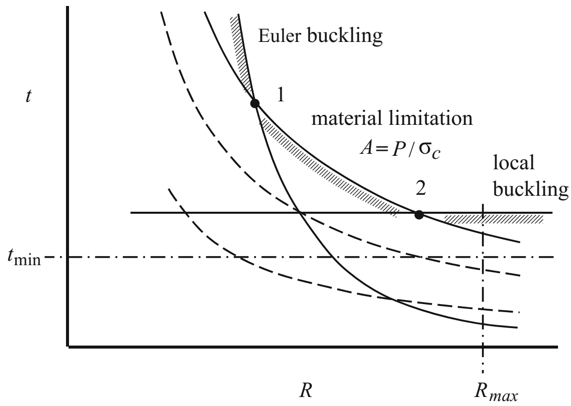


Fig. 2.5 Design space for a *circular* section *tube* with compressive stress limitation

generally constraint *selection*, in other words to determine which constraints will prove to be active at the optimum and which will be more than satisfied. Of course, there can be no single answer to this—in the case of the circular tube considered here, this depends entirely on the actual values of the parameters P, L and σ_c, and on other limitations such as $t_{\min}$ and $R_{\max}$.

2.2 Criterion for Maximum Stiffness

The two optimality criteria already discussed are concerned only with the *strength* of a structure, whether this be determined by material stress limits or by buckling. Other than up to now, the design of a structure for maximum stiffness cannot be achieved in a member-by-member, iterative resizing process as in a fully stressed design, since its stiffness depends on the properties of all parts of the structure. To establish a criterion for maximum stiffness, we again consider a simple truss structure, such as in Fig. 2.6, loaded by a single force P. If the deflection at the point of loading and in the same direction as the applied load is δ, then the stiffness of the structure is P/δ. It is this stiffness that we wish to maximize. By simply

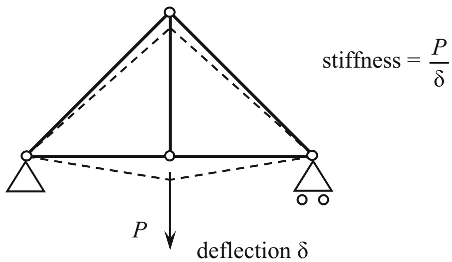


Fig. 2.6 Deflection under a single applied load

increasing the amount of material in the structure, its stiffness can be increased without limit, therefore we have to maximize the stiffness for some given total volume V of the structure.

The deflection δ can conveniently be found by the principle of conservation of energy—the work done by the applied load P is equal to the total elastic strain energy stored in the members of the truss:

$$\frac{1}{2}P\delta = \sum \frac{\sigma_i^2}{2E} V_i, \tag{2.7}$$

where σ_i is the stress in each member, V_i is the corresponding volume of each member and E is the elastic modulus of the material, for the present assumed to be the same for all members. Substituting for the stress

$$\sigma_i = \frac{F_i}{A_i} = \frac{F_i l_i}{V_i} \tag{2.8}$$

we obtain

$$\delta = \frac{1}{PE}\sum \frac{(F_i l_i)^2}{V_i}, \tag{2.9}$$

where F_i is the force in a member and A_i, l_i are its cross-sectional area and length, respectively. Note that forces F_i are constant if, as assumed here, the truss is statically determinate.

Putting $V = \sum V_i$ for a given volume of material, we can substitute

$$V_1 = V - V_2 - V_3 - \cdots$$

into Eq. (2.9) to give

$$\delta = \frac{1}{PE}\left[\frac{(F_1 l_1)^2}{(V - V_2 - V_3 - \cdots)} + \frac{(F_2 l_2)^2}{V_2} + \frac{(F_3 l_3)^2}{V_3} + \cdots\right]. \tag{2.10}$$

(Choice of V_1 for elimination in the formula above is entirely arbitrary.) Differentiating with respect to the remaining variables V_i for minimum δ:

$$\frac{\partial(\delta)}{\partial V_2} = \frac{1}{PE}\left[\frac{(F_1 l_1)^2}{(V - V_2 - V_3 - \cdots)^2} - \frac{(F_2 l_2)^2}{V_2^2}\right] = 0,$$

$$\frac{\partial(\delta)}{\partial V_3} = \frac{1}{PE}\left[\frac{(F_1 l_1)^2}{(V - V_2 - V_3 - \cdots)^2} - \frac{(F_3 l_3)^2}{V_3^2}\right] = 0,$$

and so on, from which

$$\frac{(F_1 l_1)^2}{V_1^2} = \frac{(F_2 l_2)^2}{V_2^2} = \frac{(F_3 l_3)^2}{V_3^2} = \cdots.$$

Note that elimination of V_1 above, then substituting back, is simply a device to ensure that differentiation takes account of a required total volume V. A more elegant way would be by use of Lagrange multipliers, to be introduced in the next chapter. Referring back to Eq. (2.8), the above condition becomes simply

$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \cdots = \sigma_0^2,$$

where σ_0 denotes here an arbitrarily chosen stress level. Uniform stress in all members is now the criterion for maximum stiffness, implying in principle a statically determinate structure, as was assumed at the start.

With the same stress in all members, Eq. (2.9) can be simplified to

$$\delta = \frac{V}{PE} \cdot \sigma_0^2. \tag{2.11}$$

For any fully stressed design of truss, with a maximum stress σ_0, its total volume V is given by Eq. (1.6):

$$V = n \cdot \frac{PL}{\sigma_0},$$

where as before, coefficient n depends only on the layout of the truss. For given P, V and span L, we can rewrite this as:

$$\sigma_0 = nPL/V,$$

and substituting for σ_0 into Eq. (2.11) gives

$$\delta = \frac{n^2 L^2}{VE} \cdot P.$$

For minimum deflection or, in other words, maximum stiffness P/δ for given volume V of material, it is clear that we require the smallest n value. We can conclude, therefore, that the optimum layout of truss (maximum strength-to-weight ratio) also has maximum stiffness, with regard to deflection at the point of loading, provided of course that the members are indeed uniformly stressed.

It should be noted that the criterion for maximum stiffness developed above applies strictly to a truss under a single applied load and to deflection at the point of loading in the direction of the applied load. If there is more than one load, such as an additional load applied at the top of the truss as in Fig. 2.7, the left-hand side of Eq. (2.7) has to be replaced by $\frac{1}{2}(P_1\delta_1 + P_2\delta_2)$, or in general by $\frac{1}{2}\sum P_i\delta_i$. The rest

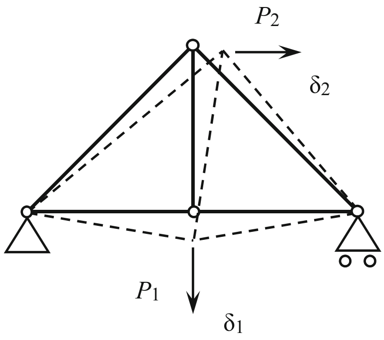


Fig. 2.7 Deflection under two applied loads

of the analysis is then unchanged, which implies that the condition of uniform stress in all members leads to a minimum of $\sum P_i\,\delta_i$. This is the minimum of the sum of the individual deflections weighted in proportion to the magnitude of the applied loads, but does not in practice provide any useful definition of maximum stiffness.

If the members of the truss are of different materials, each with elastic modulus E_i, Eq. (2.7) for a single load P becomes

$$\frac{1}{2}P\delta = \sum \frac{\sigma_i^2}{2E_i} V_i. \tag{2.12}$$

Following the same procedure as before, but by keeping the individual elastic moduli within each term of Eq. (2.10), we replace the condition of uniform stress by one of the uniform strain energy densities (strain energy per unit volume)

$$\frac{\sigma_1^2}{E_1} = \frac{\sigma_2^2}{E_2} = \frac{\sigma_3^2}{E_3} = \cdots.$$

If the members also have different densities ρ_i, and it is required to maximize the stiffness of the structure for a given mass rather than for a given volume, then Eq. (2.12) becomes

$$\frac{1}{2}P\delta = \sum \frac{\sigma_i^2}{2E_i\rho_i} M_i,$$

where M_i is the mass of an individual member. Following again the same procedure, but differentiating now with respect to mass rather than volume, the criterion for maximum stiffness becomes

$$\frac{\sigma_1^2}{\rho_1 E_1} = \frac{\sigma_2^2}{\rho_2 E_2} = \frac{\sigma_3^2}{\rho_3 E_3} = \cdots.$$

This is the condition of uniform specific strain energy (strain energy per unit mass). This means that, when made of different materials (different modulus or density), a uniformly stressed design is now no longer the optimum design for stiffness. Our third optimality criterion, for maximum stiffness for a given total mass of the structure, has finally become one of the uniform strain energies per unit mass throughout the structure. The above criterion, while developed here for truss structures, can be expected to apply more generally to other types of structure. However, it should be borne in mind that, even for a structure made of a single material, in a two- or three-dimensional state of stress, the condition must be that of uniform strain energy, not simply uniform stress. Furthermore, it has to be remembered that the condition applies strictly to stiffness measured at the point of loading under a single applied load. In practice, of course, material strength, buckling and other constraints on the design may override this criterion for maximum stiffness to a greater or lesser extent.

2.3 Spreadsheet Programs

The spreadsheets illustrate the inclusion of behavioural constraints such as buckling and material stress limits into the optimization process. In Sect. 2.3.1, we take the now familiar problem of a thin tube loaded in compression, for which we already have a theoretical efficiency formula based on simultaneous buckling modes. By including maximum stress and dimensional limits as well as buckling constraints, the spreadsheets offer a more general solution. In Sect. 2.3.2, the spreadsheet for a seven-bar truss in the previous chapter is extended by taking into account the buckling of the compression members.

2.3.1 'Circular and Square Tubes'

The spreadsheets use Solver to optimize the cross-sectional dimensions of a circular or square tube loaded in axial compression, with buckling and allowable stress constraints, also allowing practical limits to be set on the dimensions (outer diameter or side of the square and thickness). The two tubes are on separate sheets of the workbook, as shown in Figs. 2.8 and 2.9.

The applied compressive load P, effective simply supported length L (depending on the required end conditions), elastic modulus E and maximum allowable compressive stress σ_c of the material have to be inserted in the appropriate cells. Maximum and minimum values of the outer diameter d or side b and thickness t have to be specified (these cells may not be left blank). Constraints to be satisfied are Euler buckling and local buckling, using the formulae in Sect. 2.1, and the maximum allowable stress of the material. The local buckling coefficient for a circular tube is set to $K = 0.605$ (maximum theoretical value) on the spreadsheet,

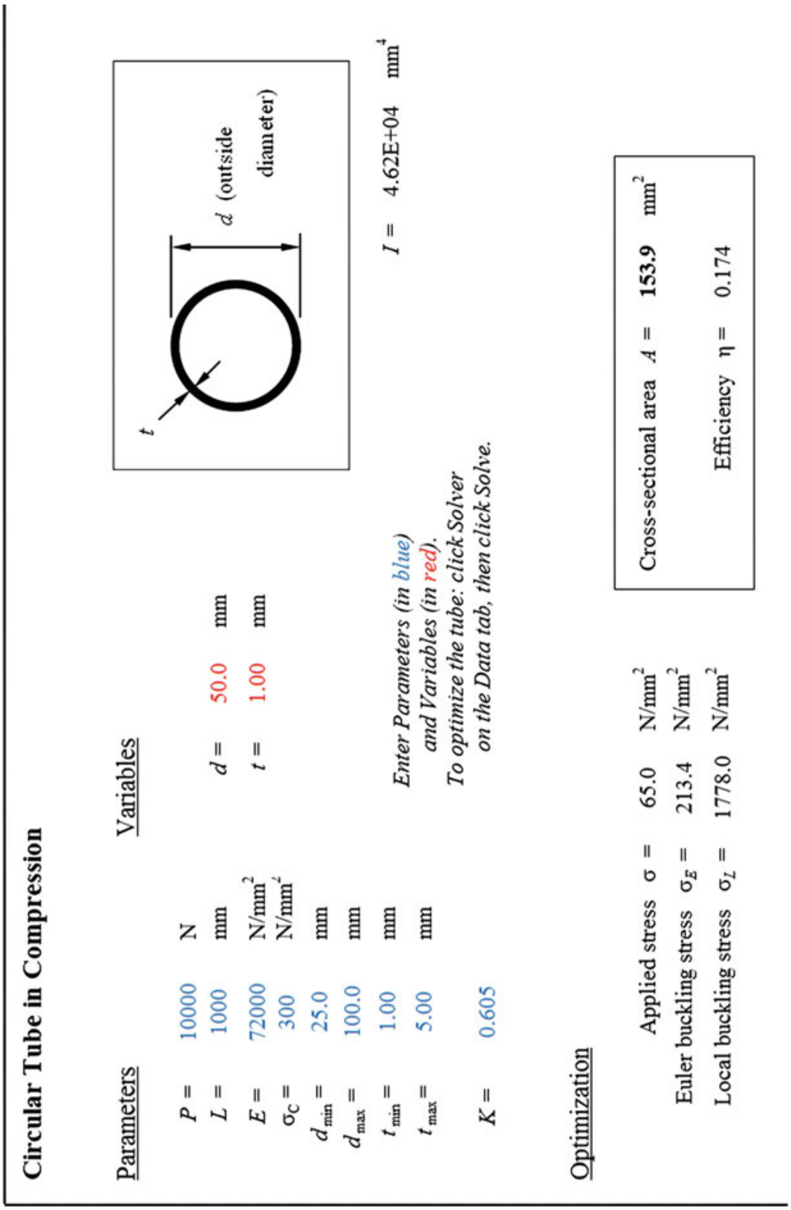


Fig. 2.8 Spreadsheet 'Circular Tube in Compression'

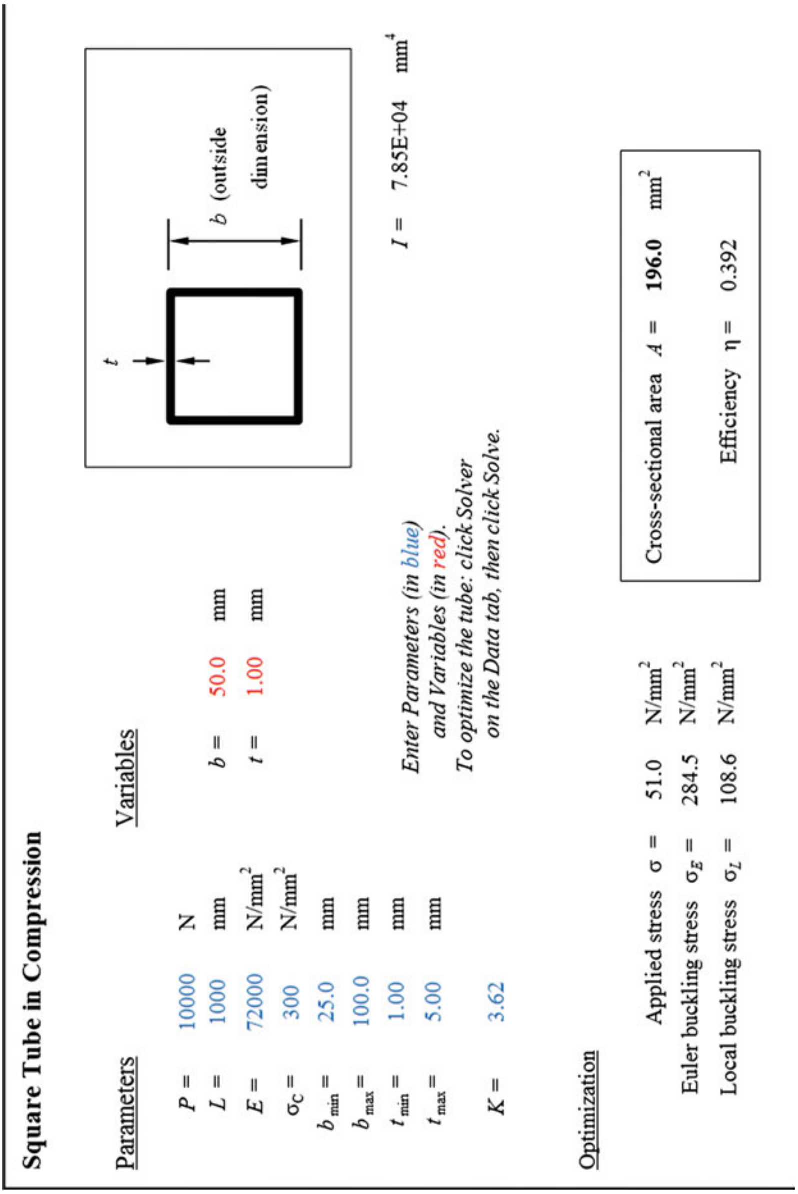


Fig. 2.9 Spreadsheet 'Square Tube in Compression'

but may be reduced to allow for imperfections if required. For a square tube, the local buckling coefficient is $K = 3.62$. The material is assumed to be perfectly elastic up to the maximum allowable stress. These constraints, together with limits on cross-sectional dimensions, are already set up in the Solver dialog box on the relevant sheet. Suitable initial values of the outer dimensions d or b and thickness t have to be entered in the appropriate cells. With these as variables, and subject to the above constraints, the cross-sectional area of the section is minimized by the GRG nonlinear method. To represent a solid circular or square bar, an extra constraint $t = d/2$ or $t = b/2$, respectively, may be added to the list of constraints in the Solver dialog box (in which case the local buckling stress has to be ignored). Parameters and design variables to be entered in the spreadsheets are listed in Table 2.3.

After optimization, the optimum cross-sectional dimensions replace the initial values of d or b and t, and corresponding values of stress σ, Euler buckling stress σ_E, local buckling stress σ_L and the achieved efficiency η (see Sect. 2.1.1) are calculated. Note that the efficiency is not calculated if either buckling stress exceeds the allowable compressive stress of the material, or if t/d or t/b exceeds 0.1. Comparison of the achieved efficiency with the known maximum efficiency for the circular or square tube shows the loss of efficiency due to the practical limits set on dimensions. The spreadsheets can be used to plot figures similar to Fig. 2.4 for any range of materials and structural index. If limits on dimensions do not intervene, and at a compressive load below that at which the material stress limit is reached, values of efficiency are obtained in agreement with those in Table 2.1.

An extended version of the present spreadsheet for a circular tube, with eccentrically applied compressive load and the effect of yielding of the material before the critical buckling load is reached, is presented in the next chapter.

Table 2.3 Data entry for spreadsheet programs 'Circular and Square Tubes in Compression'

Parameters	
Compressive load P	Enter the value in cell C6 as a *positive* number
Effective simply supported length L	Enter the value in cell C7
Elastic modulus E, allowable compressive stress σ_c	Enter values in cells C8:C9 as *positive* numbers
Min. and max. *outer* diameter d (or min. and max. *outer* width b)	Enter values in cells C10:C11 (cells may not be left blank)
Min. and max. thickness t	Enter values in cells C12:C13 (cells may not be left blank)
Local buckling coefficient K	Enter a reduced value in cell C15 for a circular tube if required
Variables	
Diameter d (or side b) and thickness t	Enter initial values in cells F7:F8

2.3.2 'Truss with Tubular Members'

The spreadsheet illustrates the use of Solver to optimize the seven-bar truss in Sect. 1.4, now with compression members made of circular tubes subject to both buckling and material stress limitations. Whereas in Example 2.2 an efficiency formula is used to predict the maximum stress in the compression members of a truss, in the spreadsheet Euler and local buckling and the maximum allowable stress are treated as separate constraints in each member. This allows upper and lower limits to be specified for the dimensions of any of the members, if so required. The spreadsheet is shown in Fig. 2.10.

The applied load P, span L, material allowable stresses σ_t and σ_c in tension and compression, elastic modulus E, specific weight ρ_w and local buckling coefficient K have to be entered in the spreadsheet. Constraints are the allowable stresses of the material, and Euler and local buckling of the compression members using the formulae in Sect. 2.1. These constraints are already set up in the Solver dialog box. Design variables are the dimensions D and H of the truss, the diameter d and thickness t of the compression members (1 and 3 in the diagram) and the cross-sectional area of the tension members (2 and 4). Ratios D/L and H/L are limited to not less than 0.01. With suitable initial values of the design variables, the volume of the truss is minimized by the GRG nonlinear method, subject to the above constraints. Parameters and design variables to be entered in the spreadsheet are listed in Table 2.4.

After optimization, initial values of the design variables are replaced by their optimized ones, together with the volume of the truss, its strength-to-weight ratio P/W, the stress σ in each member, the maximum forces in the bars in the different failure modes and the deflection δ of the truss at the point of loading. The spreadsheet shows the influence of buckling of the compression members on the optimum layout and strength-to-weight ratio of the truss. Since the truss is statically determinate, unless dimensional constraints imposed the tension members will reach the allowable tensile stress. Depending on the magnitude of the applied load, and again if no dimensional constraints are imposed, the compression members will reach either the allowable compressive stress or a reduced stress due to buckling. In the latter case, for maximum efficiency, buckling will occur simultaneously in Euler and local buckling. As in the spreadsheets in Sect. 2.3.1, no account is taken of reduction in modulus with yielding, the allowable compressive stress being treated as a simple cut-off for the buckling stress. It will be observed that the thickness of the tubes in compression frequently comes out impractically small. Only limits on the minimum thickness of the compression members are specified in the spreadsheet. Additional constraints for maximum or minimum dimensions can readily be added in the Solver dialog box, if required. The tubular members can be replaced by solid circular bars by adding constraints to specify the thickness equal to one-half of the diameter of the tubes.

Finally, a comment on the initial values of the design variables is worthwhile at this stage. As in all optimization problems, these should be chosen as far as possible

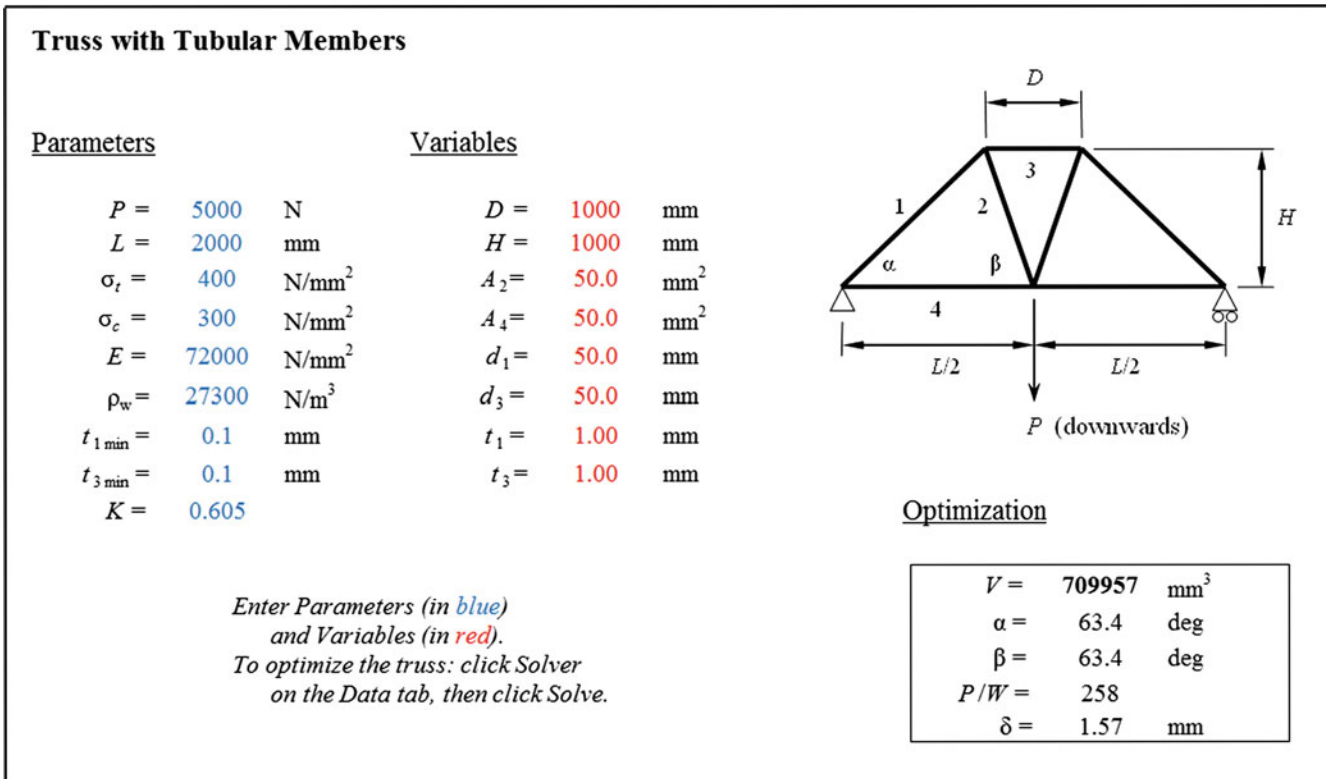


Fig. 2.10 Spreadsheet 'Truss with Tubular Members'

Table 2.4 Data entry for spreadsheet 'Truss with Tubular Members'

Parameters	
Applied load P (vertical downwards only)	Enter a positive value in cell C6
Span of truss L	Enter the value in cell C7
Allowable tensile stress σ_t, allowable compressive stress σ_c	Enter values in cells C8:C9 as *positive* numbers
Elastic modulus E, specific weight ρ_w	Enter values in cells C10:C11
Minimum thickness t_1, t_3 of bars 1 and 3	Enter values in cells C12:C13 (may not be left blank)
Local buckling coefficient K	Enter a reduced value in cell C14 for a circular tube if required
Design variables	
Dimensions D and H of truss (see figure on spreadsheet)	Enter initial values in cells F6:F7 (both positive, nonzero)
Cross-sectional area A_2, A_4 of bars 2 and 4	Enter initial values in cells F8:F9
Outer diameter d_1, d_3 of bars 1 and 3	Enter initial values in cells F10:F11
Thickness t_1, t_3 of 1 and 3	Enter initial values in cells F12:F13

within the range of the expected outcome of the optimization. Large differences can lead to Solver finding some other local minimum away from the true optimum, or it may fail to reach any solution at all. In the latter case, this is may be due to the starting point being far enough removed that the solution moves away from the optimum until it is no longer in the valid range of the original problem. Good practice is to repeat the optimization from different starting points, to gain confidence that the true optimum has been found.

2.4 Summary

Of the three optimality criteria we have now seen, the first—that of the fully stressed design—needs little further attention here, as it was treated extensively in the previous chapter. It will only be reiterated that a fully stressed design cannot in general be guaranteed to be a true optimum, although in the great majority of cases the progressive resizing of a structure to reach the maximum allowable stress in all its parts will lead to at least an improved design. The need for numerical optimization arises when constraints other than simple strength limitations are imposed, and when it is no longer possible to associate particular constraints with specific design variables in order to perform the necessary resizing. This last aspect is taken up in detail in the following chapter.

The second criterion—that of simultaneous modes of buckling—applies to a structure, or some part of it, liable to buckling in two or more different modes. A thin circular tube loaded in compression, which can buckle either in Euler buckling or in local buckling, is used to demonstrate that the optimum corresponds to buckling simultaneously in both of these modes. With this condition, efficiency formulae are derived by means of which the maximum stress that can be achieved is readily predicted. Again, it cannot be guaranteed that the condition of simultaneous buckling modes will be valid in all cases. Numerical optimization becomes necessary when a structure is subject to buckling constraints as well as material strength limitations, and to dimensional and other constraints.

The third criterion—that of uniform (specific) strain energy density for maximum stiffness—was demonstrated for a truss structure, but under highly restricted conditions. These are that the stiffness relates to deflection under a single load, measured at the point of application of that load. For a truss structure made of a single material, uniform strain energy density amounts to uniform stress throughout the structure, in other words a fully stressed design. When the conditions above are not met, for example if the stiffness is measured at some point other than the point of loading, or if buckling and other constraints intervene, numerical optimization is again necessary.

The three optimality criteria introduced in this and the previous chapter, deduced for simple truss structures, might be assumed to apply in principle to any type of structure. In fact, they are most likely what we would intuitively have assumed in the first place. However, while the use of optimality criteria may be quite appealing, it is seen that they are limited in application and their validity cannot be guaranteed. Numerical optimization methods avoid the limitations of optimality criteria and, above all, provide a consistent, general approach to the design of a structure. One of the main reasons for studying optimality criteria in some detail here is that, not surprisingly, many of the same characteristics reappear in designs obtained by numerical optimization. A good understanding of optimality criteria is therefore of great assistance in assessing the results of a numerical optimization and in understanding what has led to the solution obtained. The application of numerical optimization methods, together with the underlying numerical procedures, will be the principal task of the further chapters of this book.

Exercises

2.1 Verify the efficiency formula for a square section tube in Table 2.1. For a typical aluminium alloy, with an allowable compressive stress of 300 N/mm^2 and elastic modulus 72,000 N/mm^2, at what value of structural index is the maximum stress limited by the allowable stress of the material?
To derive the efficiency formula, assume that the thickness of the tube is small compared with its width, i.e. use simplified formulae for A and I similar to those for the circular tube in Sect. 2.1.

2.2 Derive the efficiency of a thin tube of hexagonal cross section (6 equal sides) and simply supported length *L* under a compressive load *P*. To compare the hexagonal tube with circular and square tubes, plot the maximum stress of the three sections for a chosen material over a realistic range of structural index.

Verify that the second moment of area of a thin hexagonal section about any axis through its middle point is $5b^3t/2$*, where t is the thickness of the tube and b is the mean width of each side. For the local buckling stress, use the formula:* $\sigma_L = 3.62E\ (t/b)^2$.

2.3 Verify the relation $W \propto (P/L^2)^{-1/3}PL$ in Sect. 2.1.1 for the minimum weight of a circular tube in compression. Derive a similar relation for a square tube.

Use the efficiency formulae in Table 2.1.

2.4 Draw the design space for the three-bar truss made of different materials in Sect. 1.1.1.

Draw the design space with variables A_1 *and* A_2*. Use the formulae in* Eqs. (1.1) and (1.2) *for the stress in the bars, with allowable stresses as in* Table 1.2 *and other data in* Fig. 1.7*. Plot the maximum stress constraints for both the single material and for the two different materials, Notice that in the second case, the stress in the outer bars will be the critical design condition unless these two bars are removed altogether.*

2.5 Draw the design space for the three-bar truss under alternative loads in Sect. 1.1.2. Verify the minimum volume of the truss given in that section.

Draw the design space with variables A_1 *and* A_2*. Use the formulae in* Eqs. (1.3)–(1.5) *for the stress in the bars. 'Goal Seek' can be used in Excel to solve for* A_2 *for a series of values of* A_1 *in* Eq. (1.3)*. Take* $P = 100$ kN, $\sigma_0 = 300$ N/mm^2 *and* $L = 1000$ mm*, as in* Sect. 1.1.2.

2.6 Use the spreadsheet 'Circular Tube in Compression' with different values of minimum thickness to explore the effect of this on the achieved efficiency of the tube.

Use the values of P, L and E already on the spreadsheet (*or any other convenient values*)*. Set* $d_{\min} = 0$*, and* $d_{\max}$ *and* σ_c *large enough to ensure they have no effect. Make a plot of* η *and d over a range of minimum thickness.*

2.7 Use the spreadsheet 'Circular Tube in Compression' to make a plot of maximum stress against structural index for circular tubes made of various different materials.

The allowable compressive stress and elastic modulus of different grades of aluminium alloy, steel, titanium and other materials are widely available. Choose P and L for values of structural index that give realistic stress levels for the materials.

2.8 Modify the spreadsheet 'Square Tube in Compression' for the optimization of a hexagonal tube in compression.

Use the formulae in Exercise 2.2*. Compare the efficiency with the previously calculated value.*

2.9 Use the spreadsheet 'Truss with Tubular Members' to find the optimum layout of the seven-bar truss under buckling constraints.

Use the material and other data already on the spreadsheet. Try different starting points for the optimization to verify that the same result is obtained. Note the deflection of the truss and the stress in each member of the optimized truss.

2.10 Verify the deflection δ given in the spreadsheet for the optimized truss in the previous exercise.
Deflection δ can be calculated from Eqs. (2.7) *to* (2.9).

2.11 Use the spreadsheet 'Truss with Tubular Members' to verify that the optimum design for stiffness, in the absence of constraints, has equal stress in all members and has the same shape as the optimum design for minimum weight.
Use the material and other data already on the spreadsheet (*or any other convenient values*). *First remove all constraints currently in the Solver dialog box. Add a new constraint to set the volume V of the truss to any reasonable value. Deflection δ at the point of loading is given on the spreadsheet. Change the objective to deflection δ in the dialog box, and use Solver to optimize the truss for minimum δ.*

2.12 Modify the spreadsheet 'Truss with Tubular Members' for a load applied vertically upwards at mid-span.
Members 2 *and* 4 *are now in compression, members* 1 *and* 3 *in tension. Compare the strength-to-weight ratio P/W with that for a downwards applied load.*

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http://www.springer.com/978-3-319-55196-8

Optimization Methods in Structural Design
Rothwell, A.
2017, XVI, 314 p. 113 illus., Hardcover
ISBN: 978-3-319-55196-8