

Chapter 2

Lagrangian Mechanics

Abstract In this chapter, we lay out the foundations of Lagrangian Mechanics. We introduce the basic concepts of Lagrangian mechanical systems, namely the Lagrangian, the action, and the equations of motion, also known as the Euler–Lagrange equations. We also discuss important examples, such as the free particle, the harmonic oscillator, as well as motions in central force potentials, such as Newton’s theory of gravity and Coulomb’s electrostatic theory. Highlighting the importance of symmetries, we study integrals of motion and Noether’s theorem. As an application, we consider motions in radial potentials and, further specializing to motions in Newton’s gravitational potential, we conclude this section with a derivation of Kepler’s laws of planetary motion.

2.1 Lagrangian Mechanical Systems and their Equations of Motion

Definition 2.1 A *Lagrangian mechanical system* is a pair $(M, \mathcal{L})$ consisting of a smooth manifold M and a smooth function $\mathcal{L}$ on TM . The manifold M is called the *configuration space* and the function $\mathcal{L}$ is called the *Lagrangian function* (or simply the *Lagrangian*) of the system.

More generally, one can allow for the Lagrangian to depend explicitly on an extra variable interpreted as time.

Example 2.2 The Lagrangian of a point particle of mass $m > 0$ moving in Euclidean space $\mathbb{R}^n$ under the influence of a *potential* $V \in C^\infty(\mathbb{R}^n)$ is

$$\mathcal{L}(v) = \frac{1}{2}m\langle v, v \rangle - V(x), \quad v \in T_x\mathbb{R}^n, \quad x \in \mathbb{R}^n,$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean scalar product. Given a smooth curve $t \mapsto \gamma(t)$ in $\mathbb{R}^n$, the quantities $E_{\text{kin}}(t) = \frac{1}{2}m\langle \gamma'(t), \gamma'(t) \rangle$ and $E_{\text{pot}}(t) = V(\gamma(t))$ are called the *kinetic energy* and the *potential energy* at time t , respectively. Their sum

$$E(t) = E_{\text{kin}}(t) + E_{\text{pot}}(t)$$

is the *total energy* at time t .

By specializing V one obtains many important mechanical systems:

1. The *free particle*: $V = 0$.
2. The *harmonic oscillator*: $n = 1$ and $V = \frac{1}{2}kx^2$, where k is a positive constant known as *Hooke's constant*.
3. *Newton's theory of gravity*: $n = 3$ and

$$V = -\frac{\kappa m M}{r},$$

where $r = |x|$ denotes the Euclidean norm of the vector x , $M > 0$ is the mass of the particle (placed at the origin $0 \in \mathbb{R}^3$) generating the gravitational potential, and κ is a positive constant known as *Newton's constant*. (Notice that in this case V is not defined at the origin, so the configuration space is $\mathbb{R}^3 \setminus \{0\}$ rather than $\mathbb{R}^3$.)

4. *Coulomb's electrostatic theory*: $n = 3$ and

$$V = \frac{k_e q_1 q_2}{r},$$

where q_1 is the charge of the particle of mass m moving under the influence of the electric potential generated by a particle of charge q_2 and k_e is a positive constant known as *Coulomb's constant*. Notice that up to a constant factor Coulomb's potential is of the same type as Newton's potential. However, contrary to Newton's potential, the Coulomb potential can have either sign. As we shall see, this corresponds to the fact that electric forces can be attractive or repulsive, depending on the sign of the product $q_1 q_2$, whereas gravitational forces are always attractive. Another important difference is that the gravitational potential contains the mass m as a factor whereas the electric potential does not. As we shall see, the former property implies that the acceleration γ'' of the particle in Newton's theory of gravity is independent of its mass.

Example 2.3 More generally, given a smooth function V on a pseudo-Riemannian manifold (M, g) , one can consider the Lagrangian

$$\mathcal{L}(v) = \frac{1}{2}g(v, v) - V(x), \quad v \in T_x M, \quad x \in M.$$

Definition 2.4 Let $(M, \mathcal{L})$ be a Lagrangian mechanical system. The *action* of a smooth curve $\gamma : [a, b] \rightarrow M$ is defined as

$$S(\gamma) := \int_a^b \mathcal{L}(\gamma'(t)) dt.$$

A *motion* of the system is a critical point of S under smooth variations with fixed endpoints. (This statement is a mathematical formulation of *Hamilton's principle of least action*, which should better be called principle of stationary action.)

Next we will derive the *equations of motion*, which are the differential equations describing critical points of the action functional. These are known as the *Euler–Lagrange equations* and are given in (2.3). For this we introduce coordinates $(x^1, \dots, x^n)$ on an open subset $U \subset M$. They induce a system of coordinates $(q, \hat{q}) = (q^1, \dots, q^n, \hat{q}^1, \dots, \hat{q}^n)$ on the open subset $TU = \pi^{-1}(U) \subset TM$, where $\pi : TM \rightarrow M$ denotes the canonical projection. The *induced coordinates* are defined by

$$q^i := x^i \circ \pi, \quad \hat{q}^i(v) := dx^i(v), \quad v \in TU.$$

Let $\gamma_s : [a, b] \rightarrow M$, $-\varepsilon < s < \varepsilon$, $\varepsilon > 0$, be a smooth variation of $\gamma : [a, b] \rightarrow M$ with fixed endpoints. Then in local coordinates as above we compute

$$\begin{aligned} \frac{d}{ds} \Big|_{s=0} \mathcal{L}(\gamma'_s) &= \sum \left(\frac{\partial \mathcal{L}}{\partial q^i}(\gamma') \frac{d}{ds} \Big|_{s=0} q^i(\gamma'_s) + \frac{\partial \mathcal{L}}{\partial \hat{q}^i}(\gamma') \frac{d}{ds} \Big|_{s=0} \hat{q}^i(\gamma'_s) \right) \\ &= \sum \left(\frac{\partial \mathcal{L}}{\partial q^i}(\gamma') \frac{d}{ds} \Big|_{s=0} \gamma_s^i + \frac{\partial \mathcal{L}}{\partial \hat{q}^i}(\gamma') \frac{d}{ds} \Big|_{s=0} \dot{\gamma}_s^i \right), \end{aligned}$$

where $\gamma^i := x^i \circ \gamma$ and $\dot{\gamma}^i := (\gamma^i)'$. Note that here and throughout the rest of the book, we use an adapted version of *Einstein's summation convention*, where repeated upper and lower indices are to be summed over. This is indicated by a plain sum symbol (see also p. xi). If we denote by $\mathcal{V} := \frac{d}{ds} \Big|_{s=0} \gamma_s$ the variation vector field along γ , then we can rewrite this as

$$\begin{aligned} \frac{d}{ds} \Big|_{s=0} \mathcal{L}(\gamma'_s) &= \sum \left(\frac{\partial \mathcal{L}}{\partial q^i}(\gamma') \mathcal{V}^i + \frac{\partial \mathcal{L}}{\partial \hat{q}^i}(\gamma') \dot{\mathcal{V}}^i \right) \\ &= \sum \left(\frac{\partial \mathcal{L}}{\partial q^i}(\gamma') - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \hat{q}^i}(\gamma') \right) \mathcal{V}^i + f', \end{aligned}$$

where $\mathcal{V}^i := dx^i(\mathcal{V}) = \hat{q}^i(\mathcal{V})$ are the components of $\mathcal{V}$ and

$$f := \sum \frac{\partial \mathcal{L}}{\partial \hat{q}^i}(\gamma') \dot{\mathcal{V}}^i.$$

Notice that the local vector field $\mathcal{V}^{\text{ver}} = \sum \mathcal{V}^i \frac{\partial}{\partial \hat{q}^i}$ along γ' is independent of the choice of local coordinates and, hence, defines a global vector field along γ' . This follows from the fact that it corresponds to $\mathcal{V}$ under the canonical identification of $T_{\gamma(t)}M$ with the vertical tangent space

$$T_{\gamma'(t)}^{\text{ver}}(TM) := \ker d\pi|_{\gamma'(t)} \subset T_{\gamma'(t)}(TM).$$

Using this vector field we can rewrite the function f as

$$f = \mathcal{V}^{\text{ver}}(\mathcal{L})|_{\gamma'}.$$

This shows that also f is globally defined on the interval $[a, b]$ and vanishes at the endpoints. Therefore we have

$$\left. \frac{d}{ds} \right|_{s=0} S(\gamma_s) = \left. \frac{d}{ds} \right|_{s=0} \int_a^b \mathcal{L}(\gamma'_s(t)) dt = \int_a^b \left. \frac{d}{ds} \right|_{s=0} \mathcal{L}(\gamma'_s(t)) dt \quad (2.1)$$

$$= \int_a^b \left(\left. \frac{d}{ds} \right|_{s=0} \mathcal{L}(\gamma'_s(t)) - f'(t) \right) dt, \quad (2.2)$$

where the integrand is a globally defined function on the interval $[a, b]$ with the following local expression:

$$\left. \frac{d}{ds} \right|_{s=0} \mathcal{L}(\gamma'_s(t)) - f'(t) = \sum \left(\frac{\partial \mathcal{L}}{\partial q^i}(\gamma'(t)) - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}^i}(\gamma'(t)) \right) \mathcal{V}^i(t).$$

Theorem 2.5 *Let $(M, \mathcal{L})$ be a Lagrangian mechanical system, $n = \dim M$. The motions of the system are the solutions $\gamma : [a, b] \rightarrow M$ of the following system of ordinary differential equations:*

$$\alpha_i := \frac{\partial \mathcal{L}}{\partial q^i}(\gamma') - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}^i}(\gamma') = 0, \quad i = 1, \dots, n. \quad (2.3)$$

Proof Let us first remark that for every vector field $\mathcal{V}$ along γ the function $\sum \alpha_i \mathcal{V}^i = \left. \frac{d}{ds} \right|_{s=0} \mathcal{L}(\gamma'_s) - f'$ is coordinate independent and thus globally well-defined. From this we can deduce that $\alpha = \sum \alpha_i dx^i|_{\gamma}$ is a well-defined one-form along γ (cf. Appendix A, Exercise 1). In virtue of (2.2), we see that γ is a motion if and only if

$$\int_a^b \alpha(\mathcal{V})|_t dt = 0.$$

for all vector fields $\mathcal{V}$ along γ vanishing at the endpoints. To see that this implies that $\alpha(t_0) = 0$ for all $t_0 \in (a, b)$ we take $\delta > 0$ such that $\gamma([t_0 - \delta, t_0 + \delta])$ is contained in a coordinate domain and consider $\mathcal{V}^i = h\alpha_i$, where $h \geq 0$ is a smooth function on $[a, b]$ with support contained in $(t_0 - \delta, t_0 + \delta)$ such that $h(t_0) > 0$. This defines a smooth vector field $\mathcal{V}$ along γ vanishing at the endpoints and $0 = \int_a^b \alpha(\mathcal{V})|_t dt = \int_{t_0-\delta}^{t_0+\delta} h \sum \alpha_i^2 dt$ now implies that $\alpha(t_0) = 0$. This proves that $\alpha|_{(a,b)} = 0$ and, by continuity, $\alpha = 0$. $\square$

Definition 2.6 The one-form α along γ will be called the *Euler–Lagrange one-form*.

Example 2.7 Let (M, g) be a pseudo-Riemannian metric and consider the Lagrangian $\mathcal{L}(v) = \frac{1}{2}g(v, v)$, $v \in TM$. In canonical local coordinates $(q^1, \dots, q^n, \hat{q}^1, \dots, \hat{q}^n)$ on TM associated with local coordinates $(x^1, \dots, x^n)$ on M it is given by

$$\mathcal{L} = \frac{1}{2} \sum \tilde{g}_{ij} \hat{q}^i \hat{q}^j,$$

where $g = \sum g_{ij} dx^i dx^j$ and $\tilde{g}_{ij} = g_{ij} \circ \pi$. We compute

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial q^i} &= \frac{1}{2} \sum \frac{\partial \tilde{g}_{kj}}{\partial q^i} \hat{q}^k \hat{q}^j = \frac{1}{2} \sum \left(\frac{\partial g_{kj}}{\partial x^i} \circ \pi \right) \hat{q}^k \hat{q}^j, \\ \frac{\partial \mathcal{L}}{\partial \hat{q}^i} &= \sum \tilde{g}_{ij} \hat{q}^j, \\ \frac{\partial \mathcal{L}}{\partial q^i}(\gamma') &= \frac{1}{2} \sum \frac{\partial g_{kj}}{\partial x^i}(\gamma) \dot{\gamma}^k \dot{\gamma}^j, \\ \frac{\partial \mathcal{L}}{\partial \hat{q}^i}(\gamma') &= \sum g_{ij}(\gamma) \dot{\gamma}^j, \\ \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \hat{q}^i}(\gamma') &= \sum \frac{\partial g_{ij}}{\partial x^k}(\gamma) \dot{\gamma}^k \dot{\gamma}^j + \sum g_{ij}(\gamma) \ddot{\gamma}^j, \\ -\alpha_i &= \sum g_{ij}(\gamma) \ddot{\gamma}^j + \frac{1}{2} \sum \left(\frac{\partial g_{ij}}{\partial x^k}(\gamma) + \frac{\partial g_{ik}}{\partial x^j}(\gamma) - \frac{\partial g_{kj}}{\partial x^i}(\gamma) \right) \dot{\gamma}^k \dot{\gamma}^j \\ &= \sum_{\ell} g_{i\ell}(\gamma) \left(\ddot{\gamma}^{\ell} + \sum_{j,k} \Gamma_{jk}^{\ell}(\gamma) \dot{\gamma}^j \dot{\gamma}^k \right). \end{aligned}$$

This shows that the Euler–Lagrange equations are equivalent to the geodesic equations

$$\ddot{\gamma}^{\ell} + \sum \Gamma_{jk}^{\ell}(\gamma) \dot{\gamma}^j \dot{\gamma}^k = 0, \quad \ell = 1, \dots, n.$$

Using the covariant derivative ∇ they can be written as $\nabla_{\gamma'} \gamma' = 0$ or $\frac{\nabla}{dt} \gamma' = 0$.

Proposition 2.8 *Let V be a smooth function on a pseudo-Riemannian manifold (M, g) and consider the Lagrangian*

$$\mathcal{L}(v) = \frac{1}{2}g(v, v) - V(\pi(v)), \quad v \in TM, \quad (2.4)$$

of Example 2.3. Then a curve $\gamma : I \rightarrow M$ is a motion if and only if it satisfies the following equation

$$\frac{\nabla}{dt} \gamma' + \text{grad } V|_{\gamma} = 0 \quad (2.5)$$

Proof We have already shown that the Euler–Lagrange one-form in the case $V = 0$ is given by $\alpha = -g\left(\frac{\nabla}{dt}\gamma', \cdot\right)$. It remains to compute the contribution of the potential V to the Euler–Lagrange one-form, which is

$$-\sum \frac{\partial(V \circ \pi)}{\partial q^i}(\gamma') dx^i = -\sum \frac{\partial V}{\partial x^i}(\gamma) dx^i = -dV_\gamma.$$

So in total we obtain

$$-\alpha = g\left(\frac{\nabla}{dt}\gamma', \cdot\right) + dV|_\gamma$$

and, hence,

$$-g^{-1}\alpha = \frac{\nabla}{dt}\gamma' + \text{grad } V|_\gamma,$$

where $g^{-1} : T^*M \rightarrow TM$ denotes the inverse of the map

$$g : TM \rightarrow T^*M, \quad v \mapsto g(v, \cdot).$$

□

2.2 Integrals of Motion

Definition 2.9 Let $(M, \mathcal{L})$ be a Lagrangian mechanical system. A smooth function f on TM is called an *integral of motion* if it is constant along every motion of the system, that is for every motion $\gamma : I \rightarrow M$ the function $f(\gamma')$ is constant.

Obviously, the integrals of motion are completely determined by the equations of motion and do not depend on the precise Lagrangian. Therefore, the notion of an integral of motion is meaningful if we are just given a system of second order differential equations¹ for a curve $\gamma : I \rightarrow M$ in a smooth manifold M .

Proposition 2.10 (Conservation of energy) *Let V be a smooth function on a pseudo-Riemannian manifold (M, g) and consider the Lagrangian*

$$\mathcal{L}(v) = \frac{1}{2}g(v, v) - V(\pi(v)), \quad v \in TM, \quad (2.6)$$

of Example 2.3. Then the total energy

$$v \mapsto E(v) := \frac{1}{2}g(v, v) + V(\pi(v))$$

¹Such a system will be usually given by a consistent specification of a system of second order differential equations for the components of the curve in each local coordinate system. A typical example is (2.3).

is conserved, that is it is an integral of motion. If V is constant then the motion is geodesic.

Proof We compute the derivative of $t \mapsto E(\gamma'(t))$ along a motion γ :

$$\frac{d}{dt} \left(\frac{1}{2} g(\gamma', \gamma') + V(\gamma) \right) = g \left(\frac{\nabla}{dt} \gamma', \gamma' \right) + dV|_{\gamma} \gamma' \stackrel{(2.5)}{=} 0.$$

If V is constant then the equation of motion (2.5) reduces to the geodesic equation. $\square$

Remark The last statement of the proposition is related to conservation of momentum in Newtonian mechanics. Recall from Chap. 1 that the momentum of a particle $\gamma : I \rightarrow \mathbb{R}^n$ of mass m moving in Euclidean space is $\mathbf{p} = m\gamma'$. Newton's second law of mechanics has the form $\mathbf{p}' = F$, where F is the force acting on the particle, which may depend on the position γ and momentum $\mathbf{p}$ of the particle. Obviously, $F = 0$ implies that the momentum is constant along every motion. The conservation of energy holds in Newtonian mechanics if the force field F depends only on the position and is a gradient vector field. In fact, then we can write $F = -\text{grad } V$ for some function V on $\mathbb{R}^n$ and hence, Newton's equation is a special case of (2.5).

Definition 2.11 Let $(M, \mathcal{L})$ be a Lagrangian mechanical system. A diffeomorphism $\varphi : M \rightarrow M$ is called an *automorphism* of the system if

$$\mathcal{L}(d\varphi v) = \mathcal{L}(v),$$

for all $v \in TM$. A vector field $X \in \mathfrak{X}(M)$ is called an *infinitesimal automorphism* if its flow consists of local automorphisms of $(M, \mathcal{L})$.

Theorem 2.12 (Noether's theorem) *With every infinitesimal automorphism X of a Lagrangian mechanical system $(M, \mathcal{L})$ we can associate an integral of motion $f = d\mathcal{L}X^{ver}$, where $X^{ver} \in \mathfrak{X}(TM)$ denotes the vertical lift of X . (In local coordinates we have $X^{ver} = \sum (X^i \circ \pi) \frac{\partial}{\partial \dot{q}^i}$ if $X = \sum X^i \frac{\partial}{\partial x^i}$.)*

Proof Let us denote by φ_s the flow of X . For every motion $\gamma : I \rightarrow M$ of the system with values in the domain of definition of φ_s the curve $\gamma_s = \varphi_s \circ \gamma$ is again a motion and

$$\gamma'_s = d\varphi_s \gamma', \quad \left. \frac{\partial}{\partial s} \right|_{s=0} \gamma'_s = \tilde{X}|_{\gamma'},$$

where

$$TM \ni v \mapsto \tilde{X}(v) = \left. \frac{\partial}{\partial s} \right|_{s=0} d\varphi_s(v)$$

denotes the vector field on TM the flow of which is $d\varphi_s$. Differentiating the equation $\pi \circ d\varphi_s = \varphi_s \circ \pi$ with respect to s , we obtain $d\pi \circ \tilde{X} = X \circ \pi$. In local coordinates (x^i) on M and corresponding local coordinates $(q^i, \dot{q}^i)$ on TM this means that

$$\tilde{X} = \sum \left((X^i \circ \pi) \frac{\partial}{\partial \hat{q}^i} + Y^i \frac{\partial}{\partial \hat{q}^i} \right), \quad X = \sum X^i \frac{\partial}{\partial x^i}.$$

The locally defined functions Y^i on TM can be computed at $v \in TM$ as follows:

$$\begin{aligned} Y^i(v) &= d\hat{q}^i \tilde{X}(v) = \frac{\partial}{\partial s} \Big|_{s=0} \hat{q}^i(d\varphi_s v) = \frac{\partial}{\partial s} \Big|_{s=0} \hat{q}^i \left(\sum_{j,k} v^j \frac{\partial \varphi_s^k}{\partial x^j} \frac{\partial}{\partial x^k} \right) \\ &= \frac{\partial}{\partial s} \Big|_{s=0} \sum_j v^j \frac{\partial \varphi_s^i}{\partial x^j} = dX^i(v), \end{aligned}$$

where $\varphi_s^i := x^i \circ \varphi_s$ and we have used that $\frac{\partial}{\partial s} \Big|_{s=0} \varphi_s^i = X^i$. The above implies that

$$0 = \frac{\partial}{\partial s} \Big|_{s=0} \mathcal{L}(\gamma'_s) = d\mathcal{L}\tilde{X}|_{\gamma'} = \sum \left(\frac{\partial \mathcal{L}}{\partial \hat{q}^i} \Big|_{\gamma'} X^i(\gamma) + \frac{\partial \mathcal{L}}{\partial \hat{q}^i} \Big|_{\gamma'} dX^i(\gamma') \right). \quad (2.7)$$

Using the Euler–Lagrange equations we can now compute

$$\begin{aligned} \frac{d}{dt} f(\gamma') &= \frac{d}{dt} \sum \frac{\partial \mathcal{L}}{\partial \hat{q}^i} \Big|_{\gamma'} X^i(\gamma) \\ &= \sum \underbrace{\left(\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \hat{q}^i} \Big|_{\gamma'} \right)}_{= \frac{\partial \mathcal{L}}{\partial \hat{q}^i} \Big|_{\gamma'}} X^i(\gamma) + \sum \frac{\partial \mathcal{L}}{\partial \hat{q}^i} \Big|_{\gamma'} dX^i(\gamma') \stackrel{(2.7)}{=} 0. \end{aligned}$$

□

Proposition 2.13 *The group of automorphisms of the Lagrangian system of Example 2.3 is given by the Lie subgroup*

$$\text{Aut}(M, \mathcal{L}) = \{\varphi \in \text{Isom}(M, g) \mid V \circ \varphi = V\} \subset \text{Isom}(M, g).$$

Its Lie algebra consists of all Killing vector fields X such that $X(V) = 0$.

Proof See Appendix A, Exercise 6. □

Corollary 2.14 *Let V be a smooth function on a pseudo-Riemannian manifold (M, g) and consider the Lagrangian $\mathcal{L}(v) = \frac{1}{2}g(v, v) - V(\pi(v))$, $v \in TM$, of Example 2.3. Then every Killing vector field X such that $X(V) = 0$ gives rise to an integral of motion $f(v) = g(v, X(\pi(v)))$, $v \in TM$.*

Corollary 2.15 (Conservation of momentum) *Consider, as in the previous corollary, the Lagrangian of Example 2.3. Assume that with respect to some coordinate system*

on M the metric g and the potential V are both invariant under translations in one of the coordinates x^i . Then the function

$$p_i := \sum \tilde{g}_{ij} \dot{q}^j$$

is an integral of motion on the coordinate domain.

Since the Euclidian metric is translational invariant we have the following special case of Corollary 2.15.

Corollary 2.16 *Let $V = V(x^2, \dots, x^n)$ be a smooth function on Euclidean space $\mathbb{R}^n$ which does not depend on the first coordinate and consider the Lagrangian $\mathcal{L}(v) = \frac{1}{2}m\langle v, v \rangle - V(\pi(v))$, $v \in T\mathbb{R}^n$, of Example 2.2. Then the first component p_1 of the momentum vector $\mathbf{p} = \sum p_i e_i$ is an integral of motion, where $p_i(v) := m\langle v, e_i \rangle$, $v \in T\mathbb{R}^n$.*

Note that the result still holds if we replace the Euclidean scalar product by a pseudo-Euclidean scalar product, such as the Minkowski scalar product.

Corollary 2.17 (Conservation of angular momentum) *Let V be a smooth radial function on $\mathbb{R}^3 \setminus \{0\}$, that is V depends only on the radial coordinate r , and consider the Lagrangian $\mathcal{L}(v) = \frac{1}{2}m\langle v, v \rangle - V(\pi(v))$, $v \in T(\mathbb{R}^3 \setminus \{0\})$. Then the components of the angular momentum vector*

$$\mathbf{L}(v) = \pi(v) \times \mathbf{p}(v) = m \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \times \begin{pmatrix} v^1 \\ v^2 \\ v^3 \end{pmatrix}$$

are integrals of motion. Here, x^1, x^2, x^3 denote the components of $x = \pi(v)$.

Proof This can be proven either directly from the equations of motion or by applying Noether's theorem (see Appendix A, Exercise 9). The first proof uses the fact that the moment of force $\gamma \times F(\gamma)$, $F = -\text{grad } V$, is zero for every curve $\gamma : I \rightarrow \mathbb{R}^3 \setminus \{0\}$ and yields also the following proposition. $\square$

Definition 2.18 A vector field F on $\mathbb{R}^3 \setminus \{0\}$ is called *radial* if there exists a radial function f on $\mathbb{R}^3 \setminus \{0\}$ such that $F(x) = f(x) \frac{x}{|x|}$ for all $x \in \mathbb{R}^3 \setminus \{0\}$.

Proposition 2.19 *Let F be a smooth radial vector field on $\mathbb{R}^3 \setminus \{0\}$. Then the angular momentum is constant for every solution of Newton's equation $\frac{d}{dt}\mathbf{p}(t) = F(t)$, where we are using the usual notation $\mathbf{p}(t) = \mathbf{p}(\gamma'(t))$ and $F(t) = F(\gamma(t))$.*

Remark A function or vector field on $\mathbb{R}^3 \setminus \{0\}$ is radial if and only if it is *spherically symmetric*, that is invariant under $\text{SO}(3)$ (or, equivalently, $\text{O}(3)$), see Appendix A, Exercises 11 and 12. The name is due to the fact that $\text{SO}(3)$ is the group of orientation preserving isometries of the sphere S^2 and $\text{O}(3) = \text{Isom}(S^2)$.

In the next section we will see how to use the conservation of energy and angular momentum to analyze the motion in a radial potential.

2.3 Motion in a Radial Potential

We consider the Lagrangian $\mathcal{L}(v) = \frac{1}{2}m\langle v, v \rangle - V(r)$ on $\mathbb{R}^3 \setminus \{0\}$, where V is a smooth function of the radial coordinate $r = |x|$ alone. Since the mass can be absorbed into the definitions of V and $\mathcal{L}$ (denoting $\mathcal{L}/m$ and V/m again by $\mathcal{L}$ and V) we may as well put $m = 1$. So from now on we consider a particle of unit mass.

We know by the results of the previous section that the energy $E(t) = \frac{1}{2}\langle \gamma'(t), \gamma'(t) \rangle + V(|\gamma(t)|)$ and angular momentum $\mathbf{L}(t) = \gamma(t) \times \gamma'(t)$ are constant for every motion $\gamma : I \rightarrow \mathbb{R}^3 \setminus \{0\}$ of the system.

The first observation is that the conservation of angular momentum implies that the motion is *planar*, that is contained in a plane.

Proposition 2.20 *If the vector $\mathbf{L} = \gamma \times \gamma'$ is constant along the curve $\gamma : I \rightarrow \mathbb{R}^3 \setminus \{0\}$ then γ is a planar curve.*

Proof If $\mathbf{L} = 0$ then γ is a radial curve (see Appendix A, Exercise 7) and thus planar. Therefore we can assume that $\mathbf{L}$ is a nonzero constant vector. Since the cross product of two vectors is always perpendicular to both of them, we know that $\gamma(t) \in \mathbf{L}^\perp$ for all t . So γ is contained in the plane $\mathbf{L}^\perp$. $\square$

By a change of coordinates we can assume that the motion is restricted to the plane $e_3^\perp$ and use polar coordinates (r, φ) in that plane rather than Cartesian coordinates (x^1, x^2) to describe the motion. In these coordinates, the Euclidean metric is $dr^2 + r^2 d\varphi^2$ and, hence,

$$E = \frac{1}{2} (\dot{r}^2 + r^2 \dot{\varphi}^2) + V(r). \quad (2.8)$$

To compute the cross product $\gamma \times \gamma'$ in polar coordinates we recall that

$$\partial_r = \cos \varphi \partial_1 + \sin \varphi \partial_2, \quad \partial_\varphi = -r \sin \varphi \partial_1 + r \cos \varphi \partial_2,$$

where we have abbreviated $\partial_r = \partial/\partial r$, $\partial_\varphi = \partial/\partial \varphi$ and $\partial_i = \partial/\partial x^i$. From these relations we easily obtain

$$\partial_r \times \partial_\varphi = r e_3$$

and hence

$$\mathbf{L} = \gamma \times \gamma' = r \partial_r \times (\dot{r} \partial_r + \dot{\varphi} \partial_\varphi) = r^2 \dot{\varphi} e_3.$$

Changing the time parameter t to $-t$, if necessary, we can assume that the constant $r^2 \dot{\varphi} \geq 0$ and, hence,

$$L := |\mathbf{L}| = r^2 \dot{\varphi}. \quad (2.9)$$

Substituting $\dot{\varphi} = L/r^2$ into the energy equation (2.8) we arrive at

$$E = \frac{1}{2} \dot{r}^2 + V_{\text{eff}}(r), \quad V_{\text{eff}}(r) := V(r) + \frac{L^2}{2r^2}, \quad (2.10)$$

where V_{eff} is called the *effective potential*. As a consequence, we obtain the following result.

Theorem 2.21 *Consider a particle of unit mass moving in a radial potential $V(r)$ in $\mathbb{R}^3$. Then the radial coordinate obeys the equation of motion $\ddot{r} = -V'_{\text{eff}}(r)$ of a particle $t \mapsto r(t) > 0$ moving in the Euclidean line under the potential*

$$V_{\text{eff}}(r) := V(r) + \frac{L^2}{2r^2}, \quad (2.11)$$

which reduces to the first order equation

$$\dot{r} = \pm \sqrt{2(E - V_{\text{eff}}(r))} \quad (2.12)$$

solvable by separation of variables. The angular coordinate is then given by

$$\varphi(t) = L \int_0^t \frac{ds}{r(s)^2} + \varphi(0),$$

where the initial value $\varphi(0)$ is freely specifiable or, as a function of r , by

$$\varphi(r) = \pm L \int_{r_0}^r \frac{ds}{s^2 \sqrt{2(E - V_{\text{eff}}(s))}} + \varphi(r_0).$$

Proof The energy equation (2.10) immediately implies (2.12), which can be solved by separation of variables:

$$\pm \frac{dr}{\sqrt{2(E - V_{\text{eff}}(r))}} = dt.$$

Integration yields t as a function of r . Then r as a function of t is the inverse of that function. Differentiating (2.12) gives Newton's equation $\ddot{r} = -V'_{\text{eff}}(r)$. Given r as a function of t , the angular coordinate is now determined from $\dot{\varphi} = L/r^2$ by integration as claimed. Finally, to express φ as a function of r one does not need to know the inverse of the function $r \mapsto t(r)$ but can proceed as follows:

$$\frac{d}{dr} \varphi = \dot{\varphi} \frac{dt}{dr} = \pm \frac{L}{r^2 \sqrt{2(E - V_{\text{eff}}(r))}}.$$

□

Next we specialize the discussion to Newton's potential $V(r) = -\frac{M}{r}$, from which we will derive, in particular, Kepler's laws of planetary motion.

2.3.1 Motion in Newton's Gravitational Potential

By (2.11) we know that the motion in Newton's gravitational potential is governed by the following effective potential

$$f(r) := V_{\text{eff}}(r) = -\frac{M}{r} + \frac{L^2}{2r^2}. \quad (2.13)$$

We will assume that $L > 0$; the case $L = 0$ is treated in Appendix A, Exercise 8. The qualitative behavior of the effective potential is essential for the qualitative analysis of the possible orbits. The proof of the next proposition is elementary.

Proposition 2.22 *The function $f : \mathbb{R}^{>0} \rightarrow \mathbb{R}$ defined in (2.13) has the following properties:*

1. f has a unique zero, at $r_0 := \frac{L^2}{2M}$.
2. It is positive for $r < r_0$ and negative for $r > r_0$.
3. $\lim_{r \rightarrow 0} f(r) = \infty$, $\lim_{r \rightarrow \infty} f(r) = 0$.
4. f has a unique critical point, at $r_{\min} := \frac{L^2}{M} > r_0$, where f attains its global minimum $f(r_{\min}) = -\frac{M^2}{2L^2}$.
5. f is strictly decreasing for $r < r_{\min}$ and strictly increasing for $r > r_{\min}$.

Proposition 2.23

- (i) *The range of possible energies for motions with nonzero angular momentum in Newton's gravitational potential is given by the interval $\left[-\frac{M^2}{2L^2}, \infty\right)$. The motions are unbounded if $E \geq 0$ and bounded if $E < 0$. The orbit of minimal energy $E_{\min} = -\frac{M^2}{2L^2}$ is a circle of radius $\frac{L^2}{M}$.*
- (ii) *The perihelion distance r_{per} , that is the minimal distance to the origin, for a given energy $E \geq 0$ (and $L > 0$) is given by the unique positive solution of the equation $r^2 + \frac{M}{E}r - \frac{L^2}{2E} = 0$, which is $r_{\text{per}} = \frac{M}{2E} \left(-1 + \sqrt{1 + 2\frac{L^2 E}{M^2}}\right)$. For $E_{\min} < E < 0$ (and $L > 0$) the distance to the origin varies between the two positive solutions of the equation $r^2 + \frac{M}{E}r - \frac{L^2}{2E} = 0$, which are $r_{\text{per}} = \frac{M}{2|E|} \left(1 - \sqrt{1 + 2\frac{L^2 E}{M^2}}\right)$ and $r_{\text{aph}} = \frac{M}{2|E|} \left(1 + \sqrt{1 + 2\frac{L^2 E}{M^2}}\right)$ (the aphelion distance, that is the maximal distance to the origin).*

Proof The first part follows immediately from Proposition 2.22. For the second part note that at the points of minimal or maximal distance to the origin we have $\dot{r} = 0$ and, hence, $E = V_{\text{eff}} = f$. The last equation is equivalent to the quadratic equation $r^2 + \frac{M}{E}r - \frac{L^2}{2E} = 0$, which has a unique solution for $E = E_{\min}$ (the radius of the circular orbit), two positive solutions for $E_{\min} < E < 0$, and only one positive solution for $E \geq 0$. $\square$

Next we will solve the equations of motion. The function $t \mapsto r(t)$ is determined by

$$\ddot{r} = -f'(r) = -\frac{M}{r^2} + \frac{L^2}{r^3}.$$

To solve this equation we make the substitution $u = 1/r$ and compute

$$\frac{du}{d\varphi} = \frac{du}{dt} \bigg|_{t(\varphi)} \frac{dt}{d\varphi} = - \frac{\dot{r}}{r^2 \dot{\varphi}} \bigg|_{t(\varphi)} = - \frac{\dot{r}}{L} \bigg|_{t(\varphi)},$$

which implies

$$\frac{d^2 u}{d\varphi^2} = -\frac{\ddot{r}}{L} \frac{dt}{d\varphi} = -\frac{\ddot{r}}{L \dot{\varphi}} = -\frac{\ddot{r} r^2}{L^2} = \frac{f'(r) r^2}{L^2} = \frac{M}{L^2} - u.$$

So we obtain the equation

$$\frac{d^2 u}{d\varphi^2} + u = \frac{M}{L^2},$$

the general solution of which is

$$u = k \cos(\varphi - \varphi_0) + \frac{M}{L^2},$$

where $k \geq 0$ and φ_0 are constants. By choosing $\varphi_0 = 0$, this allows us to write

$$r = \frac{p}{1 + \varepsilon \cos \varphi}, \quad p := \frac{L^2}{M}, \quad \varepsilon := \frac{k L^2}{M}. \quad (2.14)$$

For $0 \leq \varepsilon < 1$ this is an ellipse of eccentricity ε .

For $\varepsilon = 1$ it is a parabola and for $\varepsilon > 1$ a component of a hyperbola. (Observe that for the parabola and hyperbola the angle φ is constrained by the condition that $\varepsilon \cos \varphi > -1$.) Recall that for an ellipse with major half-axis a and minor half-axis b the distance between the two focal points is $2c$, where $c^2 = a^2 - b^2$, and the eccentricity and parameter are given by

$$\varepsilon = \frac{c}{a}, \quad p = \frac{b^2}{a},$$

see e.g. [10, VIII.43] for a detailed discussion of conic sections.

Summarizing we obtain:

Theorem 2.24 *The motions of a particle (with nonzero angular momentum) in Newton's gravitational potential are along conic sections with a focus at the origin. The bounded motions are ellipses.*

The latter statement is the content of *Kepler's first law* (of planetary motion): The planets move along ellipses with the sun at a focal point. *Kepler's second law* asserts that the area swept out during a time interval $[t, t + s]$ by the line segment connecting the sun and the planet depends only on its duration s and not on the initial time t . This is a simple consequence of the conservation of angular momentum as shown in Appendix A, Exercise 13:

Proposition 2.25 *Let $\gamma : I \rightarrow \mathbb{R}^3 \setminus \{0\}$ be the motion of a particle in a radial force field F according to Newton's law $m\gamma'' = F(\gamma)$. Then the angular momentum vector $\mathbf{L}$ is constant, the motion is planar and the area $A(t_0, t_1)$ swept out by the vector γ during a time interval $[t_0, t_1]$ is given by*

$$A(t_0, t_1) = \frac{L}{2}(t_1 - t_0),$$

where L is the length of the angular momentum vector.

As a corollary we obtain *Kepler's third law*:

Corollary 2.26 *The elliptic orbits in Newton's gravitational potential are periodic with period*

$$T = \frac{2\pi}{\sqrt{M}}a^{3/2},$$

where a is the major half-axis of the ellipse.

Proof For a given initial time $t_0 \in \mathbb{R}$ let us denote by $T = T(t_0)$ the smallest positive real number such that $(r, \varphi)|_{t_0+T} = (r, \varphi)_{t_0}$, where $t \mapsto (r(t), \varphi(t))$ is an elliptic motion of the system. Since $A(t_0, t_0 + T) = \frac{L}{2}T$ is the area of the ellipse, T is clearly independent of t_0 , which shows that the motion is periodic. The area of the ellipse is $A = \pi ab$ and, hence,

$$T = \frac{2}{L}A = \frac{2\pi ab}{L} \stackrel{(2.14)}{=} \frac{2\pi ab}{\sqrt{\mu M}} = \frac{2\pi a^{3/2}}{\sqrt{M}}.$$

□

Mathematical Methods of Classical Physics

Cortés, V.; Haupt, A.S.

2017, XI, 99 p., Softcover

ISBN: 978-3-319-56462-3