Chapter 2
Linear Stability of Inviscid Plane-Parallel Flows of Vibrationally Excited Diatomic Gases

Abstract This chapter is devoted to investigations of linear stability of plane-parallel flows of an inviscid nonheat-conducting vibrationally excited gas. Some classical results of the theory of linear stability of ideal gas flows, such as the first and second Rayleigh's theorems and Howard's theorem, are generalized. An equation of the energy balance of disturbances is derived, which shows that vibrational relaxation generates an additional dissipative factor, which enhances flow stability. Calculations of the most unstable inviscid modes with the maximum growth rates in a free shear layer are described. It is shown that enhancement of excitation of vibrational modes leads to reduction of the growth rates of inviscid disturbances.

Linear stability of plane-parallel shear flows are traditionally studied within the framework of the hydrodynamic stability theory. Such studies for an ideal incompressible fluid were performed in the classical works of Helmholtz, Kelvin, and Rayleigh. Later on, their results were extended to more realistic problems of ideal compressible gas flows and inhomogeneous stratified and conducting fluid flows in fields of various mass forces [1].

This chapter describes the results of studies of linear stability of plane-parallel shear flows of an inviscid non-heat-conducting vibrationally excited compressible gas. Linear equations for inviscid disturbances derived by linearization of the original system (1.27) with respect to a spatially homogeneous steady flow are formulated in Sect. 2.1.

In Sect. 2.2, it is proved by using energy integrals that vibrational relaxation is an additional dissipation factor, which enhances flow stability. Generalization of the Rayleigh's classical first and second theorems is obtained as necessary conditions for instability enhancement in the flows considered. Under certain conditions, a range of eigenvalues of unstable perturbations is specified in the upper complex half-plane as a counterpart of Howard's semicircle theorem. In the limit there is a continuous transition to well-known results for an ideal fluid as the Mach number and the vibrational relaxation time τ approach zero and for an ideal compressible gas as τ approaches zero.

The results of numerical calculations of eigenvalues and eigenfunctions of the most unstable inviscid modes in a free shear layer are presented in Sect. 2.3. Their

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dependencies on the Mach number of the carrier flow, τ, and the degree of thermal nonequilibrium are analyzed. The vorticity eigenfunctions of these modes are used as initial data for numerical calculations of nonlinear evolution of the Kelvin–Helmholtz waves presented in Chap. 7.

2.1 Equations of the Linear Stability Theory

In the $(x,\ y)$ coordinate plane we consider a shear flow in which the main (carrier) flow of uniform density, ρ_0, and temperature, T_0, is directed along the abscissa axis x and has a velocity profile $U_s = U_s(y)$. The perturbed flow is described by the system of equations of two-temperature gas dynamics (1.27) (see also [2–4]). In dimensionless variables the system has the form

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_i}{\partial x_i} = 0, \quad \rho\left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}\right) = -\frac{\partial p}{\partial x_i},$$

$$\frac{\partial T}{\partial t} + u_i \frac{\partial T}{\partial x_i} + (\gamma - 1) T \frac{\partial u_i}{\partial x_i} + \gamma_v \frac{(T - T_v)}{\tau} = 0,$$

$$\frac{\partial T_v}{\partial t} + u_i \frac{\partial T_v}{\partial x_i} = \frac{T - T_v}{\tau}, \quad \gamma \mathrm{M}^2 p = \rho T, \quad i,\ j = 1,\ 2. \tag{2.1}$$

Here, $x_1 = x$ and $x_2 = y$ and summation over repeated subscripts is implied, ρ is the density, u_i are the velocity components, p is the pressure, and T and T_v are the static and vibrational temperatures of the gas. The parameters in the equations of system (2.1) are defined as follows: $\gamma = c_P/c_V$ is the ratio of specific heats, $c_V = c_{V,tt} + c_{V,rt}$ and $c_P = c_V + R$ are the specific heats at constant volume and constant pressure, due to the translational and rotational motions of molecules, respectively, the parameter $\gamma_v = c_{V,v}/c_V$ represents the degree of excitation of the vibrational mode, $c_{V,v}$ is the specific heat at constant volume corresponding to the relaxing vibrational mode, $\mathrm{M} = U_0/\sqrt{\gamma R T_0}$ is the Mach number of the carrier flow, and R is the gas constant. All the specific heats defined here are assumed to be constant. The scaling parameters are chosen to be a certain characteristic length, L, and a characteristic velocity, U_0, density ρ_0, temperature, T_0, a characteristic time, $\tau_0 = \delta_0/U_0$, and pressure, $p_0 = \rho_0 U_0^2$, derived from these parameters.

System (2.1) describes a situation, widespread in aerodynamics, where the characteristic times of microscopic processes are estimated by the following system of inequalities:

$$\tau_{tt} \sim \tau_{rt} \ll \tau_{vv} \ll \tau_{vt} \sim \tau_0. \tag{2.2}$$

In this case the translational and rotational degrees of freedom, with comparable short relaxation times $\tau_{tt} \sim \tau_{rt}$, form a quasi-equilibrium thermostat with a flow temperature T within times of the order of the characteristic flow time τ_0. Correspondingly, in the subsystem of vibrational levels, a quasi-equilibrium distribution with a vibrational temperature T_v is established within the time τ_{vv}. Relaxation of vibrational degree of freedom to the equilibrium state is described by the Landau-Teller equation with the characteristic time $\tau_{vt} \equiv \tau$. Here and in successive chapters this equation is rewritten in terms of vibrational and translation temperatures T_v, T using relation $E_v = c_{V,v} T_v$ with $c_{V,v} =$ const.

The instantaneous values of the perturbed flow fields are represented as the sum of steady values of the carrier flow and small fluctuations, which depend on the time and the coordinates:

$$u_1 = U_s + u', \quad u_2 = v', \quad \rho = 1 + \rho',$$

$$T = 1 + T', \quad T_v = 1 + T_v{}', \quad p = \frac{1}{\gamma \mathrm{M}^2} + p'. \tag{2.3}$$

By substituting expressions (2.3) into system (2.1) and linearizing it with respect to the steady flow, we obtain a system of equations for small perturbations.

For our further analysis it is more convenient to pass to an equivalent system of equations for perturbations, in which the equations for density and temperature fluctuations are replaced, with the help of the linearized equation of state, by an equation for pressure fluctuations. The converted system, in which the primes in the notation of the dependent variables are omitted, has the form

$$\frac{\partial u}{\partial t} + U_s \frac{\partial u}{\partial x} + v \frac{\partial U_s}{\partial y} = -\frac{\partial p}{\partial x} = 0, \quad \frac{\partial v}{\partial t} + U_s \frac{\partial v}{\partial x} = -\frac{\partial p}{\partial y} = 0,$$

$$\mathrm{M}^2 \left(\frac{\partial p}{\partial t} + U_s \frac{\partial p}{\partial x} \right) + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{\gamma_v}{\gamma} \frac{T - T_v}{\tau},$$

$$\frac{\partial T_v}{\partial t} + U_s \frac{\partial T_v}{\partial x} = \frac{T - T_v}{\tau}, \quad \gamma \mathrm{M}^2 p = \rho + T. \tag{2.4}$$

2.2 Some General Necessary Conditions of Instability Growth

We assume that all the perturbations are periodic along the x axis. This enables us to consider the solutions of system (2.4) in a certain strip along the ordinate axis, finite or unbounded. Then, we can impose zero boundary conditions, both on the finite boundaries of the strip along the y axis and in the asymptotic limits for all the unknown functions. Under these assumptions system (2.4) yields the perturbation energy balance equation

$$\frac{dE}{dt} = -\int uv \frac{\partial U_s}{\partial y} \, d\Omega$$

$$-\frac{\gamma_v}{\gamma^2 \mathrm{M}^2} \int \frac{\rho\,(T - T_v)}{\tau}\, d\Omega - \frac{3}{2}\frac{1}{\mathrm{Re}_r} \int \frac{(T - T_v)^2}{\gamma\,\tau}\, d\Omega, \tag{2.5}$$

where $\mathrm{Re}_r = U_0 L_r/\nu_b$. The integrals here and henceforth are taken over the entire flow region.

The energy integral of the perturbations can be represented in the following quadratic form:

$$E = \frac{1}{2} \int \left(u^2 + v^2 + \mathrm{M}^2\, p^2 + \frac{\gamma_v T_v^2}{\gamma^2 \mathrm{M}^2} \right) d\Omega.$$

Compared to the case of an ideal gas [5], the energy integral E, in addition to the term with the pressure p, defining the perturbation of the internal energy of the gas in the local thermodynamic equilibrium, contains a term with T_v associated with the perturbation of vibrational mode.

The last integral on the right-hand side of Eq. (2.5) is positive definite and shows, in explicit form, the dissipative effect of thermodynamic relaxation of the vibrational mode. In order to emphasize this, by analogy with the equations of energy balance in viscous media [6], we separate the parameter Re_r in front of the integral, which can be defined as the Reynolds relaxation number. It is calculated from the characteristic relaxation "length" $L_r = U_0 \tau$ and the coefficient of kinematic bulk viscosity $\nu_b = \eta_b/\rho_0$, where the coefficient of dynamic bulk viscosity, related to the relaxation vibrational mode, is defined by the relation [2–4]

$$\eta_b = \frac{2}{3} \frac{c_{V,v}}{c_V}\, p_0 \tau.$$

Hence, the presence of thermal relaxation enhances the stability of a compressible plane-parallel flow compared to the case of an ideal gas in local thermodynamic equilibrium [5]. Obviously, a continuous transition from Eq. (2.5) to the analogous equation for an ideal gas occurs. In fact if there is no energy pumping into the vibrational mode and $\tau < \tau_0$, local thermodynamic equilibrium is reached (at $T_v = T$); then, by taking the limit as $\gamma_v \to 0$ and $T_v \to T$, Eq. (2.5) transforms to the equation [5]

$$\frac{dE'}{dt} = -\int u\, v\, \frac{\partial U_s}{\partial y}\, d\Omega, \quad E' = \frac{1}{2} \int \left(u^2 + v^2 + \mathrm{M}^2\, p^2 \right) d\Omega,$$

and only the integral corresponding to energy exchange between the carrier flow and perturbations (Reynolds stresses) remains on the right-hand side.

Because of the presence of the dissipative process in system (2.4), there is no integral of motion similar to that obtained in [5] for an ideal gas. Introducing the perturbation of the generalized potential vorticity in the form

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + \mathrm{M}^2 \frac{\partial U_s}{\partial y}\, p + \frac{\gamma_v}{\gamma} \frac{\partial U_s}{\partial y} T_v,$$

from system (2.4) we obtain the equation

$$\frac{\partial \omega}{\partial t}+\frac{\partial U_s \omega}{\partial x}-v\frac{d^2 U_s}{dy^2}=0$$

which is identical to the potential vorticity equation [5]. The energy equation for the vorticity ω is also identical to the corresponding equation derived previously [5]:

$$\frac{d}{dt}\int U_s \frac{\omega^2}{2}\left(\frac{d^2 U_s}{dy^2}\right)^{-1} d\Omega=\int v\omega U_s\, d\Omega. \tag{2.6}$$

However, further transformations, i.e., integration by parts of the right-hand side of Eq. (2.6) with allowance for the equations of system (2.4), yield the equation

$$\frac{d}{dt}\int\left[U_s \frac{\omega^2}{2}\left(\frac{d^2 U_s}{dy^2}\right)^{-1}+uU_s\left(\mathrm{M}^2 p+\frac{\gamma_v}{\gamma}\cdot T_v\right)\right] d\Omega$$
$$=\int u\, v\,\frac{\partial U_s}{\partial y}\, d\Omega-\int U_s T_v \frac{\gamma_v}{\gamma}\frac{\partial p}{\partial x}\, d\Omega. \tag{2.7}$$

Unlike the case considered earlier [5], the resultant equation contains additional terms related to the nonequilibrium of the vibrational mode. Addition of the energy equations (2.5) and (2.7) and rearrangement aimed at obtaining complete squares on the left-hand side of the resultant equation lead to the equation

$$\frac{1}{2}\frac{d}{dt}\int\left\{\left[1-(1+\gamma_v)\,\mathrm{M}^2 U_s^2\right] u^2+v^2+\mathrm{M}^2\,(p+U_s u)^2\right.$$
$$\left.+\,U_s\omega^2\left(\frac{d^2 U_s}{dy^2}\right)^{-1}+\gamma_v \mathrm{M}^2\left(U_s u+\frac{T_v}{\gamma \mathrm{M}^2}\right)^2\right\} d\Omega$$
$$=-\frac{\gamma_v}{\gamma^2 \mathrm{M}^2}\int \rho\,\frac{T-T_v}{\tau}\, d\Omega-\int \frac{\gamma_v U_s T_v}{\gamma}\,\frac{\partial p}{\partial x}\, d\Omega-\frac{3}{2\mathrm{Re_r}}\int \frac{(T-T_v)^2}{\gamma\,\tau}\, d\Omega. \tag{2.8}$$

Here we eliminated the integral describing the exchange of energy between the carrier flow and perturbations. Nevertheless, the time derivative, separated on the left-hand side of Eq. (2.8), cannot vanish, because its right-hand side contains nonzero terms related to the relaxation process, in particular the sign-definite integral, which inhibits the increase in energy of the perturbations with time. However, if local thermodynamic equilibrium is reached in the flow at $\tau < \tau_0$, equality (2.8) in the limit as $T_v \to T$ and $\gamma_v \to 0$ transforms into an integral of a system of linear dynamics of the perturbations in an ideal gas [5]:

$$\frac{1}{2}\frac{d}{dt}\int\left\{\left(1-\mathrm{M}^2U_s^2\right)\,u^2+v^2\right.$$

$$\left.+\ \mathrm{M}^2\,(p+U_s u)^2+U_s\omega^2\left(\frac{d^2U_s}{dy^2}\right)^{-1}\right\}d\Omega=0.$$

Hence, following the well-known approach [5], we find that the perturbation energy E' in the local thermodynamic equilibrium state does not increase if

$$U_s(d^2U_s/dy^2)^{-1}>0,\quad \mathrm{M}\,|U_s| =< 1,$$

i.e., under the conditions formulated above, a compressible subsonic flow is stable to small perturbations.

We further consider the stability of wave perturbations in the form of plane waves

$$\boldsymbol{q}=\widetilde{\boldsymbol{q}}\,\exp\left[i\alpha(x-ct)\right],$$

$$\boldsymbol{q}=(u,\ v,\rho,\ p,\ T,\ T_v),\quad \widetilde{\boldsymbol{q}}=(\widetilde{u}(y),\ \widetilde{v}(y),\ \widetilde{\rho}(y),\ \widetilde{p}(y),\ \widetilde{\theta}(y),\ \widetilde{\theta}_v(y)).\tag{2.9}$$

Here $\boldsymbol{q}$ is the vector of the dependent variables in system (2.4), $\widetilde{\boldsymbol{q}}$ is the vector of the complex amplitudes of the perturbations, $\alpha>0$ is the real wavenumber, and $c=c_r+ic_i$ is the complex phase velocity. Substituting expression (2.9) into system (2.4), we obtain a system of ordinary differential equations for the perturbation amplitudes, for which the following homogeneous boundary conditions are imposed both on the boundaries of the finite interval

$$y_1\le y\le y_2$$

and as $y\to\pm\infty$:

$$\widetilde{u}=\widetilde{v}=\widetilde{\rho}=\widetilde{\theta}=\widetilde{\theta}_v=\widetilde{p}=0.\tag{2.10}$$

From here we obtain equations for the amplitudes of the pressure perturbations $\widetilde{p}(y)$ and the transverse components of the velocity $\widetilde{v}(y)$, which have the form

$$\left(W^{-2}\widetilde{p}\,'\right)'-\xi^2\widetilde{p}=0,\quad \xi=\alpha\sqrt{W^{-2}-m^2\mathrm{M}^2},\tag{2.11}$$

$$\left[\frac{W\,\widetilde{v}\,'-U_s'\,\widetilde{v}}{1-\mathrm{M}^2W^2}\right]'-\alpha^2\,W\,\widetilde{v}=0.\tag{2.12}$$

Here

$$W=U_s-c,\quad m^2=m_r^2+im_i^2,$$

$$m_r^2 = \frac{R(1+\gamma_v+\alpha\tau c_i)+\Delta^2}{R^2+\Delta^2}, \quad m_i^2 = -\frac{\gamma_v}{\gamma}\frac{(\gamma-1)\Delta}{R^2+\Delta^2},$$

$$R = 1+(\gamma_v/\gamma)+\alpha\tau\, c_i, \quad \Delta = \alpha\tau(U_s - c_r).$$

The prime denotes the operation of differentiation with respect to the y variable.

If there is no excited vibrational mode Eqs. (2.11) and (2.12) are converted by taking the limit as $\gamma_v \to 0$ into equations for the amplitudes of the pressure and transverse velocity perturbations [5]. Although the parameter m^2 is complex, nevertheless, $|m| > 1$, and from the form in which it occurs in Eq. (2.11), we can assume that the action of the relaxation process is similar to the stabilizing action of compressibility. As is well known [5], compressibility reduces the growth rates of the perturbations as compared to the case of an incompressible fluid [7].

An investigation of Eq. (2.11) enables us to obtain some additional stability characteristics. Obviously, for perturbations of the form (2.9) to be unstable, it is necessary that $c_i > 0$. Here $W \neq 0$ and, for the unstable solutions of Eq. (2.11), as in the well-known approach [8], we can introduce a new variable $\widetilde{p} = W^n\, H$. The converted equation in self-conjugate form is

$$\left[W^{2(n-1)}\; H'\right]' + \left[n\, W^n\, \left(W^{n-3}W'\right)' - \xi^2 W^{2n}\right] H = 0. \qquad (2.13)$$

The quadratic form of this equation is obtained by multiplying it by the complex conjugate function $\overline{H}$ and integrating over the range of variation of y, taking the boundary conditions (2.11) into account:

$$\int \left\{W^{2(n-1)}|H'|^2 - \left[n W^n \left(W^{n-3}W'\right)' - \xi^2 W^{2n}\right]|H|^2\right\} dy = 0. \qquad (2.14)$$

At $n = 0$, it follows from the imaginary part of Eq. (2.14) that

$$2\, c_i \int (U_s - c_r)\left[Q + \alpha^2 \mathrm{M}^2\, Q_1\, |H|^2\right] dy = 0, \qquad (2.15)$$

where

$$Q = \frac{|H'|^2 + \alpha^2 |H|^2}{|W|^4}, \quad Q_1 = \frac{\gamma_v(\gamma-1)\alpha\,\tau}{2\gamma c_i (R^2+\Delta^2)} > 0.$$

As the expression in the square brackets in the integrand in Eq. (2.15) is non-negative, this equality is satisfied provided that the difference $(U_s - c_r)$ changes its sign in the flow field. Consequently, for developing an instability in the flow of a vibrationally excited gas considered in this study, it is necessary that the Rayleigh condition be satisfied in the same form as for the cases of homogeneous and stratified incompressible fluids and an ideal gas [5]:

$$\min U_s \equiv u < c_r < U \equiv \max U_s. \qquad (2.16)$$

Hence, for any enhancement of perturbations the complex phase velocity, c, should lie in the upper half-plane $c_i > 0$ in a half-strip whose width is given by condition (2.16).

However, a more rigid constraint on the phase velocity c, known as the semicircle theorem [1, 5], can only be obtained here under additional conditions.

The real part of Eq. (2.14) has the following form at $n = 0$:

$$\int \left[(U_s - c_r)^2 - c_i^2\right] Q d\,y - K = 0, \quad K = \alpha^2 \mathrm{M}^2 \int m_r^2 |H|^2 d\,y. \tag{2.17}$$

In order to obtain the necessary lower estimate, we consider [1] the inequality

$$0 \geq \int (U_s - u)(U_s - U) Q\,d\,y = I_2 - (U + u) I_1 + U u I_0,$$

$$I_2 = \int U_s^2 Q\,d\,y, \quad I_1 = \int U_s Q\,d\,y, \quad I_0 = \int Q\,d\,y\,. \tag{2.18}$$

The integrals on the right-hand side of the inequality are expressed from Eqs. (2.15) and (2.17) in the form

$$I_1 = c_r I_0 - J, \quad J = \int (U_s - c_r)\, Q_1\, d\,y,$$

$$I_2 = 2 c_r I_1 - \left(c_r^2 - c_i^2\right) I_0 + K\,.$$

Substituting these expressions into inequality (2.18), we obtain

$$0 \geq \left\{(c_r - \overline{U}\,)^2 + c_i^2 - \frac{(U - u)^2}{4}\right\} I_0 + K + 2(\overline{U} - c_r) J, \quad \overline{U} = \frac{U + u}{2}\,. \tag{2.19}$$

Here $\overline{U}$ is the mean velocity of the carrier flow. If the inequality

$$K + 2(\overline{U} - c_r) J \geq 0, \tag{2.20}$$

holds, we have

$$(c_r - \overline{U}\,)^2 + c_i^2 \leq \frac{(U - u)^2}{4},$$

and the semicircle theorem is also valid [1]: “For any unstable mode at $c_i > 0$, the complex phase velocity lies in the upper half-plane in a semicircle of radius $r = |U - u|/2$ with the center at the point $c_r = \overline{U}$.”

In an ideal gas $m_r^2 = 1$ and $Q_1 = 0$; hence, inequality (2.20) is obviously satisfied [5]. Because of the above-mentioned continuity of the transition to the case of an ideal gas, inequality (2.20) and the semicircle theorem also hold for low but finite levels of excitation, as long as the non-sign-definite term in inequality (2.20) does not change the sign of the entire expression. For an arbitrary level of excitation satisfaction of inequality (2.20) must be verified for specific values of the parameters occurring in it.

At $n = 1/2$, the imaginary part of Eq. (2.14) has the form

$$\int |W|^{-2}|H'|^2 dy + \int \left(|W|^{-2} + \alpha^2 m_r^2 \mathrm{M}^2\right)|H|^2 dy + \int \Psi(y)|H|^2 dy = 0,$$

$$\Psi(y) = \frac{1}{c_i}\alpha^2 |m_i^2| \mathrm{M}^2 \left(U_s - c_r\right)$$

$$+\frac{5}{4}\left[3(U_s - c_r)^2 - c_i^2\right]|W|^{-6}U_s'^2 - (U_s - c_r)\,|W|^{-4}]U''.$$

Hence, it follows that, at $c_i > 0$, in order for instability to develop, the function $\Psi(y)$ should change its sign in the interval of integration. Correspondingly, the sufficient condition for stability is expressed (see also [8]) by the requirement that it should be nonnegative.

At $n = 1$ it follows from the imaginary part of Eq. (2.14)

$$c_i \int \Big\{ U_s'' - \big[4\,|W|^{-2} U_s'$$

$$+\alpha^2 \mathrm{M}^2 \left(m_i^2 |W|^4 c_i + 2m_r^2 |W|^2\right)\big](U_s - c_r)\Big\}\,|H|^2 |W|^{-2}\,dy = 0\,,$$

that the necessary condition for instability is a change in the sign of the expression in the square brackets, within the limits of integration. This requirement extends the well-known Rayleigh condition [1] of the necessity for a point of inflection to exist on the unstable velocity profile in an ideal fluid to the case of a compressible vibrationally excited gas.

2.3 Growth Rates and Eigenfunctions of Unstable Inviscid Modes in a Free Shear Flow

2.3.1 Formulation of the Problem

It is of interest to find the most unstable disturbances of the form (2.15), for which $c_r = U_s(0) = 0$, in a free shear flow with a velocity profile

$$U_s = \tanh\, y, \quad y \subset (-\infty;\ \infty).$$

An eigenvalue problem then arises for Eq. (2.13) with the conditions

$$\widetilde{p}\big|_{y=\pm\infty} = 0. \tag{2.21}$$

The eigenvalues are c_i, whereas the parameters of the problem are the wavenumber α, the Mach number M, the coefficient γ_v, representing the degree of perturbation of the vibrational mode, and the relaxation time τ. The wavenumber was varied in the range $0 \leq \alpha \leq 1$ in steps of $\Delta\alpha = 0.1$. The Mach number was varied in the subsonic range. The coefficient γ_v took the values $\gamma_v = 0$, 0.111, 0.250, and 0.667. The relaxation time covered a range of three orders of magnitude: $\tau = 0.01$, 0.1, and 1. The calculations were confined to the case of diatomic gases with $\gamma = 1.4$.

We use the methodology of numerical calculations developed earlier when investigating similar eigenvalue problems for an ideal incompressible fluid [7] and an ideal gas [5]. For this velocity profile we have

$$U_s\big|_{y\to\pm\infty} = \pm 1, \quad U_s{}'\big|_{y\to\pm\infty} = 0,$$

and for the asymptotic behavior of Eq. (2.13) as $y \to \pm\infty$ we obtain

$$\widetilde{p}'' - \zeta_\pm^2 \widetilde{p} = 0,$$

$$\zeta_\pm = \alpha\sqrt{1 - m_\pm^2 \mathrm{M}^2 (1 \mp c_i)^2}, \quad m_\pm^2 = \frac{1 + \gamma_v \pm i\alpha\tau(1 \mp c_i)}{1 + \gamma_v/\gamma \pm i\alpha\tau(1 \mp c_i)}.$$

Hence, the asymptotic solution is defined as

$$\widetilde{p} \sim \exp(\mp\zeta_\pm y). \tag{2.22}$$

Using the well-known approach [7], we introduce new dependent and independent variables in problem (2.11) and (2.21):

$$\Pi(y) = d\ln\widetilde{p}/dy = \Pi_r + i\Pi_i, \quad z = \mathrm{th}\, y.$$

By replacing the variables we transit to a finite interval of integration $z \in [-1;\ 1]$ and reduce the eigenvalue problem to the form

$$\frac{d\Pi}{dz} = \frac{\zeta^2 - \Pi^2}{1 - z^2} + \frac{2\Pi}{z - c_i}, \tag{2.23}$$

$$\Pi\big|_{z=\pm 1} = \mp\zeta_\pm. \tag{2.24}$$

The boundary values (2.24) for the function Π are found from the asymptotic Eq. (2.22). For the calculations it is also necessary to know values of the derivatives of the function Π at the ends of the integration interval. They are found directly from Eq. (2.23) as $z \to \pm 1$. Then, in the first term on the right-hand side of this equation, there is a singularity, which is removed by using L'Hopital's rule. As a result, we have

$$\left.\frac{d\Pi}{dz}\right|_{z=\pm 1} = \frac{\alpha^2 (1 \mp c_i)^2 \left[2\, m_\pm^2 \mathrm{M}^2 \mp B_\pm (1 \mp c_i)\right] - 4\zeta_\pm}{2(1 \mp c_i)(1 \pm \zeta_\pm)},$$

$$B_\pm = \frac{\gamma_v}{\gamma} \frac{i\alpha\tau(\gamma - 1)\mathrm{M}^2}{\left[1 + \gamma_v/\gamma \pm i\alpha\tau(1 \mp c_i)\right]^2}. \qquad (2.25)$$

2.3.2 Numerical Method and Results

To calculate the eigenvalues of the unstable modes Eq. (2.23) and boundary conditions (2.24) and (2.25) are replaced by equations and boundary conditions for the real part Π_r and the imaginary part Π_i of the function Π. The system obtained in this way for fixed sets of parameters is solved numerically by the shooting method. Integration with the Cauchy data (2.24) and (2.25) is carried out by using the fourth-order Runge–Kutta procedure in the ranges $z \in [-1;\ 0]$ and $z \in [0;\ 1]$ in steps of $\Delta z = 10^{-3}$. The "aiming" point is $z = 0$. The values of c are chosen so that the values of the functions Π_r and Π_i calculated "on the left" and "on the right" of the point $z = 0$ are identical to within 10^{-8}. The value of c_i corresponding to this identical value is taken as the eigenvalue for a specified set of parameters. To check the adequacy of the numerical algorithm, we calculate the eigenvalues of the unstable modes in an ideal gas for $\gamma_v = 0$ and $\alpha^2 + \mathrm{M}^2 \leq 1$. The results agree with the data obtained earlier [5] within the doubled accuracy of the computer calculations.

Figure 2.1 shows the isolines of the growth rates αc_i in the (M, α) plane for $\tau = 1$ and $\gamma_v = 0.667$ (solid curves). The dashed curves represent the data [5] for an ideal gas. Curves *1* and *2* show the change in the maximum growth rates for an ideal gas and a vibrationally excited gas as a function of the Mach number M. Some numerical values for these curves are given in Table 2.1. The greatest growth rate $\alpha c_i = 0.1897$ is obtained for an ideal fluid with $\mathrm{M} = 0$ and $\alpha = 0.4446$ [7]; for $\mathrm{M} = 1$ the growth rates αc_i are equal to zero.

It can be seen that relaxation, similar to compressibility, reduces the growth rates of the unstable modes. As the Mach number increases, the decrease in the growth rates due to relaxation becomes more and more noticeable. The calculations show that the relaxation effect is enhanced as the excitation level increases, which is specified here by the coefficient γ_v. It should be noted that we used moderate values of γ_v in the calculations. At the same time the model of two-temperature gas dynamics is applicable for higher levels of excitation [2, 3]. We can assume that the relaxation effect on excitation levels close to the onset of dissociation will be comparable in order of magnitude to the decrease in the growth rates with the compressibility effect.

Figure 2.2 shows graphs of the growth rates of the most unstable modes against the relaxation time parameter τ for $\mathrm{M} = 0.5$. In the range of variation of τ and γ_v

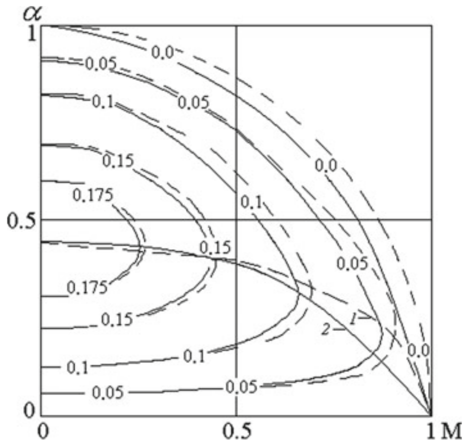

Fig. 2.1 Isolines of the growth rates αc_i at $\tau = 1$. The *dashed* and *solid curves* show the results for $\gamma_v = 0$ and 0.667, respectively. Curves *1* and *2* show the growth rates for $\gamma_v = 0$ and 0.667, respectively

Table 2.1 Spectral characteristics and growth rates of the most unstable inviscid modes for $\tau = 1$, $\gamma_v = 0$ and 0.667

M	α		c_i		$\alpha\, c_i$	
	$\gamma_v = 0$	0.667	0	0.667	0	0.667
0	0.4446	0.4446	0.4266	0.4266	0.1897	0.1897
0.2	0.4260	0.4377	0.4255	0.4115	0.1813	0.1801
0.5	0.3970	0.3890	0.3556	0.3449	0.1413	0.1341
0.8	0.2790	0.2895	0.2790	0.2142	0.0778	0.0620
1	0	0	0	0	0	0

considered in this study the growth rates are almost independent of the relaxation time.

We calculate the eigenfunctions for certain sets of parameters and the eigenvalues of the most unstable modes obtained. We use problem (2.13) and (2.21) to find the eigenfunctions of the pressure. The equation for the eigenfunction of the pressure $\widetilde{p}$ is split into a real and imaginary parts, which are reduced to a first-order system of equations. The asymptotic conditions at $y = \pm\infty$ are transferred to the ends of the interval $[-y_0;\ y_0]$, and the y_0 coordinate is determined from the condition

$$\left|\tanh\, y_0 - 1\right| \le 10^{-12}.$$

The value $y_0 = 20$ is used in the calculations. The system is integrated by using the fourth-order Runge–Kutta procedure.

As the eigenfunctions are determined, except for a constant factor, the normalization of the eigenfunctions of the pressure $\widetilde{p}$ is chosen in the same way as that used

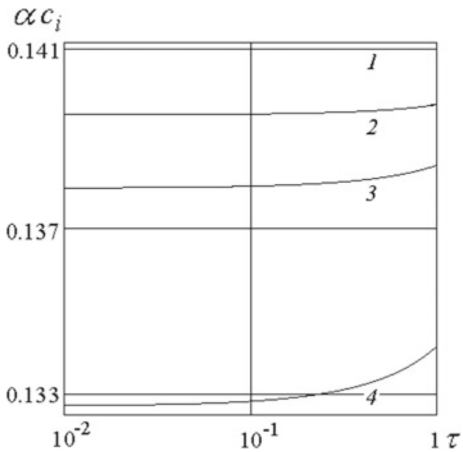

Fig. 2.2 Dependencies of the growth rates αc_i of the most unstable modes against the relaxation time parameter τ for M = 0.5 and $\gamma_v = 0$ (*1*), 0.111 (*2*), 0.250 (*3*), and 0.667 (*4*)

previously [5], which enables direct comparisons of the calculated results. Other eigenfunctions are found in terms of the calculated eigenfunctions of the pressures $\widetilde{p}$ and $\widetilde{p}'$ by using the relations

$$\widetilde{u} = \frac{U_s{}'\widetilde{p}'}{D^2} - \frac{i\alpha\widetilde{p}}{D}, \quad \widetilde{v} = -\frac{\widetilde{p}'}{D}, \quad \widetilde{\rho} = \frac{1+\gamma_v+\tau D}{1+\gamma_v/\gamma+\tau D}\,\widetilde{p}\,\mathrm{M}^2,$$

$$\widetilde{\theta} = \big(1+\tau D\big)\widetilde{\theta}_v, \quad \widetilde{\theta}_v = \frac{\gamma-1}{1+\gamma_v/\gamma+\tau D}, \quad D = i\alpha\ (U_s - c_i) \tag{2.26}$$

following from the system of equations for the perturbation amplitudes.

The curves in Fig. 2.3 illustrate the behavior of the real and imaginary parts of the eigenfunctions of the perturbations of the vibrational temperature $\widetilde{\theta}_v$ of the most unstable modes for $\mathrm{M} = 0.5$ and $\tau = 1$. The eigenfunctions of the pressure $\widetilde{p}$ and the eigenfunctions $\widetilde{\rho}$ and $\widetilde{\theta}$, which depend linearly on it, behave in the same way. Their real parts are symmetrical about $y = 0$, while the imaginary parts are antisymmetrical about $y = 0$. The eigenfunctions of the velocity components $\widetilde{u}$ and $\widetilde{v}$, which depend on $\widetilde{p}'$, conversely, have antisymmetrical real parts and symmetrical imaginary parts, due to the evenness of the function Π_r and the oddness of the function Π_i. This property, in turn, can be established directly from the system of differential equations for these functions, as was done previously [5, 7].

It is of interest to analyze the effect of excitation on the generalized vorticity perturbation amplitude

$$Real(\widetilde{\omega}) = Real(\widetilde{\omega}_0) + Real(\widetilde{\omega}_1) + Real(\widetilde{\omega}_2)\,.$$

Here

$$Real(\widetilde{\omega}_0) = -(\alpha\,\widetilde{v}_i + \widetilde{u}_r')\cos\alpha\,x - (\alpha\,\widetilde{v}_r - \widetilde{u}_i')\sin\alpha\,x,$$

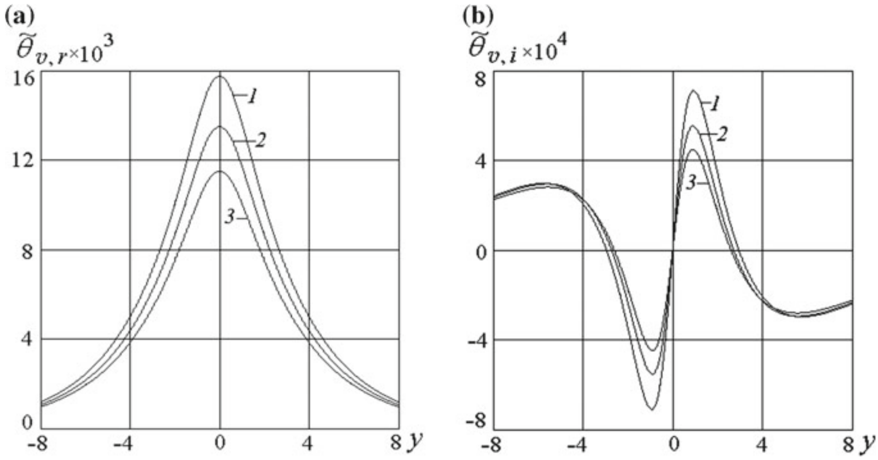

Fig. 2.3 Dependencies of the real $\widetilde{\theta}_{v,r}(y)$ and imaginary $\widetilde{\theta}_{v,i}(y)$ parts of the perturbation of the vibrational temperature $\widetilde{\theta}_v$ for M $= 0.5$, $\tau = 1$, and $\gamma_v = 0$ (*1*), 0.250 (*2*), and 0.667 (*3*). **a** shows dependencies $\widetilde{\theta}_{v,r}(y)$. **b** shows dependencies $\widetilde{\theta}_{v,i}(y)$

Table 2.2 Numerical absolute values of the real parts of the generalized vorticity $\widetilde{\omega}$ and additive contributions that determine it at $\tau = 1$, $\gamma_v = 0$ and 0.667

M	$\lvert Real(\tilde{\omega}_0)\rvert$		$\lvert Real(\tilde{\omega}_1)\rvert$		$\lvert Real(\tilde{\omega}_2)\rvert$		$\lvert Real(\tilde{\omega})\rvert$	
	$\gamma_v = 0$	0.667	0	0.667	0	0.667	0	0.667
0.2	0.3992	0.3923	0.0078	0.0082	0	0.0012	0.4072	0.4018
0.5	0.3828	0.3479	0.0412	0.0386	0	0.0056	0.4242	0.3922

$$Real(\widetilde{\omega}_1) = \mathrm{M}^2\, U_S' \,(\widetilde{p}_r \cos\alpha\, x - \widetilde{p}_i\, \sin\alpha\, x),$$

$$Real(\widetilde{\omega}_2) = \frac{\gamma_v\, U_S'}{\gamma}\,(\widetilde{\theta}_{v,r} \cos\alpha\, x - \widetilde{\theta}_{v,i}\, \sin\alpha\, x).$$

The corresponding data are presented in Table 2.2.

The isolines of the fluctuations of the generalized (potential) vorticity $\widetilde{\omega}$ in an ideal gas ($\gamma_v = 0$) and in a vibrationally excited gas ($\gamma_v = 0.667$) for M $= 0.5$ and $\tau = 1$ are compared in Fig. 2.4. Both patterns of the isolines are antisymmetric about the $x = 0$ axis.

According to the data in Table 2.2, excitation and compressibility act in the same direction. Correspondingly, the vorticity decreases as γ_v and M increase, whereas the addition terms due to the internal energy increase. The contribution of excitation of the vibrational mode becomes more significant with increasing Mach number, which was already noted for the growth increments. It is $\widetilde{\omega}_0$ whose absolute value undergoes the greatest change. On the whole, the absolute value of the generalized vorticity decreases as the depth of excitation increases.

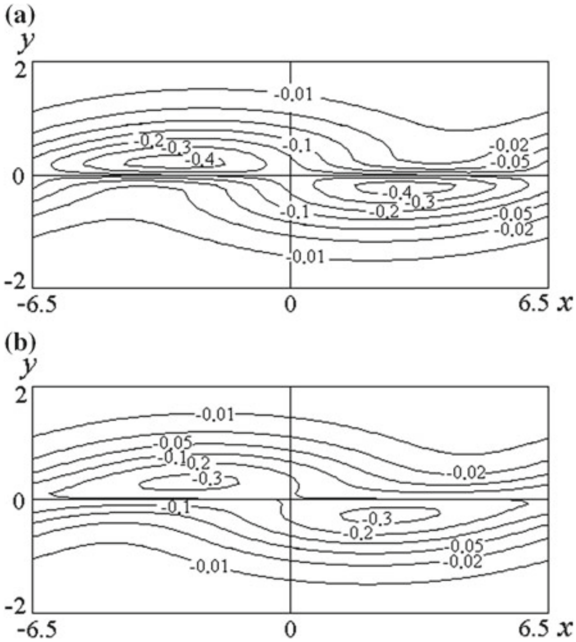

Fig. 2.4 Isolines of fluctuations of the generalized vorticity $\widetilde{\omega}$ for $M = 0.5$ and $\tau = 1$. **a** is $\gamma_v = 0$. **b** is $\gamma_v = 0.667$

The dissipative effect of excitation can be clearly seen in Fig. 2.4. As compared to the case of an ideal gas, not only the maximum intensity of vortices decreases, but also the gradients of the perturbation field of the generalized vorticity are smoothed.

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