Chapter 2
Uncertainty

2.1 Introduction

Every measurement has the purpose of obtaining the numerical value of a physical quantity. However, this statement that seems to be so clear and obvious requires careful scrutiny. In the first place, the quantity to be measured, called the *measurand*, is not always perfectly defined. In order for the definition to be exhaustive, we should specify all the conditions that could influence the measurand's value. In the acoustic case, for instance:

(1) Atmospheric conditions (atmospheric pressure, temperature, humidity, wind velocity and direction, altitude, presence of particles),
(2) Geometry and materials of the environment (presence of acoustic barriers or screens, sound absorbing or reflecting surfaces, etc.),
(3) Operational conditions of the source (for instance, temperature, load conditions, power supply voltage),
(4) Opportunity (time, kind of process currently taking place, time interval since the beginning of the process, time interval since last maintenance),
(5) Presence of other simultaneous sources,
(6) Location of measuring instrument and its operator.[1]

In the second place, even if all influencing conditions could be identified and specified, it would be physically impossible to warrant they are satisfied during the measurement; among other reasons, because it would be impossible to know the exact values of the associated parameters. This means that in a real-life situation, there is no single value of the measurand.

However, we can conceive the existence of a hypothetical *true value*, i.e. the value that would have the measurand if it would be measured in ideal conditions by

[1]The measuring instrument itself introduces reflection and diffraction phenomena that may change the sound field, which in turn changes the measurand. The same is true for the operator conducting the measurement.

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means of an ideal instrument. Because of the previous comments, it is not possible to find exactly the true value.[2] We can only expect to get an estimate along with some expression of *uncertainty*, i.e. a way to quantify how far can be the true value from our estimate.

The way to express the uncertainty has changed over time. At the beginning, it was assumed that it was possible to provide some interval around the estimated value where the true value lies. Nowadays, since the publication by the ISO and other organizations of the Guide for the Expression of Uncertainty in Measurements, known by its partial acronym GUM (ISO-GUM 1993; JCGM 2008; Taylor and Kuyayt 1994), the paradigm has shifted towards a statistical expression, specifying the uncertainty as an interval where we can find the true value with a given probability, for instance 95 %.

2.2 Resolution, Precision, Accuracy

Every measurement instrument has a limit for the minimum difference between two readings that are not identical called *resolution*. In an instrument with an analog display (which are now rare), it is given by the distance between two consecutive scale divisions. In a digital instrument, it corresponds to a unit of the least significant digit. In most sound level meters the resolution is 0.1 dB, but some of them allow to configure the resolution as 0.01 dB (as we shall see, this can make sense for equivalent level measurements).

The resolution of a measurement procedure may be better than that of the instrument used to conduct the basic measurements. An example is when the outcome of the measurement is the average of several readings. For instance, if the measurement procedure involves averaging 10 readings of an instrument with a resolution of 0.1 dB, the measurement procedure will have a resolution of 0.01 dB.

When several measurements of the same measurand are conducted in identical conditions[3] (i.e. several *realizations of the measurand* are measured), the readings are not always identical, but they present some statistical dispersion instead. This means that it is more correct to consider the reading as a random variable instead of a single value that is a function of the true value.[4] We define the *precision* of an instrument as the statistical dispersion of readings. It can be expressed as the

[2]According to the GUM, there may be more than one true values. This is because when the specification of the measurand is not exhaustive, there are many different values that agree with the definition.

[3]*Identical conditions* means to control as far as possible all possible variables; for instance, the same physical system, the same instrument, the same operator, the same environmental conditions, and the minimum reasonable time between measurements to prevent drifts in the instrument's characteristics as well as those of the physical system where the measurand manifests itself.

[4]We say "that is a function of" instead of "equal to" because the instrument could be uncalibrated or have some nonlinearity that should be corrected.

standard deviation σ of the readings or as half the width of a confidence interval, where lies a given percentage of the readings (typically, 95 %; for normal distribution, it equals 2σ).

A more controversial concept is that of *accuracy*, defined, according to the GUM, as the "degree of closeness of the agreement between the result of a measurement and a true value of the measurand." Pursuant to the GUM, it is a qualitative concept, so one should not fall into the temptation of assigning a numerical value. However, the GUM itself and other metrological documents use frequently the term *accuracy* as a quantitative concept. For instance, in subclause 3.1.3, the GUM says:

> In practice, the required specification or definition of the measurand is dictated by the required accuracy of measurement.

We can ask how it is possible to specify the "required accuracy" in a non-quantitative form, since a qualitative expression would make it impossible to check whether the specification has been satisfied. In fact, an example in the same subclause reads:

> If the length of a nominally one-metre long steel bar is to be determined to micrometre accuracy, its specification should include the temperature and pressure at which the length is defined (…).

Despite what the GUM asserts, it would be possible to specify the accuracy quantitatively as a dispersion, for instance the standard deviation of the difference between the observed value and the true value (or an equivalent confidence interval).

2.3 Measurement Method and Procedure

Every measurement is carried out following some *measurement procedur*e that implements a *measurement method* and arrives at a result directly or through a *measurement model*.

The measurement method describes in a general way the structure of the measuring process. In its simplest version, it reduces to reading the instrument's display. In more elaborate cases, it can involve the use of several transducers, the processing of the magnitudes they generate and the application of corrections, formulas, etc. The measurement method rests on a *measurement principle*, i.e. one or more physical phenomena on which the measurement is based (for instance, the electroacoustic transduction in a condenser microphone) and is embodied in a *measurement procedure* where all the steps of the measurement are described in detail, covering the selection of instruments and samples, a survey of environmental variables or any other circumstance that could influence the measurement outcome, the specification of the measurement conditions, precautions and recommendations to observe during the measurement, as well as formulas and calculations, and the structure and content of the *measurement report*.

For typical and frequent measurements, the measurement procedure is usually stated in national or international standards. In more specific cases, it may be part of internal laboratory documents, or part of the measurement report itself.

Example: The measurement of the acoustic power radiated by a machine in a semi-anechoic chamber is based on the principle of electroacoustic transduction of pressure into electric voltage and in the simple relationship that is held between sound intensity and sound pressure in a free field. The measurement method consists in taking pressure measurements at several points of an imaginary surface around the machine, converting sound pressure into sound intensity, computing a weighted sum of the intensities, and multiplying by the total area of the surface. The measurement procedure indicates the characteristics of the measurement environment, type of instrument to use (for instance, one-third octave spectrum analyzer), environmental conditions, operational conditions of the machine (for instance installation, supply voltage, warming time), coordinates of the measuring points, type, number and duration of measurements, microphone orientation, formulas to use, atmospheric condition corrections, effect of background noise, and uncertainty calculation.

2.3.1 Direct and Indirect Measurement Methods

Measurement methods can be classified into *direct* and *indirect*. Direct methods consist in just reading and recording the instrument's display. An example is the measurement of the equivalent level with an integrating sound level meter.

Indirect methods require some transformation of the direct data in order to arrive at the desired result. For instance, starting from the measurement of the sound pressure level in some point of a plane progressive wave (free field), it is possible to estimate the particle velocity by means of the equation.

$$U_{\mathrm{ef}} = \frac{1}{\rho_{\mathrm{o}} c} P_{\mathrm{ref}} 10^{\frac{L_{\mathrm{p}}}{20}}, \tag{2.1}$$

where ρ_{o} is the air density at equilibrium and c is the velocity of sound.

2.4 Measurement Model

Except in very simple cases, in general, the measurement result is not the simple reading of the instrument's display but the outcome of a series of operations that often condense into an explicit or implicit functional relationship between two or more variables, or even a software algorithm.

Calling the measurand Y and the quantities obtained by direct reading or comparison X_k, we can write the explicit form

$$Y = f(X_1, \ldots, X_N), \tag{2.2}$$

or the implicit form,

$$h(Y, X_1, \ldots, X_N) = 0. \tag{2.3}$$

In some cases, it will be possible to directly solve for Y, arriving at an explicit form such as (2.2). In other cases, a numerical solution can be attempted, for instance, applying the Newton–Raphson algorithm.

Example 1 If L_1, …, L_{10} are the octave band levels corresponding to frequencies f_1 = 31.5 Hz, …, f_{10} = 16 kHz, then the A-weighted sound level can be computed by means of the formula.

$$L_{\mathrm{pA}} = 10 \log \sum_{k=1}^{10} 10^{\frac{L_k + A_k}{10}}, \tag{2.4}$$

where A_k is the A-weighting correction for the band centered at f_k.

Example 2 Let $L_{\mathrm{Aeq},\ Tn}$ be the A-weighted equivalent level of a given noise for N intervals of durations T_n covering the total duration T of a desired interval (for instance, a working day). Then we can compute the global equivalent level as

$$L_{\mathrm{Aeq},T} = 10 \log\left(\frac{1}{T}\sum_{n=1}^{N} T_n 10^{\frac{L_{\mathrm{Aeq},T_n}}{10}}\right). \tag{2.5}$$

Example 3 In order to measure the sound transmission loss *STL* of a partition in a sound transmission facility, one-third octave levels are measured at the source room (generated by a proper sound source) as well as at the receiving room (the result of the transmission through the partition and flanking transmission). Reverberation time at each one-third octave band is also measured. If we call these values L_{k1}, L_{k2}, and T_k, we can obtain the sound transmission class for each one-third octave band k as

$$\mathrm{STL}_k = L_{k1} - L_{k2} + 10 \log_{10} \frac{T_k S_{12}}{0.161 V_2}, \tag{2.6}$$

where S_{12} is the partition area and V_2 is the receiver room volume.

Example 4 Suppose we want to measure the sound pressure level at some location and at some environmental reference conditions defined as a temperature of 23 °C, a relative humidity of 50 %, and an atmospheric pressure of 1013.25 hPa, but the actual measurement conditions are 27 °C, 75 %, and 1005.3 hPa. Measurement is conducted using a Brüel and Kjær microphone Type 4189.

It will be necessary to apply some correction to refer the actual measurement to the desired conditions. Since the variations respect the reference conditions are small, their effect is assumed to be linear:

$$L_{\mathrm{p}}(T_1, h_{\mathrm{r}1}, P_{\mathrm{a}1}) = L_{\mathrm{p}}(T_0, h_{\mathrm{r}0}, P_{\mathrm{a}0}) + K_T \Delta T + K_{h_{\mathrm{r}}} \Delta h_{\mathrm{r}} + K_{P_{\mathrm{a}}} \Delta P_{\mathrm{a}} \tag{2.7}$$

For a Brüel and Kjær microphone Type 4189, we have (Brüel and Kjær 1995, 1996):

$$\begin{aligned} K_T &= -0.001\,\mathrm{dB}/{}^\circ\mathrm{C} \\ K_{h_r} &= 0.001\,\mathrm{dB}/\% \\ K_{P_{\mathrm{a}}} &= -0.001\,\mathrm{dB}/\mathrm{hPa} \end{aligned}$$

The effect of the environmental conditions turns out to be 0.02895 dB, so the correction to add to the measured value is −0.02895 dB. In most cases this correction can be ignored. One exception may be when performing calibrations by the reciprocity method (Popescu 2007; Brüel and Kjær 1996).

Example 5 Consider the following method to find the resonant frequency f_{o} of a system. The response Y_k at three frequencies f_k close to the resonant frequency is measured (select the frequencies such that Y_1, $Y_3 < Y_2$). Then a parabolic interpolation[5] is computed and finally the maximum is computed analytically. The associated model is as follows:

$$\begin{aligned} Y_{\max} + K(f_1 - f_{\mathrm{o}})^2 &= Y_1 \\ Y_{\max} + K(f_2 - f_{\mathrm{o}})^2 &= Y_2 \\ Y_{\max} + K(f_3 - f_{\mathrm{o}})^2 &= Y_3 \end{aligned} \tag{2.8}$$

This is a system of three equations with three unknowns, $Y_{\max}$, K, f_{o}, of which only f_{o} is relevant to our problem. We can get rid of $Y_{\max}$ by subtracting the second equation from the first and the third from the first. Then we can remove K by dividing the resulting equations. This yields

$$\frac{(f_1 - f_{\mathrm{o}})^2 - (f_2 - f_{\mathrm{o}})^2}{(f_1 - f_{\mathrm{o}})^2 - (f_3 - f_{\mathrm{o}})^2} = \frac{Y_1 - Y_2}{Y_1 - Y_3}, \tag{2.9}$$

where we have reduced the problem to a single equation with a single unknown, f_{o}. This is an implicit formulation of the measurement model. Clearly, in this case it is easy to solve for f_{o} since it is basically a second-degree equation (which is easily

[5]The resonance curve is not exactly parabolic, but in the neighborhood of its maximum it can be approximated by a parabola. The approximation will be better if the frequencies where the response is measured are close to the resonant frequency.

reduced to a first-degree equation). In more complicated cases it might not exist as an explicit solution, or it could be prohibitively complicated.[6]

2.5 Uncertainty

When measuring any measurand, the measurement result will differ not only from the true value, but also from the results of repeated measurements. This characteristic of any measurement process is known as *uncertainty*. Among the causes of uncertainty we can mention:

(1) Causes attributable to the measurand: An incomplete definition of the measurand (for instance lack of specification of some variable that has a significant effect on the measurand); impossibility to warrant that the measurand's realization meets all specifications (for instance, the definition specifies the room temperature but it is known only approximately).
(2) Causes attributable to the measurement procedure: The model underlying the measurement method is incomplete, insufficient, or inaccurate (for example, there is a correction of the effect of temperature but the temperature coefficient is known only approximately); Finite number of repetitions; presence of interferences or disturbances (for instance, background noise); presence of factors that alter the measurand (such as a reflective surface close to the measurement site); excessive effect of environmental conditions.
(3) Causes attributable to the measurement instrument: Finite resolution, lack of precision, manufacturing tolerance, lack of calibration, inadequate, or insufficient frequency response.
(4) Causes attributable to the operator: Alteration of the measurand due to their presence; parallax errors while reading analog instruments; indecision on what value should be recorded from a digital instrument when the least significant digits fluctuate over time.

These sources of uncertainty should not be confused with gross errors, mistakes, or incorrect application of procedures (Bell 2001), for instance, confusing figures when recording readings, confusing measurands, departing from the procedure without a reason (and ignoring the effects that this could cause). In what follows, we assume that all these defects have been prevented and all measurements are conducted by knowledgeable personnel.

As a general rule, uncertainty is the dispersion that can be expected for the results of different measurements of the measurand in specified conditions. Note that in general, it is not necessary to make any explicit mention of the true value.

[6] An example where there are explicit solutions but numerical methods are preferred is the solution of linear equation systems of high order.

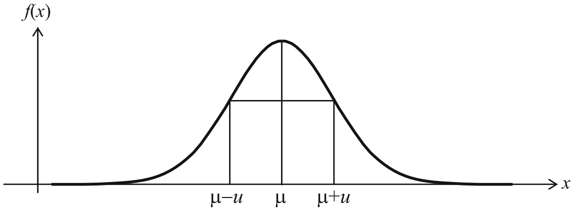


Fig. 2.1 Standard uncertainty example for a normal probability density function

More specifically, we define the *standard uncertainty*, u, as the standard deviation of different measurements of the measurand, i.e.[7]

$$u = \sigma = \sqrt{\int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx}, \tag{2.10}$$

where x is a generic result of the measurement conducted in specified conditions, μ is the mean or expected value of x and $f(x)$ is its probability density function. Figure 2.1 shows an example.

Often, the standard deviation of measurement results, understood as the population[8] standard deviation, is not known. However, it is possible to estimate the population parameter from a *sample*, i.e. a finite number n of measurements. In that case, the standard uncertainty is the sample standard deviation:

$$u = s = \sqrt{\frac{1}{n-1} \sum_{k=1}^{n} (x_k - \bar{x})^2}, \tag{2.11}$$

where x_k is the k-th measured value and $\bar{x}$ is the mean of $\{x_k\}$.

Both ways of getting the standard uncertainty are considered correct. Moreover, as we shall see later, the uncertainty may have several components. This gives rise to a general classification of uncertainty components into two categories, called type A uncertainty and type B uncertainty, described in the next section.

2.5.1 Type A Uncertainty

It is any uncertainty component that is determined empirically by applying statistical methods to the results of several measurements. For instance, suppose that we have measured the following ten sound pressure level values:

[7]Appendix 2 introduces basic statistical concepts. In the Glossary of Appendix 1, there is a summary of definitions of common use in metrology.

[8]The term "population" is used in Statistics to refer to all of the homogeneous elements to which some given statistical parameters apply.

73.6	80.2	76.7	75.5	76.7	75.3	75.4	77.8	77.7	77.7

Applying (2.11), we get a mean level of 76.66 dB and a standard deviation of 1.83 dB. A second ten-value sample of measurements corresponding to the same population yields:

76.6	73.8	76.7	78.0	76.3	77.2	76.7	75.1	76.0	74.4

In this case the mean turns out to be 76.08 dB and the standard deviation, 1.29 dB. As can be seen, the estimated values can be rather different between different samples of the same population.

2.5.2 Type B Uncertainty

It is any component of uncertainty that is obtained by any method other than the statistical treatment of measurement results. For example,

(a) Previous knowledge or estimation by scientific or technical agreement, for instance the uncertainty in physical constants or arising from procedures;
(b) Information provided in standards;
(c) Instrument specifications;
(d) Information provided in calibration certificates;
(e) Software simulations;
(f) Application of physical laws.

Note: Probably the body of knowledge involved in most components of Type B uncertainty has been gathered originally using statistical methods. The difference is that those methods have been applied to general data or research material, not to the measurement results in a specific occasion, which are what Type A uncertainty refers to.

Examples:

(a) The gas constant R appears in the formula of the velocity of sound c. According to CODATA (Mohr et al. 2012), R = 8.314 462 75 J/(mol·K) with a standard uncertainty u = 0.000 000 91 J/(mol·K). This uncertainty must be considered when calculating c.
(b) International Standard ISO 1996 indicates that in the absence of better data, the standard uncertainty in traffic noise measurements is given by $10/\sqrt{n}$ dB, where n is the number of vehicles that have passed by the observer during the measurement.
(c) A sound level meter rated as Class 1 will have, at 1 kHz, a standard uncertainty of 0.2 dB (IEC 61672-1 2002).
(d) The calibration certificate of a sound calibrator indicates a standard uncertainty of 0.15 dB. Therefore, a sound level meter that has been checked with this calibrator includes an uncertainty component of 0.15 dB due to calibration.

(e) If we measure some noise whose spectrum has a known frequency behavior (for instance, pink noise), it is possible to obtain, by software simulations, the uncertainty in the result of an A-weighted measurement if we know the uncertainty of the instrument's responses at each particular frequency. To this end one could generate many A-weighted frequency responses (a method called Montecarlo) and apply each of them to different realizations of the noise. Finally, we can apply statistical analysis to get the standard deviation. This result could be applied later to measurements of similar noises.

(f) If the quantity to be measured can be computed by means of a known mathematical expression as a function of other quantities to be measured directly, it is possible to propagate the uncertainties to the desired result. We shall see later how it is done (Sect. 2.5.4).

2.5.3 Expanded Uncertainty

Standard uncertainty u, as the standard deviation of measurement results, cannot provide by itself a definite idea of how spread can be the possible measurement results. We could ask, for instance, what is the probability that the result lies in a range $\pm u$ around the mean. The answer depends on the type of statistical distribution. If it is a normal distribution, the probability will be 66.3 %. If it is, instead, a uniform distribution, it will be 57.7 %.

To circumvent this problem, we introduce the *expanded uncertainty*, U, so that the interval $[\mu - U, \mu + U]$ covers a specified percentage of the possible measurement results. In other words, the measurement result will lie in that interval with a specified probability p (which may be expressed in %).

This probability, called *coverage probability*, is not normalized, so one is free to specify it according to the specific need. In any case, the adopted value should be specified and sometimes it is included as a subindex: U_p.

The factor k that converts u into U, which depends on the distribution, is called *coverage factor*:

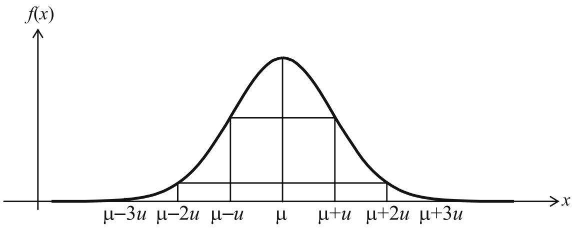
$$U = k\,u. \tag{2.12}$$

In the frequent case, where the distribution is normal (or approximately normal), it is customary to adopt coverage probabilities of 95.4 % and 99.7 %, which correspond, respectively to $2u$ and $3u$. In this case we can write, for example,

$$U_{95} = 2u, \tag{2.13}$$

where, as is common practice, to alleviate notation, we have written U_{95} instead of $U_{95.4}$. Some examples are presented in Fig. 2.2.

Fig. 2.2 Some examples of intervals corresponding to different coverage probabilities for a normal distribution



2.5.4 Combined Standard Uncertainty

We have already seen that measurements often require to measure several intermediate or subsidiary quantities, and combine them according to a given measurement model. We can ask now to what extent the uncertainty of those intermediate results will contribute to the general uncertainty.

To start with, let us analyze the simplest case, in which the model of the measurand is a linear combination of several independent variables:

$$Y = a_1 X_1 + \cdots + a_N X_N. \tag{2.14}$$

In the first place, note that if μ_k are the means of X_k, then the mean of Y, i.e. μ_Y, is obtained by the same linear combination:

$$\mu_Y = a_1 \mu_1 + \cdots + a_N \mu_N. \tag{2.15}$$

It can also be shown that the standard deviation σ_Y is given by

$$\sigma_Y^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} a_i a_j \operatorname{cov}(X_i, X_j), \tag{2.16}$$

where $\operatorname{cov}(X_i, X_j)$ is the covariance between X_i and X_j, given by

$$\operatorname{cov}(X_i, X_j) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x_i - \mu_i)(x_j - \mu_j) f(x_i, x_j) \mathrm{d}x \mathrm{d}y, \tag{2.17}$$

where $f(x_i, x_j)$ is the joint probability density function (see Appendix 1).

We define the *combined standard uncertainty*, u_c, as the total standard deviation given in (2.16).

$$u_c = \sigma_Y. \tag{2.18}$$

In the case of a type A uncertainty, where the covariance is estimated from a finite sample, we shall have

$$u_c = s_Y = \sqrt{\sum_{i=1}^{N}\sum_{j=1}^{N} a_i a_j c_{ij}}, \tag{2.19}$$

where c_{ij} is the sample covariance between x_i and x_j calculated from an n-sized sample of both variables according to

$$c_{ij} = \frac{1}{n-1}\sum_{k=1}^{n}(x_{ik} - \bar{x}_i)(y_{jk} - \bar{y}_j). \tag{2.20}$$

In many cases, the variables can be considered statistically independent, thus the cross-covariances (where $i \neq j$) are 0. Since the covariance between a variable and itself is the variance s^2, Eq. (2.19) may be written as

$$s_Y = \sqrt{\sum_{i=1}^{N} a_i^2 s_i^2}. \tag{2.21}$$

The variance of each variable is the square of the standard deviation, so we can rewrite the preceding equation as

$$u_c = \sqrt{\sum_{i=1}^{N} a_i^2 u_i^2}. \tag{2.22}$$

Let us consider now the general case where the measurement model (i.e., the relationship between the measurand and the intermediate quantities that have been directly measured) is not necessarily linear, as in Eq. (2.2),

$$Y = f(X_1, \ldots, X_N). \tag{2.23}$$

This case may be analyzed in a similar fashion. First, we perform a first-order Taylor expansion of f around the measured values. As is shown in Appendix 8, the sample variance of the measurand is given by

$$s_y^2 = \sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\partial f}{\partial x_i}(\bar{x}_i)\frac{\partial f}{\partial x_j}(\bar{x}_j)c_{ij}, \tag{2.24}$$

where c_{ij} is the covariance between variables x_i and x_j which can be estimated by means of the sample covariance expression

$$c_{ij} = \frac{1}{n-1}\sum_{k=1}^{n}(x_{ik} - \bar{x}_i)(x_{jk} - \bar{x}_j). \tag{2.25}$$

In the case in which the random variables $X_1, \ldots, X_N$ are independent, the covariance between pairs of variables vanishes, so the only terms that are preserved are those with $i = j$:

$$s_y^2 = \sum_{i=1}^{N} \left(\frac{\partial f}{\partial x_i} (\bar{x}_i) \right)^2 s_i^2, \tag{2.26}$$

or, in terms of uncertainties,

$$u_c = \sqrt{\sum_{i=1}^{N} \left(\frac{\partial f}{\partial x_i} (\bar{x}_i) \right)^2 u_i^2}, \tag{2.27}$$

The partial derivatives can be computed analytically or numerically. They can be calculated even in the case of implicit functions (see Appendix 9 for an example).

2.6 Examples

In this section, we analyze several situations that appear frequently in acoustical measurements, either at the field or at the laboratory.

2.6.1 *Relationship Between Uncertainties in Level and Pressure*

Although a microphone generates an electric signal that is proportional to sound pressure, sound level meters calculate and display the sound pressure level L_p. However, sometimes it is necessary to know the sound pressure. It can be computed by means of the inverse formula

$$P_\mathrm{ef} = P_\mathrm{ref} 10^{L_p/20}. \tag{2.28}$$

In order to get the uncertainty in P_ef, it is better to convert to base e exponential:

$$P_\mathrm{ef} = P_\mathrm{ref} e^{\frac{\ln 10}{20} L_\mathrm{p}}. \tag{2.29}$$

Derivation is now easier. Once calculated and replaced into (2.27), we arrive at the following expression:

$$u_{P_\mathrm{ef}} = \frac{\ln 10}{20} P_\mathrm{ef} u_{L_\mathrm{p}}. \tag{2.30}$$

We can see that if the uncertainty in L_p is constant with L_p, then the uncertainty in P_ef will be proportional to sound pressure.

2.6.2 Uncertainty in the Correction of Environmental Conditions

Consider the case of example 4 in Sect. 2.4, where the sound pressure level must be corrected to refer it to different environmental conditions from those that prevailed when conducting the measurement. The measurement model was

$$L_\mathrm{p}(T_1, h_{\mathrm{r}1}, P_{\mathrm{a}1}) = L_\mathrm{p}(T_0, h_{\mathrm{r}0}, P_{\mathrm{a}0}) + K_T\Delta T + K_{h_\mathrm{r}}\Delta h_\mathrm{r} + K_{P_\mathrm{a}}\Delta P_\mathrm{a}. \tag{2.31}$$

The problem is that there is some uncertainty in the determination of the environmental conditions. Suppose that the standard uncertainty in the measurement of the respective atmospheric variables is:

$$u_\mathrm{T} = 0.8\,^\circ\mathrm{C}$$
$$u_{\mathrm{h}_\mathrm{r}} = 3\ \%$$
$$u_{\mathrm{P}_\mathrm{a}} = 3\,\mathrm{hPa}$$

The combined standard uncertainty due to these factors is a type B uncertainty, since it depends on specifications and/or standards fulfilled by the measuring instruments (thermometer, hygrometer and barometer). Its value, calculated by means of Eq. (2.22) under the reasonable assumption that the three variables are independent, is:

$$u_\mathrm{c,amb} = \sqrt{(-0.001 \times 0.8)^2 + (0.001 \times 3)^2 + (-0.001 \times 3)^2}\mathrm{dB} = 0.0043\,\mathrm{dB}.$$

We see that the contribution from the measurement errors in the atmospheric variables to total uncertainty is negligible. This is the reason why in general there is no need of sophisticated instruments to measure these variables.

In order to get the total combined uncertainty, we should add the uncertainty due to the instrument, as we shall see later.

2.6.3 Uncertainty in the Calculation of the Equivalent Level

Suppose that we have measured or estimated theequivalent level L_{Aeq,T_i} in N time intervals T_i and we want to get the total equivalent level by means of the formula

$$L_{\mathrm{Aeq},T} = 10\ \log\left(\frac{1}{T}\sum_{i=1}^{N} T_i\, 10^{\frac{L_{\mathrm{Aeq},T_i}}{10}}\right). \tag{2.32}$$

This problem arises when we want to determine the equivalent level during a work day from the measurement of a short sample of each individual process into which the daily sequence of activities can be split. These measurements are conducted during relatively short intervals and are extrapolated to the total duration of each process.

We need to calculate the combined standard uncertainty, which has two types of components: one due to levels and the other due to interval durations. Note that the uncertainty in interval durations is attributable not only to the measurement process (including the measurement instrument, in this case a chronometer) but also to some indefinition of the measurand itself, since the duration of a process might depend on factors of difficult control that may adjust according to the intermediate results.We have, therefore,

$$L_{\mathrm{Aeq},T} = g(L_1, \ldots, L_N, T_1, \ldots, T_N) = 10 \log\left(\frac{1}{T}\sum_{i=1}^{N} T_i 10^{\frac{L_{\mathrm{Aeq},T_i}}{10}}\right)$$

In order to calculate the partial derivatives, we express the logarithm as a natural logarithm. The derivatives with respect to L_{Aeq,T_i} are

$$\frac{\partial g}{\partial L_{\mathrm{Aeq},T_i}} = \frac{10}{\ln 10} \frac{\frac{\partial}{\partial L_{\mathrm{Aeq},T_i}}\left(\frac{1}{T}\sum_{i=1}^{N} T_i e^{\ln 10 \frac{L_{\mathrm{Aeq},T_i}}{10}}\right)}{\frac{1}{T}\sum_{i=1}^{N} T_i\, 10^{\frac{L_{\mathrm{Aeq},T_i}}{10}}}$$

$$= \frac{10}{\ln 10}\frac{T_i}{T}\frac{\frac{\ln 10}{10} 10^{\frac{L_{\mathrm{Aeq},T_i}}{10}}}{10^{\frac{L_{\mathrm{Aeq},T}}{10}}},$$

which, after simplification, yields

$$\frac{\partial g}{\partial L_{\mathrm{Aeq},T_i}} = \frac{T_i}{T} 10^{\frac{L_{\mathrm{Aeq},T_i} - L_{\mathrm{Aeq},T}}{10}}. \qquad (2.33)$$

The derivatives respect to T_i are, taking into account that $T = \Sigma T_i$,

$$\frac{\partial g}{\partial T_i} = \frac{10}{\ln 10} \frac{\frac{T - T_i}{T^2} 10^{\frac{L_{\mathrm{Aeq},T_i}}{10}}}{\frac{1}{T}\sum_{i=1}^{N} T_i\, 10^{\frac{L_{\mathrm{Aeq},T_i}}{10}}} = \frac{10}{\ln 10}\frac{T - T_i}{T^2} 10^{\frac{L_{\mathrm{Aeq},T_i} - L_{\mathrm{Aeq},T}}{10}}. \qquad (2.34)$$

As a numerical example, consider three intervals of durations 2 h, 3 h, and 3 h measured with a standard uncertainty of 12 min. Their equivalent levels were 85.1 dBA, 83.0 dBA and 80.2 dBA, respectively, with a standard uncertainty of 1 dB. Then the total equivalent level turns out to be

$$L_{\mathrm{Aeq},T} = 10 \log\left(\frac{2}{8}10^{\frac{85.1}{10}} + \frac{3}{8}10^{\frac{83.0}{10}} + \frac{3}{8}10^{\frac{80.2}{10}}\right) \mathrm{dBA} = 82.9\, \mathrm{dBA}$$

The derivatives with respect to the levels are

$$\frac{\partial g}{\partial L_{\mathrm{Aeq},T_1}} = \frac{2}{8} 10^{\frac{85.1-82.9}{10}} = 0.415,$$
$$\frac{\partial g}{\partial L_{\mathrm{Aeq},T_2}} = \frac{3}{8} 10^{\frac{83.0-82.9}{10}} = 0.384,$$
$$\frac{\partial g}{\partial L_{\mathrm{Aeq},T_3}} = \frac{3}{8} 10^{\frac{80.2-82.9}{10}} = 0.201.$$

To find the time derivatives we multiply by $10/(T_k \ln 10)$, i.e.,

$$\frac{\partial g}{\partial T_1} = 0.676 \frac{\mathrm{dBA}}{\mathrm{h}}.$$
$$\frac{\partial g}{\partial T_2} = 0.347 \frac{\mathrm{dBA}}{\mathrm{h}}.$$
$$\frac{\partial g}{\partial T_3} = 0.182 \frac{\mathrm{dBA}}{\mathrm{h}}.$$

Therefore,

$$u_{\mathrm{c}} = \sqrt{(0.415^2 + 0.384^2 + 0.201^2) \times 1^2 + (0.676^2 + 0.347^2 + 0.182^2) \times 0.2^2}$$
$$= 0.62\,\mathrm{dBA}$$

Notice that the uncertainty is smaller than that of each single measurement (1 dB). This is due to the effect of dispersion reduction in an average of several variables.

In the absence of temporal errors, if the three levels were equal and had the same duration, the uncertainty would be $1/\sqrt{3}$ dB = 0.58 dB. The duration uncertainty makes it slightly larger.

2.6.4 Uncertainty in A-Weighting Computed from Octave Spectrum

Suppose we have measured the octave band levels $\{L_1, \ldots, L_{10}\}$ of a given noise. The A-weighted sound level can be obtained from

$$L_{\mathrm{A}} = 10 \log\left(\sum_{i=1}^{10} 10^{\frac{L_i + A_i}{10}}\right), \tag{2.35}$$

where A_i is the A-weighting correction for the i-th band. The octave band levels are random variables that are not necessarily independent. In order to determine the uncertainty, we need to find the partial derivatives of L_A with respect to the band levels L_i, as well as the covariances of such levels.

As in the previous examples, the partial derivatives can be calculated converting previously the logarithm and exponentials to base e:

$$L_\mathrm{A} = \frac{10}{\ln 10} \ln \left(\sum_{i=1}^{10} e^{\ln 10 \frac{L_i + A_i}{10}} \right).$$

Applying the chain rule,

$$\frac{\partial L_\mathrm{A}}{\partial L_i} = \frac{10}{\ln 10} \frac{1}{\left(\sum_{i=1}^{10} e^{\ln 10 \frac{L_i + A_i}{10}} \right)} e^{\ln 10 \frac{L_i + A_i}{10}} \frac{\ln 10}{10} = \frac{10^{\frac{L_i + A_i}{10}}}{10^{\frac{L_\mathrm{A}}{10}}}$$

from where we finally get

$$\frac{\partial L_\mathrm{A}}{\partial L_i} = 10^{\frac{L_i + A_i - L_\mathrm{A}}{10}}. \tag{2.36}$$

The covariance matrix for calculating a type B uncertainty can only be obtained if some properties of the phenomenon originating the noise to be measured are known. Otherwise, it will be necessary to conduct a series of measurements and estimate the covariance matrix from them. If $L_{i,k}$ and $L_{j,k}$ are the octave band levels of the i-th and j-th bands corresponding to the k-th measurement, then the covariances can be estimated by

$$u(L_i, L_j) = \frac{1}{n-1} \sum_{k=1}^{n} (L_{ik} - \overline{L}_i)(L_{jk} - \overline{L}_j), \tag{2.37}$$

where $\overline{L}_i$ and $\overline{L}_j$ are the mean levels of the i-th and j-th bands respectively, and n, the number of measured spectra. Finally, we can obtain the combined standard uncertainty as

$$u_c = \sqrt{\sum_{i=1}^{10} \sum_{j=1}^{10} \frac{\partial L_\mathrm{A}}{\partial L_i} \frac{\partial L_\mathrm{A}}{\partial L_j} u(L_i, L_j)}. \tag{2.38}$$

This uncertainty corresponds to the estimation of the A-weighted level from a single measurement. However, having taken n measurements, it is natural to estimate the A-weighted as an average of all measurement results. In such case the covariance are divided by n, increasing the confidence of L_A.

Note that for specific noises (such as traffic noise) the different spectral bands are not independent, so the cross-correlations ($i \neq j$) do not vanish.

2.6.5 Uncertainty in the Measurement of Sound Transmission Loss

The sound transmission loss of a sound insulating material is measured in the laboratory through the measurement of the octave or one-third octave band levels L_{1k} and L_{2k} at the source and receiver chambers, respectively, and the reverberation time T_k of the latter. Also needed are the area S_{12} of the sample and the volume V_2 of the receiver chamber. For the k-th band we have

$$R_k = L_{1k} - L_{2k} + 10 \log_{10} \frac{T_k S_{12}}{0.161\ V_2}, \tag{2.39}$$

The derivatives with respect to L_{1k} and L_{2k} are, respectively, 1 and -1. The derivatives with respect to T_k, S_{12} and V_2 are, respectively,

$$\frac{\partial R_k}{\partial T_k} = \frac{10}{\ln\ 10} \frac{1}{T_k}. \tag{2.40}$$

$$\frac{\partial R_k}{\partial S_{12}} = \frac{10}{\ln\ 10} \frac{1}{S_{12}}. \tag{2.41}$$

$$\frac{\partial R_k}{\partial T_k} = -\frac{10}{\ln\ 10} \frac{1}{V_2}. \tag{2.42}$$

Example: Suppose that the k-th band has been measured, with the following results: T_k = 2.3 s, L_{1k} = 87.3 dB, L_{2k} = 45.6 dB, S_{12} = 10.4 m^2, V_2 = 51.84 m^3. As regards the uncertainties, it is known that u_L = 1 dB, u_T = 0.2 s, u_S = 0.099 m^2, u_V = 0.38 m^3. Replacing into the previous equations, we have

$$\frac{\partial R_k}{\partial T_k} = 1.888\ \frac{\text{dB}}{\text{s}}$$
$$\frac{\partial R_k}{\partial S_{12}} = 0.418 \frac{\text{dB}}{\text{m}^2}$$
$$\frac{\partial R_k}{\partial V_2} = -0.0838 \frac{\text{dB}}{\text{m}^3}$$

Hence, the combined standard uncertainty in R_k will be

$$
\begin{aligned}
u_{c,R_k} &= \sqrt{\frac{\partial R_k}{\partial L_{1k}}^2 u_{L_{1k}}^2 + \frac{\partial R_k}{\partial L_{2k}}^2 u_{L_{2k}}^2 + \frac{\partial R_k}{\partial T_k}^2 u_{T_k}^2 + \frac{\partial R_k}{\partial S_{12}}^2 u_{S_{12}}^2 + \frac{\partial R_k}{\partial V_2}^2 u_{V_2}^2} \\
&= \sqrt{1^2 \times 1^2 + 1^2 \times 1^2 + 1.888^2 \times 0.2^2 + 0.418^2 \times 0.099^2 + 0.0838^2 \times 0.38^2} \\
&= \sqrt{1 + 1 + 0.143 + 0.0017 + 0.00101} = 1.46\,\mathrm{dB}
\end{aligned}
$$

The intermediate results are detailed in order to show that the greatest contribution to uncertainty corresponds to the acoustic components, i.e., the measurement of levels at both sides of the partition under test.

2.7 Uncertainty and Resolution

The finite resolution of a measuring instrument implies an uncertainty, since every true value of the measurand that falls between two consecutive divisions or between two least significant digits is assigned the same value. We can assume that the statistical distribution of the error is uniform, extended to an interval whose amplitude is equal to the resolution. As detailed in Appendix 2, if we call the resolution Δx_{min}, the standard uncertainty will be

$$u_{\mathrm{resol}} = \frac{\Delta x_{min}}{\sqrt{12}}. \tag{2.43}$$

Since we are dealing with a uniform distribution, the expanded uncertainty is smaller than for a normal distribution. For instance, U_{95} is attained with a coverage factor k such that

$$k \frac{\Delta x_{\mathrm{min}}}{\sqrt{12}} \frac{1}{\Delta x_{\mathrm{min}}} = \frac{0.95}{2},$$

Hence,

$$k = \frac{0.95\sqrt{12}}{2} = 1.65.$$

However, this component of uncertainty is not isolated but is part of several components. The *central limit theorem* implies that when adding several random variables together the resulting distribution tends to be normal. Besides, the resolution component of sound level meters uncertainty is much smaller than the other components.

Example: If a sound level meter has a resolution of 0.1 dB, then the uncertainty due to resolution will be

$$u_{\mathrm{resol}} = \frac{0.1\,\mathrm{dB}}{\sqrt{12}} = 0.029\,\mathrm{dB}.$$

It is vanishingly small compared to the uncertainty of the instrument itself, which is of the order of 1 dB.

2.8 Uncertainty and Systematic Error

Many measurements present the problem of the presence of a *systematic error*, i.e. some constant or predictable bias or offset that affect equally all measurements conducted under given conditions. It can be due to the operator, the instrument, the procedure, or a combination of them.

In the case of the operator, the cause is usually a parallax error when reading an analog (needle) instrument as well as a biased decision while reading a fluctuating digital display. In the case of the instrument itself, the typical reason is the lack of calibration. In the case of the measurement procedure, the reason may be some uncorrected effect (known or not), for instance, some temperature drift.

The systematic error may be additive (zero error), multiplicative (scale error) or nonlinear (linearity error).

2.8.1 *Additive Systematic Error*

We can define the additive systematic error as the difference between the mean of the measured values, μ_o and the true value x of the measurand,[9] i.e.

$$\varepsilon_{\mathrm{sys}} = \mu_o - x. \tag{2.44}$$

In order to apply (2.44), we would need to know the population mean of all possible measurements of the measurand, μ_o, and its true value, x, which is not possible. Instead, we can apply (2.44) in particular to a *standard reference*, for instance, the length of a standard bar, the mass of a standard weight or the level of a calibration tone. In any case, it is a quantity with a known conventional value and with an also known standard uncertainty (in general, substantially smaller than a

[9] Equivalently, it can be defined as the mean of the difference between the measured value and the true value.

typical instrument's uncertainty). If we call that value x_{cal}, we can estimate the mean of the observed values $\mu_{x\mathrm{cal,o}}$ with the mean $\bar{x}_{\mathrm{cal,o}}$ of a sample of m measurements, and adopt the conventional value as the true value. Our estimate of the systematic error will be, hence,

$$\hat{\varepsilon}_{\mathrm{sys}} = \bar{x}_{\mathrm{cal,o}} - x_{\mathrm{cal}}. \tag{2.45}$$

where $\bar{x}_{\mathrm{cal,o}}$ is the mean of m measurements of the reference. The uncertainty of this estimate will be

$$u_{\hat{\varepsilon}_{\mathrm{sys}}} = \sqrt{u^2_{\bar{x}_{\mathrm{cal,o}}} + u^2_{x_{\mathrm{cal}}}}. \tag{2.46}$$

If u_{o} is the uncertainty of a single measurement,

$$u_{\hat{\varepsilon}_{\mathrm{sys}}} = \sqrt{\frac{u_{\mathrm{o}}^2}{m} + u^2_{x_{\mathrm{cal}}}}, \tag{2.47}$$

Once the systematic error has been estimated, we can correct the value of the measurand to remove the systematic effects:

$$x_{\mathrm{c}} = \bar{x}_{\mathrm{o}} - \hat{\varepsilon}_{\mathrm{sys}}. \tag{2.48}$$

where $\bar{x}_{\mathrm{o}}$ is the mean of n observations of the measurand (n is not necessarily the same as the previous m).

The uncertainty in this measurement will be

$$u_{x_{\mathrm{c}}} = \sqrt{u^2_{\bar{x}_{\mathrm{o}}} + u^2_{\hat{\varepsilon}_{\mathrm{sys}}}}, \tag{2.49}$$

or, in terms of the general uncertainty of a single measurement,

$$u_{x_{\mathrm{c}}} = \sqrt{\frac{u_{\mathrm{o}}^2}{n} + u^2_{\hat{\varepsilon}_{\mathrm{sys}}}}. \tag{2.50}$$

It is interesting to note that although it is possible to correct the systematic error (and it *must* be corrected), it is not possible to remove its uncertainty.

Example: Suppose that we take four measurements of a noise in identical conditions to get 83.3 dB, 83.6 dB, 84.1 dB and 83.0 dB with an instrument whose standard uncertainty is 1 dB. Previously, the calibration is checked with an acoustic calibrator whose nominal value is 93.85 dB and whose recent calibration certificate indicates a standard uncertainty of 0.15 dB and an error of −0.1 dB. It is checked three times, getting in all cases the same value 94.1 dB.

Let us first calculate the correction and its uncertainty. The nominal value of the reference level is 93.85 dB, but as the systematic error is −0.1 dB, a perfect instrument would read

$$L_{\mathrm{cal}} + \varepsilon_{L_{cal},\mathrm{sys}} = 93.75\,\mathrm{dB}.$$

Our instrument reads, instead,

$$\bar{x}_{\mathrm{cal,o}} = \frac{94.1 + 94.1 + 94.1}{3}\,\mathrm{dB} = 94.1\,\mathrm{dB}$$

The estimated systematic error due to lack of calibration will be, hence

$$\hat{\varepsilon}_{\mathrm{sys}} = 94.1\,\mathrm{dB} - 93.75\,\mathrm{dB} = 0.35\,\mathrm{dB}.$$

The uncertainty of this estimate will be

$$\begin{aligned} u_{\hat{\varepsilon}_{\mathrm{sys}}} &= \sqrt{\frac{1^2 + (0.1/\sqrt{12})^2}{3} + 0.15^2} \\ &= \sqrt{0.3333 + 0.0003 + 00.0225} = 0.597\,\mathrm{dB} \end{aligned}$$

The second term inside the root symbol corresponds to the finite resolution uncertainty. We can see that in this case this effect is negligible.

As regards the measured and corrected value of the measurand, we have:

$$x_{\mathrm{c}} = \frac{83.3 + 83.6 + 84.1 + 83.0}{4} - 0.35 = 83.15\,\mathrm{dB}$$

The standard uncertainty is

$$\begin{aligned} \varepsilon_{x_c} &= \sqrt{\frac{1^2 + (0.1/\sqrt{12})^2}{4} + 0.597^2} \\ &= \sqrt{0.25 + 0.0002 + 0.3564} = 0.78\,\mathrm{dB} \end{aligned}$$

Once more, the uncertainty due to finite resolution is not important compared to the instrument's and calibrator's uncertainty. We can also see that the latter prevails when we average several measurements. In this example, by averaging only four measurements the instrument's uncertainty drops below the uncertainty due to the systematic error of the calibrator.

2.8.2 *Multiplicative Systematic Error*

Also known as *scale error* or *relative systematic error*, it is present when the physical principle of the measurement has the form,

$$y = Kx, \tag{2.51}$$

where x is a measurand that cannot be directly measured (for instance, a sound pressure), y, an associated quantity that can be directly measured (for instance, a voltage) and K, the transducer sensitivity that converts the measurand into a measurable quantity.

What is actually measured is, thus, y, after which it is possible to get the desired value as

$$x = \frac{1}{K} y, \tag{2.52}$$

The error arises when the real constant K in (2.51) differs from the one used in (2.52). In general, in (2.51) there is an unknown constant K_{real}, but when we compute x in (2.52) we assume a nominal or ideal constant K_{ideal}. In consequence, the relationship between the observed value x_{o} and the true value will be

$$x_{\text{o}} = \frac{K_{\text{real}}}{K_{\text{ideal}}} x. \tag{2.53}$$

A new difficulty arises: the measured value of y has also some uncertainty since the instrument is not an ideal one. Hence, (2.51) must be rewritten as

$$y = K_{\text{real}} x + \varepsilon_y, \tag{2.54}$$

where ε_{y} is a random variable centered at 0 that represents the dispersion in the measurement of y. Hence, the observed value of x will be

$$x_{\text{o}} = \frac{1}{K_{\text{ideal}}} y = \frac{K_{\text{real}}}{K_{\text{ideal}}} x - \frac{1}{K_{\text{ideal}}} \varepsilon_y. \tag{2.55}$$

Since ε_y has mean 0, we could remove the random measurement error taking the population mean μ_{o} of all observed values x_{o} of x:

$$\mu_{\text{o}} = \frac{K_{\text{real}}}{K_{\text{ideal}}} x. \tag{2.56}$$

Finally, we define the *relative systematic error* ε_{rel} as

$$\varepsilon_{\text{rel}} = \frac{\mu_{\text{o}}}{x} - 1 = \frac{K_{\text{real}}}{K_{\text{ideal}}} - 1. \tag{2.57}$$

As can be noted, it represents the relative error between the real and ideal constants. The relative error is dimensionless and is 0 in the ideal case in which $K_{\text{real}} = K_{\text{ideal}}$. For acceptable measurements it should be much less than 1.

Since μ_o is the population mean of x_o, it cannot be known, so we shall attempt to estimate ε_{rel}. To that end, instead of an arbitrary value of x, we chose a reference calibration value x_{cal}, and take the average of m observations of x_{cal}. We assume that x_{cal} and its uncertainty are known. We get

$$\hat{\varepsilon}_{rel} = \frac{\bar{x}_{cal,o}}{x_{cal}} - 1. \tag{2.58}$$

The combined standard uncertainty is computed by means of Eq. (2.27), yielding

$$\begin{aligned} u_{\hat{\varepsilon}_{rel}} &= \sqrt{u_{\bar{x}_{cal,o}}^2 \left(\frac{1}{x_{cal}}\right)^2 + u_{x_{cal}}^2 \left(\frac{-\bar{x}_{cal,o}}{x_{cal}^2}\right)^2} \\ &= \frac{1}{x_{cal}} \sqrt{u_{\bar{x}_{cal,o}}^2 + u_{x_{cal}}^2 (1 + \hat{\varepsilon}_{rel})^2} \end{aligned} \tag{2.59}$$

Now that we have estimated the value of the relative systematic error, it is possible to compute the corrected value of a measurand x in general:

$$x_c = \bar{x}_o (1 + \hat{\varepsilon}_{rel}). \tag{2.60}$$

The combined standard uncertainty turns out to be

$$u_{x_c} = \sqrt{u_{\bar{x}_o}^2 (1 + \hat{\varepsilon}_{rel})^2 + \bar{x}_o^2 u_{x_{cal}}^2} \tag{2.61}$$

Or, making explicit the reduction of the mean with respect to that of a single measurement,

$$u_{x_c} = \sqrt{\frac{u_{x_o}^2}{n} (1 + \hat{\varepsilon}_{rel})^2 + \bar{x}_o^2 u_{x_{cal}}^2}. \tag{2.62}$$

Note that since

$$\bar{x}_o = \frac{x_c}{1 + \hat{\varepsilon}_{rel}}, \tag{2.63}$$

the uncertainty in x_c increases with x_c, unlike the relative systematic error, which is independent of x_c.

Example: Consider the measurement of a sound pressure with a microphone whose sensitivity S differs from the nominal one. In that case, if v is the voltage generated in the microphone when excited with a pressure p, we will have [10]

[10] It is interesting to note that in this case, we are interested in measurements of sound pressure instead of sound pressure level.

$$p = \frac{1}{S} v. \tag{2.64}$$

Suppose a Rion UC-53A microphone whose nominal (ideal) sensitivity is −28.0 dB referred to 1 V/Pa, and whose calibration certificate indicates a real sensitivity (at 1 kHz) of −26.2 dB with a standard uncertainty of 0.2 dB. We are interested in obtaining the relative systematic error. We need the ideal and real sensitivities and uncertainty in V/Pa. The ideal sensitivity is

$$S_{\text{ideal}} = 10^{\frac{S_{\text{ideal,dB}}}{20}} 10^{\frac{-28.0}{20}} \text{V/Pa} = 0.03981 \text{ V/Pa}$$

The real sensitivity,

$$S_{\text{real}} = 10^{\frac{S_{\text{real,dB}}}{20}} = 10^{\frac{-26.2}{20}} \text{V/Pa} = 0.04898 \text{ V/Pa}$$

The standard uncertainty can be obtained by applying Eq. (2.27), in this case with a single variable:

$$u_{S_{\text{real}}} = \frac{\partial S_{\text{real}}}{\partial S_{\text{real,dB}}} u_{S_{\text{real,dB}}} = \frac{\ln 10}{20} S_{\text{real}} u_{S_{\text{real,dB}}} = 0.0011 \text{ V/Pa}$$

If we suppose that the microphone generates a voltage v, the observed pressure will be

$$p_{\text{o}} = \frac{v}{S_{\text{ideal}}},$$

while the real pressure is

$$p = \frac{v}{S_{\text{real}}}.$$

The estimated relative error turns out to be

$$\hat{\varepsilon}_{\text{rel}} = \frac{p_{\text{o}}}{p} - 1 = \frac{S_{\text{real}}}{S_{\text{ideal}}} - 1 = 0.2303.$$

The standard uncertainty of this error is

$$u_{\hat{\varepsilon}_{\text{rel}}} = \frac{u_{S_{\text{real}}}}{S_{\text{ideal}}} = 0.0276.$$

If, for example, we measured a sound pressure p_o = 1 Pa, the corrected value would be

$$p_{\text{c}} = 1 \text{ Pa} \times (1 + 0.2303) = 1.2303 \text{ Pa}$$

The uncertainty component due to the relative error will be

$$u_{p_{\rm c},{\rm rel}} = 1\,\mathrm{Pa} \times 0.0276 = 0.0276\,\mathrm{Pa}$$

The total uncertainty must include, besides, the uncertainty due to nonsystematic random effects, inherited from the uncertainty in the measurement of v.

Note: The multiplicative systematic error is converted into additive when the quantities are presented in logarithmic version, since the error in the scale factor is transformed into an additive term.

2.8.3 Nonlinear Systematic Error

The *nonlinear systematic error* or *linearity error* is present in those situations in which a linear model is assumed:

$$y = Kx, \tag{2.65}$$

but it is actually nonlinear and presents dispersion:

$$y = g(x) + \varepsilon_y. \tag{2.66}$$

In the preceding equations y is a quantity that can be directly measured, and x, the measurand to be computed from y. To determine x it is generally assumed that (2.65) holds with a nominal or ideal constant $K_{\rm o}$, so that the observed value, $x_{\rm o}$, will be

$$x_{\rm o} = \frac{y}{K_{\rm o}} = \frac{g(x)}{K_{\rm o}} + \frac{\varepsilon_y}{K_{\rm o}}. \tag{2.67}$$

Suppose, for instance, that the function $g(x)$ can be approximated by a third-degree polynomial without independent term,

$$y = Kx + K_1x^2 + K_2x^3. \tag{2.68}$$

For a cubic equation such as this one, it is possible to solve for x, but in the most interesting case where K_1 and K_2 are small (a quasi-linear relationship), we can approximate

$$x \cong Ay + By^2 + Cy^3. \tag{2.69}$$

Although the actually measured quantity is y, the instrument in general provides an observed value $x_{\rm o}$ of the measurand, obtained internally from y with the proportional relationship of Eq. (2.67). Thus, the true value of x can be approximated by

$$x \cong a x_o + b x_o^2 + c x_o^3. \tag{2.70}$$

If we knew K, K_1, and K_2, we could obtain a, b, and c as follows. First we replace x_o by y/K_o according to (2.67) and y as a function of x using (2.68). The second member is a ninth degree polynomial. Keeping up to the cubic term and equating coefficients, we arrive at the following correction for the instrument's reading x_o:

$$x_c = \frac{K_o}{K} x_o - \frac{K_1 K_o^2}{K^3} x_o^2 + \frac{2K_1^2 - K_2 K}{K^5} K_o^3 x_o^3. \tag{2.71}$$

Note that we have replaced the sign $\cong$ by an equality. This is because we do not claim that x_c is equal to x, but only a corrected value that, by taking into account the nonlinear systematic effect, is a better approximation of the true value.

In this case, we do not attempt to introduce a specific quantification of the error since it would require several parameters. However, it is still possible to use the concept of relative systematic error with respect to the linear term.

In general, K, K_1, and K_2 are unknown, so the model must be obtained empirically. Instead of the model (2.68) we will directly use (2.70), since it provides a corrected value directly from the observed value (the instrument's reading). In order to find a, b, and c, we need to measure three reference values whose conventional values x_1, x_2, x_3 and uncertainties u_{x1}, u_{x2}, u_{x3} are known.[11] Let x_{1o}, x_{2o}, x_{3o} be the measured (observed) values (or the respective means of m measurements). We can pose the equation system

$$\begin{cases} a x_{1o} + b x_{1o}^2 + c x_{1o}^3 = x_1 \\ a x_{2o} + b x_{2o}^2 + c x_{2o}^3 = x_2 \\ a x_{3o} + b x_{3o}^2 + c x_{3o}^3 = x_3 \end{cases} \tag{2.72}$$

This system of three equations and three unknowns, a, b, and c can be solved numerically. We first rewrite it in matrix form,

$$\begin{bmatrix} x_{1o} & x_{1o}^2 & x_{1o}^3 \\ x_{2o} & x_{2o}^2 & x_{2o}^3 \\ x_{3o} & x_{3o}^2 & x_{3o}^3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \tag{2.73}$$

from where

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} x_{1o} & x_{1o}^2 & x_{1o}^3 \\ x_{2o} & x_{2o}^2 & x_{2o}^3 \\ x_{3o} & x_{3o}^2 & x_{3o}^3 \end{bmatrix}^{-1} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}. \tag{2.74}$$

[11] Note that only the conventional values are known, not the real ones. However, the conventional values have less dispersion and do not have any known systematic effect.

This completes the data that are necessary to correct for the nonlinear systematic effects:

$$x_c = ax_o + bx_o^2 + cx_o^3. \tag{2.75}$$

Now let us calculate the uncertainty. Note, first, that once the constants a, b, and c are obtained we have a model that can be applied to many observed data x_o, but that model is not the only one that we could have arrived to using the same procedure, because the measurements of the reference values are subject to uncertainty. So what we really have is

$$x_c = g(x_o, x_{1o}, x_{2o}, x_{3o}). \tag{2.76}$$

The functional dependency of x_{1o}, x_{2o} and x_{3o} is through the coefficients a, b, and c:

$$x_c = a(x_{1o}, x_{2o}, x_{3o})x_o + b(x_{1o}, x_{2o}, x_{3o})x_o^2 + c(x_{1o}, x_{2o}, x_{3o})x_o^3. \tag{2.77}$$

The uncertainty is, then,

$$u_{x_c}^2 = \left(\frac{\partial g}{\partial x_o}\right)^2 u_{x_o}^2 + \left(\frac{\partial g}{\partial x_{1o}}\right)^2 u_{x_{1o}}^2 + \left(\frac{\partial g}{\partial x_{2o}}\right)^2 u_{x_{2o}}^2 + \left(\frac{\partial g}{\partial x_{3o}}\right)^2 u_{x_{3o}}^2. \tag{2.78}$$

The basic uncertainties u_{xo}, u_{x1o}, u_{x2o}, u_{x3o}, are those corresponding to the measurement (may be a direct reading or an average of several readings). They might depend on the measurand value in each case. The derivatives are:

$$\frac{\partial g}{\partial x_o} = a + 2bx_o + 3cx_o^2 \tag{2.79}$$

$$\frac{\partial g}{\partial x_{io}} = \frac{\partial a}{\partial x_{io}}x_o + \frac{\partial b}{\partial x_{io}}x_o^2 + \frac{\partial c}{\partial x_{io}}x_o^3, \tag{2.80}$$

where i = 1, 2, 3. The functions a, b, and c are defined implicitly through system (2.72). Even if it is possible to derive an analytic expression, it is preferable to apply implicit derivation. Let us rewrite (2.72) as:

$$\begin{cases} ax_{1o} & + & bx_{1o}^2 & + & cx_{1o}^3 & - & x_1 & = & 0 \\ ax_{2o} & + & bx_{2o}^2 & + & cx_{2o}^3 & - & x_2 & = & 0 \\ ax_{3o} & + & bx_{3o}^2 & + & cx_{3o}^3 & - & x_3 & = & 0 \end{cases} \tag{2.81}$$

Deriving each expression with respect to x_{1o} we get

$$\begin{cases} \frac{\partial a}{\partial x_{1o}} x_{1o} & + & \frac{\partial b}{\partial x_{1o}} x_{1o}^2 & + & \frac{\partial c}{\partial x_{1o}} x_{1o}^3 & = & -(a + 2bx_{1o} + 3cx_{1o}^2) \\ \frac{\partial a}{\partial x_{1o}} x_{2o} & + & \frac{\partial b}{\partial x_{1o}} x_{2o}^2 & + & \frac{\partial c}{\partial x_{1o}} x_{2o}^3 & = & 0 \\ \frac{\partial a}{\partial x_{1o}} x_{3o} & + & \frac{\partial b}{\partial x_{1o}} x_{3o}^2 & + & \frac{\partial c}{\partial x_{1o}} x_{3o}^3 & = & 0 \end{cases} \tag{2.82}$$

This is a system of three equations and three unknowns that allow to solve for the derivatives of a, b, and c with respect to x_{1o}. In matrix version:

$$\begin{bmatrix} x_{1o} & x_{1o}^2 & x_{1o}^3 \\ x_{2o} & x_{2o}^2 & x_{2o}^3 \\ x_{3o} & x_{3o}^2 & x_{3o}^2 \end{bmatrix} \begin{bmatrix} \partial a/\partial x_{1o} \\ \partial b/\partial x_{1o} \\ \partial c/\partial x_{1o} \end{bmatrix} = - \begin{bmatrix} a + 2bx_{1o} + 3cx_{1o}^2 \\ 0 \\ 0 \end{bmatrix} \tag{2.83}$$

Interestingly, we see that the system's matrix is the same as for system (2.73), originally used to solve for a, b, and c. This is advantageous, since it suffices to compute the inverse only once.

Proceeding similarly, we get and solve systems for the derivatives respect to x_{2o} and x_{3o}. After replacing in (2.80) and similar equations the values obtained from (2.83) and similar systems, we substitute (2.79) and (2.80) in (2.78), arriving finally at the desired uncertainty. The calculations are rather lengthy so they are implemented by software. In Appendix 9 there is an example.

2.8.4 Uncorrected Systematic Errors

Sometimes there is a systematic error that cannot be fully corrected. A typical example is when an instrument's calibration is checked with a calibrator whose actual level does not coincide with the nominal one. This situation is dealt with taking into account the calibrator's uncertainty.

A similar situation happens when the calibrator suffers long-term drifts. For instance, the specification chart of calibrator Brüel and Kjær Type 4231 (Brüel and Kjær 2014) indicates that the long-term stability (one year) is better than 0.05 dB with a confidence level of 96 %. We can assume that this implies a coverage factor $k = 2$. Hence, within a year, the standard uncertainty will increase by 0.025 dB with respect to the standard uncertainty attributable to the reference level of the calibrator, so the uncertainty due to the uncorrected systematic error will be

$$u_{\text{cal}} = \sqrt{u_{\text{initial}}^2 + u_{1\,\text{yr}}^2} \tag{2.84}$$

With the data of the example given in 2.8.1, it turns out to be $u_{\text{cal}} = 0.152$ dB. This represents a negligible increment compared to the uncertainty of 0.15 dB indicated in the most recent calibration certificate.

2.9 Chain Calculation of Uncertainty

In complex uncertainty calculations, it is often possible to split the problem into a series of chained calculations through the use of nested composite functions. Suppose, for instance, that we have a measurand that depends on several variables which in turn depend on other variables and so on, for instance:

$$
\begin{aligned}
w &= f(z_1, \ldots, y_L),\\
z_k &= z_k(y_1, \ldots, y_M),\\
y_j &= y_j(x_1, \ldots, x_N).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
u_w^2 = \sum_{i=1}^{L} \left(\frac{\partial w}{\partial x_i}\right)^2 u_{x_i}^2 &= \sum_{i=1}^{L} \left(\sum_{j=1}^{M} \frac{\partial w}{\partial y_j}\frac{\partial y_j}{\partial x_i}\right)^2 u_{x_i}^2\\
&= \sum_{i=1}^{L} \left(\sum_{j=1}^{M} \left(\sum_{k=1}^{N} \frac{\partial f}{\partial z_k}\frac{\partial z_k}{\partial y_j}\right)\frac{\partial y_j}{\partial x_i}\right)^2 u_{x_i}^2
\end{aligned}
$$

i.e.,

$$
u_w^2 = \sum_{i=1}^{L} \left(\sum_{j=1}^{M}\sum_{k=1}^{N} \frac{\partial f}{\partial z_k}\frac{\partial z_k}{\partial y_j}\frac{\partial y_j}{\partial x_i}\right)^2 u_{x_i}^2
$$

However, even if x_i are statistically independent, y_j and z_k not necessarily are, since they depend functionally on the same variables. In cases as this one, it is important that the intermediate variables depend on different independent variables, which sometimes can be achieved grouping together in a single function variables that depend on the same variable.

Problems

(1) Imagine you are measuring traffic noise in a street across a park. Identify all possible factors of uncertainty for your measurement and try to figure out how you would minimize their effects.

(2) In the same scenario of problem (1), try to define as thoroughly as possible your measurand. Remember that the measurement conditions are a part of the definition.

(3) Describe in as much detail as possible the measurement method and procedure in the case of problem (1).

(4) Repeat problems (1) to (3) for a different type of measurement of your choice, such as one you perform frequently.

(5) Explain why a high precision measurement instrument could yield very inaccurate results.

(6) Indicate which of the following situations correspond to direct measurement methods: (a) Measuring the equivalent level of a room's ambient noise using an integrating sound level meter. (b) Measuring the A-weighted equivalent level extended to 8 h in a factory which has three well-identified processes by measuring each process during a short interval and then combining the results according to the specified duration of each process. (c) Measuring the octave band spectrum of the immission noise in an office using a spectrum analyzer. (d) Measuring the statistical level L_{90} by taking 100 measurements at 5 s intervals and selecting the largest of the 10 smallest results. (e) Measuring the equivalent noise level of a laboratory chamber applying a correction for temperature. (f) Measuring the statistical level L_{10} in a street using an instrument with statistical measurement capabilities. (g) Measuring the peak level of conversational speech at 1 m with an instrument capable of measuring peak levels.

(7) Indicate the measurement model used to measure the D-weighted sound pressure level from the measurement of the one-third octave spectrum of the signal.

(8) The noise reduction coefficient (*NRC*) is defined as the average of the absorption coefficients at 250, 500, 1000, and 2000 Hz. Describe a possible measurement model for the *NRC*.

(9) Find the resolution of a method of measurement that consists in averaging 10 direct measurements with a sound level meter whose resolution is 0.1 dB

(10) Find the type A uncertainty of the following series of 15 s measurements of traffic noise: 70.5, 79.3, 74.7, 73.1, 74.6, 72.8, 73.0, 76.2, 76.0, 76.0.

(11) Find the type B uncertainty of a Class 2 sound level meter associated to the weighting network for different frequencies.

(12) Consider a class of noise sources that have a 1/3 octave noise spectrum increasing at a rate of 6 dB/oct until a peak is reached at 250 Hz, after which it starts decaying at −3 dB/oct. Assume that the standard deviation for all bands is 1 dB and that the deviation from the preceding model is independent for all bands. (a) Find the type B uncertainty in the indirect measurement of the A-weighted sound level from the 1/3 octave spectrum using the partial derivatives approach. (b) Simulate the outcome of a large number of spectrum measurements by adding to each band a different random number with a normal distribution with standard deviation 1 dB, then compute the A-weighted sound level for each measurement and find the uncertainty as the standard deviation of all the results. Compare with the result of (a).

(13) In order to measure the sound power of an acoustic source in a semi-anechoic room, a hemispherical surface equidistant to the projection on the floor of the center of the source is first divided into regions of area S_k. Then the spectrum is measured with a microphone located at the geometric center of each region

yielding levels L_{ik} where i corresponds to the different bands and k to the different regions. Then the following formula is applied.

$$\overline{L_{pi}} = 10 \log\left(\frac{1}{S}\sum_{k=1}^{N} S_k 10^{L_{ik}/10}\right),$$

where $\overline{L_{pi}}$ is the mean sound pressure level of the i-th band on the whole surface S. Finally the sound power level is computed with the formula

$$L_{\mathrm{w}} = \overline{L_{pi}} + 10 \log\frac{2\pi r^2}{S_{\mathrm{o}}},$$

where r is the hemisphere radius and S_{o} = 1 m^2. Find the uncertainty in L_{w} given the uncertainty u_{Li} of each band level and the uncertainty on the radius r, u_r. Note: The uncertainty may differ between the different bands.

(14) A sound level meter has temperature, relative humidity, and ambient pressure coefficients K_T = −0.001 dB/ °C, $K_{h\mathrm{r}}$ = 0.001 dB/%, $K_{P\mathrm{a}}$ = − 0.001 dB/hPa. Suppose the environmental conditions cannot be measured but it is known from past climate statistics that the mean conditions at the location and current season are T = 22 °C, h_{r} = 60 %, P_{a} = 1010 hPa with standard deviations σ_T = 5 °C, $\sigma_{h\mathrm{r}}$ = 15 % and $\sigma_{P\mathrm{a}}$ = 10 hPa. Find the measurement uncertainty due to lack of knowledge of the environmental conditions.

(15) A sound level meter has a resolution of 0.1 dB. A measurement method is implemented to measure random noise by averaging 20 instantaneous measurements of L_{p} using fast time constant taken at regular intervals. (a) Indicate which is the method resolution. (b) Find the uncertainty due to the resolution. Hint: The distribution is not uniform but approximately normal (Can you tell why?).

(16) Suppose we have a sound calibrator whose nominal sound pressure level within its acoustic coupler is 94.0 dB. The calibration certificate states that the calibrator has an error of 0.2 dB and an expanded uncertainty U_{95} of 0.3 dB. We perform 10 independent measurements of the calibrator tone with a free-field sound level meter (removing each time the microphone from the coupler) getting eight times a value of 94.2 dB and twice a value of 94.1 dB. Afterwards we measure 10 times the noise of an electric pump at a distance of 1 m, getting the following values: 87.3, 86.8, 86.7, 86.9, 87.1, 87.5, 87.3, 87.4, 87.2, 87.2 dBA. Assuming the sound level meter has an expanded uncertainty of 1.4 dB and a resolution of 0.1 dB, find: (a) The systematic error of the calibrator tone and its uncertainty; (b) The systematic error of the sound level meter and its uncertainty; (c) The average sound level of the pump noise corrected for systematic effects and its uncertainty.

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