

## Chapter 2

# Fuzzy Coalitions and Fuzziness of Games

### 2.1 Introduction

In a cooperative game the mean tool to analyze the situation is the knowledge of the worths of the coalitions. In an coalition it is supposed that all the players cooperate at the same level or certainty. Zadeh [20] introduced fuzzy sets to represent different degrees of membership for the elements of a set. Later Aubin [1] proposed to use fuzzy sets to define fuzzy coalitions allowing an asymmetry of the participation of the players. This fact supposes to replace the discrete scope by the continuous one. Several interesting surveys about fuzzy games (games with fuzzy coalitions) are given in Butnariu and Klement [7], Branzei et al. [5] and Borkotokey and Mesiar [3]. There is another different way in the analysis of fuzzy games, using vague payoffs. The reader can use Mares [12] to study this model. Classical games are an specific family of fuzzy games, that we name crisp games. This book works with crisp games but some information about fuzzy games helps to comprehend better the developing of our study. The mean difficulty to define a solution for a fuzzy game is how to incorporate the whole information of the game. The diagonal value, (Aumann and Shapley [2] and Aubin [1]) and the crisp Shapley value (Branzei et al. [4]) are two different approach to solve a fuzzy game in the Shapley way. But neither of them uses all the information of the fuzzy game.

A fuzziness of a crisp game is a fuzzy game determined from the crisp one. Remember that we deal with non-flexible games, therefore we look for fuzziness following the classical behavior of the players: to form the maximal feasible coalition. Fuzzy cooperation can be used as probabilistic data or real membership of the players. In the second point of view the rule of the maximal cooperation is reached by two different ways: maximal level of cooperation of maximal group of players involved. Three fuzziness following all these comments have been studied in the literature, the multilinear extension [15], the proportional extension [6] and the Choquet extension [18].

## 2.2 Brief Overview About Fuzzy Sets

Fuzzy sets were introduced by Zadeh [20]. In this book we only deal with fuzzy sets in a finite set of points. Let  $K$  be a finite set. A *fuzzy set*  $A$  in  $K$  is given by a membership function over  $K$ , namely  $\tau_A : K \rightarrow [0, 1]$ . We denote the family of fuzzy sets in  $K$  as  $[0, 1]^K$ . Here the fuzzy set  $A$  is identified to its membership function  $\tau_A$ , and then we will use  $\tau \in [0, 1]^K$  to refer a fuzzy set. Each non-empty subset  $Q \subseteq K$  is identified with the fuzzy set  $e^Q \in [0, 1]^K$  given by  $e^Q(i) = 1$  if  $i \in Q$  and  $e^Q(i) = 0$  otherwise. Particularly for the complete set  $K$  we will use  $e^K$ . The empty-set is also a fuzzy set that we denote as  $0$  with  $0(i) = 0$  for all  $i \in K$ . The classical subsets are named *crisp sets* in order to distinguish them into the fuzzy sets.

A fuzzy set  $\tau'$  is contained in another one  $\tau \in [0, 1]^K$  if  $\tau'(i) \leq \tau(i)$  for all  $i \in K$ , we say that  $\tau'$  is a *fuzzy subset* of  $\tau$  and type  $\tau' \leq \tau$ . If  $\tau' \leq \tau$  then the *difference* is a new fuzzy set  $\tau - \tau'$  with  $(\tau - \tau')(i) = \tau(i) - \tau'(i)$ . The *complement* of a fuzzy set  $\tau$  is  $e^K - \tau \in [0, 1]^K$ .

A *T-norm* is a binary relation  $T : [0, 1] \times [0, 1] \rightarrow [0, 1]$  verifying the following properties: (1) Commutativity  $T(a, b) = T(b, a)$ , (2) Associativity  $T(a, T(b, c)) = T(T(a, b), c)$ , (3) Monotonicity  $T(a, b) \leq T(c, d)$  if  $a \leq b, c \leq d$  and (4) Identity element  $T(1, a) = a$ . The corresponding *T-conorm* to the T-norm  $T$  is the dual binary relation

$$(1 - T)(a, b) = 1 - T(1 - a, 1 - b).$$

Using different T-norms we introduced different operations for fuzzy sets. If  $T$  is a T-norm then  $T(\tau, \tau')(i) = T(\tau(i), \tau'(i))$  for all  $\tau, \tau' \in [0, 1]^N$  and  $i \in N$ . We consider three of them.

- *Intersection and union.*  $T(a, b) = a \wedge b, (1 - T)(a, b) = a \vee b$ . If  $\tau, \tau' \in [0, 1]^K$  then, the intersection and the union are respectively  $\tau \wedge \tau'$  and  $\tau \vee \tau'$ . These operations satisfy the Morgan laws in a fuzzy sense:

$$\begin{aligned} e^K - (\tau \vee \tau') &= (e^K - \tau) \wedge (e^K - \tau'), \\ e^K - (\tau \wedge \tau') &= (e^K - \tau) \vee (e^K - \tau'). \end{aligned}$$

- *Cosum and sum.*<sup>1</sup>  $T(a, b) = 0 \vee (a + b - 1), (1 - T)(a, b) = 1 \wedge (a + b)$ . If  $\tau, \tau' \in [0, 1]^K$  then, the cosum and the sum are respectively  $\tau \oplus \tau'$  and  $\tau + \tau'$ .
- *Product and coproduct.*<sup>2</sup>  $T(a, b) = ab$  and  $(1 - T)(a, b) = a + b - ab$ . The product and the coproduct of  $\tau, \tau'$  are the fuzzy sets respectively  $\tau \times \tau', \tau \otimes \tau'$ .

<sup>1</sup>This T-norm is usually named Lukasiewicz norm.

<sup>2</sup>Product is understood as the usual probabilistic intersection.

The *support* of a fuzzy set  $\tau \in [0, 1]^K$  is the set

$$\text{supp}(\tau) = \{i \in K : \tau(i) > 0\}.$$

The *image* of  $\tau$  is the set of numbers

$$\text{im}(\tau) = \{t \in (0, 1] : \exists i \in K \text{ with } \tau(i) = t\}.$$

The maximum and the minimum non-zero numbers in the image of  $\tau \neq 0$  are denoted as  $\vee \tau$  and  $\wedge \tau$  respectively. Usually the image of a fuzzy set is given as an ordered set including zero or also one although these numbers are not in the image, we take

$$\text{im}_0(\tau) = \{0 = \lambda_0 < \lambda_1 < \dots < \lambda_m\} = \text{im}(\tau) \cup \{0\}, \quad (2.1)$$

$$\text{im}_0^1(\tau) = \{0 = \lambda_0 < \lambda_1 < \dots < \lambda_m = 1\} = \text{im}(\tau) \cup \{0, 1\}. \quad (2.2)$$

The *total membership* of a fuzzy set  $\tau$  is something as the cardinality in a set, namely

$$|\tau| = \sum_{i \in K} \tau(i).$$

Two fuzzy sets  $\tau_1, \tau_2$  are *comonotone* if for all  $i, j \in K$

$$[\tau_1(i) - \tau_1(j)][\tau_2(i) - \tau_2(j)] \geq 0. \quad (2.3)$$

Let  $\tau \in [0, 1]^K$  be a fuzzy set over the finite set  $K$ . For each  $t \in (0, 1]$ , the *t-cut* is the crisp set of all the elements in  $K$  with level greater or equal than  $t$  in  $\tau$ , namely

$$[\tau]_t = \{i \in K : \tau(i) \geq t\}. \quad (2.4)$$

As  $K$  is finite there is a finite quantity of different cuts. If  $\text{im}_0(\tau) = \{\lambda_0 < \dots < \lambda_k\}$  then we use  $[\tau]_k = [\tau]_{\lambda_k}$  for  $k = 1, \dots, m$ . So,  $[\tau]_t = [\tau]_k$  if  $t \in (\lambda_{k-1}, \lambda_k]$  and  $k = 1, \dots, m$ . The cuts of  $\tau$  generate a decreasing sequence of sets,  $[\tau]_m \subset \dots \subset [\tau]_1$ .

Choquet [9] defined a *capacity* over  $K$  as a monotone set function  $v : 2^K \rightarrow \mathbb{R}$  verifying  $v(\emptyset) = 0$ , actually a monotone game (see Definition 1.3) over  $K$ . If we drop monotonicity,  $v$  is a game, then is named *signed capacity*. The Choquet integral [9] was first defined for capacities and later Schmeidler [16] and Waegenaere and Wakker [10] extended the concept to signed capacities. If  $\tau \in [0, 1]^K$  and  $v$  is a signed capacity over  $K$  then the (signed) *Choquet integral* of  $\tau$  regard to  $v$  is

$$\int_c \tau dv = \sum_{k=1}^m (\lambda_k - \lambda_{k-1}) v([\tau]_k), \quad (2.5)$$

with  $\text{im}_0(\tau) = \{\lambda_0 < \lambda_1 < \dots < \lambda_m\}$ . Next properties of the integral will be used in the book.

- (C1)  $\int_c \tau d(v_1 + v_2) = \int_c \tau dv_1 + \int_c \tau dv_2$
- (C2)  $\int_c \tau dv_1 \leq \int_c \tau dv_2$  if  $v_1 \leq v_2$
- (C3)  $\int_c \tau d(av) = a \int_c \tau dv$
- (C4)  $\int_c t \tau dv = t \int_c \tau dv$  if  $t \in [0, 1]$
- (C5)  $\int_c (\tau_1 + \tau_2) dv = \int_c \tau_1 dv + \int_c \tau_2 dv$  if  $\tau_1, \tau_2$  are comonotone
- (C6)  $\int_c \tau dv = a \bigvee_{i \in K} \tau(i)$  if  $v([\tau]_t) = a$  for all  $t \in (0, 1]$
- (C7)  $\int_c e^Q dv = v(Q)$
- (C8)  $\int_c \tau_1 dv \leq \int_c \tau_2 dv$  when  $\tau_1 \leq \tau_2$  and  $v$  monotone
- (C9)  $\int_c \tau dv$  is a continuous function regard to  $\tau$ .
- (C10)  $\int_c \tau dv = \sum_{q=1}^p (t_q - t_{q-1})v([\tau]_{t_q})$ , for any finite set of numbers in  $[0, 1]$  containing the image of  $\tau$  (and zero), namely  $\{0 = t_0 < t_1 \cdots < t_p\} \supseteq im_0(\tau)$ .

### 2.3 Fuzzy Coalitions

Fuzzy coalitions were defined by Aubin [1]. Given the finite set of players  $N$ , a fuzzy coalition is a collective decisional entity where members may have gradual degrees of membership, as Butnariu and Klement says in [7]. So, a fuzzy coalition introduces an asymmetric relation among the players beyond the mapping of the game. A fuzzy set in the set of players is used to define a fuzzy coalition.

**Definition 2.1** A fuzzy coalition of the set of players  $N$  is any fuzzy set in  $N$ . Therefore  $[0, 1]^N$  denotes the family of fuzzy coalitions.

The support of a fuzzy coalition represents the set of active players and the image the different levels of participation.

Fuzzy coalitions can be interpreted in other ways. In a probabilistic sense, if  $\tau \in [0, 1]^N$  then  $\tau(i)$  is the probability of being in the coalition. Thus each fuzzy coalition represents a particular situation of possibility.

Aubin [1] also introduced the concept of game with fuzzy coalitions or fuzzy game.<sup>3</sup>

**Definition 2.2** A fuzzy game (game with fuzzy coalitions) over  $N$  is a real mapping over the fuzzy coalitions  $v : [0, 1]^N \rightarrow \mathbb{R}$  with  $v(0) = 0$ .

Branzei et al. [5] proposed the next example of fuzzy game about a public good.

*Example 2.1* Suppose  $N$  a set of agents who cooperate in order to get a new facility for joint use. The cost of the facility depends on the total participation of the agents by monotone increasing function  $c$ . The profit obtained from the facility is determined for each player using an individual monotone increasing function  $p_i$  depending on her participation. This situation can be described by the fuzzy game over  $N$

$$v(\tau) = \sum_{i \in N} p_i(\tau(i)) - c(|\tau|).$$

*Example 2.2* In a university committee there two groups of delegates, professors and students. Next system to adopt a rule incorporates more options for the teachers, independently on the number of elements in each group. We have  $N = \{1, 2\}$  with 1 the team of professors and 2 the student's one. In a fuzzy coalition  $\tau \in [0, 1]^N$  we interpret

$$\tau(i) = \frac{|\text{delegates in team } i \text{ in favor of the rule}|}{|\text{delegates in team } i|}.$$

So, the proposition is

$$v(\tau) = \begin{cases} 1, & \text{if } \tau(1) \geq 2/3 \text{ and } \tau(2) \geq 1/2 \\ 0, & \text{otherwise.} \end{cases}$$

*Example 2.3* Linear production games (Example 1.6) can be better explained as fuzzy games. In the crisp case players use their whole endowments when they cooperate. The fuzzy version permits to use the initial endowments partially. So, considering  $c, b, A$  as in Example 1.6, the *fuzzy linear production game* is defined for each  $\tau \in [0, 1]^N$  as

$$v(\tau) = \bigvee \{c \cdot x : Ax \leq \tau \cdot b\}$$

*Example 2.4* Butnariu and Klement [7] proposed to represent rate problems for services (electricity or water) in bulk. The individual services are the players and the bulks of services are the coalitions. Each customer (or kind of customers) is identified

<sup>3</sup>The notion of fuzzy game is also used for cooperative games with fuzzy payoffs, see Mares [12]. The concept of game on  $[0, 1]^N$  can be studied in a different way as game with overlapping coalitions, see Chalkiadakis et al. [8].

with the particular bulk she consumes. Fuzzy coalitions represent bulks for services where the degree membership of an individual one is exactly the share of this service consumed by a particular customer. The worth of each fuzzy coalition is the cost of the particular bulk of consume.

The idea of value was extended also for fuzzy coalitions.

**Definition 2.3** A value for fuzzy games over  $N$  is a mapping assigning a payoff vector in  $\mathbb{R}^N$  to each fuzzy game over  $N$ . Generally a value  $f$  for fuzzy games determines a value  $f^N$  for each finite set  $N$ .

There are several intents to extend the Shapley value to fuzzy games. However it is not an easy problem in the sense that it is not possible to consider all the feasible variations (marginal contributions) of a player in the game, namely we cannot take all the information in a fuzzy game. We show two examples of these mappings. The first option is defined taking the crisp version of a fuzzy game.

**Definition 2.4** If  $v$  is a fuzzy game over  $N$  then the crisp version is given by  $v^{cr} \in \mathcal{G}^N$  with

$$v^{cr}(S) = v(e^S),$$

for all  $S \subseteq N$ .

Branzei et al. [4] introduced a Shapley value for fuzzy games.

**Definition 2.5** The crisp Shapley value is defined for each  $v$  fuzzy game over  $N$  as

$$\phi^{cr}(v) = \phi(v^{cr}).$$

This concept is very limited because it only uses the crisp information of the game, thus if two fuzzy games  $v, v'$  verify  $v^{cr} = v'^{cr}$  then  $\phi^{cr}(v) = \phi^{cr}(v')$ .

To describe axioms for the crisp Shapley value is not complicated from the usual axioms of the classical Shapley value. But the uniqueness cannot be obtained from the unanimity games (the set of fuzzy games over  $N$  is not a finite vectorial space). We use the axiomatization in Theorem 1.5. If  $v$  is a fuzzy game over  $N$  and  $S \subset N$  then the *fuzzy subgame*  $v|_S$  is a new fuzzy game over  $S$  that we denote usually again as  $v$  where  $v(\tau) = v(\tau^0)$  for each  $\tau \in [0, 1]^S$  and

$$\tau^0(i) = \begin{cases} \tau(i) & \text{if } i \in S \\ 0 & \text{if } i \in N \setminus S. \end{cases}$$

**Theorem 2.1** *The crisp Shapley value is the only value for fuzzy games satisfying efficiency (over  $e^N$ ) and balanced contributions*

*Proof* As the Shapley value is efficient (Proposition 1.9) for games over  $N$  then

$$\phi^{cr}(v)(N) = \phi(v^{cr})(N) = v^{cr}(N) = v(e^N).$$

Observe that if  $v$  is a fuzzy game over  $N$  then the crisp version of  $v$  as fuzzy subgame over  $S$  is the same that the subgame over  $S$  of the crisp version of  $v$ , namely  $(v^{cr})|_S = (v|_S)^{cr}$ . Balanced contributions follows since the Shapley value satisfies it (Proposition 1.16) and the above fact,

$$\begin{aligned} (\phi^{cr})_i^N(v) - (\phi^{cr})_j^N(v) &= \phi_i^N(v^{cr}) - \phi_j^N(v^{cr}) = \phi_i^{N \setminus \{j\}}(v^{cr}) - \phi_j^{N \setminus \{i\}}(v^{cr}) \\ &= (\phi^{cr})_i^{N \setminus \{j\}}(v) - (\phi^{cr})_j^{N \setminus \{i\}}(v). \end{aligned}$$

The proof of the uniqueness is exactly the same that in Theorem 1.5.  $\square$

The Shapley value was extended also to the class of fuzzy games by the own Aubin [1]. This extension is named the diagonal value. The first problem is that the diagonal value was only defined for a particular class of fuzzy games, those continuously differentiable. Later this value has been extended to another more general family of fuzzy games, see Butnariu and Klement [7] or Mertens [14].

**Definition 2.6** Let  $v$  be a continuously differentiable fuzzy game. The diagonal value is defined for  $v$  and  $i \in N$  as

$$\phi_i^d(v) = \int_0^1 D_i v(te^N) dt.$$

The diagonal value supposes that coalitions are formed by aggregation of small differences for one player but taking a symmetric membership of all the players. As we will see in Sect. 2.4 the expression of the diagonal value is based in another formulation previously known of the usual Shapley value. Aubin [1] provided the value with an axiomatization. Let  $f$  be a value for games with fuzzy coalitions. Besides efficiency<sup>4</sup>, anonymity and linearity he introduced next axioms.

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<sup>4</sup>Aubin named it pareto optimality.

**Continuously.**  $f$  is a continuous operator.

Consider  $P = \{S_1, \dots, S_m\}$  a partition of  $N$  in groups and  $M = \{1, \dots, m\}$ . For each fuzzy coalition  $\tau \in [0, 1]^M$  of groups we induce a fuzzy coalition  $\tau^P \in [0, 1]^N$  for players as  $\tau^P(i) = \tau(k)$  if  $i \in S_k$ . For each fuzzy game  $v$  over  $N$  a new fuzzy game over  $M$  is defined as

$$v^P(\tau) = v(\tau^P).$$

**Atomicity.** Let  $v$  be a fuzzy game over  $N$  and  $P$  a partition of  $N$ . It holds for all  $k \in M$ ,

$$f_k^M(v^P) = \sum_{i \in S_k} f_i^N(v).$$

**Theorem 2.2** *The diagonal value is the only value for continuously differentiate fuzzy games satisfying efficiency, linearity, anonymity, continuously and atomicity.*

*Proof* The diagonal value verifies continuously and linearity by construction. Adding all the payoffs of the players we get by the Barrow law for the linear integral

$$\sum_{i \in N} \int_0^1 D_i v(te^N) dt = \int_0^1 Dv(te^N) \cdot e^N dt = v(e^N).$$

Anonymity follows from  $\theta(te^N) = te^N$  and  $D_{\theta(i)}\theta v(te^N) = D_i v(te^N)$ . We test atomicity. Let  $P = \{S_1, \dots, S_m\}$  be a partition of  $N$  in groups and  $M = \{1, \dots, m\}$ . The fuzzy coalition  $\tau = te^M \in [0, 1]^M$  induces  $\tau^P = te^N \in [0, 1]^N$ . Thus, for each  $k \in M$  the chain rule applied to  $y_k = e^{S_k} \cdot x$  with  $x = (x_i)_{i \in N}$  for each  $k$  implies

$$D_k(v^P)(te^M) = Dv(te^N) \cdot e^{S_k} = \sum_{i \in S_k} D_i v(te^N).$$

To prove the uniqueness we consider a value  $f$  satisfying all the axioms. Polynomial functions are dense in the set of continuous differentiate fuzzy games, therefore by continuously and linearity we only have to study fuzzy games as  $v(\tau) = \tau_1^{p_1} \cdots \tau_n^{p_n}$ . First consider the game  $v(\tau) = \tau_1 \cdots \tau_n$ . Anonymity and efficiency imply  $f_i(v) = \frac{1}{n}$  following the proof in Proposition 1.7. Now, for the general case, suppose  $p_k$  copies of  $\tau_k$  for each  $k \in N$ . If  $S_k$  is set of copies of  $k$  and  $N' = \bigcup_{k \in N} S_k$  then  $\{S_1, \dots, S_n\}$  is a partition of  $N'$ . So, atomicity says that if  $i \in S_k$  then

$$f_i(v) = \frac{p_k}{p_1 + \cdots + p_n}.$$

□

These solutions, the crisp Shapley value and the diagonal one, are different as is showed in [5].

*Example 2.5* Suppose  $v$  a fuzzy game over  $N = \{1, 2\}$  given by

$$v(\tau) = \tau(1)\tau^2(2) \quad \forall \tau \in [0, 1].$$

The crisp version is  $cr(v)(\{1\}) = cr(v)(\{2\}) = 0$ ,  $cr(v)(\{1, 2\}) = 1$ . Hence

$$\phi^{cr}(v) = (1/2, 1/2).$$

Now  $Dv(\tau) = (\tau^2(2), 2\tau(1)\tau(2))$  and  $Dv(te^N) = (t^2, 2t^2)$ . Thus,

$$\phi_1^d(v) = \int_0^1 t^2 dt = \frac{1}{3}, \quad \phi_2^d(v) = \int_0^1 2t^2 dt = \frac{2}{3},$$

and  $\phi^d(v) = (1/3, 2/3)$ .

## 2.4 Fuzziness of Games

Let  $v \in \mathcal{G}^N$  be a game. We look for a process to evaluate a fuzzy coalition by  $v$ . In a crisp coalition  $e^S$  all the players participate in the coalition at the same level 1. Considering that the worth of a coalition is proportional to the membership of the players we know how to evaluate coalitions as  $te^S$ ,  $v(te^S) = tv(S)$ . With this premise, Aubin [1] proposed a way to estimate the worth of a fuzzy coalition.

**Definition 2.7** Let  $\tau \in [0, 1]^N$  be a fuzzy coalition. A partition by levels of  $\tau$  is a finite sequence  $\{(S_k, s_k)\}_{k=1}^m$  satisfying:

- (1)  $S_k \subseteq N$  and  $s_k > 0$  for each  $k = 1, \dots, m$ ,
- (2)  $\sum_{k=1}^m s_k e^{S_k} = \tau$ .

A partition function is a mapping  $pl$  over  $[0, 1]^N$  taking a partition by levels  $pl(\tau)$  for each fuzzy coalition  $\tau$ .

Observe that if  $\{(S_k, s_k)\}_{k=1}^m$  is a partition by levels of  $\tau \in [0, 1]^N$  then

$$\sum_{\{k:i \in S_k\}} s_k = \tau(i), \quad (2.6)$$

for every player  $i \in N$ . Moreover each  $S_k \subseteq \text{supp}(\tau)$ . If  $\tau = 0$  then no partition is necessary but if we technically need to take one we will consider  $\{(\emptyset, t)\}$  with any  $t \in [0, 1]$ . Fixed a game, we can give a partition function determining a fuzzy game.

**Definition 2.8** Let  $v \in \mathcal{G}^N$  and let  $pl$  be a partition function. The fuzziness of  $v$  obtained from  $pl$  is the fuzzy game  $v^{pl}$  verifying for all  $\tau \in [0, 1]^N$  and  $pl(\tau) = \{(S_k, s_k)\}_{k=1}^m$ ,

$$v^{pl}(\tau) = \sum_{k=1}^m s_k v(S_k).$$

Generally, these fuzziness of a game are different as we will see later but they coincide for additive games.

*Example 2.6* If  $v \in \mathbb{R}^N$  is an additive game over  $N$  then for all partition function and  $\tau \in [0, 1]^N$  it holds  $v^{pl}(\tau) = \tau \cdot v$ . We get using (2.6) with  $pl(\tau) = \{(S_k, s_k)\}_{k=1}^m$ ,

$$v^{pl}(\tau) = \sum_{k=1}^m s_k \sum_{i \in S_k} v_i = \sum_{i \in N} \left[ \sum_{\{k: i \in S_k\}} s_k \right] v_i = \sum_{i \in N} \tau(i) v_i = \tau \cdot v.$$

The own Aubin [1] considered a fuzziness of games. If  $v \in \mathcal{G}^N$  then for any coalition  $\tau \in [0, 1]^N$ ,

$$\bar{v}(\tau) = \bigvee \left\{ \sum_{k=1}^m s_k v(S_k) : \{(S_k, s_k)\}_{k=1}^m \text{ partition by levels of } \tau \right\}. \quad (2.7)$$

In this case we take a partition function  $pl$  choosing an optimal partition by levels in the above supremum. But this definition is really a total flexible option, furthermore if  $S$  is a coalition then  $\bar{v}(e^S) = v^{sa}(S)$ , see (1.6). We are interested in non-flexible fuzziness (as far as we can control) thus it seems natural to require  $v^{pl}(e^S) = v(S)$  for all coalition  $S$ . But, as we will see later in the next chapters it is better to specify more this idea.

**Definition 2.9** A partition function  $pl$  in  $N$  is an extension if  $pl(S) = \{(S, 1)\}$  for all non-empty coalition  $S$ . The fuzziness of a game obtained by an extension is named also an extension of the game.

The fuzziness (2.7) proposed by Aubin is not an extension. Partition functions can depend on the game, as the Aubin's one. When the partition function is independent on the game we guarantee that the fuzziness working well with the vectorial space  $\mathcal{G}^N$ . So, the sum and scalar product satisfy

$$(av_1 + bv_2)^{pl} = av_1^{pl} + bv_2^{pl}.$$

Suppose a partition function independent of the game  $pl$  from now on. Obviously the crisp Shapley value of an extension of a game is the Shapley value of the original game, because  $(v^{pl})^{cr} = v$  for all  $v \in \mathcal{G}^N$ .

**Theorem 2.3** *If  $pl$  is an extension then for all game  $v \in \mathcal{G}^N$*

$$\phi^{cr}(v^{pl}) = \phi(v).$$

But about the diagonal value we cannot say anything in general because fuzziness are not always continuously differentiate functions.

Three interesting extensions which do not depend on the game have been studied in the literature. One of them from the probabilistic point of view and the others following the membership interpretation in games. Suppose that we interpret number  $\tau(i)$  in a fuzzy coalition  $\tau$  as the membership of player  $i$ . Following the classical model players should look for the maximal cooperation. But in a fuzzy situation this fact can raise in two different ways:

- They look for the biggest level of cooperation, or
- they look for the biggest crisp coalition.

We will analyze these three options in the next subsections.

### ***The Multilinear Extension***

The first one was introduced by Owen [15] outside the context of fuzzy coalitions, and later by Meng and Zhang [13]. In this case component  $\tau(i)$  in a fuzzy coalition is interpreted as the probability for player  $i$  to cooperate, and then  $1 - \tau(i)$  the probability of non cooperating. Following the philosophy of the classical game if  $S$  was formed then the profit of this coalition is  $v(S)$  and this fact is not variable.

**Definition 2.10** Let  $v \in \mathcal{G}^N$  be a game. The multilinear extension of  $v$  is a fuzziness of  $v$  defined for each  $\tau \in [0, 1]^N$  as

$$v^{ml}(\tau) = \sum_{S \subseteq N} \left[ \prod_{i \in S} \tau(i) \prod_{i \notin S} (1 - \tau(i)) \right] v(S).$$

We consider the mapping  $ml$  given for every  $\tau \in [0, 1]^N$  by

$$ml(\tau) = \left( S, \prod_{i \in S} \tau(i) \prod_{i \notin S} (1 - \tau(i)) \right)_{\{S \subseteq N : [\tau]_1 \subseteq S \subseteq \text{supp}(\tau)\}}. \quad (2.8)$$

Namely we use any coalition containing player for sure, those in  $[\tau]_1$ , and not containing players out of the support, those impossible for cooperating. The worth  $v^{ml}(\tau)$  is the expecting one of cooperation, particularly each level for a coalition is the probability of forming this coalition.

**Lemma 2.1** *Let  $S$  be a coalition in  $N$ . If  $\tau \in [0, 1]^N$  is a fuzzy coalition then*

$$\sum_{T \subseteq S} \prod_{i \in T} \tau(i) \prod_{i \in S \setminus T} (1 - \tau(i)) = 1.$$

*Proof* We prove the equality by induction in the cardinality of  $S$ . If  $S = \{i\}$  then there are two options,  $T = \emptyset$  or  $T = S$ , hence  $1 - \tau(i) + \tau(i) = 1$ . Suppose true when  $|S| < k$  and take  $S$  with  $|S| = k$ . Let any  $j \in S$ ,

$$\begin{aligned} \sum_{T \subseteq S} \prod_{i \in T} \tau(i) \prod_{i \in S \setminus T} (1 - \tau(i)) &= \sum_{\{T \subseteq S : j \in T\}} \tau(j) \prod_{i \in T \setminus \{j\}} \tau(i) \prod_{i \in S \setminus T} (1 - \tau(i)) \\ &\quad + \sum_{\{T \subseteq S : j \notin T\}} (1 - \tau(j)) \prod_{i \in T} \tau(i) \prod_{i \in S \setminus (T \cup \{j\})} (1 - \tau(i)) \\ &= \sum_{T \subseteq S \setminus \{j\}} \left[ \tau(j) \prod_{i \in T} \tau(i) \prod_{i \in (S \setminus \{j\}) \setminus T} (1 - \tau(i)) \right. \\ &\quad \left. + (1 - \tau(j)) \prod_{i \in T} \tau(i) \prod_{i \in (S \setminus \{j\}) \setminus T} (1 - \tau(i)) \right] \\ &= \prod_{i \in T} \tau(i) \prod_{i \in (S \setminus \{j\}) \setminus T} (1 - \tau(i)) = 1, \end{aligned}$$

because  $|S \setminus \{j\}| < k$ . □

**Remark 2.1** The reader can think  $\tau$  as a set of probability distributions, one for each player, thus the probability to obtain a set of them is 1. If we take the distributions of a particular coalition then the probability to obtain a subset of this coalition is also 1.

We test now that  $ml$  gets a partition by levels for each fuzzy coalition and it is an extension.

**Proposition 2.1** *Mapping  $ml$  is an extension.*

*Proof* First we see that  $ml$  is a partition function, namely it obtains a partition by levels for each fuzzy coalition  $\tau \in [0, 1]^N$ . Every level in  $ml(\tau)$  is non-zero by the election of  $S$ . Moreover, if coalition  $S$  does not verify  $[\tau]_1 \subseteq S \subseteq \text{supp}(\tau)$  then

$$\prod_{i \in S} \tau(i) \prod_{i \notin S} (1 - \tau(i)) = 0.$$

Now, for each player  $j \in N$  we have

$$\begin{aligned} \sum_{\{[\tau]_1 \subseteq S \subseteq \text{supp}(\tau), j \in S\}} \prod_{i \in S} \tau(i) \prod_{i \notin S} (1 - \tau(i)) &= \sum_{\{S \subseteq N: j \in S\}} \prod_{i \in S} \tau(i) \prod_{i \notin S} (1 - \tau(i)) \\ &= \tau(j) \sum_{\{S \subseteq N: j \in S\}} \prod_{i \in S \setminus \{j\}} \tau(i) \prod_{i \notin S} (1 - \tau(i)) \\ &= \tau(j) \sum_{\{S \subseteq N \setminus \{j\}\}} \prod_{i \in S} \tau(i) \prod_{i \in (N \setminus \{j\}) \setminus S} (1 - \tau(i)) \\ &= \tau(j), \end{aligned}$$

applying the above lemma to  $N \setminus \{j\}$ . Finally we test that for any coalition  $S$  we get  $ml(e^S) = \{(S, 1)\}$ , the only coalition in  $ml(e^S)$  is  $S$  because  $[e^S]_1 = S = \text{supp}(e^S)$  and  $\prod_{i \in S} e^S(i) \prod_{i \notin S} (1 - e^S(i)) = 1$ .  $\square$

*Example 2.7* Suppose  $v \in \mathcal{G}^N$  any game over  $N = \{1, 2, 3, 4, 5\}$ . Consider the fuzzy coalition  $\tau = (0.2, 0, 0.7, 1, 0.2)$ . The information contained in  $\tau$  says that for instance the probability of cooperating player 3 is 0.7 and so her probability of non cooperating is 0.3. What is certain in  $\tau$  is that player 2 will never cooperate and player 4 is always willing to cooperate. The expecting worth of  $\tau$  is

$$\begin{aligned} v^{ml}(\tau) &= 0.192v(\{4\}) + 0.048v(\{1, 4\}) + 0.448v(\{3, 4\}) + 0.048v(\{4, 5\}) \\ &\quad + 0.112v(\{1, 3, 4\}) + 0.012v(\{1, 4, 5\}) + 0.112v(\{3, 4, 5\}) \\ &\quad + 0.028v(\{1, 3, 4, 5\}) \end{aligned}$$

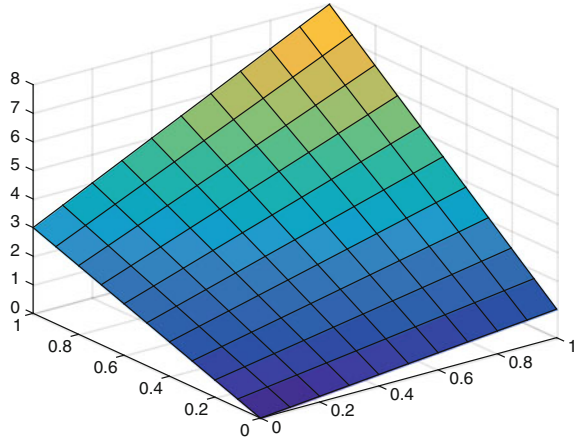
*Example 2.8* Suppose the game  $v$  over  $N = \{1, 2\}$  defined as  $v(\{1\}) = 1$ ,  $v(\{2\}) = 3$  and  $v(\{1, 2\}) = 8$ . Let  $\tau = (x, y) \in [0, 1] \times [0, 1]$  be any fuzzy coalition. The multilinear function is the quadratic polynomial (the hyperbolic paraboloid in Fig. 2.1),

$$v^{ml}(x, y) = 4xy + x + 3y,$$

with the unit square as domain.

*Example 2.9* Let  $T \subseteq N$  be a non-empty set. We calculate the multilinear extension of the unanimity game  $u_T$ . For each  $\tau \in [0, 1]^N$ ,

**Fig. 2.1** The multilinear extension  $v^{ml}$



$$\begin{aligned}
 (u_T)^{ml}(\tau) &= \sum_{T \subseteq S} \left[ \prod_{j \in S} \tau(j) \prod_{j \notin S} (1 - \tau(j)) \right] = \prod_{i \in T} \tau(i) \sum_{T \subseteq S} \left[ \prod_{j \in S \setminus T} \tau(j) \prod_{j \notin S} (1 - \tau(j)) \right] \\
 &= \prod_{i \in T} \tau(i) \sum_{S \subseteq N \setminus T} \left[ \prod_{j \in S} \tau(j) \prod_{j \in (N \setminus T) \setminus S} (1 - \tau(j)) \right] = \prod_{i \in T} \tau(i),
 \end{aligned}$$

using Lemma 2.1 with  $N \setminus T$ . Thus  $(u_T)^{ml}(\tau)$  is the probability of containing  $T$ .

Using the above example we get another formula of the multilinear extension using the dividends (Proposition 1.1) of the games.

**Proposition 2.2** For all game  $v \in \mathcal{G}^N$  it holds

$$v^{ml}(\tau) = \sum_{\{S \subseteq \text{supp}(\tau) : S \neq \emptyset\}} \Delta_S^v \prod_{i \in S} \tau(i).$$

*Proof* Given two games  $v_1, v_2$  we know that

$$(av_1 + bv_2)^{ml} = av_1^{ml} + bv_2^{ml},$$

because the extension is independent on the game. Proposition 1.1 and Example 2.9 imply the result. Observe that if a coalition  $S$  contains a player  $i$  with  $\tau(i) = 0$  then  $u_S^{ml}(\tau) = 0$ .  $\square$

The properties of games in Definition 1.3 can be extended to fuzzy games. In Branzei et al. [5] there is an extended analysis of these properties in the fuzzy context. We focus the idea only about what happens with the fuzziness if we take a monotone,

superadditive or convex game. Example 2.6 proved that the fuzziness of any additive game  $v$  is the linear function with  $v$  as coefficients.

**Proposition 2.3** *Let  $\tau, \tau' \in [0, 1]^N$  be two fuzzy coalitions.*

- (1) *If  $v \in \mathcal{G}_m^N$  and  $\tau \leq \tau'$  then  $v^{ml}(\tau) \leq v^{ml}(\tau')$ .*
- (2) *If  $v \in \mathcal{G}_{sa}^N$  and  $\tau \wedge \tau' = 0$  then  $v^{ml}(\tau \vee \tau') \geq v^{ml}(\tau) + v^{ml}(\tau')$ .*
- (3) *If  $v \in \mathcal{G}_c^N$  then  $v^{ml}(\tau \vee \tau') + v^{ml}(\tau \wedge \tau') \geq v^{ml}(\tau) + v^{ml}(\tau')$ .*

*Proof* (1) Let  $v$  be a monotone game. Suppose  $\tau \in [0, 1]^N$  and  $t \in [0, 1 - \tau(i)]$  for any player  $i$ . We get

$$\begin{aligned}
 v^{ml}(\tau + te^{(i)}) &= \sum_{\{S \subseteq N: i \in S\}} (\tau(i) + t) \prod_{j \in S \setminus \{i\}} \tau(j) \prod_{j \notin S} (1 - \tau(j)) v(S) \\
 &\quad + \sum_{\{S \subseteq N: i \notin S\}} (1 - \tau(i) - t) \prod_{j \in S} \tau(j) \prod_{j \notin S \cup \{i\}} (1 - \tau(j)) v(S) \\
 &= v^{ml}(\tau) + t \sum_{\{S \subseteq N: i \in S\}} \prod_{j \in S \setminus \{i\}} \tau(j) \prod_{j \notin S \cup \{i\}} (1 - \tau(j)) [v(S) - v(S \setminus \{i\})] \\
 &\geq v^{ml}(\tau),
 \end{aligned}$$

because, as  $v$  is monotone then  $v(S) \geq v(S \setminus \{i\})$  for all coalition  $S$ . Now if  $\tau \leq \tau'$  then

$$\tau' = \tau + \sum_{i \in N} [\tau'(i) - \tau(i)] e^{(i)}.$$

Applying sequentially the above reasoning we have  $v^{ml}(\tau) \leq v^{ml}(\tau')$ .

(2) Let  $v$  be a superadditive game. If  $\tau, \tau' \in [0, 1]^N$  with  $\tau \wedge \tau' = 0$  then for each player  $i$  one of them  $\tau(i)$  or  $\tau'(i)$  is null. So, we have  $\text{supp}(\tau) \cap \text{supp}(\tau') = \emptyset$ . We denote  $\text{supp}(\tau) = N_1$  and  $\text{supp}(\tau') = N_2$ . Moreover  $(\tau \vee \tau')(i) = \tau(i)$  if  $i \in N_1$ ,  $(\tau \vee \tau')(i) = \tau'(i)$  if  $i \in N_2$  and  $(\tau \vee \tau')(i) = 0$  otherwise. Each coalition into  $\text{supp}(\tau \vee \tau')$  can be written as  $S \cup T$  with  $S \subseteq N_1$  and  $T \subseteq N_2$ . We obtain using the superadditivity of  $v$  and Lemma 2.1,

$$\begin{aligned}
 v^{ml}(\tau \vee \tau') &= \\
 &= \sum_{S \subseteq N_1} \sum_{T \subseteq N_2} \left[ \prod_{i \in S} \tau(i) \prod_{i \in T} \tau'(i) \prod_{i \in N_1 \setminus S} (1 - \tau(i)) \prod_{i \in N_2 \setminus T} (1 - \tau'(i)) \right] v(S \cup T) \\
 &\geq \sum_{S \subseteq N_1} \sum_{T \subseteq N_2} \left[ \prod_{i \in S} \tau(i) \prod_{i \in T} \tau(i) \prod_{i \in N_1 \setminus S} (1 - \tau(i)) \prod_{i \in N_2 \setminus T} (1 - \tau'(i)) \right] (v(S) + v(T))
 \end{aligned}$$

$$\begin{aligned}
&= \sum_{S \subseteq N_1} \prod_{i \in S} \tau(i) \prod_{i \in N_1 \setminus S} (1 - \tau(i)) \left[ \sum_{T \subseteq N_2} \prod_{i \in T} \tau(i) \prod_{i \in N_2 \setminus T} (1 - \tau'(i)) \right] v(S) \\
&\quad + \sum_{T \subseteq N_2} \prod_{i \in T} \tau(i) \prod_{i \in N_2 \setminus T} (1 - \tau(i)) \left[ \sum_{S \subseteq N_1} \prod_{i \in S} \tau(i) \prod_{i \in N_1 \setminus S} (1 - \tau'(i)) \right] v(T) \\
&= \sum_{S \subseteq N_1} \prod_{i \in S} \tau(i) \prod_{i \in N_1 \setminus S} (1 - \tau(i)) v(S) + \sum_{T \subseteq N_2} \prod_{i \in T} \tau(i) \prod_{i \in N_2 \setminus T} (1 - \tau(i)) v(T) \\
&= v^{ml}(\tau) + v^{ml}(\tau').
\end{aligned}$$

(3) Now suppose  $v$  a convex game. As multilinear function  $v^{ml}$  is twice continuously differentiate for each player  $i$  it holds

$$\begin{aligned}
D_i(v^{ml})(\tau) &= \sum_{\{S \subseteq N : i \in S\}} \prod_{j \in S \setminus \{i\}} \tau(j) \prod_{j \notin S} (1 - \tau(j)) v(S) \\
&\quad - \sum_{\{S \subseteq N : i \notin S\}} \prod_{j \in S} \tau(j) \prod_{j \notin S \cup \{i\}} (1 - \tau(j)) v(S \cup \{i\}) \\
&= \sum_{\{S \subseteq N : i \in S\}} \prod_{j \in S \setminus \{i\}} \tau(j) \prod_{j \notin S \cup \{i\}} (1 - \tau(j)) [v(S \cup \{i\}) - v(S)]
\end{aligned}$$

Therefore  $D_{ii}(v^{ml})(\tau) = 0$ . If  $k \neq i$  then using the same reasoning in the first derivate we have

$$\begin{aligned}
D_{ik}(v^{ml})(\tau) &= \sum_{\{S \subseteq N : i, k \in S\}} \prod_{j \in S \setminus \{i, k\}} \tau(j) \prod_{j \notin S \cup \{i, k\}} (1 - \tau(j)) \cdot \\
&\quad \cdot [v(S \cup \{i, k\}) - v(S \cup \{i\}) - v(S \cup \{k\}) + v(S)].
\end{aligned}$$

Since  $v$  is convex then  $v(S \cup \{i, k\}) + v(S) \geq v(S \cup \{i\}) + v(S \cup \{k\})$ , thus

$$D_{ik}(v^{ml})(\tau) \geq 0.$$

Hence  $D_i(v^{ml})$  is an increasing function and so are the increments regard to  $i$ , if  $\tau \leq \tau'$  and  $t \in [0, 1 - \tau'(i)]$  then

$$v^{ml}(\tau + te^{(i)}) - v(\tau) \leq v^{ml}(\tau' + te^{(i)}) - v(\tau').$$

Now we take any  $\tau, \tau' \in [0, 1]^N$  and  $R = \{i \in N : \tau'(i) < \tau(i)\} = \{i_1, \dots, i_p\}$ . We set  $h_p = \tau(i_p) - \tau'(i_p)$  and then

$$\tau = \tau \wedge \tau' + \sum_{q=1}^p h_p e^{(i_p)}, \quad \tau \vee \tau' = \tau' + \sum_{q=1}^p h_p e^{(i_p)}.$$

Applying sequentially the above condition to  $\tau'$ ,  $\tau \wedge \tau'$  we get

$$v^{ml}(\tau) - v^{ml}(\tau \wedge \tau') \leq v^{ml}(\tau \vee \tau') - v^{ml}(\tau').$$

□

*Remark 2.2* (1) Conditions (2) and (3) in the above proposition are true also with product and coproduct, namely a probabilistic convexity.

$$v^{ml}(\tau \otimes \tau') + v^{ml}(\tau \times \tau') \geq v^{ml}(\tau) + v^{ml}(\tau').$$

(2) A fuzzy game satisfying condition (2) in the proposition is known as a fuzzy superadditive game.

(3) A fuzzy game satisfying condition (3) is named supermodular fuzzy game. Branzei et al. [4] introduced the concept of fuzzy convex game as a fuzzy game which is supermodular and also a convex function for each component.

Obviously the multilinear extension of a game is a continuously differentiate fuzzy game as we said before. Owen [15] had showed that the Shapley value is the diagonal value of the multilinear extension before the diagonal value was defined.

**Theorem 2.4** *Let  $v \in \mathcal{G}^N$  be a game. It holds:*

$$\phi^d(v^{ml}) = \phi(v).$$

*Proof* We calculate the diagonal value of  $v^{ml}$ . If  $i \in N$  then using Proposition 2.2,

$$D_i(v^{ml})(\tau) = \sum_{\{S \subseteq N: i \in S\}} \Delta_S^v \prod_{j \in S \setminus \{i\}} \tau(j). \quad (2.9)$$

Hence for each  $t \in [0, 1]$  we have

$$D_i(v^{ml})(te^N) = \sum_{\{S \subseteq N: i \in S\}} \Delta_S^v t^{|S|-1}.$$

Finally we do the integral and from Theorem 1.2,

$$\begin{aligned} \int_0^1 D_i(v^{ml})(te^N) dt &= \sum_{\{S \subseteq N: i \in S\}} \Delta_S^v \int_0^1 t^{|S|-1} dt \\ &= \sum_{\{S \subseteq N: i \in S\}} \frac{\Delta_S^v}{|S|} = \phi_i(v). \end{aligned}$$

□

Straffin [17] used the multilinear extension of a simple game (Definition 1.3) to explain the Shapley-Shubik index since a probabilistic point of view. If  $v \in \mathcal{G}_s^N$  then

$$v^{ml}(\tau) = \sum_{S \in W(v)} \prod_{j \in S} \tau(j) \prod_{j \notin S} (1 - \tau(j)),$$

where  $W(v)$  is the set of winning coalitions. The fuzzy set  $\tau$  is interpreted as a set of probability distributions, one for each player, determining the possibility to support a motion. So,  $v^{ml}(\tau)$  is the probability to join a winning coalition. The Shapley-Shubik index, following the above theorem, corresponds to the diagonal value of the multilinear extension of our simple game. For each player  $i$  number

$$D_i(v^{ml})(\tau) = \sum_{S \in SW_i(v)} \prod_{j \in S \setminus \{i\}} \tau(j) \prod_{j \notin S \cup \{i\}} (1 - \tau(j))$$

(obtained in the proof of Proposition 2.3) represents the probability to get a swing (Definition 1.11) for this player. The equality

$$\phi_i(v) = \int_0^1 D_i(v^{ml})(te^N) dt$$

means to calculate the expectation to be the critical person in the voting under the homogeneity assumption (all the probability distributions are the same).

### The Proportional Extension

We focus now on the fuzziness proposed by Butnariu [6]. Players consider the maximal level of cooperation, and the biggest coalition with this level. Then the second level and go on. This condition guarantees that each player only play once as in the classical model. If  $\tau \in [0, 1]^N$  is a fuzzy coalition over  $N$  we set for each  $t \in [0, 1]$

$$S_t^\tau = \{i \in N : \tau(i) = t\}. \quad (2.10)$$

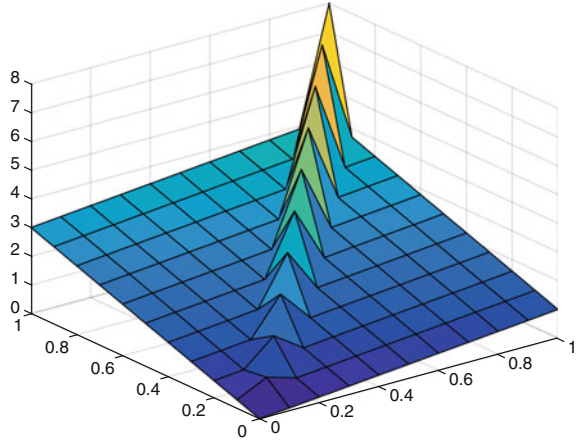
**Definition 2.11** Let  $v \in \mathcal{G}^N$  be a game. The proportional extension of  $v$  is a fuzziness of  $v$  defined for each  $\tau \in [0, 1]^N$  as

$$v^{pr}(\tau) = \sum_{t \in im(\tau)} t v(S_t^\tau).$$

Obviously family

$$pr(\tau) = (S_t^\tau, t)_{t \in im(\tau)} \quad (2.11)$$

**Fig. 2.2** The proportional extension  $v^{pr}$



is a partition by levels of  $\tau$ . If  $\tau = e^S$  then  $im(\tau) = \{1\}$  and  $S_1^\tau = S$ , therefore we can enunciate the following result.

**Proposition 2.4** *Mapping  $pr$  is an extension.*

*Example 2.10* Consider Example 2.7. The meaning now of the levels in the fuzzy coalition  $\tau = (0.2, 0, 0.7, 1, 0.2)$  is complete different. In this case, for player 3, number 0.7 is the real membership and not a probability. Players look first for the maximal level of cooperation in  $\tau$ , thus player 1 is interested to cooperate only with other players who are able to play at level 0.2. The same with player 3 and hence player 3 is not interested in cooperating with player 1. So, in this case,

$$v^{pr}(\tau) = v(\{4\}) + 0.7v(\{3\}) + 0.2v(\{1, 5\}).$$

*Example 2.11* Suppose again the game  $v$  in Example 2.8 over  $N = \{1, 2\}$  defined as  $v(\{1\}) = 1$ ,  $v(\{2\}) = 3$  and  $v(\{1, 2\}) = 8$ . Let  $\tau = (x, y) \in [0, 1] \times [0, 1]$  be any fuzzy coalition. The proportional extension is a piecewise linear function (see Fig. 2.2) which is discontinuous on the diagonal of the square,

$$v^{pr}(x, y) = \begin{cases} 8x, & \text{if } x = y \\ x + 3y, & \text{if } x \neq y. \end{cases}$$

*Example 2.12* Let  $T \subseteq N$  be a non-empty set. The proportional extension of the unanimity game  $u_T$  is given for each  $\tau \in [0, 1]^N$  as

$$(u_T)^{pr}(\tau) = \begin{cases} t, & \text{if } T \subseteq S_t^\tau \\ 0, & \text{otherwise.} \end{cases}$$

Using the worths of the fuzzy coalitions for unanimity games obtained in the above example we give another formula for the proportional extension.

**Proposition 2.5** *For all game  $v \in \mathcal{G}^N$  it holds*

$$v^{pr}(\tau) = \sum_{\{S \subseteq N : S \neq \emptyset, \exists t_S \text{ with } \tau(i) = t_S \forall i \in S\}} t_S \Delta_S^v.$$

*Proof* We get for two games  $v_1, v_2$ ,

$$(av_1 + bv_2)^{pr} = av_1^{pr} + bv_2^{pr}.$$

Proposition 1.1 and Example 2.12 imply the result.  $\square$

Not always the properties of a game are transmitted to the fuzziness. Next example shows that monotonicity and convexity are not transmitted from the game to its fuzziness.

*Example 2.13* Consider the game  $v \in \mathcal{G}_m^N$  in Example 2.11. We have  $v^{pr}(\tau) = 2.4$  with  $\tau = (0.3, 0.3)$  and  $v^{pr}(\tau') = 1.8$  with  $\tau' = (0.3, 0.5)$ , but  $\tau \leq \tau'$ .

Our game  $v$  is also convex (in this case, being convex coincides with being super-additive). Now take  $\tau = (0.5, 0.5)$  and  $\tau' = (0.4, 0.6)$ . We have  $\tau \vee \tau' = (0.5, 0.6)$  and  $\tau \wedge \tau' = (0.4, 0.5)$ . But

$$v(\tau \vee \tau') + v(\tau \wedge \tau') = 4.2 \leq 6.2 = v(\tau) + v(\tau').$$

At least superadditivity is transmitted by the proportional extension.

**Proposition 2.6** *Let  $v \in \mathcal{G}_{sa}^N$  be a superadditive game. For each two fuzzy coalitions  $\tau, \tau'$  with  $\tau \wedge \tau' = 0$  it holds*

$$v^{pr}(\tau \vee \tau') \geq v^{pr}(\tau) + v^{pr}(\tau')$$

*Proof* Let  $\tau, \tau' \in [0, 1]^N$  with  $\tau \wedge \tau' = 0$ . In that case  $\text{supp}(\tau) \cap \text{supp}(\tau') = \emptyset$ , moreover  $\text{im}(\tau \vee \tau') = \text{im}(\tau) \cup \text{im}(\tau')$ . This fact implies that for each  $t \in \text{im}(\tau \vee \tau')$  we have

$$S_t^{\tau \vee \tau'} = S_t^\tau \cup S_t^{\tau'}, \quad S_t^\tau \cap S_t^{\tau'} = \emptyset.$$

So, using that  $v$  is superadditive

$$\begin{aligned}
v^{pr}(\tau \vee \tau') &= \sum_{t \in im(\tau \vee \tau')} t v(S_t^{\tau \vee \tau'}) = \sum_{t \in im(\tau \vee \tau')} t v(S_t^\tau \cup S_t^{\tau'}) \\
&\geq \sum_{t \in im(\tau) \cup im(\tau')} t v(S_t^\tau) + \sum_{t \in im(\tau) \cup im(\tau')} t v(S_t^{\tau'}) = v^{pr}(\tau) + v^{pr}(\tau').
\end{aligned}$$

Observe that if  $t \in im(\tau') \setminus im(\tau)$  (or  $t \in im(\tau) \setminus im(\tau')$ ) then  $S_t^\tau = \emptyset$  ( $S_t^{\tau'} = \emptyset$ ).  $\square$

### The Choquet Extension

The third option uses the Choquet integral. Tsurumi et al. [18] proposed the following fuzziness. Players look for setting the largest coalition and for that they play with the smallest level.

**Definition 2.12** Let  $v \in \mathcal{G}^N$  be a game. The Choquet extension of  $v$  is a fuzziness of  $v$  defined for each  $\tau \in [0, 1]^N$  as

$$v^{ch}(\tau) = \int_c \tau dv.$$

The definition of the Choquet integral in a finite set shows that the above definition is indeed a fuzziness of  $v$ ,

$$v^{ch}(\tau) = \sum_{k=1}^m (\lambda_k - \lambda_{k-1}) v([\tau]_k),$$

if  $im_0(\tau) = \{\lambda_0 < \lambda_1 < \dots < \lambda_m\}$ . So, the partition by levels is

$$ch(\tau) = (\lambda_k - \lambda_{k-1}, [\tau]_k)_{k=1}^m. \quad (2.12)$$

If  $i \in N$  and  $\tau(i) = \lambda_{k_0}$  then

$$\sum_{\{k: i \in [\tau]_k\}} (\lambda_k - \lambda_{k-1}) = \sum_{k=1}^{k_0} (\lambda_k - \lambda_{k-1}) = \tau(i).$$

*Example 2.14* Following to Examples 2.7 and 2.10 we determine now the Choquet extension of  $v$  in  $\tau$ . In this case, players look for the biggest coalition, so they have to use the smallest level in the image of  $\tau$ , namely 0.2. Coalition  $\{1, 3, 4, 5\}$  is formed. Now players still can cooperate choosing level  $0.7 - 0.2 = 0.5$  and coalition  $\{3, 4\}$  (using actually  $(0, 0, 0.5, 0.8, 0)$ ) and go on.

$$\begin{aligned} v^{ch}(\tau) &= (0.2 - 0)v(\{1, 3, 4, 5\}) + (0.7 - 0.2)v(\{3, 4\}) + (1 - 0.7)v(\{4\}) \\ &= 0.2v(\{1, 3, 4, 5\}) + 0.5v(\{3, 4\}) + 0.3v(\{4\}). \end{aligned}$$

Next example shows graphically the proposed fuzziness with two players.

*Example 2.15* Suppose again the game  $v$  in Examples 2.8 and 2.11 over  $N = \{1, 2\}$  defined as  $v(\{1\}) = 1$ ,  $v(\{2\}) = 3$  and  $v(\{1, 2\}) = 8$ . Let  $\tau = (x, y) \in [0, 1] \times [0, 1]$  be any fuzzy coalition. The Choquet extension is also a piece linear function (see Fig. 2.3) but it is continuously although it is not differentiable,

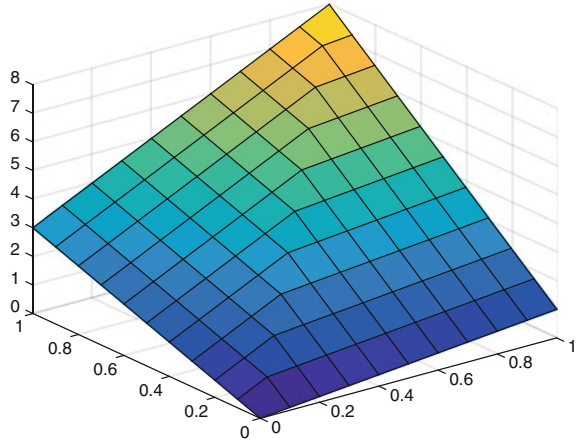
$$v^{ch}(x, y) = \begin{cases} x + 7y, & \text{if } x \geq y \\ 5x + 3y, & \text{if } x \leq y. \end{cases}$$

*Example 2.16* Let  $T \subseteq N$  be a non-empty set and  $\lambda_{k_0} = \bigwedge_{i \in T} \tau(i)$ . The Choquet extension of the unanimity game  $u_T$  is given for each  $\tau \in [0, 1]^N$  with  $im_0 = \{\lambda_0 < \lambda_1 < \dots < \lambda_m\}$  as

$$(u_T)^{ch}(\tau) = \sum_{\{k: T \subseteq [\tau]_k\}} (\lambda_k - \lambda_{k-1}) = \sum_{k=1}^{k_0} (\lambda_k - \lambda_{k-1}) = \bigwedge_{i \in T} \tau(i).$$

The above example allows again to describe the extension by the dividends of the game.

**Fig. 2.3** The Choquet extension  $v^{ch}$



**Proposition 2.7** For all game  $v \in \mathcal{G}^N$  it holds

$$v^{ch}(\tau) = \sum_{S \subseteq \text{supp}(\tau)} \Delta_S^v \bigwedge_{i \in S} \tau(i) = \sum_{k=1}^m \lambda_k \sum_{\{S \subseteq [\tau]_k : S \cap S_{i_k}^c \neq \emptyset\}} \Delta_S^v,$$

$$\text{if } im(\tau) = \{\lambda_1 < \dots < \lambda_m\}.$$

*Proof* We get for two games  $v_1, v_2$ ,

$$(av_1 + bv_2)^{ch} = av_1^{ch} + bv_2^{ch}.$$

Proposition 1.1, Example 2.16 imply both formulas. Observe that  $\bigwedge_{i \in S} \tau(i) = t$  if there exists  $i \in S$  with  $\tau(i) = t$ .  $\square$

The Choquet extension works well with the properties of the original crisp game as the multilinear extension.

**Proposition 2.8** Let  $\tau, \tau' \in [0, 1]^N$  be two fuzzy coalitions.

- (1) If  $v \in \mathcal{G}_m^N$  and  $\tau \leq \tau'$  then  $v^{ch}(\tau) \leq v^{ch}(\tau')$ .
- (2) If  $v \in \mathcal{G}_{sa}^N$  and  $\tau \wedge \tau' = 0$  then  $v^{ch}(\tau \vee \tau') \geq v^{ch}(\tau) + v^{ch}(\tau')$ .
- (3) If  $v \in \mathcal{G}_c^N$  then  $v^{ch}(\tau \vee \tau') + v^{ch}(\tau \wedge \tau') \geq v^{ch}(\tau) + v^{ch}(\tau')$ .

*Proof* (1) The result follows since property (C8) of the Choquet integral (Sect. 2.2).  
 (2) Let  $v \in \mathcal{G}_{sa}^N$ . If  $\tau \wedge \tau' = 0$  then  $im(\tau \vee \tau') = im(\tau) \cup im(\tau')$ . Moreover  $[\tau \vee \tau']_t = [\tau]_t \cup [\tau']_t$  and  $[\tau]_t \cap [\tau']_t = \emptyset$  for all  $t \in (0, 1]$ . So, if  $im_0(\tau \vee \tau') = \{0 = \lambda_0 < \lambda_1 < \dots < \lambda_m\}$

$$\begin{aligned} v^{ch}(\tau \vee \tau') &= \int_c \tau \vee \tau' dv = \sum_{k=1}^m (\lambda_k - \lambda_{k-1}) v([\tau]_k \cup [\tau']_k) \\ &\geq \sum_{k=1}^m (\lambda_k - \lambda_{k-1}) [v([\tau]_k) + v([\tau']_k)] \\ &= \sum_{k=1}^m (\lambda_k - \lambda_{k-1}) v([\tau]_k) + \sum_{k=1}^m (\lambda_k - \lambda_{k-1}) v([\tau']_k) \\ &= \int_c \tau dv + \int_c \tau' dv, \end{aligned}$$

using (C10) in the last equality. Observe that  $im(\tau), im(\tau')$  and both them contains

(3) Suppose  $v \in \mathcal{G}_c^N$ . Obviously  $im(\tau \vee \tau') \cup im(\tau \wedge \tau') = im(\tau) \cup im(\tau')$ . We denote

$$im_0(\tau) \cup im_0(\tau') = \{0 = \lambda_0 < \lambda_1 < \dots < \lambda_m\}.$$

For each  $t \in (0, 1]$  we obtain  $[\tau \vee \tau']_t = [\tau]_t \cup [\tau']_t$  and  $[\tau \wedge \tau']_t = [\tau]_t \cap [\tau']_t$ . Now, by (C10) and convexity,

$$\begin{aligned} v^{ch}(\tau \vee \tau') + v^{ch}(\tau \wedge \tau') &= \int_c \tau \vee \tau' dv + \int_c \tau \wedge \tau' dv \\ &= \sum_{k=1}^m (\lambda_k - \lambda_{k-1}) [v([\tau \vee \tau']_k) + v([\tau \wedge \tau']_k)] \\ &\geq \sum_{k=1}^m (\lambda_k - \lambda_{k-1}) [v([\tau]_k) + v([\tau']_k)] \\ &= \int_c \tau dv + \int_c \tau' dv \\ &= v^{ch}(\tau) + v^{ch}(\tau'). \end{aligned}$$

□

There are several ways to apply the diagonal formula to the Choquet extension by smoothing processes in Weiss [19].

## References

1. Aubin, J.P.: Cooperative fuzzy games. *Math. Oper. Res.* **6**(1), 1–13 (1981)
2. Aumann, R.J., Shapley, L.S.: *Values for Non-Atomic Games*. Princeton University Press, Princeton (1974)
3. Borkotokey, S., Mesiar, R.: The Shapley value of cooperative games under fuzzy settings: a survey. *Int. J. Gen. Syst.* **43**(1), 75–95 (2014)
4. Branzei, R., Dimitrov, D., Tijs, S.: Convex fuzzy games and participation monotonic allocation schemes. *Fuzzy Sets Syst.* **139**, 267–281 (2003)
5. Branzei, R., Dimitrov, D., Tijs, S.: *Models in Cooperative Game Theory*. Springer, Berlin (2008)
6. Butnariu, D.: Stability and Shapley value for a  $n$ -persons fuzzy games. *Fuzzy Sets Syst.* **4**(1), 63–72 (1980)
7. Butnariu, D., Klement, E.P.: *Triangular Norm-Based Measures and Games with Fuzzy Coalitions*. Kluwer Academic Publishers, Dordrecht (1993)
8. Chalkiadakis, G., Elkind, E., Markakis, E., Polukarov, M., Jennings, N.R.: Cooperative games with overlapping coalitions. *J. Artif. Intell. Res.* **39**, 179–216 (2010)
9. Choquet, G.: Theory of capacities. *Annales de L'Institut Fourier*, vol. 5, pp. 131–295 (1953)
10. de Waegenare, A., Wakker, P.P.: Nonmonotonic Choquet integrals. *J. Math. Econ.* **36**, 45–60 (2001)
11. Li, S.J., Zhang, Q.: A simplified expression of the Shapley function for fuzzy games. *Eur. J. Oper. Res.* **196**(1), 234–245 (2009)

12. Mares, M.: Fuzzy Cooperative Games. Studies in Fuzziness and Soft Computing vol. 72. Physica-Verlag, Heidelberg; Springer, Heidelberg (2001)
13. Meng, F.Y., Zhang, Q.: The Shapley function for fuzzy cooperative games with multilinear extension form. *Appl. Math. Lett.* **23**(5), 644–650 (2010)
14. Mertens, J.F.: The Shapley value in the non-differentiable case. *Int. J. Game Theory* **17**, 1–65 (1988)
15. Owen, G.: Multilinear extensions of games. *Manag. Sci.* **18**(5), 64–79 (1972)
16. Schmeidler, D.: Integral representation without additivity. *Proc. Am. Math. Soc.* **97**(2), 255–261 (1986)
17. Straffin, P.D.: Game Theory and Strategy. The Mathematical Association of America, vol. 36. New Mathematical Library (1993)
18. Tsurumi, M., Tanino, T., Inuiguchi, M.: A Shapley function on a class of cooperative fuzzy games. *Eur. J. Oper. Res.* **129**(3), 596–618 (2001)
19. Weiss, C.: Games with Fuzzy Coalitions, Concepts Based on the Choquet Extension. Bielefeld University, Germany (2003)
20. Zadeh, L.A.: Fuzzy sets. *Inf. Control* **8**, 338–353 (1965)

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