Chapter 2
Notes on Point Set Topology

Abstract The chapter provides a brief exposition of point set topology. In particular, it aims to make readers from the engineering community feel comfortable with the subject, especially with those topics required in latter chapters. The implicit appearance of topological concepts in the context of continuum mechanics is sketched first. Afterwards, topological concepts like interior and boundary of sets, continuity of mappings, etc., are discussed within metric spaces before the introduction of the concept of topological space.

2.1 Preliminary Remarks and Basic Concepts

Here, tensor calculus is not seen as an end in itself, but rather in the context of engineering mechanics, particularly continuum mechanics. As already discussed earlier, the latter first requires a model for space.

A first step in defining such a model is to interpret space as a set of points. However, if, for instance, different points should refer somehow to different locations, then a set of points, as such, is rather useless. Although sets already possess a kind of structure due to the notion of subsets and operations like union, intersection, etc., some additional structure is required by which a concept like location makes sense. Experience tells us that location is usually expressed with respect to a previously defined point of reference. This, on the other hand, requires relating different points with each other, for instance, by means of a distance.

Distance is primarily a quantitative concept. However, apart from quantitative characteristics like distances between points, length of a curve, etc., there are also others, commonly subsumed under the term topological characteristics. For instance, in continuum mechanics, the concept of a material body is usually introduced as follows. A material body becomes apparent to us by occupying a region in space at a certain instant in time, i.e., a subset of the set of points mentioned above. A look at the sketches commonly used to illustrate basic ideas of continuum mechanics reveals that it is tacitly assumed that this region is "connected" and usually has an "interior" and a "boundary."

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Intuitively, we have no problem accepting "connected," "interior," and "boundary" as topological properties. However, there are other properties whose topological character is not at all that obvious. Therefore, the question arises as to how to define more precisely what topological properties actually are. The following experiment might give us some initial clues. A party balloon is inflated until it is possible to draw geometric figures on it. This might be called state one. Afterwards, the balloon is inflated further up to a state two. The figures can be inspected in order to work out those properties which do not change under the transition from state one to state two. Having this experiment in mind, one might be tempted to state that topological properties are those which do not change under a continuous transformation, i.e., a continuous mapping. However, without a precise notion of continuity, this is not really a definition but, at best, a starting point from which eventually to work out a rigorous framework. Furthermore, it turns out that not all continuous mappings are suitable for distinguishing unambiguously topological properties from others, but rather only so-called homeomorphisms, bijective mappings which are continuous in both directions, to be discussed in greater detail later.

The discipline which covers the problems sketched so far is called point set topology and it can be outlined in a variety of ways. Perhaps the most intuitive one is to start with metric spaces, i.e., a set of points together with a structure which allows us to measure distances between points. Most texts on topology in metric spaces focus first on convergence of series and continuity. Properties of sets like interior or boundary are addressed only afterwards in order to prepare the next level of abstraction, namely topological spaces. Here, however, we follow Geroch [2] and start with topological properties of sets in metric spaces.

2.2 Topology in Metric Spaces

The concept of a metric space is a generalization of the notion of distance. Measuring distances consists in assigning real numbers to pairs of points by means of some ruler, hence, it is essentially a mapping $d : X \times X \to \mathbb{R}$ where d is called a distance function. A second step toward a more general scheme is to abstain from interpreting the elements of X in a particular way. In the following, the elements of X can be objects of any kind. By combining the set X with a distance function d, a space (X, d) is generated. The most challenging part is to define a minimal set of rules that a distance function must obey such that certain concepts can be defined unambiguously and facts about (X, d) can be deduced by logical reasoning.

Defining this minimal set of rules is an iterative process for a variety of reasons. The design of a particular distance function has to take into account the specific nature of the elements of X. Furthermore, as in daily life, different rulers, hence different distance functions, should be possible for the same X. A minimal set of rules must cover all these cases and should therefore be rather general. In addition, the set of rules has to account for what should be agreed upon no matter which particular distance function is used to perform a measurement on a given set X. And last but not least, for cases in which we do not even need mathematics because things are simply obvious just by looking at them, facts about (X, d) compiled in respective theorems should be in line with our intuition. The final result of this process is the following definition.

Definition 2.1 (*Metric*) Given a point set X, a metric is a mapping

$$d : X \times X \to \mathbb{R}$$

with the properties:

(i) $d(p, q) > 0$,
(ii) $d(p, q) = 0 \quad \text{implies} \quad p = q$ and vice versa,
(iii) $d(p, q) = d(q, p)$,
(iv) $d(p, q) \leq d(p, r) + d(r, q)$,

where $p, q, r \in X$.

The most common representatives of a metric are the absolute value of the difference of two real numbers in $\mathbb{R}$ and the euclidean metric in $\mathbb{R}^n$. However, there are other options (see Example 2.1). While the properties (i)–(iii) in Definition 2.1 are rather obvious, the triangle inequality (iv) is not. Although (iv) is true for absolute value and euclidean metric, it is not obvious at this point why a metric should possess this property in general. This will become clear only later (see Example 2.3).

Example 2.1 The following distance functions are commonly used for $X = \mathbb{R}^n$,

- $d_1(p, q) = \sum_{i=1}^{n} |p_i - q_i|$,
- $d_2(p, q) = \sqrt{\sum_{i=1}^{n} [p_i - q_i]^2}$,
- $d_\infty(p, q) = \max_{1 \leq i \leq n} |p_i - q_i|$,

where $|a|$ denotes the absolute value of the argument a. Regarding the notation, see Fig. 2.1 as well.

The following definitions refer to a metric space (X, d), specifically subsets $A, B \subset X$. By means of a metric, the interior of a set can now be defined unambiguously after introducing a so-called ε-neighborhood.

Definition 2.2 (*Open ε-ball*) The set of points q fulfilling $d(p, q) < \varepsilon$ is called the open ε-ball around p or ε-neighborhood of p.

Definition 2.3 (*Interior*) Given a point set A. The interior of A, int(A), is the set of all points p for which ***some*** ε-neighborhood exists which is completely in A.

It is important for further understanding to dissect Definition 2.3 thoroughly, since it illustrates a rather general idea. Although a metric allows for assigning numbers to pairs of points, what matters are not the specific values but only the possibility as such to assign numbers.

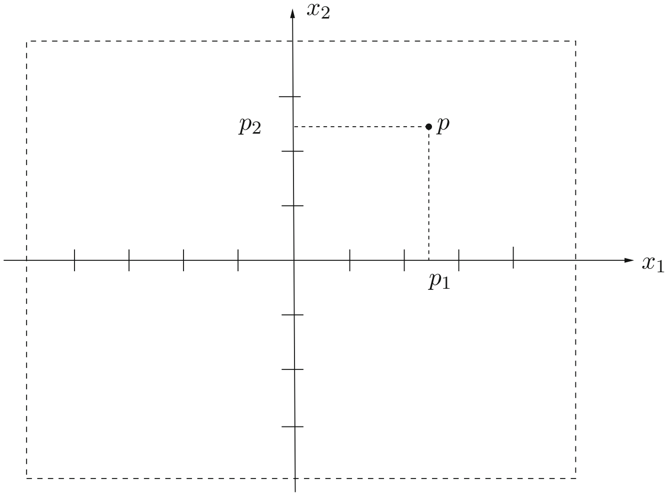

Fig. 2.1 Visualization of the $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$

Example 2.2 Given the interval $A = [0, 1] \subset \mathbb{R}$ and $d(p, q) = |p - q|$. Do the points $p_1 = 0.5$, $p_2 = 0.99$ and $p_3 = 1.0$ belong to the interior of A?
In order to check for p_1, some positive number ε has to be found such that all points q with $d(0.5, q) < \varepsilon$ belong to A. Any ε with $0 < \varepsilon \leq 0.5$ does the job. Since there exists some positive number according to Definition 2.3, p_1 belongs to the interior of A. The point p_2 belongs to the interior of A too, but now we have to chose an ε according to $0 < \varepsilon \leq 0.01$. However, p_3 does not belong to the interior of A.
More generally, the interior of A is the open interval $(0, 1)$.

Similarly, other properties of subsets of X can be defined precisely, for instance, the boundary of a subset of X.

Definition 2.4 (*Boundary*) Given a point set A. The boundary of A, $\text{bnd}(A)$, is the set of all points p such that for ***every*** $\varepsilon > 0$ there exists a point q with $d(p, q) < \varepsilon$ in A ***and*** also a point q' with $d(p, q') < \varepsilon$ not in A.

Based on the properties of the distance function and the foregoing definitions, certain facts about sets in metric spaces can be deduced. Some of these are:

1. $\text{int}(A) \subset A$,
2. $\text{int}(\text{int}(A)) = \text{int}(A)$,
3. if $A \subset X$ then every point of X is either in $\text{int}(A)$ or $\text{int}(A^c)$ or $\text{bnd}(A)$ and no point of X is in more than one of these sets, where A^c is the complement of A.

Example 2.3 In order to illustrate the need for the triangle inequality in Definition 2.1, $\text{int}(\text{int}(A)) = \text{int}(A)$ is discussed in more detail. If a point p belongs to $\text{int}(A)$, then p has an ε-neighborhood of points which all belong to A. On the

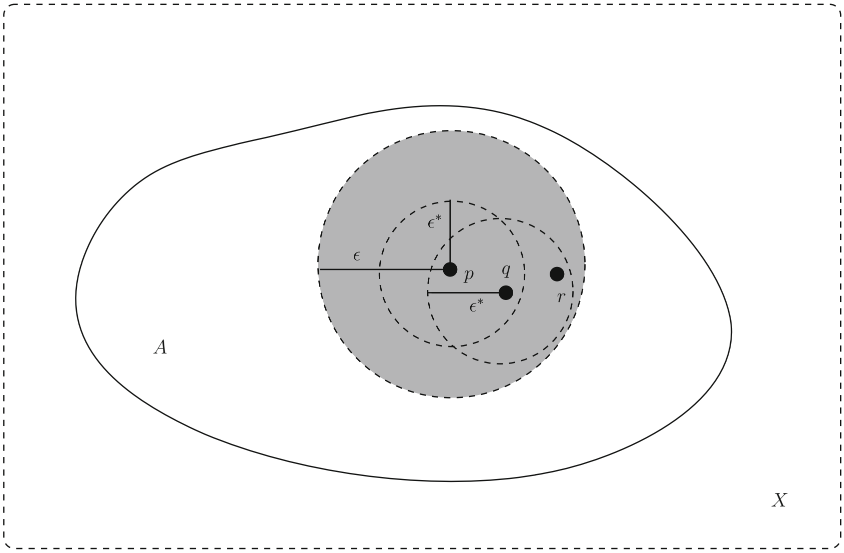

Fig. 2.2 Sketch related to Example 2.3

other hand, if p belongs to int(int(A)) there must exist an ε^*-neighborhood of p which belongs entirely to int(A). In other words, all points in an ε^*-neighborhood of p must have themselves an ε^*-neighborhood of points which belong to A. A sketch, see Fig. 2.2, based on subsets and points in a plane reveals that in this case, this is obviously true for any $p \in \text{int}(A)$. However, in order to ensure the functionality of a metric space (X, d) independently of the particular nature of the elements of X, a formal proof is required.

If $p \in \text{int}(A)$, and, for instance, $\varepsilon^* \leq \frac{1}{2}\varepsilon$ is used, then every point q with $d(p, q) \leq \varepsilon^*$ belongs to A and has itself an ε^*-neighborhood of points which are in A. The latter is ensured by the triangle inequality $d(p, r) \leq d(p, q) + d(q, r)$ where q is any point within the ε^*-neighborhood of p and r is a point within the ε^*-neighborhood of q. Since $d(p, q) \leq \varepsilon/2$ and $d(q, r) \leq \varepsilon/2$, the point r belongs to A, since $d(p, r) \leq \varepsilon$. According to our intuition, an operation which deletes the boundary of an argument should leave that argument, which no longer has a boundary, unchanged. Without triangle inequality, this cannot be assured in general, and such a space (X, d) would not fulfill its purpose.

As already mentioned, the concept of a continuous mapping plays a crucial role in topology. We first define continuity by means of a metric.

Definition 2.5 (*Continuous mapping: $\varepsilon - \delta$ version*) Given the metric spaces (X, d_X) and (Y, d_Y). A mapping $f : X \to Y$ is continuous at $a \in X$ if for ***every*** $\varepsilon > 0$, there exists a $\delta > 0$ such that $d_X(x, a) < \delta$ whenever $d_Y(f(x), f(a)) < \varepsilon$ with $x \in X$.

The following examples use cases, the reader might already know from analysis in $\mathbb{R}$, in the more general setting of this section.

Example 2.4 We consider $X = \mathbb{R}$, $Y = \mathbb{R}$, both with the absolute value distance function and $f = 2x + 14$. Is f continuous at $x = a$?
We have $d_Y(f(x), f(a)) = 2|x - a| < \varepsilon$ and $d_X(x, a) = |x - a| < \delta$. Choosing a particular value for ε, e.g., $\varepsilon = 0.01$, every $\delta < 0.05$ assures that the images of x and a are within a distance smaller than 0.01. However, this must work for every $\varepsilon > 0$, e.g., $\varepsilon = 0.001$, $\varepsilon = 0.0001$, etc. Since, in general we can set $\delta < \frac{\varepsilon}{2}$, a $\delta > 0$ can be found for every $\varepsilon > 0$, and hence, f is continuous at $x = a$. Since this is true for any a, f is continuous everywhere.

So far, this is just the very beginning of an exposition of the theory of metric spaces. However, since here the main objective is to provide a working knowledge in point set topology, we take a short cut. So far, all definitions rely on a distance function. However, a distance can hardly be a topological property, which is a rather unsatisfying situation. One alternative way of defining a minimal structure by which topological concepts can be discussed requires the definition of self-interior sets, also called open sets.

Definition 2.6 (*Open set*) A set A is said to be open or self-interior if $A = \text{int}(A)$.

Once open sets have been defined by means of a distance function d, the latter can be avoided in all subsequent definitions and theorems referring to concepts considered as purely topological. For instance, it can be shown that for metric spaces, the following definition is completely equivalent to Definition 2.7, see, e.g., Mendelson [4].

Definition 2.7 (*Continuous mapping: open sets version*) A mapping $f : X \to Y$ is continuous if for ***every*** open subset $U \subset Y$, the subset $f^{-1}(U) \subset X$ is open.

However, such a reformulation also makes use of the following properties of the collection τ of all open sets induced by the distance function:

1. The empty set $\emptyset$ and X are open.
2. Arbitrary unions of open sets are open.
3. The intersections of two open sets is open.

This leads to the conclusion that a distance function is not needed at all to impose a topological structure on a set. It is sufficient to provide an appropriate collection of sets which are defined formally as open. This is done in the following section. Prior to this, another observation which supports a metric independent generalization of point set topology should be mentioned. A closer look at Example 2.1 reveals that most distance functions are defined through the use of addition, subtraction, and multiplication. This requires the existence of a structure on the considered set which is already rather elaborated. This additional structure is by no means necessary for discussing fundamental topological concepts and it might even obscure the view as to what is important.

2.3 Topological Space: Definition and Basic Notions

Definition 2.8 (*Topological space*) A nonempty set X together with a collection τ of subsets of X with the properties:

(i) the empty set $\emptyset$ and X are in τ,
(ii) arbitrary unions of members of τ are in τ,
(iii) the intersection of any two members of τ is in τ,

is called a topological space (X, τ). The elements of τ are by definition open sets.

Remark 2.1 There is a number of equivalent definitions which can be obtained in a similar way as that sketched for open sets in the previous section. For instance, defining closed sets and working out their properties in a metric space eventually leads to the definition of a topological space based on the notion of closed sets. Another alternative can be worked out using so-called neighborhood systems.

Definition 2.9 (*Closed set*) Given a topological space (X, τ). A set $A \subset X$ is closed, if its complement A^{c} is open.

Remark 2.2 From Definitions 2.9 and 2.8 it follows that $\emptyset$ and X of (X, τ) are both open and closed since $X^{\mathrm{c}} = \emptyset$ and $\emptyset^{\mathrm{c}} = X$.

Example 2.5 The collection of all open intervals $\tau = \{(a, b)\}$, $a, b \in \mathbb{R}$ defines a topological space $(\mathbb{R}, \tau)$. On the other hand, the collection of all open intervals $\tau^* = \{(x - a, x + a)\}$, $a, b, x \in \mathbb{R}$ defines the same topological space.

Metric spaces are automatically topological spaces, because a distance function induces a topology. In this context, two questions arise. The first asks whether the reverse is true as well and the answer here is no. There are topological spaces which do not arise from metric spaces. The second question asks whether the topology of a metric space depends on the choice of a particular distance function. Here, the answer is yes, since different distance functions may induce different topologies. However, it turns out that the distance functions d_1, d_2 and d_∞ defined for $\mathbb{R}^n$ (see Example 2.1) induce the same topology, which is called the standard topology of $\mathbb{R}^n$. One way to prove this is to show that every set that is open in terms of one distance function is also open if one of the other distance functions is used, and vice versa.

Defining the notion of continuous mapping now consists essentially in repeating Definition 2.7 in the context of topological spaces according to Definition 2.8.

Definition 2.10 (*Continuous mapping*) Given two topological spaces (X, τ) and (Y, τ^*). A mapping $f : X \rightarrow Y$ is continuous if for ***every*** open subset $U \subset Y$, the subset $f^{-1}(U) \subset X$ is open.

It can be shown that the composition of continuous mappings is again continuous, which is a important fact, for obvious reasons.

Example 2.6 Consider the finite sets $X = Y = \{0, 1\}$. In order to check whether the mapping

$$\begin{aligned} f : X &\to Y \\ 0 &\mapsto 1 \\ 1 &\mapsto 0 \end{aligned}$$

is continuous or not, it has to be checked as to whether the domains of all open sets in Y are open sets in X. The first observation is that there is not enough information available to do this, since it is not clear which topologies are used. Given the topologies $\tau_1 = \{X, \emptyset\}, \tau_2 = \{X, \emptyset, \{0\}, \{1\}\}$, then $f : (X, \tau_1) \to (Y, \tau_2)$ is not continuous, since the domain of $\{1\}$ is $\{0\}$ but $\{0\}$ does not belong to τ_1.

Remark 2.3 (*Continuity*) Intuitively, one associates continuity with mappings between infinite sets like $\mathbb{R}$ or subsets of $\mathbb{R}$. But Example 2.6 illustrates that after introducing the concept of topological spaces, continuity makes sense for finite sets too. This, on the other hand, has reverse effects. For instance, it provides a reasoning for the transparent motivation of compactness, a far reaching concept to be discussed in more detail later.

Having defined what is meant by a continuous mapping, it is about time to make the notion of topological properties more precise. Intuitively accepting the property of being open as a topological property of a set, the standard example of $f = x^2$ shows that defining topological characteristics as those which are invariant under continuous mappings does not work properly. This can be seen by taking the open interval $(-1, 1)$, which is mapped by f to the half open interval $[0, 1)$, i.e., $f((-1, 1)) = [0, 1)$. Although f is continuous, the property of being open is not preserved under f. The reason for this is that f is not bijective. This leads to the notion of homeomorphism.

Definition 2.11 (*Homeomorphism*) Given two topological spaces (X, τ_X) and (Y, τ_Y). A homeomorphism is a bijective mapping $f : X \to Y$ where f and f^{-1} are continuous.

Since a homeomorphism preserves what is accepted intuitively as topological characteristics, it might serve as a general criterion for distinguishing topological characteristics from others. Eventually, this hypothesis can be confirmed, and topological spaces which can be related by such a mapping are called topologically equivalent or homeomorphic.

One of the most familiar applications of homeomorphisms are coordinate charts, the definition of which also implies a notion of dimensionality of a topological space.

Definition 2.12 (*Coordinate chart*) Given a topological space (X, τ) and an open subset $U \subset X$. A homeomorphism $\phi : U \to V$ where V is an open subset of $\mathbb{R}^n$ is called a coordinate chart.

Definition 2.13 (*Dimension of a topological space*) Given a topological space (X, τ). X has dimension n if every open subset of X is homeomorphic to some open subset of $\mathbb{R}^n$.

2.4 Connectedness, Compactness, and Separability

After introducing the most basic notions, it is rather compulsory to ask under which conditions certain operations can be safely performed. In this context, a number of additional topological concepts will be discussed in the following section, starting with the idea of subspace.

Definition 2.14 (*Subspace*) A topological space (Y, τ_Y) is a subspace of the topological space (X, τ_X), if all members of τ_Y can be derived from $o = O \cap Y$ for $o \in \tau_X$ and some $O \in \tau_Y$. The topology τ_Y is called the relative topology on Y induced by τ_X.

In order to discuss the concept of connectedness, we start with the definition of a path in a topological space X.

Definition 2.15 (*Path*) A path is a continuous function f which maps a closed interval of $\mathbb{R}$ to a topological space X,

$$\begin{aligned} f : \mathbb{R} &\to X \\ [a, b] &\mapsto \Gamma \subset X\,, \end{aligned}$$

in this way connecting the start point $f(a)$ with the end point $f(b)$.

Definition 2.16 (*Path-connected*) A topological space is path-connected if for each pair of points $x, y \in X$, there is a path which connects them.

It can be shown that a topological space is connected if it is path-connected. According to Definition 2.16, finite spaces with more than one point can obviously not be connected. The connectedness of subsets of a topological space can be discussed by adapting Definitions 2.15 and 2.16 using the relative topology according to Definition 2.14.

Remark 2.4 In continuum mechanics, the configuration of a material body at an instant of time, e.g., $\tau = t_0$ can be encoded as a mapping $\kappa(t_0)$ which maps all points of the body to a subset of the space X. A motion is now described as a sequence of such mappings, one mapping for each instant of time $\tau = t$. It is supposed that these mappings and their inverses are continuous, i.e., the motion is assembled from homeomorphisms. This ensures automatically that a connected body remains connected in the course of the motion.

Certain properties of continuous mappings are of particular interest. This will be illustrated by means of continuous functions $f : A \to \mathbb{R}$ where $A \subset \mathbb{R}$. We are often particularly interested in where a function attains its maximum or minimum values. However, existence of such properties first requires that the considered function be bounded.

Definition 2.17 (*Bounded*) A function $f : A \to \mathbb{R}$ with $A \subset \mathbb{R}$ is bounded if $|f(x)| \leq M$ for $x \in A$ and $M \in \mathbb{R}$.

The standard example $f(x) = \frac{1}{x}$ indicates that being bounded depends somehow on the properties of the domain, since f is bounded on the intervals $[0.001, 1)$ and $[0.001, 1]$ but not on $(0, 1)$ or $(0, 0.5]$. Therefore, the question arises if there are conditions regarding the domain under which a continuous function is definitely bounded. At this point, the concept of compactness enters the scene. One possible line of argument to motivate compactness takes finite sets as a role model, since functions on finite sets are bounded in any case.

Remark 2.5 Care should be be taken here regarding the interpretation of $\pm\infty$ in the context of real analysis. There is no number $\frac{1}{0}$ in $\mathbb{R}$ and ∞ is no element of $\mathbb{R}$ but only an indication for something arbitrarily big. Therefore, $A = \{0, 1\}$ and $g = \frac{1}{x}, x \in A$ does not make sense, because the element 0 is not mapped to an element of $\mathbb{R}$. For the same reason, g together with $A = [-1, 1]$ is not a proper definition of a function according to Definition 1.3, since not every element of A has an image. On the other hand, g together with $A = (0, 1)$, for example, is correct.

If A is a finite set, then a function $f : A \to \mathbb{R}$ is locally bounded, since it assigns to every element of A some real number. One of these numbers will be the one with the largest absolute value, and f is bounded globally by the latter. This argument, trivial for finite sets, does not apply if A is infinite. Here, a similar line of argument is developed by means of the concept of open covers.

Definition 2.18 (*Open cover*) A collection of open sets is a cover of a set A if A is a subset of the union of these open sets.

Suppose that an infinite set can be covered by a finite number of open sets. If, in addition f is bounded on all these open sets, the same argument used for finite sets can be adapted. However, there are many possible open covers for a given set and the finiteness argument must not depend on the choice of a particular cover. This gives raise to the following definition of compactness.

Definition 2.19 (*Compactness*) A set A of a topological space is compact if ***every*** open cover of A has a finite sub-cover which covers A.

Example 2.7 The open interval $I = (0, 1)$ can, of course, be covered by a finite number of open sets, starting with I itself. However, this does not mean that I is compact. In order to show that I is not compact, it suffices to find at least one open cover which does not have a finite sub-cover. The collection $O_n = (\frac{1}{n}, 1 - \frac{1}{n})$ covers, for $n \to \infty$, the open interval I but there is no finite n for which I is covered completely. Therefore, I is not compact.

Of course, to check compactness for a topological space or some subspace by means of Definition 2.19 might be an option for a mathematician, but will not be for nonmathematicians. However, it turns out that, for instance, all closed intervals $[a, b]$ with $a, b \in \mathbb{R}$ are compact.

Last but not least, the so-called Hausdorff property ensures that if a sequence converges in a topological space X, it converges to exactly one point of X.

Definition 2.20 (*Hausdorff*) A topological space X is separable or Hausdorff if, for every $x, y \in X$ and $x \neq y$, there exist two disjoint open sets $A, B \subset X$ such that $x \in A$ and $y \in B$.

The concept of a basis for a topology is, among other things, useful for checking if a topological space is a Hausdorff-space.

Definition 2.21 (*Basis*) Given a topological spaces (X, τ). A collection τ_B of elements of τ is called a basis if every member of τ is a union of members of τ_B.

It can be shown that every space for which a countable basis can be constructed is a Hausdorff-space. All metric spaces in particular fulfill this condition. A possible basis for the standard topology in $\mathbb{R}$ consists of all open sets (a, b) with $a, b \in \mathbb{Q}$, where $\mathbb{Q}$ is the set of rational numbers. Since $\mathbb{Q}$ is countable, $\mathbb{R}$ is a Hausdorff-space.

2.5 Product Spaces and Product Topologies

Building a new set X from two sets X_1 and X_2 by means of the Cartesian product, i.e., $X = X_1 \times X_2$, is a common and most natural thing to do. If X_1 and X_2 are equipped with respective topologies τ_1 and τ_2, the question about the topology of X inevitably emerges. Since τ_1 is the collection of all open sets in X_1 and τ_2 is the collection of all open sets in X_2, it seems reasonable to try with the Cartesian product of these sets, specifically to define a collection

$$\mathcal{B} = \{U_1 \times U_2 \,|\, U_1 \in \tau_1 \text{ and } U_2 \in \tau_2\} \tag{2.1}$$

and to check if $\mathcal{B}$ may serve as a topology.

Remark 2.6 In the following, most examples refer to $\mathbb{R}^n$. Since we started with the intention of working out a minimal structure, suitable for discussing topological concepts, this might seem inconsistent, because $\mathbb{R}^n$ has a lot of additional structure. However, $\mathbb{R}$ and $\mathbb{R}^2$ especially are quite accessible to our intuition. Furthermore, a topology on a set X can be defined, for instance, by means of a bijective mapping $g : X \to \mathbb{R}^n$. Since the existence of a topology is assured in $\mathbb{R}^n$, a topology on X can be defined as the collection of the images $g^{-1}(U)$ of all open sets U of the standard topology of $\mathbb{R}^n$. This makes X a topological space and g a continuous mapping.

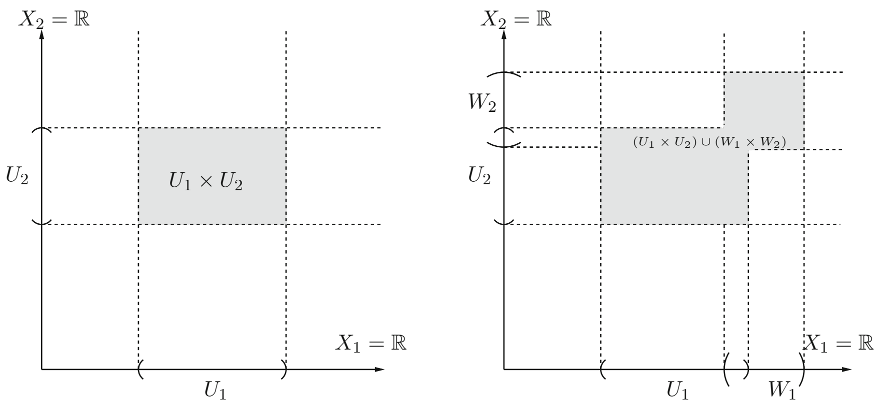

Fig. 2.3 Sketch illustrating for $X_1 = X_2 = \mathbb{R}$ that the union of members of $\mathcal{B}$, see (2.1), is not an element of $\mathcal{B}$. Therefore, $\mathcal{B}$ given by (2.1) can not be a topology

According to Definition 2.8, arbitrary unions and finite intersections of members of τ must result in an element of τ. Figure 2.3 illustrates that this does not hold for $\mathcal{B}$. Therefore, $\mathcal{B}$ is not a topology.

However, $\mathcal{B}$ can be used to construct a topology. To this end, the concept of the so-called sub-basis is introduced. The reader is encouraged to solve Exercise 2.3 after reading the following definition carefully.

Definition 2.22 (*Sub-basis*) A sub-basis $\mathcal{B}$ for a topological space (X, τ) is a collection of open sets in which the union of all members of $\mathcal{B}$ equals X. The topology of X is the collection of all unions of finite intersections of members of $\mathcal{B}$.

$\mathcal{B}$ as defined by (2.1) fulfills the requirements for being a sub-basis according to Definition 2.22. It turns out that for $X_1 = X_2 = \mathbb{R}$ the product topology coincides with the standard topology of $\mathbb{R}^2$. Considering the projections

$$\begin{aligned} \pi_1 : X_1 \times X_2 &\to X_1 \\ (x_1, x_2) &\mapsto x_1 \end{aligned} \qquad\qquad \begin{aligned} \pi_2 : X_1 \times X_2 &\to X_2 \\ (x_1, x_2) &\mapsto x_2 \,, \end{aligned}$$

more insight about the product topology can be gained. The sub-basis $\mathcal{B}$ can be written by means of the projections as follows:

$$\mathcal{B} = \left\{ \pi_1^{-1}(U_1) \cap \pi_2^{-1}(U_2) \mid U_1 \in \tau_1 \text{ and } U_2 \in \tau_2 \right\} ,$$

which shows that the product topology τ generated by $\mathcal{B}$ ensures that the projections are continuous. In fact, τ is the coarsest topology for which the projections are continuous. What coarse means depends on the topologies τ_1 and τ_2 (see Exercises 2.4 and 2.5). Furthermore, it can be shown that a function

$$\begin{aligned} f : Z &\to X_1 \times X_2 \\ z &\mapsto (\, f_1(z), \, f_2(z)\,) \end{aligned}$$

is continuous if $f_1 : Z \to X_1$ and $f_2 : Z \to X_2$ are continuous, and vice versa.

The step toward general finite products, i.e., $X_1 \times X_2 \times \cdots \times X_n$, is straight forward. Infinite products, however, require additional considerations.

2.6 Further Reading

A didactic introduction to topology, which nicely illustrates the creativity behind the formal "definition-theorem-proof" structure by showing how the formal language of topology evolves from first ideas accompanied by trial and error, is given in Geroch [2]. The classic book by Mendelson [4] is a compact and accessible introduction. Also recommendable are Conover [1] and Runde [5], the latter of which also contains a number of historical notes. For further steps in topology, we recommend, for example, Jänich [3]. In addition, summaries of point set topology can be found in most textbooks on differentiable manifolds.

Exercises

2.1 Check continuity of f in Example 2.6 for the following cases:

- $f : (X, \tau_2) \to (Y, \tau_1)$
- $f : (X, \tau_2) \to (Y, \tau_2)$

2.2 Check if (Y, τ_Y) with $Y = \{a, b\}$ and $\tau_Y = \{\emptyset, \{a\}, \{b\}\}$ is a subspace of (X, τ_X) with $X = \{a, b, c, d\}$ and $\tau_x = \{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}\}$.

2.3 Given $X_1 = \{0, 1\}$, $X_2 = \{a, b\}$ together with the topologies $\tau_1 = \{X_1, \emptyset, \{0\}, \{1\}\}$ and $\tau_2 = \{X_2, \emptyset, \{a\}, \{b\}\}$. Determine the sub-basis $\mathcal{B}$ for the product topology of $X = X_1 \times X_2$.

2.4 Given X_1, X_2 and τ_1 as in Exercise 2.3. Determine the sub-basis $\mathcal{B}$ for the topology τ of $X = X_1 \times X_2$ for $\tau_2 = \{X_2, \emptyset\}$. Determine τ and show that the projections are continuous.

2.5 Given X_1, X_2 and τ_2 as in Exercise 2.4. Determine the sub-basis $\mathcal{B}$ for the topology τ of $X = X_1 \times X_2$ for $\tau_1 = \{X_1, \emptyset\}$. Determine τ and show that the projections are continuous. Compare τ with the corresponding result of Exercise 2.4.

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