Chapter 2
Governing Equations

In the present and the subsequent chapters, we shall, either directly or indirectly, be concerned with the boundary-layer flow of an incompressible viscous fluid without any involvement of heat and mass transfer. Therefore, our governing laws will be the conservation of mass and momentum only which are commonly known as the equations of continuity and the Navier–Stokes equations, respectively. In usual notation, they are written as (in Cartesian coordinates):

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{2.1}$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + F_x, \tag{2.2}$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) + F_y, \tag{2.3}$$

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) + F_z, \tag{2.4}$$

and in cylindrical coordinates:

$$\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0, \tag{2.5}$$

$$\begin{aligned}
&\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_r}{\partial z}\right) \\
&\quad = -\frac{\partial p}{\partial r} + \mu\left(\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r}\frac{\partial v_r}{\partial r} - \frac{v_r}{r^2} + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2}\right) + F_r,
\end{aligned} \tag{2.6}$$

A. Mehmood, *Viscous Flows*, Mathematical Engineering,
DOI 10.1007/978-3-319-55432-7_2

$$\rho\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_r v_\theta}{r}+v_z\frac{\partial v_\theta}{\partial z}\right)$$
$$=-\frac{1}{r}\frac{\partial p}{\partial \theta}+\mu\left(\frac{\partial^2 v_\theta}{\partial r^2}+\frac{1}{r}\frac{\partial v_\theta}{\partial r}-\frac{v_\theta}{r^2}+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}+\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}+\frac{\partial^2 v_\theta}{\partial z^2}\right)+F_\theta, \quad (2.7)$$

$$\rho\left(\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}+v_z\frac{\partial v_z}{\partial z}\right)=-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 v_z}{\partial r^2}+\frac{1}{r}\frac{\partial v_z}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}+\frac{\partial^2 v_z}{\partial z^2}\right)+F_z, \quad (2.8)$$

where $\boldsymbol{F}=\left(F_x,F_y,F_z\right)$ or $=\left(F_r,F_\theta,F_z\right)$ denotes the body force. However, the flows considered in the subsequent chapters do not involve any body force because of which $\boldsymbol{F}$ will be taken identically equal to zero in all those chapters.

2.1 Boundary-Layer Equations

2.1.1 The Boundary-Layer Assumption

In the study of viscous flow due to the motion of a continuous surface, a natural question arises "whether the boundary-layer assumptions considered by Prandtl are applicable to this case where the flow is established by the motion of the solid surface in the absence of any potential flow?" The same question arose in the mind of Sakiadis [1] in 1960 and he first confirmed the boundary-layer character of the viscous flow established by the motion of a continuous surface. This question, however, finds some crude justification from Sect. 1.1 of Chap. 1, where the Sakiadis flow has been explained, but a theoretical proof or experimental evidence is still required. In this connection, Sakiadis [1] performed an experiment and confirmed the formation of boundary-layer near the surface of the moving continuous surface in accordance with the explanation given in Sect. 2.1. He considered the rectangular Lucite acrylic resin tank and filled it with a water-based solution of milling yellow. He passed two parallel threads issuing from the little holes in the blocks immersed completely in the tank as shown in Fig. 2.1.

For the sake of comparison, he moved the left-hand thread from top to bottom and kept the right one fixed within the tank. After a while, when the steady state reached, the photograph (see Fig. 2.2) shows the formation of the boundary-layer near the moving thread, starting right from the hole and thus developing in the direction of motion of the thread.

This experiment confirms that the viscous flow established by the motion of a continuous surface does exhibit the boundary-layer character. Therefore, the Prandtl's boundary-layer assumptions are equally valid in this case and the order of magnitude analysis also works in the same manner as in the case of finite surfaces.

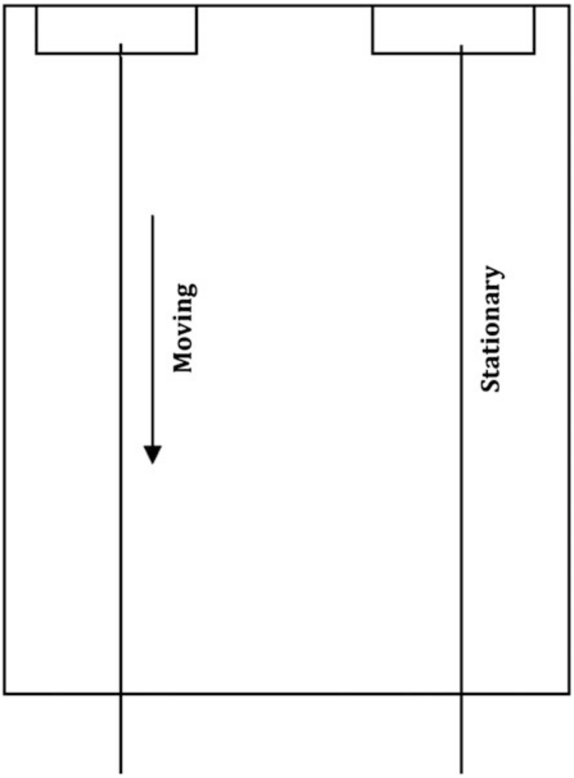

Fig. 2.1 Schematic of experimental setup of Sakiadis [1]

Fig. 2.2 Photograph of the Sakiadis' experiment [1]

2.1.2 The Pressure Gradient Term

According to the Prandtl's boundary-layer theory, the pressure within the boundary-layer does not become very much different from the pressure in the inviscid potential flow. That is, there are no significant changes in pressure across the boundary-layer because of which the term $\partial p/\partial y$ is simply ignored in the two-dimensional case; however, the changes in pressure along the lateral directions can be of any significance. Therefore, the pressure variation along the lateral directions within the boundary-layer is assumed to be the same as they are in the inviscid potential flow outside the boundary-layer and are determined with the use of Bernoulli's equation, for example, in the two-dimensional steady-state case, as

$$\frac{\mathrm{d}p}{\mathrm{d}x} = u_\infty \frac{\mathrm{d}u_\infty}{\mathrm{d}x}$$

Thus, the constancy or absence of the external potential flow makes the pressure gradient term equal to zero in the boundary-layer equations governing the flow due to a moving continuous solid surface. The external pressure in the subsequently studied flows will be considered absent, thus making the pressure gradient term equal to zero, i.e.,

$$\frac{\partial p}{\partial s_l} = 0, \tag{2.9}$$

where s_l denotes any lateral coordinate.

2.1.3 Boundary-Layer Equations in Cartesian Coordinates

However, it has now been established that the boundary-layer equations, present already in the literature, for the two-dimensional and three-dimensional flows are equally applicable to the flows due to moving continuous surfaces too. Therefore, the derivation of these equations here again does not make any sense. We, therefore, prefer to follow Schlichting [2] and write the boundary-layer equations directly here

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{2.10}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = \nu\frac{\partial^2 u}{\partial y^2}, \tag{2.11}$$

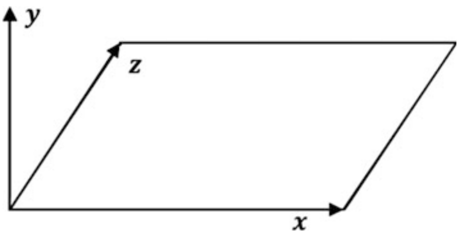

Fig. 2.3 Cartesian coordinate system

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = \nu\frac{\partial^2 w}{\partial y^2}, \tag{2.12}$$

where the system of coordinates is shown in Fig. 2.3. The velocity components u, v and w have been taken along the x-, y-, and z-axes, respectively.

2.1.4 Boundary-Layer Equations in Cylindrical Coordinates

The geometrical objects, of our interest, owing to the axial symmetry are the continuous circular cylinder and the circular disk of infinite radius. The schematic of the flow due to a continuous circular cylinder and the associated system of coordinates is shown in Fig. 2.4.

Since there is no circular rotation in the flow therefore, the boundary-layer equations in this case look like

$$\frac{\partial}{\partial z}(ru) + \frac{\partial}{\partial r}(rv) = 0, \tag{2.13}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial z} + v\frac{\partial u}{\partial r} = \nu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right), \tag{2.14}$$

whereas in the case of circular disk, the governing system takes the form

$$\frac{\partial}{\partial r}(ru) + \frac{\partial}{\partial z}(rw) = 0, \tag{2.15}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z} = \nu\frac{\partial^2 u}{\partial z^2}. \tag{2.16}$$

The chosen system of coordinates, corresponding to the disk geometry, is shown in Fig. 2.5.

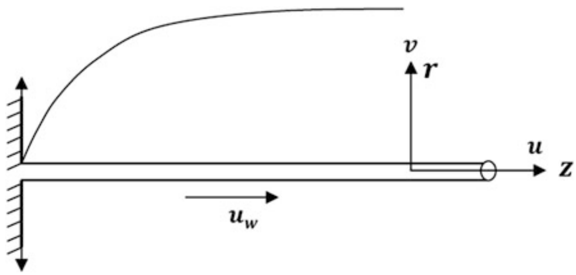

Fig. 2.4 Schematic of the continuous cylinder and the associated coordinate system

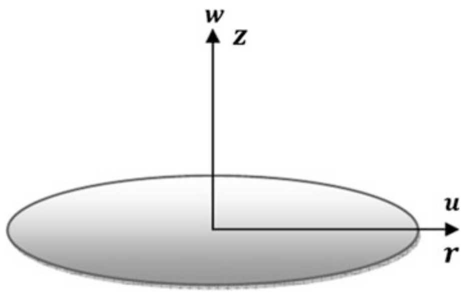

Fig. 2.5 Coordinate system for the axially symmetric disk geometry

2.2 Momentum Integral Equations

The integral form of the momentum boundary-layer equations, either in Cartesian coordinates or in cylindrical coordinates, comes directly from their respective differential forms, such as Eqs. (2.10)–(2.12), (2.13)–(2.14), (2.15)–(2.16) corresponding to the planner two- or three-dimensional flow, axially symmetric flow near a cylinder and disk, respectively.

2.2.1 In Cartesian Coordinates

Consider a steady three-dimensional flow caused due to the bilateral motion of a flexible continuous flat sheet. The appropriate boundary conditions for this flow read as:

$$\left.\begin{array}{ll} u = u_w(x,z), w = w_w(x,z), v = v_w(x,z), & \text{at } y = 0 \\ u = 0, w = 0, & \text{at } y = \infty \end{array}\right\}, \tag{2.17}$$

where $u_w(x,z)$ and $w_w(x,z)$ denote the stretching/shrinking wall velocities in x- and z-directions, respectively. In the case of porous flexible sheet, suction/injection may also be allowed through the sheet surface which is denoted by $v_w(x,z)$. Outside the boundary-layer, the fluid is supposed to be at rest in the absence of any potential flow. The momentum integral equations in this case are derived by the integration of Eqs. (2.10)–(2.12) w.r.t. y between the limits 0 and $\delta(x, z)$, where $\delta(x, z)$ denotes the boundary-layer thickness. Integration of Eq. (2.10) w.r.t. y yields

$$v = v_w - \int_0^y \left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \right) \mathrm{d}y. \tag{2.18}$$

Integration of Eqs. (2.11) and (2.12) with respect to y between the limits $0 \leq y \leq \delta$ results in the following integral equations:

$$\int_0^\delta \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) \mathrm{d}y = \frac{1}{\rho} \tau_{x,0}, \tag{2.19}$$

$$\int_0^\delta \left(u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) \mathrm{d}y = \frac{1}{\rho} \tau_{z,0}, \tag{2.20}$$

where

$$\tau_{x,0} = -\mu \left. \frac{\partial u}{\partial y} \right|_{y=0} \text{and} \quad \tau_{z,0} = -\mu \left. \frac{\partial w}{\partial y} \right|_{y=0}. \tag{2.21}$$

Substituting Eq. (2.18) in Eqs. (2.19) and (2.20) by noting that

$$\left[u \int_0^y \left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \right) \mathrm{d}y \right]_0^\delta = 0, \tag{2.22}$$

and

$$\left[w \int_0^y \left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \right) \mathrm{d}y \right]_0^\delta = 0, \tag{2.23}$$

we finally arrive at

$$\frac{\partial}{\partial x} \int_0^\delta u^2 \mathrm{d}y + \frac{\partial}{\partial z} \int_0^\delta uw \mathrm{d}y = v_w u_w + \frac{1}{\rho} \tau_{x,0}, \tag{2.24}$$

$$\frac{\partial}{\partial x}\int_0^{\delta} uw\mathrm{d}y + \frac{\partial}{\partial z}\int_0^{\delta} w^2\mathrm{d}y = v_w w_w + \frac{1}{\rho}\tau_{z,0}. \tag{2.25}$$

The system (2.24)–(2.25) represents the momentum integral equations for the steady three-dimensional viscous flow due to a moving continuous porous sheet. For the two-dimensional case, Eq. (2.25) just vanishes out and Eq. (2.24) in the absence of wall suction/injection simplifies to

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{\delta} u^2\mathrm{d}y = \frac{1}{\rho}\tau_{x,0}, \tag{2.26}$$

which is the same as derived by Sakiadis [1].

2.2.2 In Cylindrical Coordinates

In Sect. 2.1.4, the axial flow due to a moving continuous surface has been represented in cylindrical coordinates by splitting it into two particular flow geometries, namely the circular cylinder and the circular flat disk. The integral formulation for the said two cases follows immediately by the integration of Eqs. (2.13), (2.14), (2.15), and (2.16), respectively. Following the same procedure (of Sect. 2.1.4), we derive integral momentum equations for these two cases separately. The appropriate boundary conditions for a uniformly stretching/shrinking long continuous cylinder, as shown in Fig. 2.4, with porous surface read as

$$\left.\begin{array}{ll} u = u_w(z),\ v = v_w(z), & \text{at}\, r = R \\ u = 0, & \text{at}\, r = \infty \end{array}\right\}, \tag{2.27}$$

where R denotes the radius of cylinder. Integrating Eq. (2.13) w.r.t. r, we readily get

$$v = \frac{R}{r}v_w - \frac{1}{r}\int_0^{r} \frac{\partial u}{\partial z} r\mathrm{d}r. \tag{2.28}$$

Substituting Eq. (2.28) in Eq. (2.14) and integrating between the limits $R \le r \le \delta(z)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}z}\int\limits_0^{\delta} u^2 r\mathrm{d}r = R\left(u_w v_w + \frac{\tau_{z,0}}{\rho}\right), \tag{2.29}$$

which is the momentum integral equation for the uniformly stretching/shrinking cylinder. This equation immediately reduces to that, first, derived by Sakiadis [1] for a continuous cylinder of impermeable solid wall by substituting $v_w = 0$. Here, $\tau_{z,0} = -\mu \frac{\partial u}{\partial r}\big|_{r=R}$ is the tangential shear stress at the cylindrical surface. The similar procedure applies to the continuous circular disk of infinite radius. In this case, the describing conditions at the disk surface should be of the form

$$\left.\begin{array}{ll} u = u_w(r),\; w = w_w(r), & \text{at}\, z = 0 \\ u = 0, & \text{at}\, z = \infty \end{array}\right\}, \tag{2.30}$$

for a permeable stretching/shrinking disk. Notice that, Eq. (2.16) in steady-state form can be rewritten, with the utilization of Eq. (2.15), as:

$$\frac{\partial}{\partial r}\left(ru^2\right) + \frac{\partial}{\partial z}(ruw) = \nu r\frac{\partial^2 u}{\partial z^2}, \tag{2.31}$$

which upon integration between the limits $0 \le z \le \delta(r)$, in view of Eq. (2.30), simplifies to

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left[r\int\limits_0^{\delta} u^2 \mathrm{d}z\right] = u_w v_w + \frac{\tau_{r,0}}{\rho}, \tag{2.32}$$

where $\tau_{r,0} = -\mu \frac{\partial u}{\partial z}\big|_{z=0}$ is the radial component of wall tangential shear stress. Equation (2.32) constitutes the momentum integral equation for a radially stretching/shrinking disk.

References

1. B.C. Sakiadis, Boundary-layer behavior on continuous solid surface: I boundary-layer equations for two-dimensional and axisymmetric flow. AIChE **7**(1), 26–28 (1961)
2. H. Schlichting, Boundary-Layer theory, 6th edn. (McGraw-Hill Book Company, 1968). Translated by J. Kestin

Springer

http://www.springer.com/978-3-319-55431-0

Viscous Flows
Stretching and Shrinking of Surfaces
Mehmood, A.
2017, XXVII, 193 p. 73 illus., 2 illus. in color., Hardcover
ISBN: 978-3-319-55431-0