

Chapter 2

Riemann's Method in Plasticity: a Review

Sergei Alexandrov

Abstract This paper deals with models of pressure-independent and pressure dependent plasticity under plane strain conditions and provides a review of quantities satisfying the equation of telegraphy. This equation can be solved by the method of Riemann. In particular, the Green's function for the equation of telegraphy is the Bessel function of zero order. An advantage of using the method of Riemann for solving boundary value problems is a high accuracy of solutions. Therefore, solutions found by this method can be used for verifying the accuracy of other methods. Some results presented in this paper are restricted to rigid plastic solids whereas others are independent of whether elastic strains are included. The last section of the paper concerns with the theory of ideal flows.

Key words: Telegraph equation · Riemann method · Plane strain · Pressure-Independent plasticity · Pressure-Dependent plasticity · Ideal flows

2.1 Preliminary Remarks

The present paper focuses on three systems of constitutive equations of plastic solids under plane strain conditions. One of these systems is the classical theory of perfectly plastic solids. A great account on this model has been given in Hill (1950). The results for this model are valid for any yield criterion in the case of rigid plasticity. However, they are independent of whether elastic strains are included for the Tresca yield criterion. The other systems are based on pressure-dependent yield criteria. One of these criteria is the Coulomb-Mohr yield criterion. This criterion together with the stress equilibrium equations under plane strain conditions forms

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a statically determinate system. Therefore, the results presented for this system are restricted to stress analysis but are independent of any flow rule that may be chosen to calculate the deformation and also independent of whether elastic strains are included. Reviews of models of pressure-dependent plasticity based on the Coulomb-Mohr yield criterion have been provided in Cox et al (2008) and Goddard (2014). The other pressure-dependent yield criterion adopted in the present paper has been proposed in Druryanov (1993). This criterion is appropriate for powder and porous metals. The corresponding plane strain criterion is derived by means of the associated flow rule. The results for this model are restricted to rigid plasticity. The classical theory of rigid perfectly plastic solids based on the Tresca yield criterion permits a class of solutions named ideal flows. In particular, ideal flows are solenoidal smooth plastic flows in which an eigenvector field associated everywhere with the greatest principal strain rate is fixed in the material. A review of the ideal flow theory has been provided in Chung and Alexandrov (2007). Under plane strain conditions the yield criterion is immaterial. Therefore, the results given in the present paper for ideal flows are valid for any yield criterion.

All the systems of equations considered in the present article are hyperbolic. In what follows, characteristic-based coordinate systems will be denoted by (α, β) .

There are three basic boundary value problems. Those are: (i) two intersecting characteristic curves are given, (ii) the stresses are given along a certain curve (this curve should not coincide with a characteristic curve), and (iii) one characteristic curve together with a curve along with the orientation of the α -line is known is given. Many quantities involved in the aforementioned theories satisfy the equation of telegraphy:

$$\frac{\partial^2 f}{\partial \alpha \partial \beta} + f = 0. \quad (2.1)$$

This equation can be solved by the method of Riemann. In particular, the Green's function for the equation of telegraphy is (Hill, 1950)

$$G(a, b, \alpha, \beta) \equiv J_0 \left[2\sqrt{(a - \alpha)(b - \beta)} \right] \quad (2.2)$$

where $J_0(z)$ is the Bessel function of zero order. A property of the Green's function is

$$\oint \left[\left(G \frac{\partial f}{\partial \alpha} - f \frac{\partial G}{\partial \alpha} \right) d\alpha + \left(f \frac{\partial G}{\partial \beta} - G \frac{\partial f}{\partial \beta} \right) \right] d\beta = 0. \quad (2.3)$$

Here the integral is taken round any closed contour. Choosing an appropriate contour the aforementioned boundary value problems can be solved by means of equation (2.3).

The intention given in the present paper has been to define each new symbol where it first appears in the text. There are, however, certain symbols that re-appear consistently throughout the text. These symbols are defined here. In particular, σ_1 is the algebraically greatest principal stress, ψ is the angle between the axis corresponding to the principal stress σ_1 and the x of Cartesian coordinates (x, y) measured from the x -axis anticlockwise, ϕ is the angle between the α -direction of the characteristic

based coordinate system (α, β) and the x -axis measured from the x -axis anticlockwise, R is the radius of curvature of the α -characteristic lines, and S is the radius of curvature of the β -characteristic lines. Note that R and S are algebraic quantities whose signs depend on the sense of space derivatives.

2.2 Pressure-Independent Plasticity

A great account on the theory of pressure-independent rigid perfectly plastic solids has been given in Hill (1950). Plane strain deformation of such solids is described by a hyperbolic system of equations. Any plane strain yield criterion can be written in the form

$$(\sigma_{xx} - \sigma_{yy})^2 + 4\sigma_{xy}^2 = 4k^2 \quad (2.4)$$

where σ_{xx} , σ_{yy} and σ_{xy} are the components of the stress tensor in Cartesian coordinates (x, y) and k is the shear yield stress, a material constant. The results presented below are also valid for elastic/plastic deformation if the Tresca yield criterion is adopted. The flow rule associated with the yield criterion (2.4) is

$$\xi_{xx} = \lambda(\sigma_{xx} - \sigma_{yy}), \quad \xi_{yy} = -\lambda(\sigma_{xx} - \sigma_{yy}), \quad \xi_{xy} = 2\lambda\sigma_{xy}, \quad (2.5)$$

where ξ_{xx} , ξ_{yy} and ξ_{xy} are the components of the strain rate tensor in the Cartesian coordinates and λ is a non-negative multiplier. Eliminating λ in (2.5) and expressing the strain rate components in terms of the velocity components relative the Cartesian coordinates, u_x and u_y , yield

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0, \quad 2\sigma_{xy} \left[\frac{\partial u_x}{\partial x} - \frac{\partial u_y}{\partial y} \right] = (\sigma_{xx} - \sigma_{yy}) \left[\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right]. \quad (2.6)$$

The equilibrium equations are

$$\partial\sigma_{xx}/\partial x + \partial\sigma_{xy}/\partial y = 0, \quad \partial\sigma_{xy}/\partial x + \partial\sigma_{yy}/\partial y = 0. \quad (2.7)$$

The system of equations consisting of (2.4), (2.6) and (2.7) is hyperbolic. The characteristics of the stresses and the velocities coincide. Therefore, there are only two distinct characteristic directions at a point. The characteristic curves are determined by the equations:

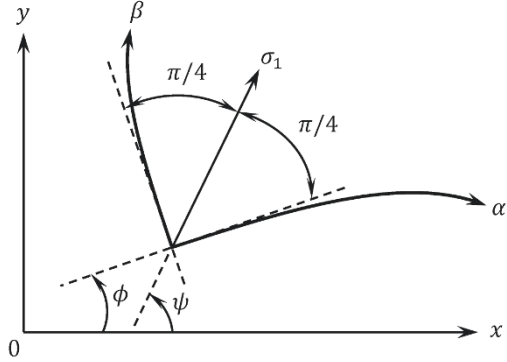
$$dy/dx = \tan(\psi - \pi/4), \quad dy/dx = \tan(\psi + \pi/4). \quad (2.8)$$

The orientation of the algebraically greatest principal stress and characteristic curves relative to the Cartesian coordinate system is illustrated in Fig. 2.1.

In what follows it is assumed that the two families of characteristics are curved. In this case the equilibrium equations (2.7) are equivalent to Hill (1950)

$$p - p_0 + 2k(\psi - \psi_0) = 4k\beta, \quad p - p_0 - 2k(\psi - \psi_0) = -4k\alpha. \quad (2.9)$$

Fig. 2.1 Orientation of the algebraically greatest principal stress σ_1 and characteristic based coordinates (α, β) relative to Cartesian coordinates (x, y) for yield criterion (2.4)



Here $p = -(\sigma_{xx} + \sigma_{yy})/2$, p_0 and ψ_0 are constant. Solving (2.9) for $p - p_0$ and $\psi - \psi_0$ yields

$$p - p_0 = 2k(\beta - \alpha), \quad \psi - \psi_0 = \beta + \alpha. \quad (2.10)$$

Let ϕ be the anticlockwise angular rotation of the α -line from the x -axis (Fig. 2.1). It is seen from (2.8) that the characteristic curves are orthogonal. Therefore, the anticlockwise angular rotation of the β -line from the x -axis is $\phi + \pi/2$. The radii of curvature R and S of the α - and β -lines are defined by the equations:

$$R^{-1} = \partial\phi/\partial s_\alpha, \quad S^{-1} = -\partial(\phi + \pi/2)/\partial s_\beta = -\partial\phi/\partial s_\beta \quad (2.11)$$

where $\partial/\partial s_\alpha$ and $\partial/\partial s_\beta$ are space derivatives taken along the α - and β -lines respectively. It is seen from (2.8) that

$$\psi = \phi + \pi/4. \quad (2.12)$$

Therefore, Eqs. (2.11) are equivalent to

$$R^{-1} = \partial\psi/\partial s_\alpha, \quad S^{-1} = -\partial\psi/\partial s_\beta. \quad (2.13)$$

It follows from the geometry of Fig. 2.1 that

$$\begin{aligned} \partial x/\partial s_\alpha &= \cos(\psi - \pi/4), \quad \partial x/\partial s_\beta = -\sin(\psi - \pi/4), \\ \partial y/\partial s_\alpha &= \sin(\psi - \pi/4), \quad \partial y/\partial s_\beta = \cos(\psi - \pi/4), \end{aligned}$$

Using (2.10) and (2.13) these equations are transformed to

$$\begin{aligned} \partial x/\partial \alpha &= R \cos(\psi - \pi/4), \quad \partial x/\partial \beta = S \sin(\psi - \pi/4), \\ \partial y/\partial \alpha &= R \sin(\psi - \pi/4), \quad \partial y/\partial \beta = -S \cos(\psi - \pi/4). \end{aligned} \quad (2.14)$$

The compatibility equations are

$$\frac{\partial^2 x}{\partial \alpha \partial \beta} = \frac{\partial^2 x}{\partial \beta \partial \alpha}, \quad \frac{\partial^2 y}{\partial \alpha \partial \beta} = \frac{\partial^2 y}{\partial \beta \partial \alpha} \quad (2.15)$$

Substituting (2.14) into these equations and taking into account (2.10) result in

$$\begin{aligned} \frac{\partial R}{\partial \beta} \cos\left(\psi - \frac{\pi}{4}\right) - R \sin\left(\psi - \frac{\pi}{4}\right) &= \frac{\partial S}{\partial \alpha} \sin\left(\psi - \frac{\pi}{4}\right) + S \cos\left(\psi - \frac{\pi}{4}\right), \\ \frac{\partial R}{\partial \beta} \sin\left(\psi - \frac{\pi}{4}\right) + R \cos\left(\psi - \frac{\pi}{4}\right) &= -\frac{\partial S}{\partial \alpha} \cos\left(\psi - \frac{\pi}{4}\right) + S \sin\left(\psi - \frac{\pi}{4}\right) \end{aligned}$$

Solving these equations for $\partial R/\partial \beta$ and $\partial S/\partial \alpha$ gives

$$\partial R/\partial \beta = S, \quad \partial S/\partial \alpha = -R. \quad (2.16)$$

It is evident from these equations that R and S separately satisfy (2.1). Once R and S have been found, the dependence of x and y on α and β is determined from (2.14) where ψ should be eliminated by means of (2.10). The dependence of the stresses on α and β is found from

$$\sigma_{xx} = -p + \cos 2\psi, \quad \sigma_{yy} = -p - \cos 2\psi, \quad \sigma_{xy} = k \sin 2\psi, \quad (2.17)$$

where ψ and p should be eliminated by means of (2.10). Thus the dependence of the stresses on x and y is given in parametric form with α and β being the parameters.

The Mikhlin coordinates $\bar{x}$ and $\bar{y}$ are defined by the equations

$$\bar{x} = x \cos \phi + y \sin \phi, \quad \bar{y} = -x \sin \phi + y \cos \phi. \quad (2.18)$$

These quantities are the coordinates of the point P under consideration referred to axes passing through the origin O and parallel to the characteristic directions at P (Fig. 2.2). Differentiating the first equation in (2.18) with respect to β and the second with respect to α leads to

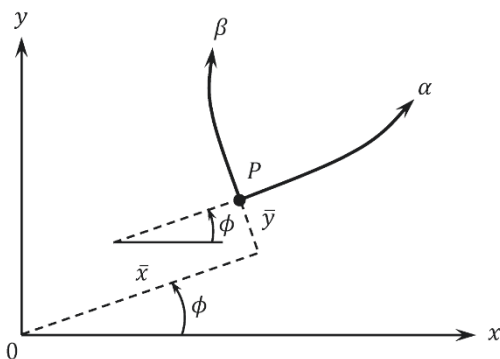


Fig. 2.2 Illustration of Mikhlin coordinates

$$\begin{aligned}\frac{\partial \bar{x}}{\partial \beta} &= \frac{\partial x}{\partial \beta} \cos \phi - x \sin \phi + \frac{\partial y}{\partial \beta} \sin \phi + y \cos \phi, \\ \frac{\partial \bar{y}}{\partial \alpha} &= -\frac{\partial x}{\partial \alpha} \sin \phi - x \cos \phi + \frac{\partial y}{\partial \alpha} \cos \phi - y \sin \phi.\end{aligned}\quad (2.19)$$

It has been taken into account here that $\partial \phi / \partial \alpha = \partial \phi / \partial \beta = 1$ due to (2.10) and (2.12). Using (2.12) equations (2.8) can be rewritten as

$$\frac{\partial y}{\partial \alpha} \cos \phi - \frac{\partial x}{\partial \alpha} \sin \phi = 0, \quad \frac{\partial y}{\partial \beta} \sin \phi + \frac{\partial x}{\partial \beta} \cos \phi = 0. \quad (2.20)$$

Substituting (2.18) and (2.20) into (2.19) gives

$$\partial \bar{x} / \partial \alpha = -\bar{x}, \quad \partial \bar{x} / \partial \beta = \bar{y}. \quad (2.21)$$

It is evident from these equations that $\bar{x}$ and $\bar{y}$ separately satisfy (2.1). Once $\bar{x}$ and $\bar{y}$ have been found, the dependence of x and y on α and β is determined from (2.18) where ϕ should be eliminated by means of (2.12) and then by means of (2.10). Thus using (2.10), (2.12) and (2.17) the dependence of the stresses on x and y is given in parametric form with α and β being the parameters.

Let us introduce a principal lines based coordinate system (ξ, η) where ξ - and η -lines are trajectories of the principal stresses σ_ξ and σ_η , respectively. Let h be the Lamé coefficient for the ξ -lines. Then, it is always possible to choose the parameterization of the η -lines such that the Lamé coefficient for these lines is $1/h$ (Sadovsky, 1941). Assume that $\sigma_\xi \equiv \sigma_1$. Then, it follows from the geometry of Fig. 2.1 that

$$\frac{\partial x}{\partial \xi} = h \cos \psi, \quad \frac{\partial x}{\partial \eta} = -\frac{1}{h} \sin \psi, \quad \frac{\partial y}{\partial \xi} = h \sin \psi, \quad \frac{\partial y}{\partial \eta} = \frac{1}{h} \cos \psi. \quad (2.22)$$

The compatibility equations are

$$\frac{\partial^2 x}{\partial \xi \partial \eta} = \frac{\partial^2 x}{\partial \eta \partial \xi}, \quad \frac{\partial^2 y}{\partial \xi \partial \eta} = \frac{\partial^2 y}{\partial \eta \partial \xi}$$

Substituting (2.22) into these equations and rotating the (x, y) coordinate system such that $\psi = 0$ result in

$$\frac{\partial h}{\partial \eta} + \frac{1}{h} \frac{\partial \psi}{\partial \xi} = 0, \quad \frac{1}{h^3} \frac{\partial h}{\partial \xi} + \frac{\partial \psi}{\partial \eta} = 0. \quad (2.23)$$

Using a standard procedure it is possible to show that this system of equations is hyperbolic. Its characteristic lines are given by

$$d\eta/d\xi = \mp h^2 \quad (2.24)$$

and the relations along these lines by

$$dh \mp h d\psi = 0. \quad (2.25)$$

The upper sign in (2.24) and (2.25) corresponds to the α -characteristic lines and the lower sign to the β -characteristic lines (Fig. 2.1). Equation (2.25) can be immediately integrated to give

$$\ln h - (\psi - \psi_0) = s(\beta), \quad \ln h + (\psi - \psi_0) = r(\alpha). \quad (2.26)$$

Here $r(\alpha)$ is an arbitrary function of α , $s(\beta)$ is an arbitrary function of β , and ψ_0 is constant. Assume that the two families of characteristic lines are curved. In this case, different choices of the functions $r(\alpha)$ and $s(\beta)$ involved in (2.26) merely change the scale of the α - and β -characteristic lines, respectively. Therefore, without loss of generality it is possible to select $r(\alpha) = 2\alpha$ and $s(\beta) = -2\beta$. Then, (2.26) becomes

$$\ln h - (\psi - \psi_0) = -2\beta, \quad \ln h + (\psi - \psi_0) = 2\alpha.$$

Solving these equations for h and $\psi - \psi_0$ yields

$$h = \exp(\alpha - \beta), \quad \psi - \psi_0 = \alpha + \beta. \quad (2.27)$$

Equation (2.24) is equivalent to

$$\partial \eta / \partial \alpha = -h^2 \partial \xi / \partial \alpha, \quad \partial \eta / \partial \beta = h^2 \partial \xi / \partial \beta. \quad (2.28)$$

Let us introduce the new variables μ and ω defined by

$$\xi = \mu \exp(\beta - \alpha), \quad \eta = \omega \exp(\alpha - \beta). \quad (2.29)$$

Substituting (2.29) into (2.28) and eliminating h by means of (2.27) yields

$$\partial(\mu + \omega) / \partial \alpha = \mu - \omega, \quad \partial(\omega - \mu) / \partial \beta = \mu + \omega. \quad (2.30)$$

Differentiating the first of these equations with respect to β and the second with respect to α and then eliminating the first derivatives with respect to α and β by means of (2.30) lead to

$$\frac{\partial^2 \mu}{\partial \alpha \partial \beta} + \frac{\partial^2 \omega}{\partial \alpha \partial \beta} = -\mu - \omega, \quad \frac{\partial^2 \omega}{\partial \alpha \partial \beta} - \frac{\partial^2 \mu}{\partial \alpha \partial \beta} = \mu - \omega$$

Solving these equations for $\partial^2 \mu / \partial \alpha \partial \beta$ and $\partial^2 \omega / \partial \alpha \partial \beta$ gives

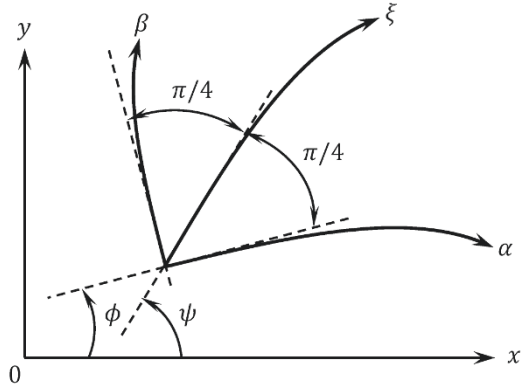
$$\frac{\partial^2 \omega}{\partial \alpha \partial \beta} + \omega = 0, \quad \frac{\partial^2 \mu}{\partial \alpha \partial \beta} + \mu = 0. \quad (2.31)$$

Thus ω and μ separately satisfy (2.1). It follows from the geometry of Fig. 2.3 and (2.13) that

$$R = \sqrt{2}h \frac{\partial \xi}{\partial \alpha}, \quad S = -\sqrt{2}h \frac{\partial \xi}{\partial \beta}$$

Substituting (2.27) and (2.29) into these equations and using (2.30) yield

Fig. 2.3 Orientation of ξ -coordinate curves relative to (α, β) and (x, y) coordinates



$$\begin{aligned} R &= \sqrt{2} \left(\frac{\partial \mu}{\partial \alpha} - \mu \right) = -\sqrt{2} \left(\frac{\partial \omega}{\partial \alpha} + \omega \right), \\ S &= -\sqrt{2} \left(\frac{\partial \mu}{\partial \beta} + \mu \right) = -\sqrt{2} \left(\frac{\partial \omega}{\partial \beta} - \omega \right). \end{aligned} \quad (2.32)$$

Using (2.27), (2.32) and the solution of (2.31) the right hand sides of the equations in (2.14) can be expressed as functions of α and β . Therefore, the dependence of x and y on α and β can be found from (2.14) by integration along any path. Then, using (2.29) and the solution of (2.31) it is possible to determine ξ and η as functions of x and y in parametric form with α and β being the parameters.

Let u and v be the velocity components referred to the α - and β - characteristic lines. The Geiringer equations are equivalent to Hill (1950)

$$\partial u / \partial \alpha - v = 0, \quad \partial v / \partial \beta + u = 0. \quad (2.33)$$

It is evident from this equation that u and v separately satisfy (2.1).

2.3 Pressure-Dependent Plasticity

The Mohr-Coulomb yield criterion is widely used in the mechanics of granular and other materials (Cox et al, 2008; Goddard, 2014). The system of equations comprising this criterion and the stress equilibrium equations under plane strain conditions forms a statically determinate system. This system is hyperbolic (Spencer, 1964). The Mohr-Coulomb yield criterion under plane strain conditions reads

$$q - p \sin \varphi = c \cos \varphi \quad (2.34)$$

where c is the cohesion, φ is the angle of internal friction and

$$2p = -(\sigma_{xx} + \sigma_{yy}), \quad 2q = \sqrt{(\sigma_{xx} - \sigma_{yy})^2 + 4\sigma_{xy}^2}.$$

Both c and ϕ are constant. The characteristic lines are given by Spencer (1964)

$$dy/dx = \tan(\psi - \pi/4 - \phi/2), \quad dy/dx = \tan(\psi + \pi/4 + \phi/2). \quad (2.35)$$

The first equation determines the α -lines and the second the β -lines. The characteristic relations along the α - and β -lines are

$$\cot \phi dq + 2q d\psi = 0, \text{ and } \cot \phi dq - 2q d\psi = 0, \quad (2.36)$$

respectively.

In what follows it is assumed that the two families of characteristics are curved. In this case equation (2.36) is equivalent to

$$\cot \phi \ln(q/q_0) + 2(\psi - \psi_0) = 4\beta \cos \phi, \quad \cot \phi \ln(q/q_0) - 2(\psi - \psi_0) = -4\alpha \cos \phi,$$

where q_0 and ψ_0 are constant. It immediately follows from these equations that

$$\ln(q/q_0) = 2(\beta - \alpha) \sin \phi, \quad \psi - \psi_0 = (\beta + \alpha) \cos \phi. \quad (2.37)$$

It is seen from (2.35) that the angle between the direction of the algebraically greatest principal stress and each of the characteristic directions is constant and is equal to $\pi/4 + \phi/2$ (Fig. 2.4). Therefore, Eqs. (2.11) and (2.13) for the radii of curvature of the characteristic lines are valid. It follows from the geometry of Fig. 2.4 that

$$\begin{aligned} \frac{\partial x}{\partial s_\alpha} &= \cos\left(\psi - \frac{\pi}{4} - \frac{\phi}{2}\right), \quad \frac{\partial x}{\partial s_\beta} = -\sin\left(\psi - \frac{\pi}{4} + \frac{\phi}{2}\right), \\ \frac{\partial y}{\partial s_\alpha} &= \sin\left(\psi - \frac{\pi}{4} - \frac{\phi}{2}\right), \quad \frac{\partial y}{\partial s_\beta} = \cos\left(\psi - \frac{\pi}{4} + \frac{\phi}{2}\right). \end{aligned}$$

Using (2.13) and (2.37) these equations are transformed to

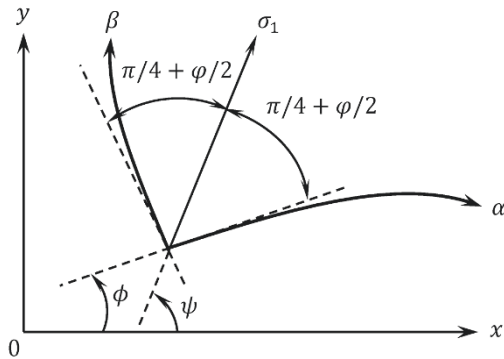


Fig. 2.4 Orientation of the algebraically greatest principal stress σ_1 and characteristic based coordinates (α, β) relative to Cartesian coordinates (x, y) for yield criterion (2.34)

$$\begin{aligned}\frac{\partial x}{\partial \alpha} &= R \cos \varphi \cos \left(\psi - \frac{\pi}{4} - \frac{\varphi}{2} \right), \quad \frac{\partial x}{\partial \beta} = S \cos \varphi \sin \left(\psi - \frac{\pi}{4} + \frac{\varphi}{2} \right), \\ \frac{\partial y}{\partial \alpha} &= R \cos \varphi \sin \left(\psi - \frac{\pi}{4} - \frac{\varphi}{2} \right), \quad \frac{\partial y}{\partial \beta} = -S \cos \varphi \cos \left(\psi - \frac{\pi}{4} + \frac{\varphi}{2} \right).\end{aligned}\quad (2.38)$$

Substituting these equations into the compatibility equations (2.15) and using (2.37) yield

$$\begin{aligned}\frac{\partial R}{\partial \beta} \cos \left(\psi - \frac{\pi}{4} - \frac{\varphi}{2} \right) - R \sin \left(\psi - \frac{\pi}{4} - \frac{\varphi}{2} \right) \cos \varphi &= \\ \frac{\partial S}{\partial \alpha} \sin \left(\psi - \frac{\pi}{4} + \frac{\varphi}{2} \right) + S \cos \left(\psi - \frac{\pi}{4} + \frac{\varphi}{2} \right) \cos \varphi, \\ \frac{\partial R}{\partial \beta} \sin \left(\psi - \frac{\pi}{4} - \frac{\varphi}{2} \right) + R \cos \left(\psi - \frac{\pi}{4} - \frac{\varphi}{2} \right) \cos \varphi &= \\ -\frac{\partial S}{\partial \alpha} \cos \left(\psi - \frac{\pi}{4} + \frac{\varphi}{2} \right) + S \sin \left(\psi - \frac{\pi}{4} + \frac{\varphi}{2} \right) \cos \varphi.\end{aligned}$$

Solving these equations for $\partial R / \partial \beta$ and $\partial S / \partial \alpha$ results in

$$\partial R / \partial \beta + R \sin \varphi = S, \quad \partial S / \partial \alpha - S \sin \varphi = -R. \quad (2.39)$$

It is evident that these equations reduce to (2.16) at $\varphi = 0$. The further analysis is facilitated by the use of the following transformation equations

$$R = R_0 \exp[(\alpha - \beta) \sin \varphi], \quad S = S_0 \exp[(\alpha - \beta) \sin \varphi]. \quad (2.40)$$

where R_0 and S_0 are new functions of α and β . Substituting (2.40) into (2.39) gives

$$\partial R_0 / \partial \beta = S_0, \quad \partial S_0 / \partial \alpha = -R_0. \quad (2.41)$$

It is evident from these equations that R_0 and S_0 separately satisfy (2.1). This result has been obtained in Alexandrov (2015). Once R_0 and S_0 have been found, the dependence of x and y on α and β is determined from (2.38) where R and S should be eliminated by means of (2.40) and then ψ by means of (2.37). The dependence of p , q and ψ on α and β immediately follows from (2.34) and (2.37). Then, the stresses are found from

$$\sigma_{xx} = -p + q \cos 2\psi, \quad \sigma_{yy} = -p - q \cos 2\psi, \quad \sigma_{xy} = q \sin 2\psi.$$

These relations and the solution for R_0 and S_0 combine to give the dependence of the stresses on x and y in parametric form with α and β being the parameters.

The yield criterion

$$\frac{|\sigma_i - \sigma_j|}{2\tau_s} + \frac{|\sigma|}{p_s} = 1, \quad i, j = 1, 2, 3, i \neq j \quad (2.42)$$

is used for porous and powder materials Druryanov (1993). In equation (2.42), σ_1 , σ_2 and σ_3 are the principal stresses, and τ_s and p_s are prescribed functions of the

porosity. In what follows, it is assumed that the porosity is uniformly distributed and therefore τ_s and p_s are constant. Assuming that the axis corresponding to the stress σ_3 is perpendicular to planes of flow, $\sigma_1 > \sigma_2$ and $\sigma_1 + \sigma_2 + \sigma_3 < 0$ the yield criterion (2.42) under plane strain conditions reduces to Druryanov (1993)

$$\sigma_1(3p_s - 4\tau_s) - \sigma_2(3p_s + 2\tau_s) = 6\tau_s p_s. \quad (2.43)$$

Using the transformation equations for stress components it is possible to find that (Fig. 2.5 a)

$$\begin{aligned} \sigma_{xx} &= \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \cos 2\psi, \quad \sigma_{yy} = \frac{\sigma_1 + \sigma_2}{2} - \frac{\sigma_1 - \sigma_2}{2} \cos 2\psi, \\ \sigma_{xy} &= \frac{\sigma_1 - \sigma_2}{2} \sin 2\psi. \end{aligned} \quad (2.44)$$

Substituting these equations into the equilibrium equations (2.7) yields to

$$\begin{aligned} \frac{\partial(\sigma_1 + \sigma_2)}{\partial x} + \frac{\partial(\sigma_1 - \sigma_2)}{\partial x} \cos 2\psi - 2(\sigma_1 - \sigma_2) \sin 2\psi \frac{\partial\psi}{\partial x} + \\ \frac{\partial(\sigma_1 - \sigma_2)}{\partial y} \sin 2\psi + 2(\sigma_1 - \sigma_2) \cos 2\psi \frac{\partial\psi}{\partial y} = 0, \\ \frac{\partial(\sigma_1 + \sigma_2)}{\partial y} - \frac{\partial(\sigma_1 - \sigma_2)}{\partial y} \cos 2\psi - 2(\sigma_1 - \sigma_2) \sin 2\psi \frac{\partial\psi}{\partial y} + \\ \frac{\partial(\sigma_1 - \sigma_2)}{\partial x} \sin 2\psi + 2(\sigma_1 - \sigma_2) \cos 2\psi \frac{\partial\psi}{\partial x} = 0, \end{aligned} \quad (2.45)$$

One can always rotate the (x, y) coordinate system so that its axes coincide with the principal stress directions at a given point (Fig. 2.5 b). Then $\psi = 0$ and equation (2.45) becomes

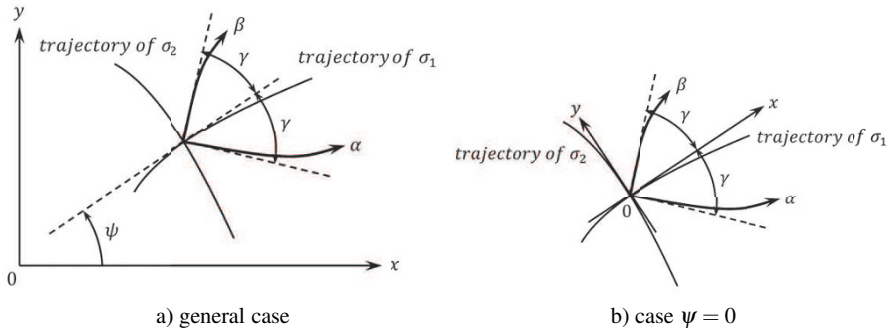


Fig. 2.5: Orientation of the algebraically greatest principal stress σ_1 and characteristic based coordinates (α, β) relative to Cartesian coordinates (x, y) for yield criterion (2.43)

$$\frac{\partial \sigma_1}{\partial x} + (\sigma_1 - \sigma_2) \frac{\partial \psi}{\partial y} = 0, \quad \frac{\partial \sigma_2}{\partial y} + (\sigma_1 - \sigma_2) \frac{\partial \psi}{\partial x} = 0.$$

Using (2.43) to eliminate σ_2 in these equations gives

$$\frac{\partial \sigma_1}{\partial x} + \frac{6\tau_s(\sigma_1 + p_s)}{3p_s + 2\tau_s} \frac{\partial \psi}{\partial y} = 0, \quad (3p_s - 4\tau_s) \frac{\partial \sigma_1}{\partial y} + 6\tau_s(\sigma_1 + p_s) \frac{\partial \psi}{\partial x} = 0. \quad (2.46)$$

Applying a standard procedure it is possible to find that the angle γ between the principal stress axis corresponding to the stress σ_1 and each of the characteristic lines is (Fig. 2.5 b).

$$\gamma = \arctan \sqrt{\frac{3p_s - 4\tau_s}{3p_s + 2\tau_s}}. \quad (2.47)$$

Thus the system (2.46) is hyperbolic if $3p_s > 4\tau_s$. This condition is usually satisfied for porous and powder materials Druryanov (1993). Returning to the (x, y) coordinate system shown in Fig. 2.5 a) the equations for characteristic lines can be written as

$$dy/dx = \tan(\psi \mp \gamma). \quad (2.48)$$

Here and in what follows the upper sign corresponds to the α -lines and the lower sign to the β -lines (Fig. 2.5). The characteristic relations are found from (2.46) and (2.47) as

$$\sqrt{\frac{(3p_s - 4\tau_s)(3p_s + 2\tau_s)}{6\tau_s(\sigma_1 + p_s)}} d\sigma_1 \mp d\psi = 0. \quad (2.49)$$

In what follows it is assumed that the two families of characteristics are curved. In this case integration in (2.49) yields

$$\begin{aligned} -\frac{\sqrt{(3p_s - 4\tau_s)(3p_s + 2\tau_s)}}{6\tau_s} \ln \left[\frac{\sigma_1 + p_s}{\sigma_0} \right] + (\psi - \psi_0) &= 2\beta \sin 2\gamma, \\ -\frac{\sqrt{(3p_s - 4\tau_s)(3p_s + 2\tau_s)}}{6\tau_s} \ln \left[\frac{\sigma_1 + p_s}{\sigma_0} \right] - (\psi - \psi_0) &= -2\alpha \sin 2\gamma, \end{aligned}$$

where σ_0 and γ are constant. It immediately follows from these equations that

$$\begin{aligned} \psi - \psi_0 &= (\alpha + \beta) \sin 2\gamma, \\ -\frac{\sqrt{(3p_s - 4\tau_s)(3p_s + 2\tau_s)}}{6\tau_s} \ln \left[\frac{\sigma_1 + p_s}{\sigma_0} \right] &= (\beta - \alpha) \sin 2\gamma. \end{aligned} \quad (2.50)$$

It is seen from (2.47) that the angle between the direction of the algebraically greatest principal stress and each of the characteristic directions is constant. Therefore, equations (2.11) and (2.13) for the radii of curvature of the characteristic lines are valid. It follows from the geometry of Fig. 2.5 a) that

$$\frac{\partial x}{\partial s_\alpha} = \cos(\psi - \gamma), \quad \frac{\partial x}{\partial s_\beta} = \cos(\psi + \gamma), \quad \frac{\partial y}{\partial s_\alpha} = \sin(\psi - \gamma), \quad \frac{\partial y}{\partial s_\beta} = \sin(\psi + \gamma).$$

Using (2.13) and (2.50) these equations are transformed to

$$\begin{aligned} \partial x / \partial \alpha &= R \sin 2\gamma \cos(\psi - \gamma), \quad \partial x / \partial \beta = -S \sin 2\gamma \cos(\psi + \gamma), \\ \partial y / \partial \alpha &= R \sin 2\gamma \sin(\psi - \gamma), \quad \partial y / \partial \beta = -S \sin 2\gamma \sin(\psi + \gamma), \end{aligned} \quad (2.51)$$

Substituting these equations into (2.15) and using (2.50) yield

$$\begin{aligned} \frac{\partial R}{\partial \beta} \cos(\psi - \gamma) - R \sin 2\gamma \sin(\psi - \gamma) &= -\frac{\partial S}{\partial \alpha} \cos(\psi + \gamma) + S \sin 2\gamma \sin(\psi + \gamma), \\ \frac{\partial R}{\partial \beta} \sin(\psi - \gamma) + R \sin 2\gamma \cos(\psi - \gamma) &= -\frac{\partial S}{\partial \alpha} \sin(\psi + \gamma) - S \sin 2\gamma \cos(\psi + \gamma). \end{aligned}$$

Solving these equations for $\partial R / \partial \beta$ and $\partial S / \partial \alpha$ results in

$$\partial R / \partial \beta - R \cos 2\gamma = S, \quad \partial S / \partial \alpha + S \cos 2\gamma = -R. \quad (2.52)$$

It is evident that these equations reduce to (2.16) at $\gamma = \pi/4$. The further analysis is facilitated by the use of the following transformation equations

$$R = R_0 \exp[(\beta - \alpha) \cos 2\gamma], \quad S = S_0 \exp[(\beta - \alpha) \cos 2\gamma], \quad (2.53)$$

where R_0 and S_0 are new functions of α and β . Substituting (2.53) into (2.52) gives

$$\partial R_0 / \partial \beta = S_0, \quad \partial S_0 / \partial \alpha = -R_0. \quad (2.54)$$

It is evident from these equations that R_0 and S_0 separately satisfy (2.1). A similar result has been obtained in Alexandrov and Lyamina (2015). Once R_0 and S_0 have been found, the dependence of x and y on α and β is determined from (2.51) where R and S should be eliminated by means of (2.53) and ψ by means of (2.50). The dependence of σ_1 and σ_2 on α and β follows from (2.43) and (2.50). Then, the stresses in the Cartesian coordinates are found from (2.44) and (2.50). These relations and the solution for R_0 and S_0 combine to give the dependence of σ_{xx} , σ_{yy} and σ_{xy} on x and y in parametric form with α and β being the parameters.

2.4 Planar Ideal Flows

Ideal plastic flows are those for which all material elements undergo minimum work paths (Chung and Richmond, 1994). The theory of bulk ideal flow has been developed for rigid perfectly plastic solids satisfying Tresca's yield condition and its associated flow rule. However, in the case of plane strain deformation any criterion can be written in the form of Eq. (2.4). The existence of steady three-dimensional ideal flows has been demonstrated in Hill (1967). This result has been extended to non-steady flows in Richmond and Alexandrov (2002). A comprehensive overview on the ideal flow theory and its applications has been provided in Chung and Alexandrov (2007).

In the case of steady flows the ideal flow condition is that the streamlines are everywhere coincident with principal strain rate (and stress) directions. The general theory for this type of ideal flow has been developed in Aleksandrov (2000). The magnitude of the velocity vector is proportional to the scale factor h involved in (2.22). Therefore, the solution for ξ and η given by (2.29) and (2.31) determines the streamlines and the velocity field.

In the case of non-steady flow the ideal flow condition is that the principal lines coordinate system (ξ, η) may be taken to be fixed in the material so that it is Lagrangian. Therefore, the solution for ξ and η given by (2.29) and (2.31) together with the solution of (2.14) where R and S should be eliminated by means of (2.32) and ψ by means of (2.27) supplies the mapping between the Lagrangian coordinates and the Eulerian coordinates (x, y) in parametric form with α and β being the parameters. The general theory for this type of ideal flow has been developed in (Richmond and Alexandrov, 2000).

2.5 Conclusions

It has been demonstrated that a number of quantities in pressure-independent and pressure-dependent plasticity satisfy the telegraph equation (2.1). These quantities are shown in (2.16), (2.21), (2.31), (2.33), (2.41), and (2.54). In the case of ideal flows ξ and η that are found by means of (2.29) and (2.31) have an additional physical meaning explained in the previous section. The solution of (2.1) can be found by means of (2.3) with a high accuracy. In particular, solutions found by this method can be used to verify the accuracy of solutions found by other methods (Hill et al, 1951).

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