Chapter 2
From a Mathematical Situation to a Problem

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Abstract The approach to problems creation starting from a mathematical situation is developed, with several examples of such situations and problems arising from this, with solutions (if the problem is not open).

Keywords Geometrical situation · Mathematical problem

2.1 Introduction

The teaching of mathematics on the basis of problem solving is a periodically repeated subject in ICMEs, as TG or WG. Within this general frame, we will consider in this chapter an approach to problems creation that we will call "From a mathematical situation to a problem".

In *Mathematical Competitions*, the journal of the WFNMC, the question of the creation of problems has been studied many times; in particular, between 1986 and 1999, more than 20 papers on this subject were been published. The paper by Engel (1987) *The creation of mathematical Olympiad Problems*, starts with the following sentence:

Dedication: To Eduardo Wagner (SBM, Brazil), from whom I learned to go from a mathematical situation to a problem or a theorem, and how to solve them.

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> It is far more difficult to create problem than to solve it. There are very few routine methods of problem creation. As far as I know no Polya among problem creators who wrote a book with the title “How to create it”.

When analyzing some examples of workshops about Learning based in problems, we notice that, although the term “situation-problem” may be used, the teacher actually presents a closed statement to the students. That is, the teacher is helping the students to find a way to gather the details of the solution of a problem from which the full statement is, sooner or later, given. It’s clear that during the discussion, students can discover some alternative statements which can became new problems, and this, no doubt, improves the enrichment of the mathematical-didactical discussion which must follow. In this sense, the treatment of the question given in the book “*Pour un enseignement problematisé des Mathématiques au Lycée*” (2 vols.), APMEP, in French, no date of publication, a collective work of the group “Problématiques Lycée”, is interesting.

To begin, we can take a look at an example included in the workshop *Aprendizaje basado en problemas* (Learning based on problem-solving), by Prof. Rolando Sáenz, from Ecuador. This example was presented in 2006 in Salinas (Ecuador), during the Iberoamerican Symposium (with emphasis in problem solving), a didactical activity prior to the Iberoamerican Mathematical Olympiad.

Example 1.1 *ABCD* is a square. We take points *M, N, O* and *P*, respectively in *AB, BC, CD* and *DA*, in a such manner that $AM = BN = CO = DP$. Determine the point *M* such that the quadrilateral *MNOP* have maximal area.

Maybe if the last sentence was changed to something like this: *Consider the quadrilateral MNOP*, some other statements, equally interesting, would emerge during the discussion. We invite the readers to try it by themselves.

Many times, the reading of a paper about problem creation will provide some very interesting problems, but there are rarely many explanations on how they were created, that is, what was the process which gave birth to the problem.

We can now take a look at some characteristics which a good problem should have. Gardiner (1992, p. 59) wrote this:

(a) *The ingredients (of the problem) should be simple and familiar, but the problem should not be of any standard type.*
(b) *No method of solution should be immediately obvious, but a careful survey of the given information should suggest one or two promising points of attack.*

(c) *And exploratory phase should then reveal how (or whether) these approaches can be exploited.*
(d) *The final solution when it emerges should, in retrospect, have an unexpected elegance or conceptual simplicity.*

Example 1.2 The positive numbers x, y and z satisfy

$$\begin{gathered} x^2+xy+\tfrac{y^2}{3}=25 \\ \tfrac{y^2}{3}+z^2=9 \\ z^2+xz+x^2=16 \end{gathered}$$

Find the value of xy + 2yz + 3zx.

Note: The sources of the problems will be included in the solutions section

The readers are invited to think about this statement and to try by themselves the "promising points of attack" in the words of Gardiner.

As last part of this introduction, here is a quote of Branko Grünbaum in his introduction to the book of Soifer (1990) *How Does One Cut a Triangle?*.

> Many people find mathematics attractive because it presents to the mind the same challenge that other activities, such as sports, present to the body. In mathematics, and specially in geometry, there are abundance of topics that are accessible without much previous knowledge. They present the exploring mind with opportunities to rise to that challenge, and to experience the joy of discovery.

2.2 What Is a Mathematical Situation?

Searching in libraries, it is possible to find—at least—two types of books which can be related to the topic of the chapter.

(1) Books where mathematical situations with problems are presented (with or without solutions).
(2) Books where mathematical problems are discussed in detail, showing what should be the way by which the solution must be presented to the audience (much more detailed than the typical way in which the solution seems to appears like riding a parachute, falling down from the sky).

One of the earliest examples of books from type 1 is *Geometry for Advanced Pupils*, by Maxwell (1949).

Dr. Maxwell presents here 47 configurations from which is possible to deduce results, many problems and geometric properties of interest. He also includes examples from the Oxford and Cambridge Examinations Papers.

An interesting paper, published in *Quantum*, January/February 2001 by the late Prof. I. Sharygin, is *Where do problems come from?* (Sharygin 2001) (*The art of problem composition*). Sharygin explains in this paper some of his own procedures for composing problems (Olympiad type): by reformulation, problems built on other problems, considering special cases of a theorem; varying the problem statement; by generalization of a problem (or some result). And he says: *However, the main source of new problems is inquisitiveness, the desire to reveal the essence of a problem, the ability to look at a well-known fact from an unusual point of view. This is when the most interesting geometric problems appear, ones that can be called discoveries.*

Sharygin ends his paper with this assertion: *You don't have to be a budding mathematical genius to make geometric discoveries—some problems show that any student can do it. And this includes you!*

Another book of type 1 is *Geometry in figures*, by Akopyan (2011) (no Editorial name, but the place is Lexington, KY). This is a collection of theorems and problems of Euclidean geometry formulated in figures, without text. This is a good illustration about what a geometrical situation is. Recently (2015), the Union of Bulgarian Mathematicians published the book by Dimitrov, Lichev and Chovanov *555 problems of Geometry* (in Bulgarian) with the solutions to the problems of the book by Akopyan.

To end with the examples of publications of type 1 it is worth mentioning the book by Monk (2009). This is a very popular book among the participating countries in the IMO since it was published. The five categories of problems of the book are E (easy), 18 problems; M (moderate difficulty), 20 problems; H (hard), 18 problems; C (Computational), 24 problems; and T (Trigonometry), 18 problems.

For the books of Type 2 the situation seems to be better. There are many publications about this subject (see the References section for more titles) and some of them are really excellent. Here are a few examples:

Burns (2000).

Gardiner (1997).

Savchev and Andreescu (2003).

Nevertheless, it seems there are not many titles in libraries and bookstores which describe what a mathematical situation is. Paraphrasing Prof. Eduardo Wagner, Brazilian expert in problem solving: *As important as teaching Mathematics is to create new problems, interesting and challenging. Problems are new questions, of different aspect to the usual one and which should stimulate the development of the reasoning. To create one problem a big effort, enough time to try many attempts, and good luck are required. With continued work and much reading, the ability to create problems is developed and the ideas can emerge in our mind more easily. This work is not different to other sciences or artistic work. To acquire any ability, everybody needs specific training.*

The "Office of creating problems", promoted by the OEI (Iberoamerican States Organization) in the years 1994 to 1997, is an introduction to the art of creating problems. With its own methodology, the participants have the opportunity of experimenting with real problem creation situations, and they then developed their own methods.

A Mathematical situation is not yet a problem. It consists of a set of mathematical objects, linked by some certain relations. With this basis, the participants (in the Office) must investigate the properties of the proposed situation, adding if necessary other elements, and to create one of more problems. In this way, with the reasoning focused in a particular situation, the activity was followed with the biggest interest by the participants and some new problems of different degrees of difficulty were created. End of the quote, taken from Wagner (1997).

Prof. Wagner was the coordinator of the "Office" in the years 1994, 1995 and 1996. The Mexican Prof. Alejandro Bravo was the coordinator in 1997.

The next section of the chapter provides examples of mathematical situations, which are deliberately left open, in order that readers can experiment by themselves creating new problems (this would be truly excellent!). In the subsequent sections, the problems arising from these situations are presented and the section of detailed solutions will follow.

2.3 Several Examples of Mathematical Situations

Situation 3.1 In the acute triangle *ABC*, let *AM* be the median (*M* belongs to the side *BC*), and let *AD* be the internal bisector of angle *A*. (*D* belongs to the side *BC*). From *B* the perpendicular to *AD* is drawn, meeting *AD* at *J*, to *AM* at *L* and to *AC* at *K*.

Situation 3.2 The most important carpet seller of Orient is very worried. His device to measure the carpets has been stolen and so he can't measure the new carpet recently received, for one of his best clients. The carpet is rectangular, but the dimensions are unknown. If he display the carpet in the floor of two of the rooms of his house, one after the other, in a convenient way, the four corners of the carpet are located on each one of the 4 walls of each room.

Situation 3.3 The quadrilateral *ABCD* has an inscribed circle, being *K*, *L*. *M* and *N* the tangency points with the sides *AB, BC, CD* and *DA*, respectively. The lines *DA* and *CB* intersect at *S*, and the lines *BA* and *CD* intersect at *P*.

Situation 3.4 Let *M* and *N* be points of the side *BC* of the triangle *ABC*, such that $BM = CN$ (point *M* is located between *B* and *N*). Let *P* and *Q* be points located respectively on *AN* and *AM* such that $\angle PMC = \angle MAB$ and $\angle QNB = \angle NAC$.

Situation 3.5 Consider the sum $\sum_{i=1}^{n} x_i y_i$, where the values of the $2n$ variables $x_1, \ldots, x_n; y_1, \ldots, y_n$ are only 0 either 1. Let $I(n)$ be the number of $2n$-tuples $x_1, \ldots, x_n; y_1, \ldots, y_n$ such that the sum is an odd number, and $P(n)$ the number of $2n$-tuples $x_1, \ldots, x_n; y_1, \ldots, y_n$ such that the sum is an even number. Consider $\frac{P(n)}{I(n)}$.

Situation 3.6 In the triangle *ABC*, *G* is the point of intersection of the medians and *K* the point of intersection of the symmedians. The lines *AG* and *AK* intersect again the circumcircle of *ABC* at *M* and *N*, respectively. Let $P = BC \cap GN$, $R = BC \cap KM$ and $S = GR \cap KP$.

Situation 3.7 The acute triangle *ABC* is inscribed in a circle. The point *P* is inside the triangle. Lines *AP*, *BP* and *CP* intersect again the circumcircle of *ABC* at *X*, *Y* and *Z*, respectively. Consider the triangle *XYZ*.

Situation 3.8 Consider the sequence of real numbers $\{x_n\}$ with x_0 arbitrary and $x_{n+1} = 2(x_n)^2 - 1$.

Situation 3.9 Lines r and s lie mutually orthogonal and do not are in the same plane. Let AB be its common perpendicular ($A \in r$, $B \in s$). Consider the sphere of diameter AB. The points $M \in r$ and $N \in s$ are variable, with the condition that MN is tangent to the sphere. Let T be the point of tangency.

Situation 3.10 Let ABC be a triangle inscribed in a circle, and I is the incenter of the triangle. Lines BI and CI intersect again the circumcircle at M and N, respectively. Line MN intersect AB at P and AC at Q, respectively.

Situation 3.11 With center in the incenter I of the triangle ABC, a circle is drawn, intersecting in two points each side of the triangle: to BC at D and P (being D the most near to B), to CA at E and Q (being E the most near to C), and to AB at F and R (being F the most near to A). Let S be the point of intersection of the diagonals of the quadrilateral $EQFR$, and T the point of intersection of the diagonals of the quadrilateral $FRDP$. Finally, let U be the intersection of the diagonals of the quadrilateral $DPEQ$.

Situation 3.12 $ABCD$ is a convex quadrilateral, and $M = AC \cap BD$. The internal bisector of $\angle ACD$ intersects BA at K. Suppose $MA \cdot MC + MA \cdot CD = MD \cdot MB$.

2.4 Some Problems Arising from the Mathematical Situations of Sect. 2.3

Problem 4.1 In the acute triangle ABC, let AM be the median (M belongs to the side BC), and let AD be the internal bisector of angle A (D belongs to the side BC). From B the perpendicular to AD is drawn, meeting AD at J, to AM at L and to AC at K. Show that AB and DM are parallel.

Problem 4.2 The most important carpet seller of Orient is very worried. His device to measure the carpets has been stolen and so he can't measure the new carpet recently received, for one of his best clients. The carpet is rectangular, but the dimensions are unknown. If he display the carpet in the floor of two of the rooms of his house, one after the other, in a convenient way, the four corners of the carpet are located on each one of the 4 walls of each room. If the sides of the first room are 55 and 50, and those of the second room are 55 and 38, find the dimensions of the carpet.

Problem 4.3 The quadrilateral $ABCD$ has an inscribed circle, being K, L. M and N the tangency points with the sides AB, BC, CD and DA, respectively. The lines DA and CB intersect at S, and the lines BA and CD intersect at P. If S, K and M are collinear, prove that P, N and L are also collinear.

Problem 4.4 Let M and N be points of the side BC of the triangle ABC, such that $BM = CN$ (point M is located between B and N). Let P and Q be points located respectively on AN and AM such that $\angle PMC = \angle MAB$ and $\angle QNB = \angle NAC$ Would it be always true that $\angle QBC = \angle PCB$?

Problem 4.5 Consider the sum $\sum_{i=1}^{n} x_i y_i$, where the values of the $2n$ variables $x_1, \ldots, x_n; y_1, \ldots, y_n$ are only 0 either 1. Let $I(n)$ be the number of $2n$-tuples $x_1, \ldots, x_n; y_1, \ldots, y_n$ such that the sum is an odd number, and $P(n)$ the number of $2n$-tuples $x_1, \ldots, x_n; y_1, \ldots, y_n$ such that the sum is an even number. Show that $\frac{P(n)}{I(n)} = \frac{2^n+1}{2^n-1}$.

Problem 4.6 In the triangle ABC, G is the point of intersection of the medians and K the point of intersection of the symmedians. The lines AG and AK intersect again the circumcircle of ABC at M and N, respectively. Let $P = BC \cap GN$, $R = BC \cap KM$ and $S = GR \cap KP$. Show that $AGSK$ is a parallelogram.

Problem 4.7 The acute triangle ABC is inscribed in a circle. The point P is inside the triangle. Lines AP, BP and CP intersect again the circumcircle of ABC at X, Y and Z, respectively. Determine the position of the point P for that XYZ be equilateral.

Problem 4.8 Consider the sequence of real numbers $\{x_n\}$ with x_0 arbitrary and $x_{n+1} = 2(x_n)^2 - 1$. Show that, if $|x_0| \leq 1$, then $|x_n| \leq 1$. Find a closed expression for x_n.

Problem 4.9.1 Lines r and s are mutually orthogonal and do not are in the same plane. Let AB be its common perpendicular ($A \in r$, $B \in s$). Consider the sphere of diameter AB. The points $M \in r$ and $N \in s$ are variable, with the condition that MN is tangent to the sphere. Let T be the point of tangency. Show that $TM.TN$ is constant.

Problem 4.9.2 Lines r and s are mutually orthogonal and do not are in the same plane. Let AB be its common perpendicular ($A \in r$, $B \in s$). Consider the sphere of diameter AB. The points $M \in r$ and $N \in s$ are variable, with the condition that MN is tangent to the sphere. Let T be the point of tangency. Determine the geometrical locus of the point T.

Problem 4.10 Let ABC be a triangle inscribed in a circle, and I is the incenter of the triangle. Lines BI and CI intersect again the circumcircle at M and N, respectively. Line MN intersect AB at P and AC at Q, respectively. Show that IA is perpendicular to MN.

Problem 4.11 With center in the incenter I of the triangle ABC, a circle is drawn, intersecting in two points each side of the triangle: to BC at D and

P (being D the most near to B), to CA at E and Q (being E the most near to C), and to AB at F and R (being F the most near to A). Let S be the point of intersection of the diagonals of the quadrilateral $EQFR$, and T the point of intersection of the diagonals of the quadrilateral $FRDP$. Finally, let U be the intersection of the diagonals of the quadrilateral $DPEQ$. Show that the circumcircles of the triangles FTR, DPU and EQS have one common point.

Problem 4.12 $ABCD$ is a convex quadrilateral and $M = AC \cap BD$. The internal bisector of $\angle ACD$ intersects BA at K. Suppose $MA \cdot MC + MA \cdot CD = MD \cdot MB$. Show that $\angle BKC = \angle CDB$.

2.5 Hints, Solutions and Comments to Some of the Problems and Examples

2.5.1 *Comment and Hint to Example 1.2*

The right hand side of the three equations are numbers of a Pythagorean triad. The left hand side of the equations represents the expressions of the cosine law for some convenient angles. So, the advice is to locate one point M inside a rectangle triangle with convenient sides in a such way the three equations be fulfilled, and from this, evaluate more easily xy + 2yz + 3zx.

Source of the problem: Zhang Jung-da et al., *Mathematics Competitions,* vol.10, number 2, 1997, pp. 52–63.

2.5.2 *Solution to Problem 4.1*

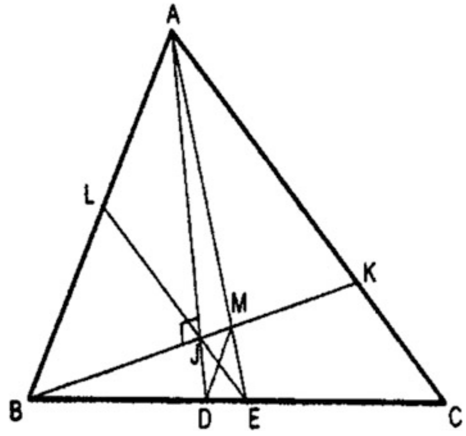
In the acute triangle ABC, let AM be the median (M belongs to the side BC), and let AD be the internal bisector of angle A. (D belongs to the side BC). From B the perpendicular to AD is drawn, meeting AD at J, to AM at L and to AC at K. Show that AB and DM are parallel (Fig. 2.1).

Solution (by F. Bellot)

There is no loss of generality if we suppose that angle B is bigger than angle C. First, being AD the internal bisector of angle A, $\angle BAD = \frac{A}{2}$. And as BJ is perpendicular to AD, $\angle ABJ = 90° - \frac{A}{2}$. The same argument in the triangle AJK gives us $\angle AKJ = 90° - \frac{A}{2}$. Then, triangle ABK is isosceles and $AK = AB = c$. From this, we get $KC = b - c$.

To prove that AB and DM are parallel, it is enough to prove that $\frac{AM}{ME} = \frac{BD}{DE}$, and the theorem of Thales will finish the problem.

Fig. 2.1 Figure for Problem 4.1



First we will compute $\frac{BD}{DE}$. From the angular bisector theorem, we have $BD = \frac{ac}{c+b}$. As E is the midpoint of BC, we have $DE = BE - BD = \frac{a(b-c)}{2(c+b)}$, therefore $\frac{BD}{DE} = \frac{2c}{b-c}$.

To compute $\frac{AM}{ME}$, we can use the Menelaus theorem in triangle AEC with the transversal KMB:

$\frac{AM}{ME} \cdot \frac{EB}{BC} \cdot \frac{CK}{KA} = 1 \Leftrightarrow \frac{AM}{ME} = \frac{2c}{b-c}$, and we are done. ∎

Source of the problem: Course on Euclidean Geometry I, University of Costa Rica, 2012.

2.5.3 *Solution of Problem 4.2*

The most important carpet seller of Orient is very worried. His device to measure the carpets has been stolen and so he can't measure the new carpet recently received, for one of his best clients. The carpet is rectangular, but the dimensions are unknown. If he display the carpet in the floor of two of the rooms of his house, one after the other, in a convenient way, the four corners of the carpet are located on each one of the 4 walls of each room. If the sides of the first room are 55 and 50, and those of the second room are 55 and 38, find the dimensions of the carpet.

Solution by María Ascensión López Chamorro, Valladolid (Spain)

We will solve the problem in a more general context, and then will apply it to the case of the carpet with the given numerical measures.

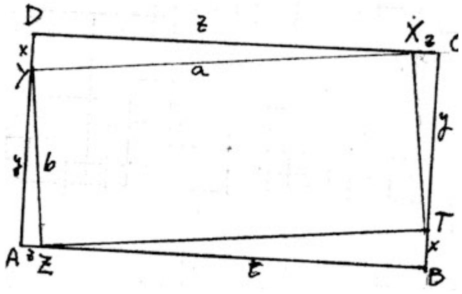


Fig. 2.2 First figure for Problem 4.2

Let $ABCD$ be a rectangle, and let $XYZT$ be another rectangle, inscribed in the first, with Z on the side AB, T on BC, X on CD and finally Y on DA (Fig. 2.2).

Suppose $AB = CD = l_1$; $AD = BC = l_2$; $XY = ZT = a$; $YZ = TX = b$. (In terms of the problem, a, b are the dimensions of the carpet; l_1, l_2 those of the room).

Triangles XDY and ZBT are congruent, also YAZ and TXC. This means

$$XC = AZ = z\,; XD = ZB = t; DY = BT = x; AY = TC = y$$

But moreover triangles XDY and YAZ are similar, and then $\frac{t}{y} = \frac{x}{z} = \frac{a}{b}$.

This proportion can be written as $bt = ay$; $bx = az$; and moreover the equalities $z + y = l_1$ and $x + y = l_2$ holds.

From this we obtain the two relations $\frac{bx}{a} + \frac{ay}{b} = l_1$; $x + y = l_2$ and solving them in the unknowns x and y gives us $\left(\frac{a}{b} - \frac{b}{a}\right)y = l_1 - \frac{b}{a}l_2$; $\left(\frac{b}{a} - \frac{a}{b}\right)x = l_1 - \frac{a}{b}l_2$.

The final expressions for x, y, z, t are the following:

$$x = \frac{a(al_2 - bl_1)}{a^2 - b^2};\ y = \frac{b(al_1 - bl_2)}{a^2 - b^2};\ z = \frac{b(al_2 - bl_1)}{a^2 - b^2};\ t = \frac{a(al_1 - bl_2)}{a^2 - b^2}.$$

But by Pythagora's Theorem, $x^2 + t^2 = a^2$; $y^2 + z^2 = b^2$. Both equalities given the same equation: $\left(\frac{al_2 - bl_1}{a^2 - b^2}\right)^2 + \left(\frac{al_1 - bl_2}{a^2 - b^2}\right)^2 = 1$; and developing, ordering and simplifying this can be written as

$$\left(a^2 + b^2\right)\left(l_1^2 + l_2^2\right) - 4l_1 l_2 ab = \left(a^2 - b^2\right)^2 (*)$$

If the rectangle *XYZT* also can be inscribed in another rectangle with dimensions m_1 and m_2, the same reasoning allows us to writing a second equation

$$\left(a^2+b^2\right)\left(m_1^2+m_2^2\right)-4m_1m_2ab=\left(a^2-b^2\right)^2(**)$$

Substracting (*) and (**) we get

$$\left(a^2+b^2\right)\left(l_1^2+l_2^2-m_1^2-m_2^2\right)-4ab(l_1l_2-m_1m_2)=0.$$

In order to simplify the notation we define

$$k=l_2^1+l_2^2-m_1^2\ -m_2^2; h=\ l_1l_2-m_1m_2;\quad \text{and}\, u=(b/a).$$

With this we have the quadratic equation in u

$$\left(1+u^2\right)k-4uh=0\Leftrightarrow ku^2-4uh+k=0$$

$$u=\frac{2h\pm\sqrt{4h^2-k^2}}{k}.$$

Now we make the computations with the data of the problem (crossing the fingers!):

$$l_1=55;\ l_2=50; m_1=55;\ m_2=38.$$

We get in sequence:

$$k=88\cdot 12;\ h=55\cdot 12$$
$$4k^2-h^2=(2h+k)(2h\ -\ k)=12^2\cdot 11^2\cdot 6^2$$
$$u=2\,\text{or}\,(1/2)$$

and from this,

$$x=20;\ t=15\Rightarrow a=25,\ b=50$$

and for the second rectangle we get

$$x_1=7; t_1=24\Rightarrow a=25,\ b=50,$$

and so the same carpet can be placed in both rooms (Fig. 2.3). ∎

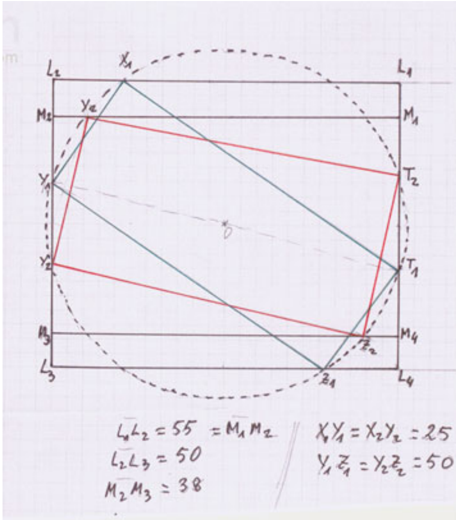


Fig. 2.3 Second figure for Problem 4.2

Source of the problem: Course on Euclidean Geometry I, University of Costa Rica, 2012.

2.5.4 Solution of the Problem 4.3

The quadrilateral *ABCD* has an inscribed circle, being *K, L. M* and *N* the tangency points with the sides *AB, BC, CD* and *DA*, respectively. The lines *DA* and *CB* intersect at *S*, and the lines *BA* and *CD* intersect at *P*. If *S, K* and *M* are collinear, prove that *P, N* and *L* are also collinear.

Source of the problem: Belarusian Math Olympiad 1996 (TST). In the booklet of this Olympiad no authorship attribution of the problem is given. In the booklet the solution of the student M. Vronski, given during the test (a long but nice metrical solution) is published. Some time after the 2002 Melbourne Conference of the WFNMC, where I presented this problem, I received the following solution:

Solution (by Andy Liu)

Let *O* be the centre of the circle and *r* its radius. Then *OS* and *LN* are perpendicular and let them meet at *R*. Also, *OP* and *SKM* are perpendicular and let them meet at *Q*. Since triangles *OLR* and *OSL* are similar, we have $OS.OR = r^2$. Similarly, $OP.OQ = r^2$. Hence *PQRS* is cyclic. Now, $\angle PRS = \angle PQS = 90° = \angle NRS$. It follows that *L, N* and *P* are collinear. ∎

2.5.5 Solution of the Problem 4.4

Let M and N be points of the side BC of the triangle ABC, such that $BM = CN$ (point M is located between B and N). Let P and Q be points located respectively on AN and AM such that $\angle PMC = \angle MAB$ and $\angle QNB = \angle NAC$ Would it be always true that $\angle QBC = \angle PCB$?

Source of the problem: National round of the Spanish Mathematical Olympiad 2015, Problem 6 (Fig. 2.4).

Solution (official solution, slightly edited by F. Bellot)

The key idea of this solution is to consider the circles *(BNQ)* and *(PMC)*. If AM meet again the circle *(BNQ)* at X, and AN meet again the circle *(PMC)* at Y, its trivial that quadrilaterals $BQNX$ and $MPCY$ are cyclic. But being $\angle QBC = \angle QBN$ and $\angle PCB = \angle PCM$, then the angles of the problem will be equal if $\angle QBN = \angle PCM$

But $\angle QBN = \angle QXN = \angle MXN$ and $\angle PCM = \angle PYM = \angle NYM$

Then, the problem will be solved in affirmative sense if we prove the equality

$$\angle MXN = \angle NYM$$

and this means than the four points M, N, Y, X belong to the same circle.

So, we will try to prove that

$$AM \cdot AX = AN \cdot AY \Leftrightarrow \frac{AM}{AN} = \frac{AY}{AX} \tag{2.5.5.1}$$

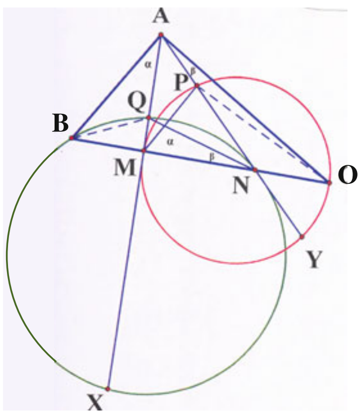


Fig. 2.4 Figure for Problem 4.4

Our argument is the following:

Triangles ABM and CAN have the same area, because their basis are equal by hypothesis and their altitudes from A are the same. So we have

$$AM \cdot AB \cdot \ \sin \alpha = AN \cdot AC \cdot \ \sin \beta \tag{2.5.5.2}$$

where $\alpha = MAB; \beta = NAC$.

For another hand, two of the angles of the triangle ABX are α, and $\angle BXQ = \angle QNB = \beta$ in circle (BNQ)

Similarly, two angles of triangle ACY are β and α. Therefore triangles ABX and ACY are similar, and we can write down the proportionality between the homologous sides as

$$\frac{AY}{AX} = \frac{CY}{AB}. \tag{2.3}$$

Finally, using the sinus law in triangle ACY, we get

$$\frac{AC}{\sin \alpha} = \frac{CY}{\sin \beta} \Leftrightarrow \frac{\sin \beta}{\sin \alpha} = \frac{CY}{AC}$$

and (2.5.5.2) can be written as

$$\frac{AM}{AN} = \frac{AC \cdot \ \sin \beta}{AB \cdot \ \sin \alpha} = \frac{AC}{AB} \cdot \frac{CY}{AC} = \frac{CY}{AB} = \text{ by(3) } = \frac{CY}{AC}$$

and we are done. ∎

2.5.6 Solution to Problem 4.5

Consider the sum $\sum_{i=1}^{n} x_i y_i$, where the values of the $2n$ variables $x_1, \ldots, x_n; y_1, \ldots, y_n$ are only 0 either 1. Let $I(n)$ be the number of $2n$-tuples $x_1, \ldots, x_n; y_1, \ldots, y_n$ such that the sum is an odd number, and $P(n)$ the number of $2n$-tuples $x_1, \ldots, x_n; y_1, \ldots, y_n$ such that the sum is an even number. Show that $\frac{P(n)}{I(n)} = \frac{2^n + 1}{2^n - 1}$.

Source of the problem: This problem, created jointly by the Mexican mathematicians Gerardo Raggi and Humberto Cárdenas, was awarded with the Second Prize in the First Iberoamerican Contest of Creation of Problems, organized by the O.E.I. (Organization of Iberoamerican States for the Education, the Science and the Culture). Before this award were announced, the problem was included in the shortlist presented to the International Jury of the XII Iberoamerican Mathematical Olympiad, held at Guadalajara, Jalisco, Mexico, September 1997, and proposed to the students as problem number 4.

Official solution

First observe that for each natural number n, the recursive formula $P(n+1) = 3P(n) + 1$ holds. This is so, because in any $2n$-uple in which the value is even, there are three possibilities of to choose the couple (x_{n+1}, y_{n+1}) to obtain one $2(n+1)$-uple such that the value still be even; and starting with one $2n$-uple such that the value is odd, there are only one way to choose the couple (x_{n+1}, y_{n+1})—*both values equal to* 1—to complete to get an even value.

Analogously we have $I(n+1) = 3I(n) + P(n)$.

We will use these recursive formulas and the induction over n to get the result.

The proposition is true if $n = 1$, because $P(1) = 3$ and $I(1) = 1$.

Suppose the result true for some $n \geq 1$ and we will prove it for $n+1$. We have

$$\frac{P(n+1)}{I(n+1)} = \frac{3P(n)+I(n)}{3I(n)+P(n)} = \frac{3\left(\frac{2^n+1}{2^n-1}\right)+1}{3+\left(\frac{2^n+1}{2^n-1}\right)} = \frac{3\cdot 2^n+3+2^n-1}{3\cdot 2^n-3+2^n+1} = \frac{4\cdot 2^n+2}{4\cdot 2^n-2} = \frac{2^{n+1}+1}{2^{n+1}-1}.$$

∎

2.5.7 Solution to Problem 4.6

In the triangle ABC, G is the point of intersection of the medians and K the point of intersection of the symmedians. The lines AG and AK intersect again the circumcircle of ABC at M and N, respectively. Let $P = BC \cap GN$, $R = BC \cap KM$ and $S = GR \cap KP$. Show that $AGSK$ is a parallelogram.

Source of the problem: Problem proposed by Spain to the International Jury of the 12th Iberoamerican Math. Olympiad. The Problem selection committee changed the statement to the problem, changing barycenter and Lemoine's point by circumcenter and orthocenter, making it more easy. This is the originally proposed problem.

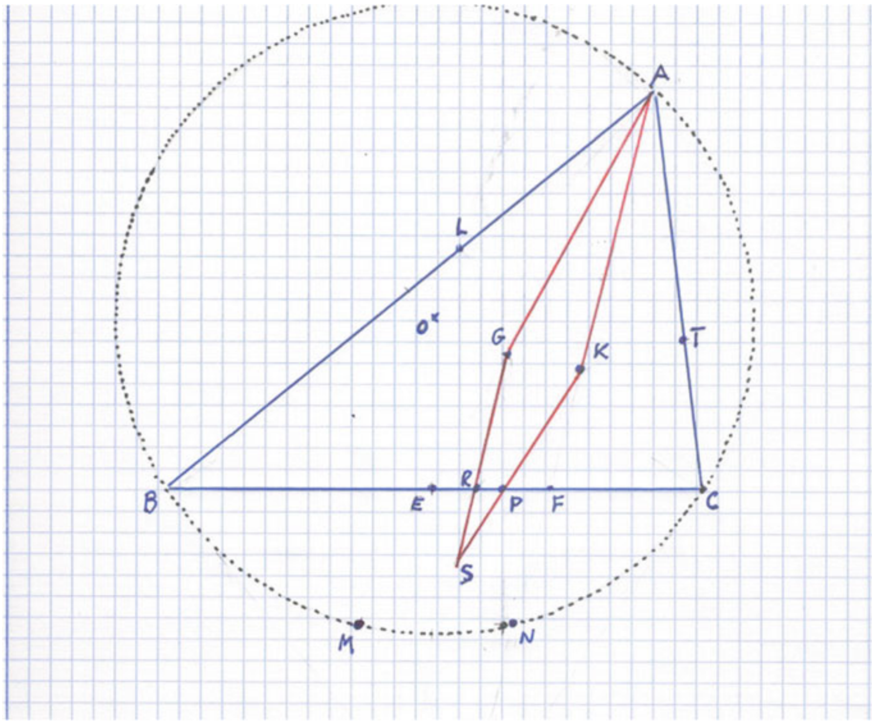


Fig. 2.5 Figure for Problem 4.6

Solution by F. Bellot (Fig. 2.5).

Let $E = BC \cap AG,\ F = BC \cap AK,\ L = AB \cap CK$ and $T = AC \cap BK$.

Taking account that cevians AE and AF are isogonal, the arcs BM and NC in the circle (ABC) are equal. From this we have that angles AEC and AMN are supplementary, due to the equalities

$$\angle AEC = \frac{1}{2}(arcAC + arcBM);\ \angle AMN = \frac{1}{2}arcAN = \frac{1}{2}(arcAC + arcNC)$$

This means that EF is parallel to MN, and as a consequence,

$$\frac{EM}{AM} = \frac{FN}{AN}. \tag{2.5.7.1}$$

For another hand, the power of point E with respect to the circle circumscribed to ABC can be written in two different ways:

$$AE \cdot EM = \frac{BC^2}{4},$$

Hence

$$EM = \frac{BC^2}{4 \cdot AE}. \tag{2.5.7.2}$$

From (2.5.7.2) we get

$$AM = \frac{4 \cdot AE^2 + BC^2}{4 \cdot AE},$$

whence, taking account that

$$AE^2 = \frac{2(AB^2 + AC^2) - BC^2}{4},$$

we get

$$AM = \frac{AB^2 + AC^2}{2 \cdot AE}. \tag{2.5.7.3}$$

From (2.5.7.2) and (2.5.7.3) we obtain

$$\frac{EM}{AM} = \frac{BC^2}{2(AB^2 + AC^2)}, \tag{2.5.7.4}$$

and by (2.5.7.2), we can write down

$$\frac{FN}{AN} = \frac{BC^2}{2(AB^2 + AC^2)}. \tag{2.5.7.5}$$

As the cevians AF, CL and BT are concurrent at K, the Van Aubel theorem allow us to write

$$\frac{AK}{KF} = \frac{AL}{LB} + \frac{AT}{TC};$$

and by the Theorem of the Symmedian,

$$\frac{AL}{LB}=\frac{AC^2}{AB^2};\ \frac{AT}{TC}=\frac{AB^2}{AC^2}.$$

So we get

$$\frac{AK}{KF}=\frac{AB^2+BC^2}{BC^2}. \tag{2.5.7.6}$$

For another hand, the Menelaus theorem applied to the triangle AEF with the transversal KM gives us

$$\frac{ER}{RF}=\frac{AK}{KF}\cdot\frac{EM}{AM}.$$

From this, with (2.5.7.4) and (2.5.7.6), we obtain $\frac{EF}{RF}=\frac{1}{2}$, and as $\frac{EG}{GA}=\frac{1}{2}$ we have

$$\frac{EF}{RF}=\frac{EG}{GA} \tag{2.5.7.7}$$

and therefore GR is parallel to AF, whence GS is parallel to AK (2.5.7.8).

Again the Menelaus theorem at AEF with GN gives us

$$\frac{EP}{PF}=\frac{AN}{FN}\cdot\frac{GE}{AG},$$

which with (2.5.7.5) gives us

$$\frac{EP}{PF}=\frac{AK}{KF},$$

and this means KP is parallel to AE, or that is the same, KS parallel to AG (2.5.7.9).

So (2.5.7.8) and (2.5.7.9) proves that $AGSK$ is a parallelogram. ∎

2.5.8 Comments and Solution to Problem 4.7

The acute triangle ABC is inscribed in a circle. The point P is inside the triangle. Lines AP, BP and CP intersect again the circumcircle of ABC at X,

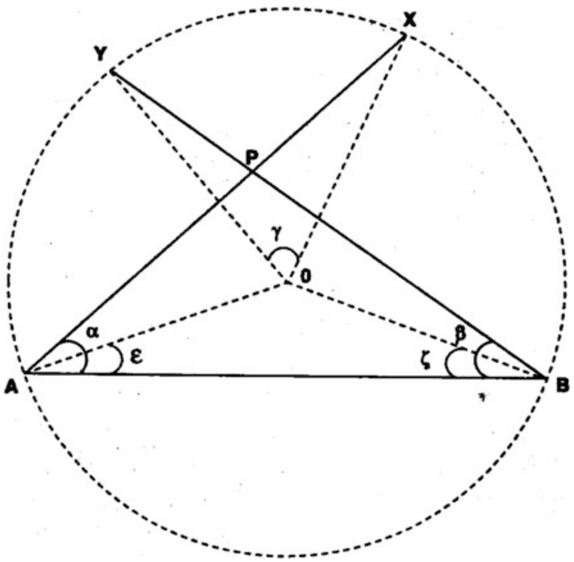


Fig. 2.6 Figure for Problem 4.7

Y and Z, respectively. Determine the position of the point P for that XYZ be equilateral.

Comments

This problem, created in the Symposium held immediately before the IXth Iberoamerican Math. Olympiad 1994, was included in the exam as problem 4. The solution below—slightly edited—was obtained by a Portuguese student, Joao Menano, during the contest (Fig. 2.6).

Solution

Consider, for instance, the side AB and the diameter of the circle (ABC) which is parallel to this side. Point C, then, must belong to the opposed semicircle to that in which A and B are located, because triangle ABC is acute. The same observation is valid for any other couple of vertices. Then we can forget the point C. We will find all the points P such that $\gamma = 120^o$. This condition is obviously necessary and sufficient for that X and Y be two vertices of the equilateral triangle XYZ (the center of the equilateral triangle must to be the center of the circle).

We have $\angle YOB = 2 \cdot \angle YAB$ and $\angle XOA = 2 \cdot \angle XBA$. In order to get $\angle XOY = 120°$. We need that

$$\angle XOY + \angle YOB + \angle AOB + \angle AOX = 360°$$

$$\angle AOB = 180° - \angle OAB - \angle OBA.$$

Observing the Fig. 2.4, this means

$$2\alpha + 2\beta = 60^\circ + \varepsilon + \delta \Rightarrow \alpha + \beta = \frac{60^\circ + \varepsilon + \delta}{2}.$$

The set of points P which verify this last equation are the points of an arc of circle through A and B with this measure. By means of this construction we get an arc of circle to which P belongs. Repeating this construction using other vertices, say B, C, we will get another arc of circle. The intersection of both arcs gives the position of searched point P. ∎

2.5.9 Solution to the Problems 4.8

Consider the sequence of real numbers $\{x_n\}$ with x_0 arbitrary and $x_{n+1} = 2(x_n)^2 - 1$. Show that, if $|x_0| \le 1$, then $|x_n| \le 1$. Find a closed formula for x_n.

Solution by F. Bellot

If $|x_0| \le 1$, we can write $x_0 = \cos\theta$, for some $\theta \in [0, \pi)$.

Then we get $x_1 = 2(\cos^2\theta) - 1 = \cos 2\theta$, and $|x_1| \le 1$. Continuing in this approach, we obtain $x_2 = \cos(2^2\theta)$, and by induction we can prove that $x_n = \cos(2^n\vartheta)$, and we are done the two proposed problems. ∎

2.5.10 Solution to Problems 4.9.1 and 4.9.2

Lines r and s are mutually orthogonal and do not are in the same plane. Let AB be its common perpendicular ($A \in r$, $B \in s$). Consider the sphere of diameter AB. The points $M \in r$ and $N \in s$ are variable, with the condition that MN is tangent to the sphere. Let T be the point of tangency. Show that $TM.TN$ is constant. Determine the geometrical locus of the point T.

Both problems were also created during the Symposium on Creating problems, previously to the 10th Iberoamerican Mathematical Olympiad, Chili 1995. The problem was chosen by the International Jury and proposed to the students as problem 3 (Fig. 2.7).

First we will prove that $TM.TN$ is constant (this part was not included in the text of the problem 3 of the Iberoamerican Olympiad 1995).The picture can be simplified a bit (Fig. 2.8):

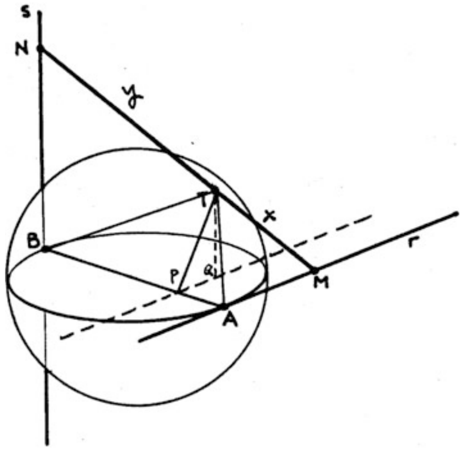


Fig. 2.7 Figure for Problem 4.9.1

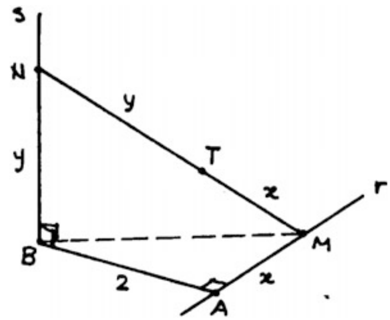


Fig. 2.8 Figure for Problem 4.9.2

The argument is by Eduardo Wagner. If we take $AB = 2$, $MA = MT = x$, $NB = NT = y$, then we get

$$NM^2 = NB^2 + BM^2 \Leftrightarrow (x+y)^2 = y^2 + 4 + x^2 \Leftrightarrow xy = 2$$

∎

Going back to the Fig. 2.7, we will give an analytical solution of the problem. (Solution by F. Bellot during the Symposium).

Suppose $AB = 2$. We will choose the midpoint O of AB as origin of a Cartesian system of coordinates in the space, the line AB will be the x axis; the line through O parallel to the line s as "y" axis; and the perpendicular to the plan xy through O (upwards) as "z" axis. OB is the positive "x" axis.

The equation of the sphere is $x^2 + y^2 + z^2 = 1$; the equations of the line r are $(x = -1, y = 0)$; the equations of the line s are $(x = 1, z = 0)$ and the coordinates of points M and N are $M(-1, 0, m), N(1, n, 0)$.

The equations of the line MN are $\frac{x+1}{-2} = \frac{y}{-n} = \frac{z-m}{m} = \mathrm{t}$.

The condition of tangency of the line MN with the sphere is $4 = m^2n^2$, that is $mn = \pm 2$.

If $mn = 2$, the coordinates of the tangency point T are $\left(\frac{m^2-2}{m^2+2}, \frac{2m}{m^2+2}, \frac{2m}{m^2+2}\right)$ and as the second and third coordinates of T are the same, this means that T belong to the plane of equation $y = z$, and so this plane contain the line AB and make an angle of 45° with the plan xy.

If $mn = -2$, the plane to which T belongs is $y = -z$, which is orthogonal to the first one. Both planes pass through the center of the sphere, and intersect it following two maximal circles through A and B, forming angles of 45° with the plan xy. ∎

2.5.11 Solution to Problem 4.10

Let ABC be a triangle inscribed in a circle, and I is the incenter of the triangle. Lines BI and CI intersect again the circumcircle at M and N, respectively. Line MN intersect AB at P and AC at Q, respectively. Show that IA is perpendicular to MN.

Source of the problem: Problem created during the Third Iberoamerican Workshop about the creation of problems, held in San José, Costa Rica, Sept. 1996, just before the 11th Iberoamerican Math Olympiad (Fig. 2.9).

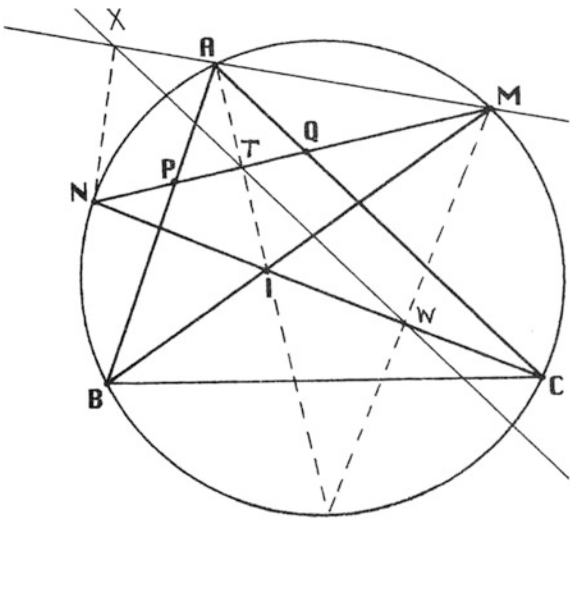


Fig. 2.9 Figure for Problem 4.10

Let L be the midpoint of the arc BC which do not contains A. The perpendicular line from N on LA intersects AL at T. The perpendicular line from N on ML intersects ML at W. Note that I is the orthocenter of the triangle LMN. The line WT is parallel to AC, and therefore is the line of the statement of the problem.

Now, if from N draw the perpendicular to AM, intersecting AM at X, the Simson line of N with respect to the triangle AML is the line which pass through T and W, that is, $X = AM \cap TW$. ∎

2.5.12 Solution to Problem 4.11

With center in the incenter I of the triangle ABC, a circle is drawn, intersecting in two points each side of the triangle: to BC at D and P (being D the most near to B), to CA at E and Q (being E the most near to C), and to AB at F and R (being F the most near to A). Let S be the point of intersection of the diagonals of the quadrilateral $EQFR$, and T the point of intersection of the diagonals of the quadrilateral $FRDP$. Finally, let U be the intersection of the diagonals of the quadrilateral $DPEQ$. Show that the circumcircles of the triangles FTR, DPU and EQS have one common point.

Source of the problem: The problem was created during the 4th Workshop of Creation of problems, held in Guadalajara, Jalisco, Mexico in September of 1997, just before the 12th Iberoamerican Mathematical Olympiad. The workshop was conducted by Prof. Alejandro Bravo. The problem was chosen by the International Jury and proposed to the students as problem number 3.

Solution by Alejandro Bravo.

As S belongs to the bisector of angle A of triangle ABC, the angles QIS and SIF are equal. But angle $QIF = 2(angle\ SIQ)$ is a central angle in the circle, and QES is inscribed and subtend the same arc FQ; therefore angle QES = angle SIQ and the four points Q, S, I and E are concyclic (Fig. 2.10).

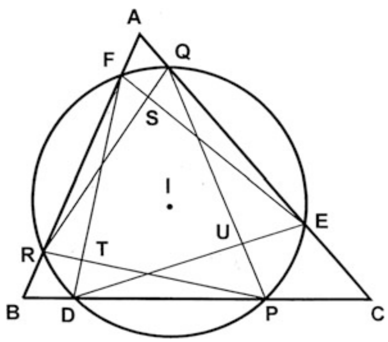


Fig. 2.10 Figure for Problem 4.11

The same argument proves that U belong to this same circle. Repeating the reasoning, the circles circumscribed to the triangles DPU, EQS and FRT pass through the incenter I of the triangle ABC. ∎

2.5.13 Solution of the Problem 4.12

$ABCD$ is a convex quadrilateral and $M = AC \cap BD$. The internal bisector of ACD intersect BA at K. Suppose $MA \cdot MC + MA \cdot CD = MD \cdot MB$. Show that $\angle BKC = \angle CDB$.

Source of the problem: Course of Euclidean Geometry 1, University of Costa Rica.

Solution by F. Bellot

First we will draw a figure in such a way that it meet the conditions of the statement of the problem (Fig. 2.11):

Drawing first the dotted circle, choose on it arbitrary points B, C and D. Choosing then the angle KCD, with K on the circle, joining K with B we will get the straight line where the point A must to be. Then, with the protractor the angle KCA equal to the angle KCD is drawn (because CK is the bisector of ACD) and so the position of the point A is determinate.

The thesis of the problem is equivalent to say that the points B, C, D and K are in the circle (and this justify the drawing) and furthermore gives an interpretation of the strange condition

$$MA \cdot MC + MA \cdot CD = MD \cdot MB \tag{2.5.13.1}$$

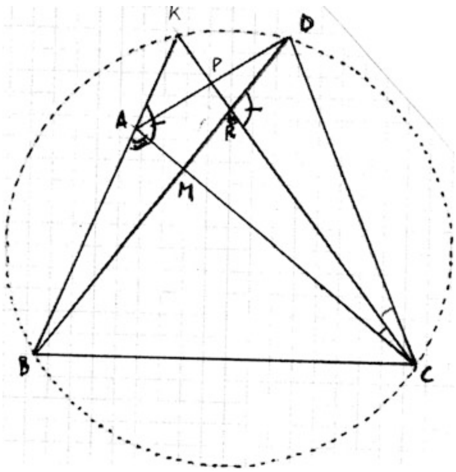


Fig. 2.11 Figure for Problem 4.12

given in the statement of the problem.

First at all, as R is the foot of the internal bisector CK of triangle MDC, we have, by the internal bisector theorem,

$$\frac{RM}{RD} = \frac{MC}{CD} \Rightarrow CD = \frac{MC \cdot RD}{MR}.$$

The value of CD is substituted in (2.5.13.1): $MA \cdot MC + MA \cdot \frac{MC \cdot RD}{MR} = MD \cdot MB$.

The left hand side can be written in the form $MA \cdot MC \cdot \left(1 + \frac{RD}{MR}\right) = MD \cdot MB$, i.e.

$$MA \cdot MC \cdot \frac{MR + RD}{MR} = MD \cdot MB \Leftrightarrow MA \cdot MC \cdot \frac{MD}{MR} = MD \cdot MB$$

which reduces to $\frac{MA \cdot MC}{MR} = MB \Leftrightarrow MA \cdot MC = MR \cdot MB$. This last equality warranty that the points B, C, A and R are in the same circle (not drawn in the picture above), and therefore the angles BAC and BRC are equal.

Consider now the triangles KAC and DRC. Both have equal the angle C (because CK is the bisector of angle ACD), and for another hand $\angle KAC = \angle DRC$, because they are supplementary of the equal angles $\angle BAC = \angle DRC$. Therefore the third angles in both triangles should to be equal, that is $\angle BRC = \angle BDC$, and we are done. ∎

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