

Chapter 2

Procedure for Designing Fractional-Order Filters

2.1 Introduction

Fractional-order differentiation and integration topologies offer attractive features in various interdisciplinary applications. A typical application is the substitution of conventional integer order parts of a system with the fractional-order parts, respectively, where the existence of derivation/integration has a decisive position, and offers important benefits. Also, they are able to be used in order to realize one of the most important circuits in fractional-order theory, which is the fractal device. On the other hand, fractional-order filters, offer more precise control of the attenuation gradient, which is an efficient feature in biomedical engineering. Fractional-order filters, differentiators, and integrators, will be presented in a systematic way that describes the most important features of these structures; realizing such circuits through the utilization of a general form enabling the capability of realizing different kind of circuits by the same topology, which is very important from the flexibility point of view. All the above will be performed through the utilization of the second-order CFE, which is an efficient tool, in terms of accuracy and circuit complexity, and has been described in Chap. 1 in detail. As a consequence, the main benefit of this procedure is that having available the design equations, which are expressed through integer-order functions, the capability of realizing these types of functions utilizing different ways of circuit design could be achieved.

2.2 Fractional-Order Generalized Filters (Order α)

Integrators and differentiators are very useful building blocks for performing signal conditioning in biomedical applications. Also, they are employed for realizing oscillators, impedance emulators, and in control systems. Fractional-order digital implementations of such circuits have been already published in the literature [1–5].

The utilization of the second-order expressions of CFE is an appropriate tool for realizing fractional-order differentiators and integrators in order to approximate the variable $(\tau s)^\alpha$ using the formula given in (2.1). In case that $\alpha = 1$, this transfer function represents a differentiator, while for $\alpha = -1$ an integrator. In the range $(0 < \alpha < 1)$, this element may generally be considered to represent a fractional-order differentiator, while in the range $(-1 < \alpha < 0)$, a fractional-order integrator.

$$(\tau s)^\alpha = \frac{\alpha_0(\tau s)^2 + \alpha_1(\tau s) + \alpha_2}{\alpha_2(\tau s)^2 + \alpha_1(\tau s) + \alpha_0} \quad (2.1)$$

2.2.1 Fractional-Order Differentiator

The transfer function, as well as the magnitude response of an integer-order differentiator is given by the formula $H(s) = \tau s$, and $H(\omega) = \omega/\omega_0$, respectively. The unity gain frequency is $\omega_0 = 1/\tau$, where τ is the corresponding time-constant. In addition the phase response is constant and equal to $\pi/2$. Thus, the transfer function of a fractional-order differentiator will be given by (2.2) as

$$H(s) = (\tau s)^\alpha \quad (2.2)$$

where $(0 < \alpha < 1)$ is the order of the differentiator. The magnitude response is given as $H(\omega) = (\omega/\omega_0)^\alpha$, from which is obvious that the unity gain frequency has the same expression as in the case of its integer-order counterpart. Also, in this case the phase response is constant but equal to $\alpha\pi/2$ predicting the total reliance of phase from the fractional-order α [6].

Comparing the above expressions of magnitude responses of fractional and integer-order differentiator, it is obvious that at the same frequency the fractional-order differentiator realizes a gain smaller than that achieved by its integer-order counterpart. As a result, with the substitution of (2.1) into (2.2), the transfer function of fractional-order differentiator is expressed as shown in (2.3), where α_i ($i = 0, 1, 2$) is given by (1.6) or (1.10). Their values depend on the type of the approximation that has been utilized.

$$H_{\text{diff}}(s) = \frac{\left(\frac{\alpha_0}{\alpha_2}\right)s^2 + \left(\frac{\alpha_1}{\alpha_2}\right)\frac{1}{\tau}s + \frac{1}{\tau^2}}{s^2 + \left(\frac{\alpha_1}{\alpha_2}\right)\frac{1}{\tau}s + \left(\frac{\alpha_0}{\alpha_2}\right)\frac{1}{\tau^2}} \quad (2.3)$$

2.2.2 Fractional-Order Integrator

The transfer function of a fractional-order lossless integrator could be written as shown in (2.4), while the magnitude response is given as $H(\omega) = (\omega_o/\omega)^\alpha$, where $\omega_o = 1/\tau$ is the unity gain frequency of the integrator. Its phase response will be a constant equal to $-\alpha\pi/2$.

$$H(s) = \frac{1}{(\tau s)^\alpha} \quad (2.4)$$

The corresponding expressions of its integer-order counterpart with the same unit gain frequency will be $H(\omega) = \omega_o/\omega$ and $-\pi/2$, respectively. By using the same order of approximation as in the case of fractional-order differentiator, the transfer function in (2.4) could be approximated as it is demonstrated in (2.5), where α_i ($i = 0, 1, 2$) is given by (1.6) or (1.10).

$$H_{\text{int}}(s) = \frac{\left(\frac{\alpha_2}{\alpha_0}\right)s^2 + \left(\frac{\alpha_1}{\alpha_0}\right)\frac{1}{\tau}s + \frac{1}{\tau^2}}{s^2 + \left(\frac{\alpha_1}{\alpha_0}\right)\frac{1}{\tau}s + \left(\frac{\alpha_2}{\alpha_0}\right)\frac{1}{\tau^2}} \quad (2.5)$$

Taking into account that the transfer function in (2.3) and (2.5) have an integer-order form, they could be easily performed either by the typical functional block diagram (FBD) of the follow-the-leader-feedback (FLF) topology depicted in Fig. 2.1a, or the inverse-follow-the-leader, multi-feedback (IFLF) topology given in Fig. 2.1b, where the notation (xG_i) implies a scaled replica of the corresponding output. The transfer function is that in (2.6). Comparing (2.3) and (2.5) with (2.6)

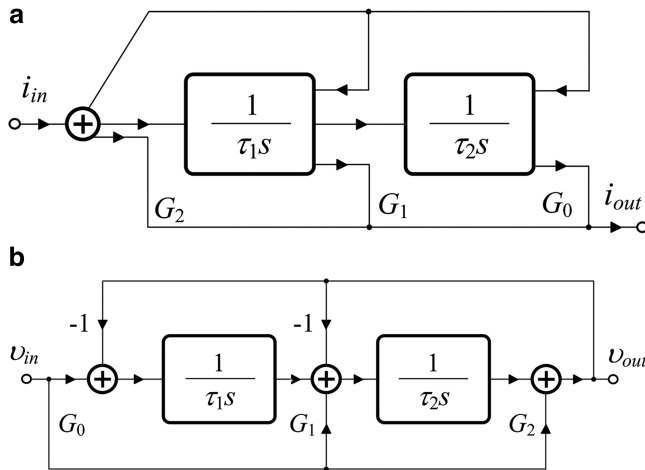


Fig. 2.1 FBD for realizing fractional-order differentiator/integrator of order α using (a) FLF current-mode topology, (b) IFLF voltage-mode topology

Table 2.1 Design expressions of time-constants τ_i for approximating fractional-order differentiator, lossless integrator with unity gain frequency ($\omega_o = 1/\tau$)

Transfer function	τ_1	τ_2
$H(s) = (\tau s)^\alpha$	$\left(\frac{\alpha_2}{\alpha_1}\right) \cdot \tau$	$\left(\frac{\alpha_1}{\alpha_0}\right) \cdot \tau$
$H(s) = \frac{1}{(\tau s)^\alpha}$	$\left(\frac{\alpha_0}{\alpha_1}\right) \cdot \tau$	$\left(\frac{\alpha_1}{\alpha_2}\right) \cdot \tau$

Table 2.2 Design expressions of gain factors G_i for approximating fractional-order differentiator, lossless integrator with unity gain frequency ($\omega_o = 1/\tau$)

Transfer function	G_2	G_1	G_0
$H(s) = (\tau s)^\alpha$	$\left(\frac{\alpha_0}{\alpha_2}\right)$	1	$\left(\frac{\alpha_2}{\alpha_0}\right)$
$H(s) = \frac{1}{(\tau s)^\alpha}$	$\left(\frac{\alpha_2}{\alpha_0}\right)$	1	$\left(\frac{\alpha_0}{\alpha_2}\right)$

the derived expressions of time-constants τ_i and gain factors G_i ($i = 0,1,2$) are as summarized in Tables 2.1, and 2.2, respectively.

$$H(s) = \frac{G_2 s^2 + \frac{G_1}{\tau_1} s + \frac{G_0}{\tau_1 \tau_2}}{s^2 + \frac{1}{\tau_1} s + \frac{1}{\tau_1 \tau_2}} \quad (2.6)$$

2.3 Fractional-Order Generalized Filters (Order α)

Fractional-order filters of order α where ($0 < \alpha < 1$) will be presented and some of the most critical frequencies have been derived in order to be fully characterized. From the stability point of view, this system is stable if and only if $\alpha > 0$ and $\alpha < 2$, while it will oscillate if and only if $\alpha > 0$ and $\alpha = 2$; otherwise it is unstable. The derived frequency responses of filters of order α exhibit a stopband attenuation proportionate to the fractional-order α , which offers a more precise control of the attenuation gradient compared to the attenuation offered in the case of integer-order filters of order n , which is $-6 \cdot n$ dB/oct [6–19].

Thus, low-pass, high-pass, band-pass, and all-pass filters of order α will be presented. Also, using a general topology, all the aforementioned type of filters could be realized, using the same core. The most important critical frequencies that will be studied are the following:

- ω_p is the frequency at which the magnitude response has a maximum or a minimum and is obtained by solving the equation $\frac{d}{d\omega} |H(j\omega)|_{\omega=\omega_p} = 0$
- ω_h is the half-power frequency at which $|H(j\omega)|_{\omega=\omega_h} = |H(j\omega)|_{\omega=\omega_p} / \sqrt{2}$
- ω_{rp} is the right-phase frequency at which the phase $\angle H(j\omega) = \pm \pi/2$

It should be mentioned that ω_{rp} exists only if $\alpha > 1$.

2.3.1 Fractional-Order Low-Pass Filter (FLPF)

The transfer function of a FLPF is that in (2.7), where κ is the low-frequency gain and $\omega_o \equiv 1/\tau$ the pole frequency. The magnitude and phase response are given by (2.8). The critical frequencies are summarized in Table 2.3, where the magnitude and phase values are also given. Using (2.8), the expressions for the ω_h and the corresponding phase are these in (2.9).

$$H(s) = \kappa \frac{\omega_o^\alpha}{s^\alpha + \omega_o^\alpha} = \kappa \frac{1}{(\tau s)^\alpha + 1} \quad (2.7)$$

$$|H(j\omega)| = \frac{\kappa}{\sqrt{\left(\frac{\omega}{\omega_o}\right)^{2\alpha} + 2\left(\frac{\omega}{\omega_o}\right)^\alpha \cos\left(\frac{\alpha\pi}{2}\right) + 1}} \quad (2.8a)$$

$$\angle H(j\omega) = \angle \kappa - \tan^{-1} \left(\frac{\left(\frac{\omega}{\omega_o}\right)^\alpha \sin\left(\frac{\alpha\pi}{2}\right)}{\left(\frac{\omega}{\omega_o}\right)^\alpha \cos\left(\frac{\alpha\pi}{2}\right) + 1} \right) \quad (2.8b)$$

In addition, the peak and right-phase frequency are found as $\omega_p = \omega_o[-\cos(\alpha\pi/2)]^{1/\alpha}$, and $\omega_{rp} = \omega_o/[-\cos(\alpha\pi/2)]^{1/\alpha}$. The stopband attenuation gradient of the fractional-order low-pass filter order α is equal to $-6 \cdot \alpha$ dB/oct.

$$\omega_h = \omega_o \left[\sqrt{1 + \cos^2\left(\frac{\alpha\pi}{2}\right)} - \cos\left(\frac{\alpha\pi}{2}\right) \right]^{1/\alpha} \quad (2.9a)$$

$$\angle H(j\omega)_{\omega=\omega_h} = \angle \kappa - \tan^{-1} \left(\frac{\sin\left(\frac{\alpha\pi}{2}\right)}{2\cos\left(\frac{\alpha\pi}{2}\right) + \sqrt{1 + \cos^2\left(\frac{\alpha\pi}{2}\right)}} \right) \quad (2.9b)$$

2.3.2 Fractional-Order High-Pass Filter (FHPF)

The transfer function of a FHPF with high-frequency gain κ and pole frequency $\omega_o \equiv 1/\tau$ is that in (2.10). The magnitude response and phase response are given by (2.11), while all the critical frequencies are summarized in Table 2.4.

Table 2.3 Magnitude and phase values at important frequencies for the FLPF

ω	$ H(j\omega) $	$\angle H(j\omega)$
$\rightarrow 0$	κ	$\angle \kappa$
ω_o	$\frac{\kappa}{2 \cos \frac{\alpha\pi}{4}}$	$\angle \kappa - \alpha\pi/4$
$\rightarrow \infty$	0	$\angle \kappa - \alpha\pi/2$
ω_h	$\kappa \cdot \frac{1}{\sqrt{2}}$	$\angle \kappa - \tan^{-1} \left(\frac{\sin\left(\frac{\alpha\pi}{2}\right)}{2\cos\left(\frac{\alpha\pi}{2}\right) + \sqrt{1 + \cos^2\left(\frac{\alpha\pi}{2}\right)}} \right)$

Table 2.4 Magnitude and phase values at important frequencies for the FHPF

ω	$ H(j\omega) $	$\angle H(j\omega)$
$\rightarrow 0$	0	$\angle \kappa + \alpha\pi/2$
ω_o	$\frac{\kappa}{2 \cos(\frac{\alpha\pi}{4})}$	$\angle \kappa + \alpha\pi/4$
$\rightarrow \infty$	κ	$\angle \kappa$
ω_h	$\kappa \cdot \frac{1}{\sqrt{2}}$	$\angle \kappa + \frac{\alpha\pi}{2} - \tan^{-1} \left(\frac{\sin(\frac{\alpha\pi}{2})}{\sqrt{1 + \cos^2(\frac{\alpha\pi}{2})}} \right)$

$$H(s) = \kappa \frac{\omega_o^\alpha s^\alpha}{s^\alpha + \omega_o^\alpha} = \kappa \frac{(\tau s)^\alpha}{(\tau s)^\alpha + 1} \quad (2.10)$$

$$|H(j\omega)| = \frac{\kappa \left(\frac{\omega}{\omega_o} \right)^\alpha}{\sqrt{\left(\frac{\omega}{\omega_o} \right)^{2\alpha} + 2 \left(\frac{\omega}{\omega_o} \right)^\alpha \cos\left(\frac{\alpha\pi}{2}\right) + 1}} \quad (2.11a)$$

$$\angle H(j\omega) = \angle \kappa + \alpha\pi/2 - \tan^{-1} \left(\frac{\left(\frac{\omega}{\omega_o} \right)^\alpha \sin\left(\frac{\alpha\pi}{2}\right)}{\left(\frac{\omega}{\omega_o} \right)^\alpha \cos\left(\frac{\alpha\pi}{2}\right) + 1} \right) \quad (2.11b)$$

The corresponding expressions for the ω_h and phase at this frequency are given by (2.12). In addition, the peak, and right-phase frequency are found as $\omega_p = \omega_o / [-\cos(\alpha\pi/2)]^{1/\alpha}$, and $\omega_{rp} = \omega_o [-\cos(\alpha\pi/2)]^{1/\alpha}$. From these expressions it is seen that both ω_p and ω_{rp} exist only if $\alpha > 1$. Also, the stopband attenuation gradient of the fractional-order high-pass filter of order α is equal to $+6 \cdot \alpha$ dB/oct.

$$\omega_h = \omega_o \left[\sqrt{1 + \cos^2\left(\frac{\alpha\pi}{2}\right)} + \cos\left(\frac{\alpha\pi}{2}\right) \right]^{1/\alpha} \quad (2.12a)$$

$$\angle H(j\omega)_{\omega=\omega_h} = \angle \kappa + \frac{\alpha\pi}{2} - \tan^{-1} \left(\frac{\sin\left(\frac{\alpha\pi}{2}\right)}{\sqrt{1 + \cos^2\left(\frac{\alpha\pi}{2}\right)}} \right) \quad (2.12b)$$

2.3.3 Fractional-Order Band-Pass Filter (FBPF)

The transfer function of a FBPF with peak-frequency gain κ and pole frequency $\omega_o \equiv 1/\tau$ is that in (2.13), where the magnitude and phase is as shown in (2.14a) and (2.14b), respectively. In order to obtain a FBPF response, the condition $\alpha > \beta$ should be fulfilled. All the critical frequencies are summarized in Table 2.5.

Table 2.5 Magnitude and phase values at important frequencies for the FBPF

ω	$ H(j\omega) $	$\angle H(j\omega)$
$\rightarrow 0$	0	$\angle \kappa + \beta\pi/2$
$\omega_p = \omega_o$	$\frac{\omega_o^\beta}{2 \cos(\frac{\alpha\pi}{4})}$	$\angle \kappa + [(2\beta - \alpha)\pi]/4$
$\rightarrow \infty$	$\omega_o^\alpha \omega^\beta \cdot \alpha$	$\angle \kappa + [(\beta - \alpha)\pi]/2$

$$H(s) = \kappa \frac{\omega_o^\alpha s^\beta}{s^\alpha + \omega_o^\alpha} = \kappa \frac{(\tau s)^\beta}{(\tau s)^\alpha + 1} \quad (2.13)$$

$$|H(j\omega)| = \frac{\kappa \left(\frac{\omega}{\omega_o}\right)^\beta}{\sqrt{\left(\frac{\omega}{\omega_o}\right)^{2\alpha} + 2\left(\frac{\omega}{\omega_o}\right)^\alpha \cos\left(\frac{\alpha\pi}{2}\right) + 1}} \quad (2.14a)$$

$$\angle H(j\omega) = \angle \kappa + \beta\pi/2 - \tan^{-1} \left(\frac{\left(\frac{\omega}{\omega_o}\right)^\alpha \sin\left(\frac{\alpha\pi}{2}\right)}{\left(\frac{\omega}{\omega_o}\right)^\alpha \cos\left(\frac{\alpha\pi}{2}\right) + 1} \right) \quad (2.14b)$$

The peak frequency (ω_p), calculated from the condition $\frac{d}{d\omega}|H(j\omega)|_{\omega=\omega_p} = 0$, is given by (2.15). In the case that $\alpha = 2\beta$, then $\omega_p = \omega_o$. Obviously, for $\alpha = \beta$, the transfer function in (2.13) is modified and corresponds to the already known FHPF.

$$\omega_p = \omega_o \left\{ \frac{\cos\left(\frac{\alpha\pi}{2}\right) \left[(2\beta - \alpha) + \sqrt{\alpha^2 + 4\beta(\alpha - \beta)\tan^2\left(\frac{\alpha\pi}{2}\right)} \right]}{2(\alpha - \beta)} \right\}^{1/\alpha} \quad (2.15)$$

All the critical frequencies are summarized in Table 2.5, where the magnitude and phase values are also given. It should be mentioned that the stopband attenuation gradient at the upper frequencies is $-6 \cdot (\alpha - \beta)$ dB/oct, while for the lower frequencies is $+6 \beta$ dB/oct, offering the capability of realizing a low- Q band pass filter with different slopes of the stopband attenuations. In case that $\alpha = 2\beta$, the slope is then equal to $-6 \cdot \beta$ dB/oct, and $+6 \beta$ dB/oct, respectively.

2.3.4 Fractional-Order All-Pass Filter (FAPF)

A FAPF with gain κ and pole frequency $\omega_o \equiv 1/\tau$ is described through the transfer function given by (2.16)

$$H(s) = \kappa \frac{s^\alpha - \omega_o^\alpha}{s^\alpha + \omega_o^\alpha} = \kappa \frac{(\tau s)^\alpha - 1}{(\tau s)^\alpha + 1} \quad (2.16)$$

Table 2.6 Magnitude and phase values at important frequencies for the FAPF

ω	$ H(j\omega) $	$\angle H(j\omega)$
$\rightarrow 0$	0	$\angle \kappa + \pi$
$\omega_p = \omega_o$	$\kappa \cdot \tan\left(\frac{\alpha\pi}{4}\right)$	$\angle \kappa + \pi/2$
ω_h	κ	$\angle \kappa$

Its frequency behavior is described by (2.17)

$$|H(j\omega)| = \kappa \sqrt{\frac{\left(\frac{\omega}{\omega_o}\right)^{2\alpha} - 2\left(\frac{\omega}{\omega_o}\right)^\alpha \cos\left(\frac{\alpha\pi}{2}\right) + 1}{\left(\frac{\omega}{\omega_o}\right)^{2\alpha} + 2\left(\frac{\omega}{\omega_o}\right)^\alpha \cos\left(\frac{\alpha\pi}{2}\right) + 1}} \quad (2.17a)$$

It can be easily seen here that $\omega_p = \omega_{rp} = \omega_o$, as well as that at this frequency a minima occurs if $\alpha < 1$ and a maxima occurs if $\alpha > 1$ while the magnitude remains flat when $\alpha = 1$ (i.e., classical integer-order all-pass filter). All the critical frequencies are summarized in Table 2.6 where the magnitude and phase values are also given.

$$\begin{aligned} \angle H(j\omega) = \angle \kappa \\ - \tan^{-1} \left(\frac{\left(\frac{\omega}{\omega_o}\right)^\alpha \sin\left(\frac{\alpha\pi}{2}\right)}{\left(\frac{\omega}{\omega_o}\right)^\alpha \cos\left(\frac{\alpha\pi}{2}\right) - 1} \right) - \tan^{-1} \left(\frac{\left(\frac{\omega}{\omega_o}\right)^\alpha \sin\left(\frac{\alpha\pi}{2}\right)}{\left(\frac{\omega}{\omega_o}\right)^\alpha \cos\left(\frac{\alpha\pi}{2}\right) + 1} \right) \end{aligned} \quad (2.17b)$$

2.3.5 Design Equations for Generalized Filters of Order α

Taking into account that the transfer functions of all the aforementioned fractional-order filters are expressed through the variables $(\tau s)^\alpha$ and/or $(\tau s)^\beta$, and the fact that they are not realizable, they should be approximated by appropriate expressions. Using the second-order expressions of the CFE given by (2.1) and substituting into (2.7), (2.10), (2.13), and (2.16), the derived transfer functions are given in (2.18).

The coefficient α_i corresponds to the approximation of variable $(\tau s)^\alpha$, while b_i to the variable $(\tau s)^\beta$. Comparing the transfer functions of the FLPF and FHPF, the numerator in $H_{HP}(s)$ is easily derived through the substitution: $\alpha_0 \leftrightarrow \alpha_2$ in the numerator of $H_{LP}(s)$.

Inspecting the transfer functions of FLPF, FHPF, and FAPF it is concluded that all of them have the same form. Consequently, they could be realized by the same topology just by changing the coefficient values. A suitable solution for this purpose has been already given in Fig. 2.1, where the realized transfer function is that given in (2.6).

$$H_a^{\text{LP}}(s) = \frac{\kappa}{\alpha_0 + \alpha_2} \cdot \frac{\alpha_2 s^2 + \alpha_1 \frac{1}{\tau} s + \alpha_0 \frac{1}{\tau^2}}{s^2 + \left(\frac{2\alpha_1}{\alpha_0 + \alpha_2} \right) \frac{1}{\tau} s + \frac{1}{\tau^2}} \quad (2.18a)$$

$$H_a^{\text{HP}}(s) = \frac{\kappa}{\alpha_0 + \alpha_2} \cdot \frac{\alpha_0 s^2 + \alpha_1 \frac{1}{\tau} s + \alpha_2 \frac{1}{\tau^2}}{s^2 + \left(\frac{2\alpha_1}{\alpha_0 + \alpha_2} \right) \frac{1}{\tau} s + \frac{1}{\tau^2}} \quad (2.18b)$$

$$H_a^{\text{AP}}(s) = \kappa \frac{\alpha_0 - \alpha_2}{\alpha_0 + \alpha_2} \cdot \frac{s^2 - \frac{1}{\tau^2}}{s^2 + \left(\frac{2\alpha_1}{\alpha_0 + \alpha_2} \right) \frac{1}{\tau} s + \frac{1}{\tau^2}} \quad (2.18c)$$

$$H_a^{\text{BP}}(s) = \frac{\kappa}{\alpha_0 + \alpha_2} \cdot \frac{\alpha_2 s^2 + \alpha_1 \frac{1}{\tau} s + \alpha_0 \frac{1}{\tau^2}}{s^2 + \left(\frac{2\alpha_1}{\alpha_0 + \alpha_2} \right) \frac{1}{\tau} s + \frac{1}{\tau^2}} \cdot \frac{1}{b_2} \cdot \frac{b_0 s^2 + b_1 \frac{1}{\tau} s + b_2 \frac{1}{\tau^2}}{s^2 + \left(\frac{b_1}{b_2} \right) \frac{1}{\tau} s + \frac{b_0}{b_2} \frac{1}{\tau^2}} \quad (2.18d)$$

Comparing the coefficients of (2.6) with those of FLPF, FHPF, and FAPF in (2.18), it is derived that the design equations about the time-constants of fractional-order low-pass, high-pass, and all-pass filters of order α are those in (2.19). The corresponding design equations for the scaling factors G_i ($i = 0, 1, 2$) are summarized in Table 2.7.

$$\tau_1 = \frac{\alpha_0 + \alpha_2}{2\alpha_1} \cdot \tau \quad (2.19a)$$

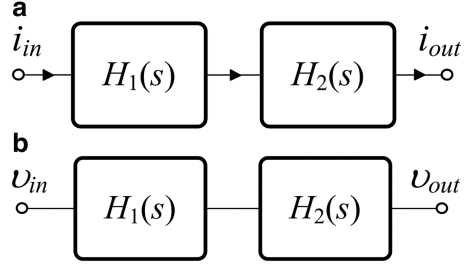
$$\tau_2 = \frac{2\alpha_1}{\alpha_2 + \alpha_0} \cdot \tau \quad (2.19b)$$

With regards to the FBPF realization, a possible solution is the FBD in Fig. 2.2, where $H_1(s)$ and $H_2(s)$ are mentioned filter blocks as that in Fig. 2.1, which realize the transfer function in (2.6). The expressions for time-constants τ_{ij} , where ($i = 1, 2$) is the number of time-constants of each state, and ($j = 1, 2$) is the number of state, are given by (2.20). The time-constants of the first stager τ_{i1} have the same values as in (2.19). The corresponding design equation for the scaling factors G_{ij} ($i = 0, 1, 2$ and $j = 1, 2$) of both stages are also summarized in Table 2.7.

Table 2.7 Values of scaling factors G_{ij} for realizing FLPF, FHPF, FAPF, and FBPF of order α in Figs 2.1, and 2.2

Filter		G_2	G_1	G_0
FLPF		$\kappa \cdot \frac{\alpha_2}{\alpha_2 + \alpha_0}$	$\frac{\kappa}{2}$	$\kappa \cdot \frac{\alpha_0}{\alpha_2 + \alpha_0}$
FHPF		$\kappa \cdot \frac{\alpha_0}{\alpha_2 + \alpha_0}$	$\frac{\kappa}{2}$	$\kappa \cdot \frac{\alpha_2}{\alpha_0 + \alpha_2}$
FAPF		$\kappa \cdot \frac{\alpha_0 - \alpha_2}{\alpha_2 + \alpha_0}$	0	$-\kappa \cdot \frac{\alpha_0 - \alpha_2}{\alpha_2 + \alpha_0}$
FBPF	($j = 1$)	$\kappa \cdot \frac{\alpha_2}{\alpha_2 + \alpha_0}$	$\frac{\kappa}{2}$	$\kappa \cdot \frac{\alpha_0}{\alpha_2 + \alpha_0}$
	($j = 2$)	$\frac{b_0}{b_2}$	1	$\frac{b_2}{b_0}$

Fig. 2.2 FBD for realizing FBPF of order α using (a) current mode topology, (b) voltage mode topology



$$\tau_{11} = \frac{\alpha_0 + \alpha_2}{2\alpha_1} \cdot \tau, \quad \tau_{12} = \frac{b_2}{b_1} \cdot \tau \quad (2.20a)$$

$$\tau_{21} = \frac{2\alpha_1}{\alpha_2 + \alpha_0} \cdot \tau, \quad \tau_{22} = \frac{b_1}{b_0} \cdot \tau \quad (2.20b)$$

2.4 Fractional-Order Generalized Filters (Order $1 + \alpha$)

Although, fractional-order filters constitute a small portion of fractional-order calculus, they have gained a growing research interest offering important features especially on behavior of the attenuation gradient. The stopband attenuation of integer order filters has been limited to increments based on the order n , but using the fractional Laplacian operator attenuations between these integer steps can be achieved creating fractional step filter of order $(n + \alpha)$, where α is the fractional step between integer orders n and $n + 1$ and is therefore limited to $(0 < \alpha < 1)$. The derived frequency responses of filters of order $1 + \alpha$ exhibit a stopband attenuation equal to $-6 \cdot (1 + \alpha)$ dB/oct, which offer a more precise control of the attenuation gradient compared to the attenuation offered in the case of integer-order filters of order n , which is $-6 \cdot n$ dB/oct [6–19]. In this section, low-pass, high-pass, band-pass, and band-stop filters of order $1 + \alpha$ are presented, and some of the most critical frequencies (ω_p , ω_h) are also given in order to be fully characterized. Finally, using the CFE method, the derived design equations results into a general topology, from which all the aforementioned type of filters could be realized.

2.4.1 Fractional-Order Low-Pass Filter (FLPF)

According to the analysis provided in [8], the direct realization of a fractional filter of the order $n + \alpha$ is stable in the case that $n + \alpha < 2$. Therefore, only fractional filters of the order $1 + \alpha$ offer realizations without stability problem. The transfer function of a $1 + \alpha$ -order fractional low-pass filter is given by

$$H(s) = \frac{\kappa_1}{s^{1+\alpha} + \kappa_2} \quad (2.21)$$

where the low-frequency gain is equal to κ_1/κ_2 , and the -3 dB frequency is given by (2.22)

$$\omega_{-3\text{dB}} = \left[\kappa_2 \left(\sqrt{1 + \cos^2 \frac{(1+\alpha)\pi}{2}} - \cos \frac{(1+\alpha)\pi}{2} \right)^{\frac{1}{1+\alpha}} \right] \quad (2.22)$$

The obtained frequency responses suffer from the presence of an undesired peaking equal to $\kappa_1/\kappa_2[(\sin(1 + \alpha)\pi/2)]$ at the frequency $\omega_p = [-\kappa_2 \cos(1 + \alpha)\pi/2]^{-(1+\alpha)}$. In order to overcome this problem, the modified transfer function given by (2.23), which intends to approximate the all-pole Butterworth response, is introduced, where an extra term equal to $\kappa_3 s^\alpha$ has been added in the denominator of the transfer function in (2.21). The transfer function of a FLPF with low-frequency gain κ_1/κ_2 and pole frequency $\omega_o = 1/\tau$ is that in (2.23)

$$H(s) = \frac{\kappa_1}{(\tau s)^{1+\alpha} + \kappa_3 (\tau s)^\alpha + \kappa_2} \quad (2.23)$$

The factors κ_i ($i = 1, 2, 3$) are calculated by appropriate expressions, in order to minimize the error in the frequency response [9]. Such expressions are recalled in (2.24).

$$\begin{aligned} \kappa_1 &= 1 \\ \kappa_2 &= 0.2937\alpha + 0.71216 \\ \kappa_3 &= 1.068\alpha^2 + 0.161\alpha + 0.3324 \end{aligned} \quad (2.24)$$

The magnitude response and phase response are given by (2.25) as

$$|H(j\omega)| = \frac{\kappa_1}{\sqrt{\left(\left(\frac{\omega}{\omega_o}\right)^{2(1+\alpha)} - 2\kappa_2\left(\frac{\omega}{\omega_o}\right)^{1+\alpha} \sin\left(\frac{\alpha\pi}{2}\right) + \kappa_3^2\left(\frac{\omega}{\omega_o}\right)^{2\alpha} + 2\kappa_2\kappa_3\left(\frac{\omega}{\omega_o}\right)^\alpha \cos\left(\frac{\alpha\pi}{2}\right) + \kappa_2^2}\right)}} \quad (2.25a)$$

$$\angle H(j\omega) = \angle \kappa_1 - \tan^{-1} \left(\frac{\left(\frac{\omega}{\omega_o}\right)^{1+\alpha} \cos\left(\frac{\alpha\pi}{2}\right) + \kappa_3\left(\frac{\omega}{\omega_o}\right)^\alpha \sin\left(\frac{\alpha\pi}{2}\right)}{-\left(\frac{\omega}{\omega_o}\right)^{1+\alpha} \sin\left(\frac{\alpha\pi}{2}\right) + \kappa_3\left(\frac{\omega}{\omega_o}\right)^\alpha \cos\left(\frac{\alpha\pi}{2}\right) + \kappa_2} \right) \quad (2.25b)$$

The peak frequency (ω_p) is calculated solving the following equation:

$$\begin{aligned} 2(1 + \alpha) \left(\frac{\omega_p}{\omega_o}\right)^{2+\alpha} - 2\kappa_2(1 + \alpha) \left(\frac{\omega_p}{\omega_o}\right) \sin\left(\frac{\alpha\pi}{2}\right) \\ + 2\alpha\kappa_3^2 \left(\frac{\omega_p}{\omega_o}\right)^\alpha + 2\alpha\kappa_2\kappa_3 \cos\left(\frac{\alpha\pi}{2}\right) = 0 \end{aligned} \quad (2.26)$$

which is derived from (2.25) under the condition $\frac{d}{d\omega}|H(j\omega)|_{\omega=\omega_p} = 0$.

The half-power (-3 dB) frequency (ω_h), defined as the frequency where there is a 0.707 drop of the passband gain, is calculated solving the following equation:

$$\begin{aligned} \left(\frac{\omega_h}{\omega_o}\right)^{2(1+\alpha)} - 2\kappa_2\left(\frac{\omega_h}{\omega_o}\right)^{1+\alpha} \sin\left(\frac{\alpha\pi}{2}\right) + \kappa_3^2\left(\frac{\omega_h}{\omega_o}\right)^{2\alpha} \\ + 2\kappa_2\kappa_3\left(\frac{\omega_h}{\omega_o}\right)^\alpha \cos\left(\frac{\alpha\pi}{2}\right) - \kappa_2^2 = 0 \end{aligned} \quad (2.27)$$

Also, the stopband attenuation gradient of the $1 + \alpha$ fractional-order low-pass filter is equal to $-6 \cdot (1 + \alpha)$ dB/oct.

2.4.2 Fractional-Order High-Pass Filter (FHPF)

The transfer function of a FHPF is given by (2.28)

$$H(s) = \frac{\kappa_1 s^{1+\alpha}}{(\tau s)^{1+\alpha} + \kappa_3 (\tau s)^\alpha + \kappa_2} \quad (2.28)$$

The magnitude response and phase response are given by (2.29).

$$|H(j\omega)| = \frac{\kappa_1 \left(\frac{\omega}{\omega_o}\right)^{1+\alpha}}{\sqrt{\left(\frac{\omega}{\omega_o}\right)^{2(1+\alpha)} - 2\kappa_2\left(\frac{\omega}{\omega_o}\right)^{1+\alpha} \sin\left(\frac{\alpha\pi}{2}\right) + \kappa_3^2\left(\frac{\omega}{\omega_o}\right)^{2\alpha} + 2\kappa_2\kappa_3\left(\frac{\omega}{\omega_o}\right)^\alpha \cos\left(\frac{\alpha\pi}{2}\right) + \kappa_2^2}} \quad (2.29a)$$

$$\begin{aligned} \angle H(j\omega) = \angle \kappa_1 + \frac{(1 + \alpha)\pi}{2} \\ - \tan^{-1} \left(\frac{\left(\frac{\omega}{\omega_o}\right)^{1+\alpha} \cos\left(\frac{\alpha\pi}{2}\right) + \kappa_3 \left(\frac{\omega}{\omega_o}\right)^\alpha \sin\left(\frac{\alpha\pi}{2}\right)}{-\left(\frac{\omega}{\omega_o}\right)^{1+\alpha} \sin\left(\frac{\alpha\pi}{2}\right) + \kappa_3 \left(\frac{\omega}{\omega_o}\right)^\alpha \cos\left(\frac{\alpha\pi}{2}\right) + \kappa_2} \right) \end{aligned} \quad (2.29b)$$

The peak frequency (ω_p) is calculated solving the equation given in (2.30), which is derived from (2.29) under the condition $\frac{d}{d\omega}|H(j\omega)|_{\omega=\omega_p} = 0$.

$$\begin{aligned} \kappa_2(1 + \alpha)\left(\frac{\omega_p}{\omega_o}\right)^{1+\alpha} \sin\left(\frac{\alpha\pi}{2}\right) - \kappa_3^2\left(\frac{\omega_p}{\omega_o}\right)^{2\alpha} \\ - (2 + \alpha)\kappa_2\kappa_3\left(\frac{\omega_p}{\omega_o}\right)^\alpha \cos\left(\frac{\alpha\pi}{2}\right) - (1 + \alpha)\kappa_2^2 = 0 \end{aligned} \quad (2.30)$$

The half-power (-3 dB) frequency (ω_h), defined as the frequency where there is a 0.707 drop of the passband gain, is calculated solving the equation given in (2.31).

Also, the stopband attenuation gradient of the fractional-order high-pass filter order $1 + \alpha$ is equal to $+6 \cdot (1 + \alpha)$ dB/oct.

$$\begin{aligned} & \left(\frac{\omega_h}{\omega_o}\right)^{2(1+\alpha)} + 2\kappa_2 \left(\frac{\omega_h}{\omega_o}\right)^{1+\alpha} \sin\left(\frac{\alpha\pi}{2}\right) - \kappa_3^2 \left(\frac{\omega_h}{\omega_o}\right)^{2\alpha} \\ & - 2\kappa_2\kappa_3 \left(\frac{\omega_h}{\omega_o}\right)^\alpha \cos\left(\frac{\alpha\pi}{2}\right) - \kappa_2^2 = 0 \end{aligned} \quad (2.31)$$

2.4.3 Fractional-Order Band-Pass Filter (FBPF)

The transfer function of a FBPF is given by (2.32)

$$H(s) = \kappa_1 \frac{\kappa_3 (\tau s)^a}{(\tau s)^{1+a} + \kappa_3 (\tau s)^a + \kappa_2} \quad (2.32)$$

Using (2.32), the magnitude and phase response are given as

$$|H(j\omega)| = \frac{\kappa_1 \kappa_3 \left(\frac{\omega}{\omega_o}\right)^\alpha}{\sqrt{\left(\frac{\omega}{\omega_o}\right)^{2(1+\alpha)} - 2\kappa_2 \left(\frac{\omega}{\omega_o}\right)^{1+\alpha} \sin\left(\frac{\alpha\pi}{2}\right) + \kappa_3^2 \left(\frac{\omega}{\omega_o}\right)^{2\alpha} + 2\kappa_2\kappa_3 \left(\frac{\omega}{\omega_o}\right)^\alpha \cos\left(\frac{\alpha\pi}{2}\right) + \kappa_2^2}} \quad (2.33a)$$

$$\begin{aligned} \angle H(j\omega) &= \angle \kappa_1 \kappa_3 + \frac{\alpha\pi}{2} \\ &- \tan^{-1} \left(\frac{\left(\frac{\omega}{\omega_o}\right)^{1+\alpha} \cos\left(\frac{\alpha\pi}{2}\right) + \kappa_3 \left(\frac{\omega}{\omega_o}\right)^\alpha \sin\left(\frac{\alpha\pi}{2}\right)}{-\left(\frac{\omega}{\omega_o}\right)^{1+\alpha} \sin\left(\frac{\alpha\pi}{2}\right) + \kappa_3 \left(\frac{\omega}{\omega_o}\right)^\alpha \cos\left(\frac{\alpha\pi}{2}\right) + \kappa_2} \right) \end{aligned} \quad (2.33b)$$

The peak frequency (ω_p) is calculated solving the equation given in (2.34), which is derived from (2.33a) under the condition $\frac{d}{d\omega} |H(j\omega)|_{\omega=\omega_p} = 0$.

$$\begin{aligned} & \left(\frac{\omega_p}{\omega_o}\right)^{2(1+\alpha)} - (1 - \alpha)\kappa_2 \left(\frac{\omega_p}{\omega_o}\right)^{1+\alpha} \sin\left(\frac{\alpha\pi}{2}\right) \\ & - \alpha\kappa_2\kappa_3 \left(\frac{\omega_p}{\omega_o}\right)^\alpha \cos\left(\frac{\alpha\pi}{2}\right) - \alpha\kappa_2^2 = 0 \end{aligned} \quad (2.34)$$

The half-power (-3 dB) frequencies (ω_{h1} , ω_{h2}) are defined as the frequencies where there is a 0.707 drop of the passband gain. The quality factor of the filter (Q) is calculated solving the following equation:

$$Q = \frac{\omega_p}{\omega_{h2} - \omega_{h1}} \quad (2.35)$$

where ω_{h2} and ω_{h1} are the upper and lower half-power frequencies, respectively.

Also, the stopband attenuation gradient at the upper frequencies is -6 dB/oct, while for the lower frequencies is $+6\alpha$ dB/oct, offering the capability of realizing a FBPF with the stopband attenuation being varied at the lower frequencies.

2.4.4 Fractional-Order Band-Stop Filter (FBSF)

A FBSF has the transfer function given by (2.36)

$$H(s) = \kappa_1 \frac{(\tau s)^{1+\alpha} + \kappa_2}{(\tau s)^{1+\alpha} + \kappa_3 (\tau s)^\alpha + \kappa_2} \quad (2.36)$$

The magnitude response and phase response are given by (2.37), while the peak frequency (ω_p) is calculated from (2.37) under the condition $\frac{d}{d\omega} |H(j\omega)|_{\omega=\omega_p} = 0$. The half-power (-3 dB) frequencies (ω_h) are calculated from the condition that at these frequencies there is a 0.707 drop of the passband gain.

$$|H(j\omega)| = \kappa_1 \sqrt{\frac{\left(\frac{\omega}{\omega_0}\right)^{2(1+\alpha)} - 2\kappa_2 \left(\frac{\omega}{\omega_0}\right)^{1+\alpha} \sin\left(\frac{\alpha\pi}{2}\right) + \kappa_2^2}{\left(\frac{\omega}{\omega_0}\right)^{2(1+\alpha)} - 2\kappa_2 \left(\frac{\omega}{\omega_0}\right)^{1+\alpha} \sin\left(\frac{\alpha\pi}{2}\right) + \kappa_3^2 \left(\frac{\omega}{\omega_0}\right)^{2\alpha} + 2\kappa_2 \kappa_3 \left(\frac{\omega}{\omega_0}\right)^\alpha \cos\left(\frac{\alpha\pi}{2}\right) + \kappa_2^2}} \quad (2.37a)$$

$$\begin{aligned} \angle H(j\omega) = \angle k_1 + \tan^{-1} & \left(\frac{\left(\frac{\omega}{\omega_0}\right)^{1+\alpha} \sin\left(\frac{\alpha\pi}{2}\right)}{-\left(\frac{\omega}{\omega_0}\right)^{1+\alpha} \cos\left(\frac{\alpha\pi}{2}\right) + \kappa_2} \right) \\ & - \tan^{-1} \left(\frac{\left(\frac{\omega}{\omega_0}\right)^{1+\alpha} \cos\left(\frac{\alpha\pi}{2}\right) + \kappa_3 \left(\frac{\omega}{\omega_0}\right)^\alpha \sin\left(\frac{\alpha\pi}{2}\right)}{-\left(\frac{\omega}{\omega_0}\right)^{1+\alpha} \sin\left(\frac{\alpha\pi}{2}\right) + \kappa_3 \left(\frac{\omega}{\omega_0}\right)^\alpha \cos\left(\frac{\alpha\pi}{2}\right) + \kappa_2} \right) \end{aligned} \quad (2.37b)$$

The quality factor of the filter is calculated according to the formula:

$$Q = \frac{\omega_p}{\omega_{h2} - \omega_{h1}} \quad (2.38)$$

where ω_{h2} and ω_{h1} are the upper and lower half-power frequencies, respectively.

2.4.5 Design Equations for Generalized Filters of Order $1 + \alpha$

The realization of the fractional-order filters of order $1 + \alpha$ has been achieved in a similar way as in the previous section, where the approximation of variable $(\tau s)^\alpha$

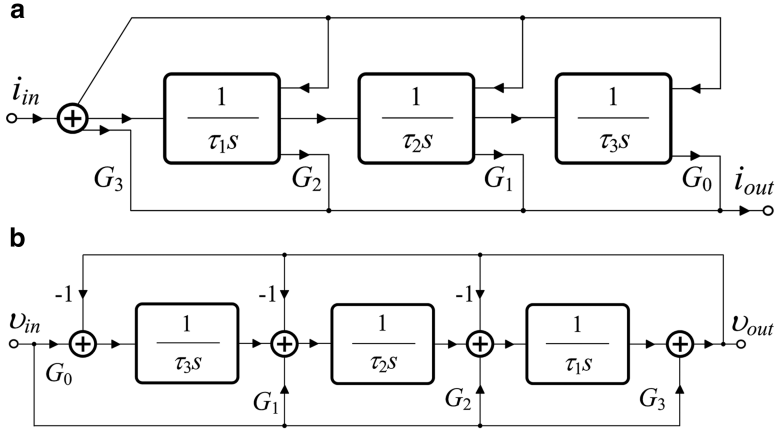


Fig. 2.3 FBD for realizing FLPF, FHFP, FBPF, and FBSF of order $1 + \alpha$ using (a) current mode topology, (b) voltage mode topology

has been performed through the utilization of the second-order expressions of the CFE given by (2.1) and substitution into (2.23), (2.28), (2.32), and (2.36). As a result, the derived transfer functions are that in (2.39).

Inspecting the transfer functions of FLPF, FHFP, FBPF, and FBSF, it is concluded that all of them have the same form. Consequently they could be realized by the same topology just by changing the coefficient values. A suitable solution for this purpose is depicted in Fig. 2.3, where a typical (FBD) of a (FLF) topology and an (ILF) topology are given in Fig. 2.3a and Fig. 2.3b, respectively. The realized transfer function is that given in (2.40).

$$H_{1+\alpha}^{\text{LP}}(s) = \frac{\kappa_1}{\alpha_0} \cdot \frac{\alpha_2 \frac{1}{\tau} s^2 + \alpha_1 \frac{1}{\tau^2} s + \alpha_0 \frac{1}{\tau^3}}{\left[s^3 + \left(\frac{\alpha_1 + \kappa_2 \alpha_2 + \kappa_3 \alpha_0}{\alpha_0} \right) \frac{1}{\tau} s^2 + \left(\frac{\alpha_2 + \kappa_2 \alpha_1 + \kappa_3 \alpha_1}{\alpha_0} \right) \frac{1}{\tau^2} s + \left(\frac{\kappa_3 \alpha_2 + \kappa_2 \alpha_0}{\alpha_0} \right) \frac{1}{\tau^3} \right]} \quad (2.39a)$$

$$H_{1+\alpha}^{\text{HP}}(s) = \frac{\kappa_1}{\alpha_0} \cdot \frac{\alpha_0 s^3 + \alpha_1 s^2 + \alpha_2 s}{\left[s^3 + \left(\frac{\alpha_1 + \kappa_2 \alpha_2 + \kappa_3 \alpha_0}{\alpha_0} \right) \frac{1}{\tau} s^2 + \left(\frac{\alpha_2 + \kappa_2 \alpha_1 + \kappa_3 \alpha_1}{\alpha_0} \right) \frac{1}{\tau^2} s + \left(\frac{\kappa_3 \alpha_2 + \kappa_2 \alpha_0}{\alpha_0} \right) \frac{1}{\tau^3} \right]} \quad (2.39b)$$

$$H_{1+\alpha}^{\text{BP}}(s) = \frac{\kappa_1 \kappa_3}{\alpha_0} \cdot \frac{\alpha_0 \frac{1}{\tau} s^2 + \alpha_1 \frac{1}{\tau^2} s + \alpha_2 \frac{1}{\tau^3}}{\left[s^3 + \left(\frac{\alpha_1 + \kappa_2 \alpha_2 + \kappa_3 \alpha_0}{\alpha_0} \right) \frac{1}{\tau} s^2 + \left(\frac{\alpha_2 + \kappa_2 \alpha_1 + \kappa_3 \alpha_1}{\alpha_0} \right) \frac{1}{\tau^2} s + \left(\frac{\kappa_3 \alpha_2 + \kappa_2 \alpha_0}{\alpha_0} \right) \frac{1}{\tau^3} \right]} \quad (2.39c)$$

Table 2.8 Values of scaling factors G_i ($i = 0, \dots, 3$) for realizing fractional low-pass, high-pass, band-pass, and band-stop filter in Fig. 2.3 of order $1 + \alpha$

Filter	G_3	G_2	G_1	G_0
FLPF	0	$\frac{\kappa_1 \alpha_2}{\kappa_2 \alpha_2 + \kappa_3 \alpha_0 + \alpha_1}$	$\frac{\kappa_1 \alpha_1}{\alpha_2 + (\kappa_2 + \kappa_3) \alpha_1}$	$\frac{\kappa_1 \alpha_0}{\kappa_2 \alpha_0 + \kappa_3 \alpha_2}$
FHPF	κ_1	$\frac{\kappa_1 \alpha_1}{\kappa_2 \alpha_2 + \kappa_3 \alpha_0 + \alpha_1}$	$\frac{\kappa_1 \alpha_2}{\alpha_2 + (\kappa_2 + \kappa_3) \alpha_1}$	0
FBPF	0	$\frac{\kappa_1 \kappa_3 \alpha_0}{\kappa_2 \alpha_2 + \kappa_3 \alpha_0 + \alpha_1}$	$\frac{\kappa_1 \kappa_3 \alpha_1}{\alpha_2 + (\kappa_2 + \kappa_3) \alpha_1}$	$\frac{\kappa_1 \kappa_3 \alpha_2}{\kappa_2 \alpha_0 + \kappa_3 \alpha_2}$
FBSF	κ_1	$\frac{\kappa_1 (\alpha_1 + \kappa_2 \alpha_2)}{\kappa_2 \alpha_2 + \kappa_3 \alpha_0 + \alpha_1}$	$\frac{\kappa_1 (\alpha_2 + \kappa_2 \alpha_1)}{\alpha_2 + (\kappa_2 + \kappa_3) \alpha_1}$	$\frac{\kappa_1 \kappa_2 \alpha_0}{\kappa_3 \alpha_2 + \kappa_2 \alpha_0}$

$$H_{1+\alpha}^{\text{BS}}(s) = \frac{\kappa_1}{\alpha_0} \cdot \frac{\alpha_0 s^3 + (\alpha_1 + \kappa_2 \alpha_2) \frac{1}{\tau} s^2 + (\alpha_2 + \kappa_2 \alpha_1) \frac{1}{\tau^2} s + (\kappa_2 \alpha_0) \frac{1}{\tau^3}}{\left[s^3 + \left(\frac{\alpha_1 + \kappa_2 \alpha_2 + \kappa_3 \alpha_0}{\alpha_0} \right) \frac{1}{\tau} s^2 + \left(\frac{\alpha_2 + \kappa_2 \alpha_1 + \kappa_3 \alpha_1}{\alpha_0} \right) \frac{1}{\tau^2} s + \left(\frac{\kappa_3 \alpha_2 + \kappa_2 \alpha_0}{\alpha_0} \right) \frac{1}{\tau^3} \right]} \quad (2.39d)$$

$$H(s) = \frac{G_3 s^3 + \frac{G_2}{\tau_1} s^2 + \frac{G_1}{\tau_1 \tau_2} s + \frac{G_0}{\tau_1 \tau_2 \tau_3}}{s^3 + \frac{1}{\tau_1} s^2 + \frac{1}{\tau_1 \tau_2} s + \frac{1}{\tau_1 \tau_2 \tau_3}} \quad (2.40)$$

Comparing the coefficients of (2.40) with these of FLPF, FHPF, and FBSF it is derived that the design equations about the time-constants τ_i ($i = 1, 2, 3$) of all the filters are given in (2.41).

$$\tau_1 = \frac{\alpha_0}{\kappa_2 \alpha_2 + \kappa_3 \alpha_0 + \alpha_1} \cdot \tau \quad (2.41a)$$

$$\tau_2 = \frac{\kappa_2 \alpha_2 + \kappa_3 \alpha_0 + \alpha_1}{\kappa_2 \alpha_1 + \kappa_3 \alpha_1 + \alpha_2} \cdot \tau \quad (2.41b)$$

$$\tau_3 = \frac{\kappa_2 \alpha_1 + \kappa_3 \alpha_1 + \alpha_2}{\kappa_2 \alpha_0 + \kappa_3 \alpha_2} \cdot \tau \quad (2.41c)$$

The corresponding design equations for the scaling factors G_i ($i = 0, \dots, 3$) are summarized in Table 2.8.

2.5 Fractional-Order Generalized Filters (Order $\alpha + \beta$)

The utilization of two different orders α and β , where $\alpha + \beta < 2$ and $\alpha, \beta > 0$ provides one more degree of freedom, which is able to vary, in order to realize fractional order filters of order $\alpha + \beta$. Such kind of filters exhibit a stopband attenuation which is proportionate to the fractional-order α, β . In this section,

FLPF, FHPF, FBPF, and FBSF of order $\alpha + \beta$ are presented, and some of the most critical frequencies have been derived in order to be fully characterized.

The most important critical frequencies that will be presented are the following:

- ω_p is the frequency at which the magnitude response has a maximum or a minimum and is obtained by solving the equation $\frac{d}{d\omega} |H(j\omega)|_{\omega=\omega_p} = 0$
- ω_h is the half-power frequency at which the power drops to half the passband power, i.e., $|H(j\omega)|_{\omega=\omega_h} = |H(j\omega)|_{\omega=\omega_p} / \sqrt{2}$

Also, using a general topology, all the aforementioned type of filters could be realized and this is very important from the flexibility point of view.

2.5.1 Fractional-Order Low-Pass Filter (FLPF)

The transfer function of the FLPF of order $\alpha + \beta$ is given by Eq. (2.42)

$$H(s) = \frac{\kappa_1}{(\tau s)^{\alpha+\beta} + \kappa_3(\tau s)^\beta + \kappa_2} \quad (2.42)$$

The magnitude response and phase response are given by (2.43). The half-power (-3 dB) frequency (ω_h), defined as the frequency where there is a 0.707 drop of the passband gain, is calculated solving the equation given by (2.44), which is derived taking into account that the maximum gain of the filter is κ_1/κ_2 . Also, the stopband attenuation gradient of the fractional-order low-pass filter of order $\alpha + \beta$ is equal to $-6 \cdot (\alpha + \beta)$ dB/oct.

$$|H(j\omega)| = \frac{\kappa_1}{\sqrt{\left(\left(\frac{\omega}{\omega_o}\right)^{2(\alpha+\beta)} + \kappa_3^2 \left(\frac{\omega}{\omega_o}\right)^{2\beta} + \kappa_2^2 + 2\kappa_3 \left(\frac{\omega}{\omega_o}\right)^{\alpha+2\beta} \cos\left(\frac{\alpha\pi}{2}\right) + 2\kappa_2 \left(\frac{\omega}{\omega_o}\right)^{\alpha+\beta} \cos\left[\frac{(\alpha+\beta)\pi}{2}\right] + 2\kappa_2\kappa_3 \left(\frac{\omega}{\omega_o}\right)^\beta \cos\left(\frac{\beta\pi}{2}\right)}}} \quad (2.43a)$$

$$\angle H(j\omega) = \angle \kappa_1 - \tan^{-1} \left(\frac{\left(\frac{\omega}{\omega_o}\right)^{\alpha+\beta} \sin\left[\frac{(\alpha+\beta)\pi}{2}\right] + \kappa_3 \left(\frac{\omega}{\omega_o}\right)^\beta \sin\left(\frac{\beta\pi}{2}\right)}{\left(\frac{\omega}{\omega_o}\right)^{\alpha+\beta} \cos\left[\frac{(\alpha+\beta)\pi}{2}\right] + \kappa_3 \left(\frac{\omega}{\omega_o}\right)^\beta \cos\left(\frac{\beta\pi}{2}\right) + \kappa_2} \right) \quad (2.43b)$$

$$\begin{aligned} &\left(\frac{\omega_h}{\omega_o}\right)^{2(\alpha+\beta)} + 2\kappa_3 \left(\frac{\omega_h}{\omega_o}\right)^{\alpha+2\beta} \cos\left(\frac{\alpha\pi}{2}\right) + 2\kappa_2 \left(\frac{\omega_h}{\omega_o}\right)^{\alpha+\beta} \cos\left[\frac{(\alpha+\beta)\pi}{2}\right] \\ &+ \kappa_3^2 \left(\frac{\omega_h}{\omega_o}\right)^{2\beta} + 2\kappa_2\kappa_3 \left(\frac{\omega_h}{\omega_o}\right)^\beta \cos\left(\frac{\beta\pi}{2}\right) - \kappa_2^2 = 0 \end{aligned} \quad (2.44)$$

2.5.2 Fractional-Order High-Pass Filter (FHPF)

The transfer function of a HLPF with maximum gain equal to κ_1/κ_2 is that in (2.45)

$$H(s) = \frac{\kappa_1 (\tau s)^{\alpha+\beta}}{(\tau s)^{\alpha+\beta} + \kappa_3 (\tau s)^\beta + \kappa_2} \quad (2.45)$$

The magnitude response and phase response are given by (2.46) as

$$|H(j\omega)| = \frac{\kappa_1 \left(\frac{\omega}{\omega_0}\right)^{\alpha+\beta}}{\sqrt{\left(\left(\frac{\omega}{\omega_0}\right)^{2(\alpha+\beta)} + \kappa_3^2 \left(\frac{\omega}{\omega_0}\right)^{2\beta} + \kappa_2^2 + 2\kappa_3 \left(\frac{\omega}{\omega_0}\right)^{\alpha+2\beta} \cos\left(\frac{\alpha\pi}{2}\right) + 2\kappa_2 \left(\frac{\omega}{\omega_0}\right)^{\alpha+\beta} \cos\left[\frac{(\alpha+\beta)\pi}{2}\right] + 2\kappa_2 \kappa_3 \left(\frac{\omega}{\omega_0}\right)^\beta \cos\left(\frac{\beta\pi}{2}\right)\right)}} \quad (2.46a)$$

$$\angle H(j\omega) = \angle \kappa_1 + \frac{(\alpha+\beta)\pi}{2} - \tan^{-1} \left(\frac{\left(\frac{\omega}{\omega_0}\right)^{\alpha+\beta} \sin\left[\frac{(\alpha+\beta)\pi}{2}\right] + \kappa_3 \left(\frac{\omega}{\omega_0}\right)^\beta \sin\left(\frac{\beta\pi}{2}\right)}{\left(\frac{\omega}{\omega_0}\right)^{\alpha+\beta} \cos\left[\frac{(\alpha+\beta)\pi}{2}\right] + \kappa_3 \left(\frac{\omega}{\omega_0}\right)^\beta \cos\left(\frac{\beta\pi}{2}\right) + \kappa_2} \right) \quad (2.46b)$$

The half-power (-3 dB) frequency (ω_h), defined as the frequency where there is a 0.707 drop of the passband gain, is calculated solving the equation given in (2.47). Also, the stopband attenuation gradient of the fractional-order high-pass filter of order $\alpha + \beta$ is equal to $6 \cdot (\alpha + \beta)$ dB/oct.

$$\begin{aligned} & \left(\frac{\omega_h}{\omega_0}\right)^{2(\alpha+\beta)} - 2\kappa_3 \left(\frac{\omega_h}{\omega_0}\right)^{\alpha+2\beta} \cos\left(\frac{\alpha\pi}{2}\right) - 2\kappa_2 \left(\frac{\omega_h}{\omega_0}\right)^{\alpha+\beta} \cos\left[\frac{(\alpha+\beta)\pi}{2}\right] \\ & - \kappa_3^2 \left(\frac{\omega_h}{\omega_0}\right)^{2\beta} - 2\kappa_2 \kappa_3 \left(\frac{\omega_h}{\omega_0}\right)^\beta \cos\left(\frac{\beta\pi}{2}\right) - \kappa_2^2 = 0 \end{aligned} \quad (2.47)$$

2.5.3 Fractional-Order Band-Pass Filter (FBPF)

The transfer function of a FBPF is

$$H(s) = \kappa_1 \frac{\kappa_3 (\tau s)^\beta}{(\tau s)^{\alpha+\beta} + \kappa_3 (\tau s)^\beta + \kappa_2} \quad (2.48)$$

Using (2.48), the magnitude and phase response is expressed through the following equations:

$$|H(j\omega)| = \frac{\kappa_1 \kappa_3 \left(\frac{\omega}{\omega_0}\right)^\beta}{\sqrt{\left(\frac{\omega}{\omega_0}\right)^{2(\alpha+\beta)} + \kappa_3^2 \left(\frac{\omega}{\omega_0}\right)^{2\beta} + \kappa_2^2 + 2\kappa_3 \left(\frac{\omega}{\omega_0}\right)^{\alpha+2\beta} \cos\left(\frac{\alpha\pi}{2}\right) + 2\kappa_2 \left(\frac{\omega}{\omega_0}\right)^{\alpha+\beta} \cos\left[\frac{(\alpha+\beta)\pi}{2}\right] + 2\kappa_2 \kappa_3 \left(\frac{\omega}{\omega_0}\right)^\beta \cos\left(\frac{\beta\pi}{2}\right)}} \quad (2.49a)$$

$$\angle H(j\omega) = \angle \kappa_1 \kappa_3 + \frac{\beta\pi}{2} - \tan^{-1} \left(\frac{\left(\frac{\omega}{\omega_0}\right)^{\alpha+\beta} \sin\left[\frac{(\alpha+\beta)\pi}{2}\right] + \kappa_3 \left(\frac{\omega}{\omega_0}\right)^\beta \sin\left(\frac{\beta\pi}{2}\right)}{\left(\frac{\omega}{\omega_0}\right)^{\alpha+\beta} \cos\left[\frac{(\alpha+\beta)\pi}{2}\right] + \kappa_3 \left(\frac{\omega}{\omega_0}\right)^\beta \cos\left(\frac{\beta\pi}{2}\right) + \kappa_2} \right) \quad (2.49b)$$

The peak frequency (ω_p) is calculated solving the equation given in (2.49), which is derived from (2.48) with the condition: $\frac{d}{d\omega} |H(j\omega)|_{\omega=\omega_p} = 0$.

$$\alpha \left(\frac{\omega_p}{\omega_0}\right)^{2(\alpha+\beta)} + \alpha \kappa_3 \left(\frac{\omega_p}{\omega_0}\right)^{\alpha+2\beta} \cos\left(\frac{\alpha\pi}{2}\right) + (\alpha - \beta) \kappa_2 \left(\frac{\omega_p}{\omega_0}\right)^{\alpha+\beta} \cos\left[\frac{(\alpha+\beta)\pi}{2}\right] - \beta \kappa_2 \kappa_3 \left(\frac{\omega_p}{\omega_0}\right)^\beta \cos\left(\frac{\beta\pi}{2}\right) - \beta \kappa_2^2 = 0 \quad (2.50)$$

The quality factor of the filter is calculated according to the formula:

$$Q = \frac{\omega_p}{\omega_{h2} - \omega_{h1}} \quad (2.51)$$

where ω_{h2} and ω_{h1} are the upper and lower half-power frequencies, respectively.

Also, the stopband attenuation gradient at the upper frequencies is $-6 \cdot \alpha$ dB/oct, while for the lower frequencies is $+6 \cdot \beta$ dB/oct, offering the capability of realizing a FBPF with the stopband attenuation being varied in both frequency regions.

2.5.4 Fractional-Order Band-Stop Filter (FBSF)

A FBSF has the transfer function given by (2.52).

$$H(s) = \kappa_1 \frac{(\tau s)^{\alpha+\beta} + \kappa_2}{(\tau s)^{\alpha+\beta} + \kappa_3 (\tau s)^\beta + \kappa_2} \quad (2.52)$$

$$|H(j\omega)| = \kappa_1 \sqrt{\frac{\left(\frac{\omega}{\omega_0}\right)^{2(\alpha+\beta)} + 2\kappa_2\left(\frac{\omega}{\omega_0}\right)^{\alpha+\beta} \cos\left[\frac{(\alpha+\beta)\pi}{2}\right] + \kappa_2^2}{\left(\frac{\omega}{\omega_0}\right)^{2(\alpha+\beta)} + \kappa_3^2\left(\frac{\omega}{\omega_0}\right)^{2\beta} + \kappa_2^2 + 2\kappa_3\left(\frac{\omega}{\omega_0}\right)^{\alpha+2\beta} \cos\left(\frac{\alpha\pi}{2}\right) + 2\kappa_2\left(\frac{\omega}{\omega_0}\right)^{\alpha+\beta} \cos\left[\frac{(\alpha+\beta)\pi}{2}\right] + 2\kappa_2\kappa_3\left(\frac{\omega}{\omega_0}\right)^\beta \cos\left(\frac{\beta\pi}{2}\right)}} \quad (2.53a)$$

$$\begin{aligned} \angle H(j\omega) = \angle \kappa_1 - \tan^{-1} & \left(\frac{\left(\frac{\omega}{\omega_0}\right)^{\alpha+\beta} \sin\left[\frac{(\alpha+\beta)\pi}{2}\right]}{\left(\frac{\omega}{\omega_0}\right)^{\alpha+\beta} \cos\left[\frac{(\alpha+\beta)\pi}{2}\right] + \kappa_2} \right) \\ & - \tan^{-1} \left(\frac{\left(\frac{\omega}{\omega_0}\right)^{\alpha+\beta} \sin\left[\frac{(\alpha+\beta)\pi}{2}\right] + \kappa_3\left(\frac{\omega}{\omega_0}\right)^\beta \sin\left(\frac{\beta\pi}{2}\right)}{\left(\frac{\omega}{\omega_0}\right)^{\alpha+\beta} \cos\left[\frac{(\alpha+\beta)\pi}{2}\right] + \kappa_3\left(\frac{\omega}{\omega_0}\right)^\beta \cos\left(\frac{\beta\pi}{2}\right) + \kappa_2} \right) \end{aligned} \quad (2.53b)$$

The magnitude response and phase response are given by (2.53), while the peak frequency (ω_p) is calculated from (2.53) under the condition $\frac{d}{d\omega} |H(j\omega)|_{\omega=\omega_p} = 0$. The half-power (-3 dB) frequencies (ω_h) are calculated using the fact that at these frequencies there is a 0.707 drop of the passband gain which is equal to κ_1 . The quality factor of the filter is calculated from (2.51).

2.5.5 Design Equations for Generalized Filters of Order $\alpha + \beta$

Utilizing the second-order expression of the CFE given by (2.1) and substituting into (2.42), (2.45), (2.48), and (2.52) the derived transfer function for FLPF, FHPF, FBPF, and FBSF is the following:

$$H_{\alpha+\beta}(s) = \frac{\kappa_1}{D_4} \cdot \frac{N_4 s^4 + N_3 \frac{1}{\tau} s^3 + N_2 \frac{1}{\tau^2} s^2 + N_1 \frac{1}{\tau^3} s + N_0 \frac{1}{\tau^4}}{\left[s^4 + \left(\frac{D_3}{D_4}\right) \frac{1}{\tau} s^3 + \left(\frac{D_2}{D_4}\right) \frac{1}{\tau^2} s^2 + \left(\frac{D_1}{D_4}\right) \frac{1}{\tau} s + \frac{D_0}{D_4 \tau^4} \right]} \quad (2.54)$$

where the coefficients of the denominator D_i and nominator N_i ($i = 1, \dots, 4$) have been defined in (2.55), and Table 2.9, respectively.

$$\begin{aligned} D_0 &\equiv \alpha_2 b_2 + \kappa_3 \alpha_0 b_2 + \kappa_2 \alpha_0 b_0 \\ D_1 &\equiv \alpha_1 b_2 + \alpha_2 b_1 + \kappa_3 \alpha_1 b_2 + \kappa_3 \alpha_0 b_1 + \kappa_2 \alpha_1 b_0 + \kappa_2 \alpha_0 b_1 \\ D_2 &\equiv \alpha_0 b_2 + \alpha_0 b_1 + \alpha_2 b_0 + \kappa_3 \alpha_2 b_2 + \kappa_3 \alpha_1 b_1 + \kappa_3 \alpha_0 b_0 \\ &\quad + \kappa_2 \alpha_2 b_0 + \kappa_2 \alpha_1 b_1 + \kappa_2 \alpha_0 b_2 \\ D_3 &\equiv \alpha_0 b_1 + \alpha_1 b_0 + \kappa_3 \alpha_2 b_1 + \kappa_3 \alpha_1 b_0 + \kappa_2 \alpha_2 b_1 + \kappa_2 \alpha_1 b_2 \\ D_4 &\equiv \alpha_0 b_0 + \kappa_3 \alpha_2 b_0 + \kappa_2 \alpha_2 b_2 \end{aligned} \quad (2.55)$$

Table 2.9 Values of the coefficients of the nominator N_i ($i = 1, \dots, 4$) in (2.54) for realizing FLPF, FHPF, FBPF, and FBSF of order $\alpha + \beta$

Filter	N_4	N_3	N_2	N_1	N_0
FLPF	$\alpha_2 b_2$	$\alpha_2 b_1 + \alpha_1 b_2$	$\alpha_2 b_0 + \alpha_1 b_1 + \alpha_0 b_2$	$\alpha_1 b_0 + \alpha_0 b_1$	$\alpha_0 b_0$
FHPF	$\alpha_0 b_0$	$\alpha_1 b_0 + \alpha_0 b_1$	$\alpha_0 b_2 + \alpha_1 b_1 + \alpha_2 b_0$	$\alpha_1 b_2 + \alpha_2 b_1$	$\alpha_2 b_2$
FBPF	$\kappa_3 \alpha_2 b_0$	$\kappa_3 (\alpha_2 b_1 + \alpha_1 b_0)$	$\kappa_3 (\alpha_2 b_2 + \alpha_1 b_1 + \alpha_0 b_0)$	$\kappa_3 (\alpha_1 b_2 + \alpha_0 b_1)$	$\kappa_3 \alpha_0 b_2$
FBSF	$\alpha_2 b_0 + \kappa_2 \alpha_2 b_0$	$\alpha_0 b_1 + \alpha_1 b_0$ $+ \kappa_2 \alpha_2 b_1 + \kappa_2 \alpha_1 b_2$	$\alpha_0 b_2 + \alpha_1 b_1$ $+ \alpha_2 b_0 + \kappa_2 \alpha_2 b_0$ $+ \kappa_2 \alpha_1 b_1 + \kappa_2 \alpha_0 b_2$	$\alpha_1 b_2 + \alpha_2 b_1$ $+ \kappa_2 \alpha_1 b_0 + \kappa_2 \alpha_0 b_1$	$\alpha_2 b_2 + \kappa_2 \alpha_0 b_0$

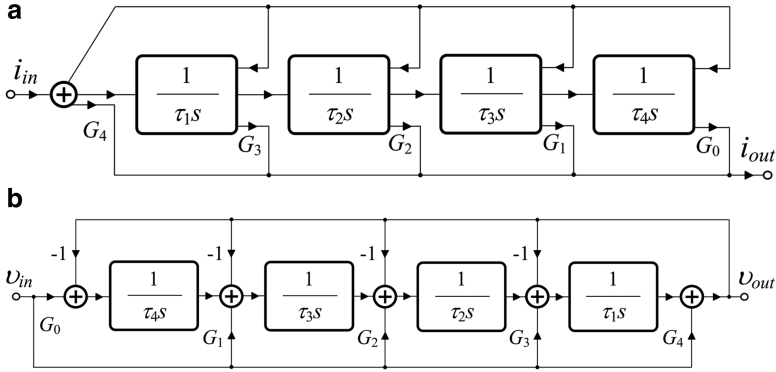


Fig. 2.4 FBD for realizing fractional FLPF, FHPF, FBPF, and FBSF of order $\alpha + \beta$ using (a) current mode topology, (b) voltage mode topology

Comparing the transfer functions of the FLPF and FHPF, it is easily derived that the numerator in $H_{HP}(s)$ is derived through the substitution $\alpha_0 \leftrightarrow \alpha_2$ and $b_0 \leftrightarrow b_2$ in the numerator of $H_{LP}(s)$. In a similar way, the numerator in $H_{BP}(s)$ is derived through the substitution $b_0 \leftrightarrow b_2$ in the numerator of $H_{LP}(s)$ multiplied by the factor κ_3 . Inspecting the transfer function given in (2.54), it is easily concluded that all of them were expressed by the same form, and consequently they could be realized by the same topology just by changing the coefficient values.

A suitable solution for this purpose is depicted in Fig. 2.4, where a typical FBD of a FLF topology and an IFLF topology are given in Fig. 2.4a and Fig. 2.4b, respectively. The realized transfer function is that given in (2.56).

Comparing the coefficients of (2.54) with those in (2.56), it is derived that the design equations about the time-constants τ_j ($j = 1, \dots, 4$) of all the filters and the corresponding design equation for the scaling factor G_i ($i = 0, \dots, 4$) are given in (2.57) and (2.58), respectively.

$$H(s) = \frac{G_4 s^4 + \frac{G_3}{\tau_1} s^3 + \frac{G_2}{\tau_1 \tau_2} s^2 + \frac{G_1}{\tau_1 \tau_2 \tau_3} s + \frac{G_0}{\tau_1 \tau_2 \tau_3 \tau_4}}{s^4 + \frac{1}{\tau_1} s^3 + \frac{1}{\tau_1 \tau_2} s^2 + \frac{1}{\tau_1 \tau_2 \tau_3} s + \frac{1}{\tau_1 \tau_2 \tau_3 \tau_4}} \quad (2.56)$$

$$\tau_j = \frac{D_{5-j}}{D_{4-j}} \cdot \tau \quad (2.57)$$

$$G_i = \kappa_1 \frac{N_i}{D_i} \quad (2.58)$$

where the values of the coefficients N_i are those given in Table 2.9, and depend on the desired filter function.

2.6 Fractional-Order Filters of Order $n + \alpha$

The procedure for realizing a high-order fractional filter of order $n + \alpha$ will be studied, where n is the integer-order of the filter and corresponds to values $n \geq 2$, and α is the order of the fractional part of the filter where $(0 < \alpha < 1)$. Such kind of filters exhibits stopband attenuation equal to $-6 \cdot (n + \alpha)$ dB/oct. The attenuation offered in the case of integer-order filters of order n , which is $-6 \cdot n$ dB/oct. Thus, low-pass, high-pass, band-pass, and band-stop filters of order $n + \alpha$ are presented, and two different design procedures are followed in order to realize these kinds of filters. According to [8, 9], the realization of a fractional-order filter of order $n + \alpha$ with Butterworth characteristics could be performed through the utilization of the polynomial ratio given by (2.59)

$$H_{n+\alpha}(s) = \frac{H_{1+\alpha}(s)}{B_{n-1}(s)} \quad (2.59)$$

where $H_{1+\alpha}(s)$ is the transfer function given by (2.23), (2.28), (2.32), (2.36) for realizing a FLPF, FHPF, FBPF, and FBSF, respectively, and $B_{n-1}(s)$ is the corresponding Butterworth polynomial of order $n-1$.

Some of these polynomials are the following:

$$\begin{aligned} B_1(s) &= s + 1 \\ B_2(s) &= s^2 + \sqrt{2}s + 1 \\ B_3(s) &= (s + 1)(s^2 + s + 1) \\ B_4(s) &= (s^2 + 0.7654s + 1)(s^2 + 1.8478s + 1) \\ B_5(s) &= (s + 1)(s^2 + 0.618s + 1)(s^2 + 1.618s + 1) \end{aligned} \quad (2.60)$$

The gain response of the $(n + \alpha)$ filter is that given in (2.61)

$$|H_{n+\alpha}(\omega)| = \frac{1}{\sqrt{\left[1 + \left(\frac{\omega}{\omega_o}\right)^{2(1+\alpha)}\right] \cdot \left[1 + \left(\frac{\omega}{\omega_o}\right)^{2(n-1)}\right]}} \quad (2.61)$$

Performing a routine algebraic procedure it is derived from (2.61) that

$$\left(\frac{\omega_h}{\omega_o}\right)^{2(n+\alpha)} + \left(\frac{\omega_h}{\omega_o}\right)^{2(n-1)} + \left(\frac{\omega_h}{\omega_o}\right)^{2(1+\alpha)} - 1 = 0 \quad (2.62)$$

2.6.1 Design Equations for Generalized Filters of Order $n + \alpha$

Utilizing the second-order expression of the CFE given by (2.1) and using the expressions in (2.39), then from (2.59) it is obtained that the general forms of the transfer functions will be the following:

$$H_{n+\alpha}^{\text{LP}}(s) = \frac{\kappa_1}{\alpha_0} \cdot \frac{\alpha_2 \frac{1}{\tau^n} s^2 + \alpha_1 \frac{1}{\tau^{n+1}} s + \alpha_0 \frac{1}{\tau^{n+2}}}{[s^{n+2} + \beta_{n+1} \frac{1}{\tau} s^{n+1} + \dots + \beta_1 \frac{1}{\tau^{n+1}} s + \beta_0 \frac{1}{\tau^{n+2}}]} \quad (2.63a)$$

$$H_{n+\alpha}^{\text{HP}}(s) = \frac{\kappa_1}{\alpha_0} \cdot \frac{\alpha_0 \frac{1}{\tau^{n-1}} s^3 + \alpha_1 \frac{1}{\tau^n} s^2 + \alpha_2 \frac{1}{\tau^{n+1}} s}{[s^{n+2} + \beta_{n+1} \frac{1}{\tau} s^{n+1} + \dots + \beta_1 \frac{1}{\tau^{n+1}} s + \beta_0 \frac{1}{\tau^{n+2}}]} \quad (2.63b)$$

$$H_{n+\alpha}^{\text{BP}}(s) = \frac{\kappa_1 \kappa_3}{\alpha_0} \cdot \frac{\alpha_0 \frac{1}{\tau^n} s^2 + \alpha_1 \frac{1}{\tau^{n+1}} s + \alpha_2 \frac{1}{\tau^{n+2}}}{[s^{n+2} + \beta_{n+1} \frac{1}{\tau} s^{n+1} + \dots + \beta_1 \frac{1}{\tau^{n+1}} s + \beta_0 \frac{1}{\tau^{n+2}}]} \quad (2.63c)$$

$$H_{n+\alpha}^{\text{BS}}(s) = \frac{\kappa_1}{\alpha_0} \cdot \frac{\alpha_0 \frac{1}{\tau^{n-1}} s^3 + (\alpha_1 + \kappa_2 \alpha_2) \frac{1}{\tau^n} s^2 + (\alpha_2 + \kappa_2 \alpha_1) \frac{1}{\tau^{n+1}} s + \kappa_2 \alpha_0 \frac{1}{\tau^{n+2}}}{[s^{n+2} + \beta_{n+1} \frac{1}{\tau} s^{n+1} + \dots + \beta_1 \frac{1}{\tau^{n+1}} s + \beta_0 \frac{1}{\tau^{n+2}}]} \quad (2.63d)$$

where β_κ ($\kappa = 0, 1, \dots, n+1$) is a function of α_i , which is a result of multiplication of the denominator of $H_{1+\alpha}(s)$ in (2.39) and the coefficients of $B_{n-1}(s)$.

Inspecting the transfer functions of FLPF, FHPF, FBPF, and FBSF, it is concluded that all of them have the same form. Consequently they could be realized by the same topology just by changing the coefficient values. A suitable solution for this purpose is depicted in Fig. 2.5, where a typical FBD of a FLF topology and an IFLF topology are given in Fig. 2.5a and Fig. 2.5b, respectively.

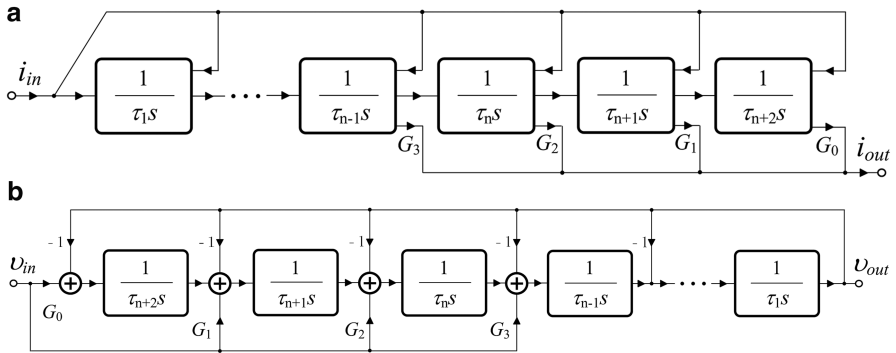


Fig. 2.5 FBD for realizing FLPF, FHPF, FBPF, and FBSF of order $n + \alpha$ using $B_{n-1}(s)$ Butterworth polynomials (a) current mode topology, (b) voltage mode topology

Table 2.10 Values of scaling factors G_i for realizing FLPF, FHFPF, FBPF, and FBSF in Fig. 2.5 of order $n + \alpha$

Filter	G_3	G_2	G_1	G_0
FLPF	0	$\frac{\kappa_1 \alpha_2}{\alpha_0 \beta_2}$	$\frac{\kappa_1 \alpha_1}{\alpha_0 \beta_1}$	$\frac{\kappa_1}{\beta_0}$
FHFPF	κ_1	$\frac{\kappa_1 \alpha_1}{\alpha_0 \beta_2}$	$\frac{\kappa_1 \alpha_2}{\alpha_0 \beta_1}$	0
FBPF	0	$\frac{\kappa_1 \kappa_3}{\beta_2}$	$\frac{\kappa_1 \kappa_3 \alpha_1}{\alpha_0 \beta_1}$	$\frac{\kappa_1 \kappa_3 \alpha_2}{\alpha_0 \beta_0}$
FBSF	$\frac{\kappa_1}{\beta_3}$	$\frac{\kappa_1 (\alpha_1 + \kappa_2 \alpha_2)}{\alpha_0 \beta_2}$	$\frac{\kappa_1 (\alpha_2 + \kappa_2 \alpha_1)}{\alpha_0 \beta_1}$	$\frac{\kappa_1 \kappa_2}{\beta_0}$

The realized transfer function is that given in (2.64). Comparing the coefficients of (2.64) with these of FLPF, FHFPF, and FBSF it is derived that, under the assumption that $\beta_{n+2} = 1$, the design equations about the time-constants τ_j ($j = 1, 2, \dots, n + 2$) of all the filters are given in (2.65). The corresponding design equations for the scaling factors G_i ($i = 0, \dots, 3$) are summarized in Table 2.10.

$$H_{n+\alpha}(s) = \frac{\frac{G_3}{\tau_1 \cdots \tau_{n-1}} s^3 + \frac{G_2}{\tau_1 \cdots \tau_n} s^2 + \frac{G_1}{\tau_1 \cdots \tau_{n+1}} s + \frac{G_0}{\tau_1 \cdots \tau_{n+2}}}{s^{n+2} + \frac{1}{\tau_1} s^{n+1} + \frac{1}{\tau_1 \tau_2} s^n + \cdots + \frac{1}{\tau_1 \tau_2 \cdots \tau_{n+2}}} \quad (2.64)$$

$$\tau_j = \frac{\beta_{n+3-j}}{\beta_{n+2-j}} \cdot \tau \quad (2.65)$$

In case time-constants are to be expressed as a function of the desired half-power frequency (ω_h), then the above equation could be modified as

$$\tau_j = \frac{\beta_{n+3-j}}{\beta_{n+2-j}} \cdot \left(\frac{\omega_h}{\omega_0} \right) \quad (2.66)$$

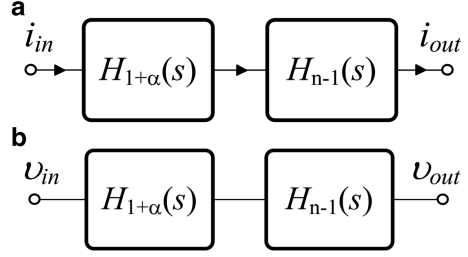
Taking into account the fact that all the aforementioned procedure is somewhat complicated due to the algebraic calculation of the coefficients β_κ especially in case of high-order filters. Thus, an alternative solution for realizing high-order fractional filters is through the cascade connection of $1 + \alpha$ and $n-1$ order filters, which is expressed in the following equation as

$$H_{n+\alpha}(s) = H_{1+\alpha}(s) \cdot H_{n-1}(s) \quad (2.67)$$

where $H_{n-1}(s) = 1/B_{n-1}(s)$ is the transfer function of the $n-1$ order Butterworth filter, the derivation of which is a trivial procedure.

A suitable topology for this purpose is depicted in Fig. 2.6, from which is obvious that having available the topology of an $1 + \alpha$ order filter and an integer-order filter of an $n-1$ order filter, it is readily obtained that one additional step is required for realizing an $n + \alpha$ order fractional filter.

Fig. 2.6 FBD for realizing FLPF, FHPF, FBPF, and FBSF of order $n + \alpha$ using $(n-1)$ Butterworth filter (a) current-mode topology, (b) voltage-mode topology



The gain response of the filter is that given in (2.68), where ω_{o1} and ω_{o2} are the -3 dB frequencies of the $1 + \alpha$ and $n-1$ order filters, respectively.

$$|H_{n+\alpha}(\omega)| = \frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_{o1}}\right)^{2(1+\alpha)}}} \cdot \frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_{o2}}\right)^{2(n-1)}}} \quad (2.68)$$

Performing a routine algebraic procedure it is derived from (2.68) that

$$\left(\frac{\omega_h}{\omega_{o1}}\right)^{2(1+\alpha)} \cdot \left(\frac{\omega_h}{\omega_{o2}}\right)^{2(n-1)} + \left(\frac{\omega_h}{\omega_{o2}}\right)^{2(n-1)} + \left(\frac{\omega_h}{\omega_{o1}}\right)^{2(1+\alpha)} - 1 = 0 \quad (2.69)$$

Taking into account that ω_{o1} and ω_{o2} are selected to be the same and equal to ω_o , then (2.69) is simplified to (2.62).

2.7 Summary

Fractional-order differentiation/integrator blocks and fractional order generalized filters have been realized in this section, through the utilization of the CFE, which enables the opportunity of realizing all the aforementioned topologies through integer-order counterparts which is a trivial procedure. Each category has been performed by using a general form, offering the capability of implementing various types of transfer functions without modifying their core, which is very important from the design flexibility point of view. Having available all this procedure, then it is easy to realize every kind of the aforementioned fractional-order circuits using the suitable or desirable way of implementing the integer-order counterparts (i.e., integrators), which thereafter depends on the designer choice. They could be realized either using a current mode procedure using for example current-mirrors, or a voltage mode using OTAs. Some of these implementations will be described and realized in detail in the next sessions.

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