On Choice Rules in Dependent Type Theory

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Abstract. In a dependent type theory satisfying the propositions as types correspondence together with the proofs-as-programs paradigm, the validity of the unique choice rule or even more of the choice rule says that the extraction of a computable witness from an existential statement under hypothesis can be performed within the same theory.

Here we show that the unique choice rule, and hence the choice rule, are not valid both in Coquand's Calculus of Constructions with indexed sum types, list types and binary disjoint sums and in its predicative version implemented in the intensional level of the Minimalist Foundation. This means that in these theories the extraction of computational witnesses from existential statements must be performed in a more expressive proofs-as-programs theory.

1 Introduction

Type theory is nowadays both a subfield of mathematical logic and of computer science. A perfect example of type theory studied both by mathematicians and by computer scientists is Martin-Löf's type theory [21], for short **ML**. This is a dependent type theory which can be indeed considered both as a paradigm of a typed functional programming language and as a foundation of constructive mathematics. The reason is that, on one hand, types can be seen to represent data types and typed terms to represent programs. On the other hand, both sets and propositions can be represented as types and both elements of sets and proofs of propositions can be represented as typed terms. These identifications are named in the literature as the propositions-as-types paradigm or the proofs-as-programs correspondence or Curry-Howard correspondence.

An important application of dependent type theory to programming is that one can use a type theory such as **ML** to construct a correct and terminating program as a typed term meeting a certain specification defined as its type. Pushing forward this correspondence one may ask whether from the proof-term $p(x)$ of an existential statement under hypothesis

$$p(x) \in \exists y \in B\ R(x, y)\ [x \in A]$$

one may extract a functional program $f \in A \rightarrow B$ whose graph is contained in the graph of $R(x, y)$, namely for which we can prove that there exists a proof-term $q(x)$ such that we can derive

$$q(x) \in R(x, f(x))\ [x \in A]$$

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This property is called *choice rule*. Then, we call *unique choice rule* the corresponding property starting from a proof-term of a unique existential statement

$$p(x) \in \exists! y \in B\ R(x, y)\ [x \in A]$$

from which we may extract a functional term $f \in A \to B$ whose graph is contained in the graph of $R(x, y)$.

It is worth noting that such choice rules *characterizes constructive arithmetics*, i.e. arithmetics within intuitionistic logic, with respect to classical Peano arithmetics (see [25,26]).

In Martin-Löf's type theory both the unique choice rule and the choice rule are valid given that they follow from the validity of the axiom of choice

$$\text{(AC)} \qquad \forall x \in A\ \exists y \in B\ R(x, y) \quad \longrightarrow \quad \exists f \in A \to B\ \forall x \in A\ R(x, f(x))$$

thanks to the identification of the **ML**-existential quantifier with the *the strong indexed sum of a set family*, which characterizes the so called *propositions-as-sets isomorphism*.

However in other dependent type theories proposed as foundations for constructive mathematics the existential quantifier is not identified with the strong indexed sum type whilst it is still a type of its proofs. As a consequence in such theories the validity of the mentioned choice rules is not evident.

A notable example of such a dependent type theory is Coquand's Calculus of Inductive Constructions [6,7] used as a logical base of the proof-assistant Coq [4,5] and Matita [2,3]. Here we consider its fragment $\mathbf{CC}^+$ extending the original system in [6] with indexed sum types, list types and binary disjoint sums. In $\mathbf{CC}^+$ propositions are defined primitively by postulating the existence of an impredicative type of propositions. In particular in $\mathbf{CC}^+$ the identification of the existential quantifier with the strong indexed sum type is not possible because it makes the typed system logically inconsistent (see [6]). In fact in $\mathbf{CC}^+$ the axiom of choice and even the axiom of unique choice

$$\text{(AC!)} \qquad \forall x \in A\ \exists! y \in B\ R(x, y) \quad \longrightarrow \quad \exists f \in A \to B\ \forall x \in A\ R(x, f(x))$$

are not generally provable as shown in [24]. In [24] it was left open whether the choice rule is validated in the original system [6]. Here we show that the choice rule is not validated in $\mathbf{CC}^+$ by proving that in $\mathbf{CC}^+$ the unique choice rule implies the axiom of unique choice and hence it is not valid. Of course, from this it follows that also the choice rule is not valid in $\mathbf{CC}^+$.

Another example of foundation for constructive mathematics based on a dependent type theory where the existential quantifier is given primitively is the Minimalist Foundation, for short **MF**, ideated by the author in joint work with G. Sambin in [16] and completed in [12]. An important feature of **MF**, which is not present in other foundations like $\mathbf{CC}^+$ or **ML**, is that it constitutes a common core among the most relevant constructive and classical foundations, introduced both in type theory, in category theory and in axiomatic set theory. Moreover it is a two-level system equipped with an *intensional level* suitable for extraction of computational contents from its proofs, an *extensional level* formulated in a language as close as possible to that of ordinary mathematics and an *interpretation of the latter in the former* showing that the extensional level has been obtained by abstraction from the intensional one according to Sambin's forget-restore principle in [23].

The two-level structure of **MF** brings many advantages in comparison to a single level foundation for constructive mathematics as $\mathbf{CC}^+$ or **ML**.

First of all the intensional level of **MF**, called **mTT** in [12] for *Minimalist Type theory*, can be used as a base for computer-aided formalization of proofs done at the extensional level of **MF**. Moreover, we can show the compatibility of **MF** with other constructive foundations at the most appropriate level: the intensional level of **MF** can be easily interpreted in intensional theories such as those formulated in type theory, for example Martin-Löf's type theory [21] or the Calculus of Inductive Constructions, while its extensional level can be easily interpreted in extensional theories such as those formulated in axiomatic set theory, for example Aczel's constructive set theory [1], or those formulated in category theory as topoi [10,11].

Both intensional and extensional levels of **MF** consist of type systems based on versions of Martin-Löf's type theory with the addition of a primitive notion of propositions: the intensional one is based on [21] and the extensional one is based on [20].

In particular **mTT** constitutes a *predicative* counterpart of $\mathbf{CC}^+$ and to which the argument disproving the validity of the unique choice in $\mathbf{CC}^+$ adapts perfectly well.

As a consequence of our results we get that the extraction of programs computing witnesses of existential statements under hypothesis proved in $\mathbf{CC}^+$ or in **mTT** needs to be performed in a more expressive proofs-as-programs theory. We also believe that the arguments presented here can be adapted to conclude the same statements even for $\mathbf{CC}^+$ with generic inductive definitions.

It is worth noting that we can choose Martin-Löf's type theory as a more expressive theory where to perform the mentioned witness extraction from proofs done in **mTT**. Another option is to perform such a witness extraction in the realizability model of **mTT** extended with the axiom of choice and the formal Church thesis constructed in [9] (note that the axiom of choice and the formal Church thesis say that we can extract computable functions from number-theoretic total relations). Furthermore, in the case we limit ourselves to extract computable witnesses from unique existential statements proven in **mTT** then we can use other realizability models such as that in [17,18] validating **mTT** extended with the axiom of unique choice and the formal Church thesis.

To perform witness extraction from proofs of existential statements done in $\mathbf{CC}^+$, an impredicative version of Martin-Löf's type theory is not available. We do not even know whether there exists a realizability model proving consistency of $\mathbf{CC}^+$ extended with the axiom of choice and the formal Church thesis.

2 The Dependent Type Theory $\mathbf{DT}_{\boldsymbol{\Sigma}}$ and the Choice Rules

Here we briefly describe a fragment, called $\mathbf{DT}_\Sigma$, of the intensional type theory **mTT** of the Minimalist Foundation in [12] which is sufficient to show that the unique choice rule implies the axiom of unique choice. This is the fragment of **mTT** needed to interpret many-sorted predicate intuitionistic logic where sorts are closed under strong indexed sums, dependent products and also comprehension.

$\mathbf{DT}_\Sigma$ is a dependent type theory written in the style of Martin-Löf's type theory [21] by means of the following four kinds of judgements:

$$A\ type\ [\Gamma] \quad A = B\ type\ [\Gamma] \quad a \in A\ [\Gamma] \quad a = b \in A\ [\Gamma]$$

that is the type judgement (expressing that something is a specific type), the type equality judgement (expressing that two types are equal), the term judgement (expressing that something is a term of a certain type) and the term equality judgement (expressing the *definitional equality* between terms of the same type), respectively, all under a context Γ.

The word *type* is used as a meta-variable to indicate two kinds of entities: sets and *small propositions*, namely

$$type \in \{set, prop_s\}$$

Therefore, in $\mathbf{DT}_\Sigma$ types are actually formed by using the following judgements:

$$A\ set\ [\Gamma] \qquad \phi\ prop_s\ [\Gamma]$$

saying that A is a set and that ϕ is a small proposition of $\mathbf{DT}_\Sigma$.

It is worth noting that the adjective *small* is there because in **mTT** we defined small propositions as those propositions closed under quantification over sets, while generic propositions may be closed under quantification over collections. In $\mathbf{DT}_\Sigma$ there are no collections and hence all "$\mathbf{DT}_\Sigma$-propositions" are small but we keep the adjective to make $\mathbf{DT}_\Sigma$ a proper fragment of **mTT**.

As in the intensional version of Martin-Löf's type theory and in **mTT**, in $\mathbf{DT}_\Sigma$ there are two kinds of equality concerning terms: one is the definitional equality of terms of the same type given by the judgement

$$a = b \in A\ [\Gamma]$$

which is decidable, and the other is the propositional equality written

$$\mathsf{Id}(A, a, b)\ prop_s\ [\Gamma]$$

which is not necessarily decidable.

We now proceed by briefly describing the various kinds of types in $\mathbf{DT}_\Sigma$, starting from small propositions and then passing to sets.

Small Propositions in **mTT** include all the logical constructors of intuitionistic predicate logic with equality and quantifications restricted to sets:

$$\phi \ prop_s \equiv \bot \mid \phi \wedge \psi \mid \phi \vee \psi \mid \phi \rightarrow \psi \mid \forall_{x \in A}\, \phi \mid \exists_{x \in A}\, \phi \mid \mathsf{Id}(A, a, b)$$

provided that A is a set. Their rules are those for the corresponding small propositions in **mTT**.

In order to close sets under comprehension and to define operations on such sets, we need to think of propositions as types of their proofs:

$$\textbf{prop}_s\textbf{-into-set)} \quad \frac{\phi \ prop_s}{\phi \ set}$$

Then *sets* in $\mathbf{DT}_\Sigma$ include the following:

$$A \ set \ \equiv \phi \ prop_s \mid \Sigma_{x \in A}\, B \mid \Pi_{x \in A}\, B$$

where the notation $\Sigma_{x \in A}\, B$ stands for the strong indexed sum of the family of sets $B \ set \ [x \in A]$ indexed on the set A and $\Pi_{x \in A}\, B$ for the dependent product set of the family of sets $B \ set \ [x \in A]$ indexed on the set A.

Their rules are those for the corresponding sets in **mTT** and we refer to [12] for their precise formulation.

Both $\mathbf{DT}_\Sigma$ as well as **mTT** can be also essentially seen as fragments of a typed system, which we call $\mathbf{CC}^+$, extending the Calculus of Constructions in [6] with the inductive rules in [4,5,7] defining binary disjoint sums, list types and strong indexed sums (see [12]).

For their crucial role to get the results in this paper, here we just recall the rules of formation, introduction, elimination and conversion of the strong indexed sum as a set:

Strong Indexed Sum

$$\text{F-}\Sigma) \ \frac{C(x) \ \ set \ [x \in B]}{\Sigma_{x \in B} C(x) \ set} \qquad \text{I-}\Sigma) \ \frac{b \in B \quad c \in C(b) \qquad C(x) \ set \ [x \in B]}{\langle b, c \rangle \in \Sigma_{x \in B} C(x)}$$

$$\text{E-}\Sigma) \ \frac{\begin{array}{l} M(z) \ set \ [z \in \Sigma_{x \in B} C(x)] \\ d \in \Sigma_{x \in B} C(x) \quad m(x, y) \in M(\langle x, y \rangle) \ [x \in B, y \in C(x)] \end{array}}{El_\Sigma(d, m) \in M(d)}$$

$$\text{C-}\Sigma) \ \frac{\begin{array}{l} M(z) \ set \ [z \in \Sigma_{x \in B} C(x)] \\ b \in B \quad c \in C(b) \quad m(x, y) \in M(\langle x, y \rangle) \ [x \in B, y \in C(x)] \end{array}}{El_\Sigma(\langle b, c \rangle, m) = m(b, c) \in M(\langle b, c \rangle)}$$

By using these rules we recall that we can define the following projections

$$\begin{array}{l} \pi_1(z) \ \equiv \ El_\Sigma(z, (x, y).x) \in B \ [z \in \Sigma_{x \in B} C(x)] \\ \pi_2(z) \ \equiv \ El_\Sigma(z, (x, y).y) \in C(\pi_1(z)) \ [z \in \Sigma_{x \in B} C(x)] \end{array}$$

satisfying

$$\pi_1(\langle b, c \rangle) = b \in B \ [\Gamma] \qquad\qquad \pi_2(\langle b, c \rangle) = c \in C(b) \ [\Gamma]$$

provided that $C(x)$ *set* $[\Gamma, x \in B]$, $b \in B$ $[\Gamma]$ and $c \in C(b)$ $[\Gamma]$ are derivable in $\mathbf{DT}_\Sigma$.

We also recall the following abbreviations: for set A *set* $[\Gamma]$ and B *set* $[\Gamma]$ we define the set of functions from A to B as

$$A \to B \equiv \Pi_{x \in A} B$$

Now we are ready to define the *choice rule*:

Definition 1. The dependent type theory $\mathbf{DT}_\Sigma$ satisfies the *choice rule* if for every small proposition $R(x, y)$ $prop_s$ $[x \in A, y \in B]$ derivable in $\mathbf{DT}_\Sigma$, for any derivable judgement in $\mathbf{DT}_\Sigma$ of the form

$$p(x) \in \exists_{y \in B} R(x, y) \ [x \in A]$$

there exists in $\mathbf{DT}_\Sigma$ a typed term

$$f(x) \in B[x \in A]$$

for which we can find a proof-term $q(x)$ and derive in $\mathbf{DT}_\Sigma$

$$q(x) \in R(x, f(x)) \ [x \in A]$$

Then we recall the definition of the *axiom of choice* by internalizing the above choice rule as follows:

Definition 2. The *axiom of choice* is the following small proposition

$$\text{(AC)} \qquad \forall x \in A\ \exists y \in B\ R(x, y) \quad \longrightarrow \quad \exists f \in A \to B\ \forall x \in A\ R(x, f(x))$$

defined for any small proposition $R(x, y)$ $prop_s$ $[x \in A, y \in B]$.

A special instance of the choice rule is the following *unique choice rule*

Definition 3. The dependent type theory $\mathbf{DT}_\Sigma$ satisfies the *unique choice rule* if for every small proposition $R(x, y)$ $prop_s$ $[x \in A, y \in B]$ derivable in $\mathbf{DT}_\Sigma$, for any derivable judgement in $\mathbf{DT}_\Sigma$ of the form

$$p(x) \in \exists!_{y \in B} R(x, y) \ [x \in A]$$

there exists a typed term $f(x) \in B[x \in A]$ for which we can find a proof-term $q(x)$ and derive in $\mathbf{DT}_\Sigma$

$$q(x) \in R(x, f(x)) \ [x \in A]$$

where

$$\begin{aligned} &\exists! y \in B\ R(x, y) \equiv \\ &\qquad \exists y \in B\ R(x, y)\ \wedge\ \forall y_1, y_2 \in B (R(x, y_1)\ \wedge\ R(x, y_1) \to \mathsf{Id}(B, y_1, y_2)) \end{aligned}$$

Then the *axiom of unique choice* is the internal form of the unique choice rule defined as follows:

Definition 4. The *axiom of unique choice* is the following small proposition

$$\text{(AC!)} \qquad \forall x \in A\ \exists! y \in B\ R(x,y) \quad \longrightarrow \quad \exists f \in A \rightarrow B\ \forall x \in A\ R(x, f(x))$$

defined for any small proposition $R(x,y)\ prop_s\ [x \in A, y \in B]$.

Observe the following obvious relation between the above rules and the corresponding axioms:

Lemma 1. If $\mathbf{DT}_\Sigma$ satisfies the choice rule then $\mathbf{DT}_\Sigma$ proves the unique choice rule.

Lemma 2. If $\mathbf{DT}_\Sigma$ proves the axiom of choice then $\mathbf{DT}_\Sigma$ proves the axiom of unique choice.

Now we are ready to show the following crucial proposition:

Proposition 1. If $\mathbf{DT}_\Sigma$ satisfies the unique choice rule then $\mathbf{DT}_\Sigma$ proves the axiom of unique choice.

Proof. Suppose that $R(x,y)\ prop_s\ [x \in A, y \in B]$ is derivable in $\mathbf{DT}_\Sigma$. Observe that we can derive in $\mathbf{DT}_\Sigma$

$$\pi_2(z) \in \exists!_{y \in B}\ R(\pi_1(z), y)\ [z \in \Sigma_{x \in A}\ \exists!_{y \in B}\ R(x,y)]$$

Suppose now that the unique choice rule is valid in $\mathbf{DT}_\Sigma$. Then, by using this rule there exists a typed term

$$f(z) \in B\ [z \in \Sigma_{x \in A}\ \exists!_{y \in B}\ R(x,y)]$$

and a proof-term $q(z)$ of $\mathbf{DT}_\Sigma$ for which we can derive

$$q(z) \in R(\pi_1(z), f(z))\ [z \in \Sigma_{x \in A}\ \exists!_{y \in B}\ R(x,y)]$$

By using these proof-terms we can derive

$$\langle m(w), h(w) \rangle \in \exists_{g \in A \rightarrow B}\ \forall x \in A\ R(x, g(x))\ [w \in \forall_{x \in A}\ \exists!_{y \in B}\ R(x,y)]$$

where

$$m(w) \equiv \lambda x'.f(\langle x', w(x') \rangle)$$

since we can derive

$$w(x') \in \exists!_{y \in B}\ R(x'\ ,\ y)\ [w \in \forall_{x \in A}\ \exists!_{y \in B}\ R(x,y), x' \in A]$$

and

$$\langle x', w(x') \rangle \in \Sigma_{x \in A}\ \exists!_{y \in B}\ R(x,y)\ [w \in \forall_{x \in A}\ \exists!_{y \in B}\ R(x,y), x' \in A]$$

and where

$$h(w) \equiv \lambda x''.q(\langle x'', w(x'')\rangle) \in \forall x \in A\ R(x\,,\, m(w)(x))$$

since

$$q(\langle x'', w(x'')\rangle) \in R(\pi_1(\langle x'', w(x'')\rangle)\,,\, f(\langle x'', w(x'')\rangle)) = R(x'',\, m(w)(x''))$$

for $x'' \in A$ and $w \in \forall_{x\in A}\ \exists!_{y\in B}\ R(x,y)$.

Finally we conclude that

$$\begin{aligned}&\lambda w.\langle m(w), h(w)\rangle \in \\ &\qquad \forall x \in A\ \exists! y \in B\ R(x,y) \quad \longrightarrow \quad \exists f \in A \to B\ \forall x \in A\ R(x, f(x))\end{aligned}$$

i.e. we conclude that the axiom of unique choice is valid in $\mathbf{DT}_\Sigma$.

Observe that the above proof can be adapted to show that also the choice rule implies its axiomatic form by simply replacing $\exists!_{y\in B^-}$ with $\exists_{y\in B^-}$ in the proof of Proposition 1 and hence we also get:

Proposition 2. If $\mathbf{DT}_\Sigma$ satisfies the choice rule then $\mathbf{DT}_\Sigma$ satisfies the axiom of choice.

By definition $\mathbf{DT}_\Sigma$ is a fragment of **mTT** and therefore by repeating the proofs above we conclude that:

Proposition 3. If **mTT** satisfies the unique choice rule then **mTT** satisfies the axiom of unique choice.

Proposition 4. If **mTT** satisfies the choice rule then **mTT** satisfies the axiom of choice.

Now, observe that $\mathbf{DT}_\Sigma$ can be seen essentially as a fragment of $\mathbf{CC}^+$ after interpreting $\mathbf{DT}_\Sigma$-sets as $\mathbf{CC}^+$-sets and $\mathbf{DT}_\Sigma$-small propositions as the corresponding $\mathbf{CC}^+$-propositions. In the same way **mTT** can be also viewed essentially as a fragment of $\mathbf{CC}^+$ as first described in [12]. Therefore, we also get the following:

Proposition 5. If $\mathbf{CC}^+$ satisfies the unique choice rule then it satisfies the axiom of unique choice.

Proposition 6. If $\mathbf{CC}^+$ satisfies the choice rule then $\mathbf{CC}^+$ satisfies the axiom of choice.

Note that the above propositions hold also for the extension of $\mathbf{CC}^+$ with inductive definitions [7].

Then, we recall the following result by T. Streicher:

Theorem 1 (T. Streicher). $\mathbf{CC}^+$ does not validate the axiom of unique choice and hence the axiom of choice.

Proof. This is based on [24] and the fact that types are interpreted as assemblies which can be organized into a lextensive regular locally cartesian closed category with a natural numbers object [8,22].

Again since $\mathbf{DT}_\Sigma$ and **mTT** can be both viewed essentially as fragments of $\mathbf{CC}^+$, we also get from Theorem 1:

Corollary 1. Both $\mathbf{DT}_\Sigma$ and **mTT** do not validate the axiom of unique choice and hence the axiom of choice.

Now from Proposition 1 and Corollary 1 we get:

Theorem 2. $\mathbf{DT}_\Sigma$ does not validate the unique choice rule and hence the choice rule.

Analogously, from Proposition 3 and Corollary 1 we also get:

Theorem 3. **mTT** does not validate the unique choice rule and hence the choice rule.

And from Proposition 5 and Theorem 1 we finally get:

Theorem 4. $\mathbf{CC}^+$ does not validate the unique choice rule and hence the choice rule.

We conclude by saying that, as suggested by T. Streicher, it is very plausible that the model in [24] can be extended to interpret generic inductive definitions and hence to show that the axiom of unique choice is not provable in $\mathbf{CC}^+$ extended with generic inductive definitions. Therefore, if this is confirmed, from Proposition 5 we can conclude that the unique choice rule is not valid even in this extension of $\mathbf{CC}^+$.

Remark 1. We believe that in the context of categorical models of dependent type theories we can prove categorical results corresponding to Propositions 1 and 2.

Indeed, the relationship between the choice rules and their axiomatic form in Propositions 1 and 2 was inspired by categorical investigations done in a series of papers [13–15] about setoid models used in dependent type theory to interpret quotient sets. In particular these papers focus their analysis on the quotient model used in [12] to interpret the extensional level of the Minimalist Foundation into its intensional level **mTT**. As shown in [19], the model used in [12] coincides with the usual exact completion in category theory if and only if the unique choice rule is valid in the completion. We expect to be able to prove also in the context of [19] that the unique choice rule implies the axiom of unique choice, as well as that the choice rule implies the axiom of choice.

3 Conclusion

From the above results it follows that when proving a statement of the form

$$\forall_{x \in A} \exists y \in B \ R(x, y)$$

in the dependent typed theory $\mathbf{CC}^+$ or in **mTT** we can not always extract a functional term $f \in A \rightarrow B$ computing the witness of the existential quantification depending on a $x \in A$, within the theory itself and we need to find it in a more expressive proofs-as-programs theory.

For **mTT** we can use Martin-Löf's type theory **ML** as the more expressive theory where to perform the mentioned witness extraction. This is done by first embedding into **ML** the proof-term

$$p \in \forall_{x \in A} \ \exists y \in B \ R(x, y)$$

derived in **mTT** and then using **ML**-projections to extract f.

Another possibility is to perform this witness extraction in the realizability model in [9] showing consistency of **mTT** extended with the Formal Church thesis and the axiom of choice. Moreover, in the case we simply want to extract computable witness from unique existential statements proved in **mTT** under hypothesis we can use also other realizability models such as that in [17,18] showing consistency of **mTT** with the axiom of unique choice and the formal Church thesis.

For $\mathbf{CC}^+$, and even more for its extension with inductive definitions, it is an open problem whether there is a realizability model showing its consistency with the axiom of choice and the formal Church thesis.

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