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General Equilibrium Theory

Here we briefly review the general equilibrium theory, which is pretty traditional: preference and the concept of ordinal utility, demand and comparative statics, the definition of Arrow–Debreu equilibrium, Pareto efficiency and welfare theorems, welfare comparison and compensation principle, and incomplete asset markets. As they are standard, they are presented without proofs. For a comprehensive treatment of general equilibrium theory, see a standard textbook, such as Mas-Colell et al. [1] as well as reputable books, such as Debreu [2], Mas-Colell [3], Magill and Quinzii [4].

2.1 Preference and Utility Function

Consider that, there are n goods in the economy. The consumption set for each individual is taken to be the nonnegative orthant \mathbb{R}_+^n .

Let \succsim denote a generic individual's *preference ordering* over \mathbb{R}_+^n , which satisfy

Completeness: for all $x, y \in \mathbb{R}_+^n$, it holds either $x \succsim y$ or $y \succsim x$.

Transitivity: for all $x, y, z \in \mathbb{R}_+^n$, $x \succsim y$ and $y \succsim z$ imply $x \succsim z$.

Continuity: for all sequences $\{x^\nu\}$ and $\{y^\nu\}$ in \mathbb{R}_+^n , such that $x^\nu \succsim y^\nu$ for all ν and $\lim_{\nu \rightarrow \infty} x^\nu = x$ and $\lim_{\nu \rightarrow \infty} y^\nu = y$ it holds $x \succsim y$.

Let \succ denote the strict preference and \sim denote indifference, which is defined by

$$x \succ y \iff x \succsim y \text{ and not } y \succsim x$$

and

$$x \sim y \iff x \succsim y \text{ and } y \succsim x$$

Through the book, we also assume that preference \succsim satisfies

Strong Monotonicity: for all $x, y \in \mathbb{R}_+^n$ it holds

$$x \geq y, x \neq y \implies x \succ y$$

Strict Convexity: for all $x, y \in \mathbb{R}_+^n$ with $x \neq y$ and $\lambda \in (0, 1)$ it holds

$$x \sim y \implies \lambda x + (1 - \lambda)y \succ x$$

although we may consider a weaker version of convexity at some point:

Convexity: for all $x, y \in \mathbb{R}_+^n$ with $x \neq y$ and $\lambda \in (0, 1)$ it holds

$$x \sim y \implies \lambda x + (1 - \lambda)y \succsim x$$

A numerical function $u : \mathbb{R}_+^n \rightarrow \mathbb{R}$ is said to *represent* \succsim if it holds

$$x \succsim y \iff u(x) \geq u(y)$$

for all $x, y \in \mathbb{R}_+^n$. It is called a *utility function*.

The concept of utility function here is *ordinal*, in the sense that such a function is no more than a representation of preference ranking, and the assigned numerical values as utilities have no quantitative meaning.

To be precise, the following statement holds.

Theorem 2.1 Fix a preference ranking \succsim .

(i) Suppose that a utility function u represents \succsim . Take any function $f : u(\mathbb{R}_+^n) \rightarrow \mathbb{R}$ that is strictly increasing. Then a function $f \circ u$ defined by

$$(f \circ u)(x) = f(u(x))$$

for each $x \in \mathbb{R}_+^l$ is also a utility function which represents \succsim .

(ii) Suppose that u and v are utility functions which represent \succsim . Then, there is a strictly increasing function $f : u(\mathbb{R}_+^n) \rightarrow \mathbb{R}$ such that

$$v = f \circ u$$

2.2 Demand and Compensated Demand

2.2.1 Demand Function and Indirect Utility Function

Consider a generic consumer, who is supposed to be price-taking throughout. Given, a price vector $p \in \mathbb{R}_{++}^n$ and income $w > 0$, she solves the utility maximization problem

$$\max_{x \in \mathbb{R}_+^n} u(x)$$

subject to

$$p \cdot x \leq w.$$

The existence of optimization point is guaranteed by Continuity and compactness of budget set $B(p, w) = \{x \in \mathbb{R}_+^n : p \cdot x \leq w\}$. Under Strong Monotonicity the budget constraint is met with equality, and under Strict Convexity the optimal consumption is unique; hence, it is denoted by $x(p, w)$ and satisfies $p \cdot x(p) = w$ for all (p, w) . The function $x : \mathbb{R}_{++}^n \times \mathbb{R}_{++} \rightarrow \mathbb{R}_+^n$ defined as above is called *demand function*.

Denote the maximal utility under (p, w) by $v(p, w)$. The function $v : \mathbb{R}_{++}^n \times \mathbb{R}_{++} \rightarrow \mathbb{R}$ defined so is called *indirect utility function*.

Proposition 2.1 *Let $x(p, w)$ and $v(p, w)$ denote the demand function and the indirect utility function defined for utility representation $u(\cdot)$, respectively.*

Let f be any monotone transformation and denote the compensated demand function and the expenditure function defined for representation $\tilde{u}(x) = f(u(x))$.

Let $\tilde{x}(p, w)$ and $\tilde{v}(p, w)$ denote the demand function and the indirect utility function defined for utility representation $\tilde{u}(\cdot)$, respectively.

Then, it holds

$$\begin{aligned}\tilde{x}(p, w) &= x(p, x) \\ \tilde{v}(p, w) &= f(v(p, w)).\end{aligned}$$

2.2.2 Compensated Demand Function, Expenditure Function, and Income Compensation Function

Given, a price vector $p \in \mathbb{R}_{++}^n$ and utility level u , consider the expenditure minimization problem

$$\min_{x \in \mathbb{R}_+^n} p \cdot x$$

subject to

$$u(x) \geq u$$

Denote the solution as a function of (p, u) by $h(p, u)$, and call it *compensated demand function*. From Strong Monotonicity and Continuity, the solution for expenditure minimization problem always exists, and from Strict convexity the solution is always unique.

Also, denote the minimized expenditure by

$$e(p, u) = p \cdot h(p, u),$$

and call it *expenditure function*.

The expenditure-minimizing point is utility-maximizing given the price when the minimized expenditure is given as the income. Thus, it holds

$$h(p, u) = x(p, e(p, u))$$

Also, the utility-maximizing point is minimizing the expenditure given the price in order to satisfy the same level of utility as it yields. Thus, it holds

$$x(p, w) = h(p, v(p, w))$$

We might be uncomfortable with taking “utility level” as an input, as it appears to contradict with the concept of ordinal utility. But it is without loss of generality to formulate compensated demand function and expenditure function with a particular representation, as different formulations obtained from different utility representations are suitably translatable to each other.

Proposition 2.2 *Let $h(p, u)$ and $e(p, u)$ denote the compensated demand function and the expenditure function defined for utility representation $u(\cdot)$, respectively.*

Let f be any monotone transformation, and denote the compensated demand function and the expenditure function defined for representation $\tilde{u}(x) = f(u(x))$.

Let $\tilde{h}(p, u)$ and $\tilde{e}(p, u)$ denote the compensated demand function and the expenditure function defined for utility representation $\tilde{u}(\cdot)$, respectively.

Then, it holds

$$\tilde{h}(p, v) = h(p, f^{-1}(v))$$

$$\tilde{e}(p, v) = e(p, f^{-1}(v)).$$

The proof follows directly from the definition.

Actually, we can get rid of utility representation in order to consider the concept of compensated demand, by introducing *income compensation function*, which is defined by

$$\mu(p'|p, w) = e(p', v(p, w))$$

Then by definition it holds

$$h(p', v(p, w)) = x(p', \mu(p'|p, w))$$

Thus, we can carry out the analysis of compensated demand just by demand function x and income compensation function μ , which turns out to be rather helpful in some cases.

2.3 Comparative Statics

To facilitate comparative statics, we will assume differentiable preferences.

Differentiable Preference: On $\mathbb{R}_{++}^n, \succsim$ allows representation $u : \mathbb{R}_{++}^n \rightarrow \mathbb{R}$ which is twice-continuously differentiable, such that

1. $Du(x) \gg \mathbf{0}$ for all $x \in \mathbb{R}_{++}^n$,
2. it has negative definite bordered Hessian at all $x \in \mathbb{R}_{++}^n$.

Under Differentiable Preference, we can define marginal rate of substitution of Good h for Good k at $x \in \mathbb{R}_{++}^l$ by

$$MRS^{k,h}(x) = \frac{\frac{\partial u(x)}{\partial x_k}}{\frac{\partial u(x)}{\partial x_h}}$$

We also impose a boundary condition, which rules out corner solutions.

Boundary Condition: For all $x \in \mathbb{R}_{++}^n$ it holds

$$\lim_{z_k \rightarrow 0} MRS^{k,h}(z_k, x_h, x_{-\{k,h\}}) = \infty$$

and

$$\lim_{z_h \rightarrow 0} MRS^{k,h}(x_k, z_h, x_{-\{k,h\}}) = 0$$

The following claims are standard.¹

Proposition 2.3 (Interior Solution) *Under Differentiable Preference and Boundary Condition, the utility maximization problem has a unique solution $x(p, w)$ in \mathbb{R}_{++}^n , which satisfy the first-order condition*

$$Du(x) = \lambda p$$

where $\lambda > 0$ is the corresponding Lagrange multiplier. Moreover, the demand function $x(p, w)$ and indirect utility function $v(p, w)$ are differentiable over $\mathbb{R}_{++}^n \times \mathbb{R}_{++}$.

Under Differentiable Preference and Boundary Condition, the expenditure minimization problem has a unique solution $h(p, u)$ in \mathbb{R}_{++}^n , which satisfy the first-order condition

$$p = \mu Du(x)$$

where $\mu > 0$ is the corresponding Lagrange multiplier. Moreover, the compensated demand function $h(p, u)$ and expenditure function $e(p, u)$ are differentiable over $\mathbb{R}_{++}^n \times u(\mathbb{R}_{++})$.

Proposition 2.4 (Shepard's Lemma) *Compensated demand function and expenditure minimization satisfy*

$$\frac{\partial e(p, u)}{\partial p_k} = h_k(p, u)$$

for all $k = 1, \dots, n$.

Proposition 2.5 (The Slutsky Equation) *Demand function and compensated demand function satisfy*

$$\frac{\partial h_l(p, u)}{\partial p_k} = \frac{\partial x_l(p, w)}{\partial p_k} + \frac{x_l(p, w)}{\partial w} x_k(p, w)$$

for all $k, l = 1, \dots, n$.

Proposition 2.6 (Roy's Identity) *Demand function and indirect utility function satisfy*

$$x_k(p, w) = - \frac{\frac{\partial v(p, w)}{\partial p_k}}{\frac{\partial v(p, w)}{\partial w}}$$

for all $k = 1, \dots, n$.

From Shepard's lemma, for income compensation function, we obtain

$$\begin{aligned} \frac{\partial \mu(p' | p, u)}{\partial p'_k} &= \frac{\partial e(p', v(p, w))}{\partial p'_k} \\ &= h_k(p', v(p, w)) \\ &= x_k(p', \mu(p' | p, w)) \end{aligned}$$

for all $k = 1, \dots, n$.

2.4 General Equilibrium in Exchange Economies

2.4.1 Setting and Definitions

Consider that, there are I individuals and n goods. Each $i = 1, \dots, I$ is characterized by

1. Consumption set \mathbb{R}_+^n
2. Preference relation \succsim_i over \mathbb{R}_+^n , which is assumed to satisfy the conditions as in the previous section.
3. Initial endowment $\omega_i \in \mathbb{R}_{++}^n$

Here is the definition of competitive equilibrium.

Definition 2.1 A *competitive equilibrium* is a pair of price vector $p \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}$ and allocation x such that $p \cdot x_i \leq p \cdot \omega_i$ and it holds

$$p \cdot x'_i \leq p \cdot \omega_i \implies x_i \succsim x'_i$$

for all $x'_i \in \mathbb{R}_+^I$ for all $i = 1, \dots, I$, and

$$\sum_{i=1}^I x_i = \sum_{i=1}^I \omega_i$$

2.4.2 Efficiency

Let us briefly review the definition of Pareto efficiency and the two welfare theorems.

Definition 2.2 Allocation $x = (x_i)_{i=1, \dots, I} \in \mathbb{R}^{In}$ is said to be *feasible* if

$$\sum_{i=1}^I x_i \leq \sum_{i=1}^I \omega_i.$$

Definition 2.3 A feasible allocation x is said to be *Pareto-efficient* if there is no feasible allocation x' such that

$$x'_i \succsim_i x_i$$

for all $i = 1, \dots, m$ and

$$x'_i \succ_i x_i$$

for at least one i .

Theorem 2.2 (First welfare theorem) *Under Strong Monotonicity of preference, any competitive equilibrium allocation is Pareto-efficient.*²

Theorem 2.3 (Second welfare theorem) *Assume Strong Monotonicity and Convexity of preference. Then, any Pareto-efficient allocation x is obtained as a competitive quasi-equilibrium allocation after suitable redistribution of initial endowments, in the following sense: there is a pair of price vector $p \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}$ and income distribution $w = (w_i)_{i=1, \dots, I} \in \mathbb{R}_+^I$ such that it holds $p \cdot x_i = w_i$ and*

$$x_i \succ x'_i \implies p \cdot x'_i \leq w_i$$

for all $x'_i \in \mathbb{R}_+^I$ for all $i = 1, \dots, I$, and

$$\sum_{i=1}^I x_i = \sum_{i=1}^I \omega_i$$

2.4.3 Differential Characterization

Under the assumption of differentiable preferences satisfying the boundary condition, an interior equilibrium allocation is characterized by the first-order condition

$$Du_i(x_i) = \lambda_i p,$$

where λ_i is the corresponding Lagrange multiplier for individual i .

Hence the marginal rate of substitution is equalized to relative price. The marginal rate of substitution of Good h for Good k for i at $x_i \in \mathbb{R}_{++}^I$ is given by

$$MRS_i^{k,h}(x_i) = \frac{\frac{\partial u_i(x_i)}{\partial x_{ik}}}{\frac{\partial u_i(x_i)}{\partial x_{ih}}}$$

Let $MRS_i(x_i) = (MRS_i^{k,h}(x_i))_{k,h=1,\dots,I}$.

Then, it holds

$$MRS_i^{k,h}(x_i) = \frac{p_k}{p_h}$$

for all $k, h = 1, \dots, n$ and for all $i = 1, \dots, I$.

Here efficiency of interior allocation is characterized by equalization of the marginal rate of substitution between individuals. In other words, efficiency imposes that subjective relative values between goods are equal for all individuals *at margin*.

Then the following claim holds.

Proposition 2.7 *Interior allocation x is Pareto-efficient if and only if*

$$MRS_i(x_i) = MRS_j(x_j)$$

for all $i, j = 1, \dots, n$.

At an interior competitive equilibrium allocation, it follows from the first-order condition that

$$MRS_i^{k,h}(x_i) = \frac{p_k}{p_h}$$

for all k, h . Hence the first welfare theorem follows.

At any interior Pareto-efficient allocation it holds

$$MRS_i(x_i) = MRS_j(x_j)$$

and the second welfare theorem follows by taking the competitive equilibrium price $p \in \mathbb{R}_{++}^l$ by

$$\frac{p_k}{p_h} = MRS_i^{k,h}(x_i)$$

for all k, h , where the definition does not depend on the choice of i because of equalization of MRSs.

The Pareto Set

There may be arbitrarily many Pareto-efficient allocations. As illustrated in Fig. 2.1, we can draw arbitrarily many pairs indifference curves which are tangent to each other. We can actually draw a continuous curve by depicting points at which such tangency holds. In the current setting in which the goods are continuously divisible, there is actually a continuum of Pareto-efficient allocations.

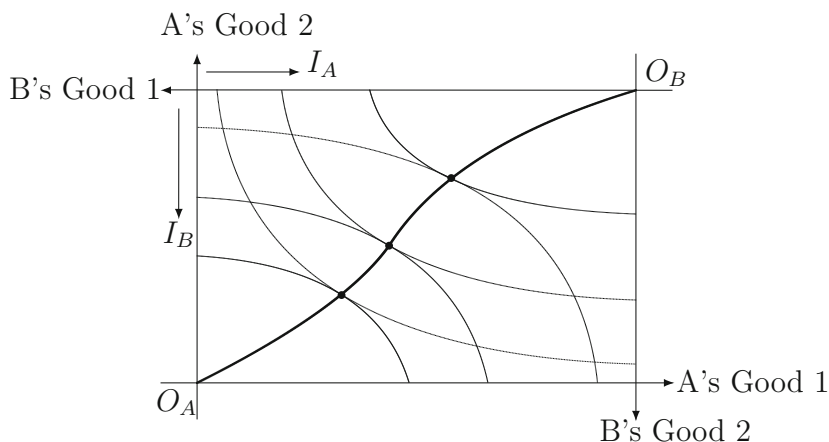


Fig. 2.1 Set of Pareto-efficient allocations

2.5 The Compensation Principle

Change in economic activity does not always make all individuals better off. Nevertheless, in the partial equilibrium analysis such change is often justified on the ground that it is maximizing social surplus or generating a larger surplus.

What normative criteria is it resorting to? Later, we will see that it is resorting to so-called compensation principles. Let me provide the general definition of the principles here. There are several criteria being proposed in the literature.

The so-called *Kaldor criterion* says that a change should be accepted if we can make everybody better off by reallocating the allocation obtained by the change than in the allocation before the change. Formally, it says

Definition 2.4 An allocation $y = (y_1, \dots, y_n)$ is a *Kaldor-improvement* of $x = (x_1, \dots, x_n)$ if there exists an allocation $y' = (y'_1, \dots, y'_n)$ with

$$\sum_{i=1}^n y'_i = \sum_{i=1}^n y_i$$

such that it holds

$$y'_i \succsim_i x_i$$

for all i and

$$y'_i \succ_i x_i$$

for at least one i .

It is obvious that if y is a Pareto-improvement of x it is a Kaldor-improvement of x .

See Fig. 2.2, in which there are two consumers A and B. Then allocation (y_A, y_B) is a Kaldor-improvement of (x_A, x_B) , since we can obtain (y'_A, y'_B) by reallocating (y_A, y_B) , which is a Pareto-improvement of (x_A, x_B) . Note that vectors $y'_A - y_A$ and $y'_B - y_B$ are exactly opposite of each other.

Or, one can explain this by using utility possibility frontiers. Fix a representation of A's preference u_A and a representation of B's preference u_B . Given, a vector of aggregate resources available to the society $e = (e_1, e_2)$, let

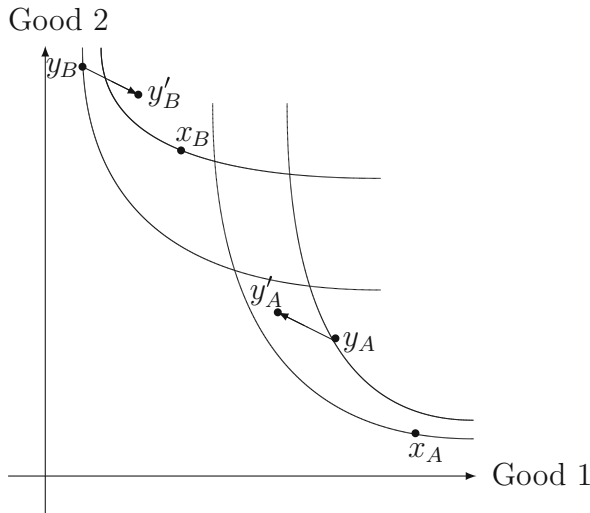


Fig. 2.2 Kaldor-improvement

$$I(e) = \{(u_A(x_A), u_B(x_B)) : x_A + x_B = e_1, x_{A2} + x_{B2} = e_2\}$$

be the set of pairs of A's utility and B's utility which are obtained by allocating e . Of course, this is only for describing trade-offs between A's gain and B's gain and utility numbers themselves have no quantitative meanings.

See Fig. 2.3, in which two utility possibility frontiers are drawn, $I(e)$ obtained from e and $I(e')$ obtained from e' . Then y on $I(e')$ makes a Kaldor-improvement of x on $I(e)$ since we can pick y' on $I(e')$ which is in the upper-right of x .

Let me state two problems of the Kaldor criterion. One is,

It says a change should be accepted if we can reallocate the allocation after the change so as to make everybody better-off. Why not just doing such reallocation? If the reallocation is indeed done it is simply a Pareto-improvement, isn't it?

The definition of Kaldor-improvement only says "we *can* reallocate the allocation," and it does not require that such reallocation is indeed done.

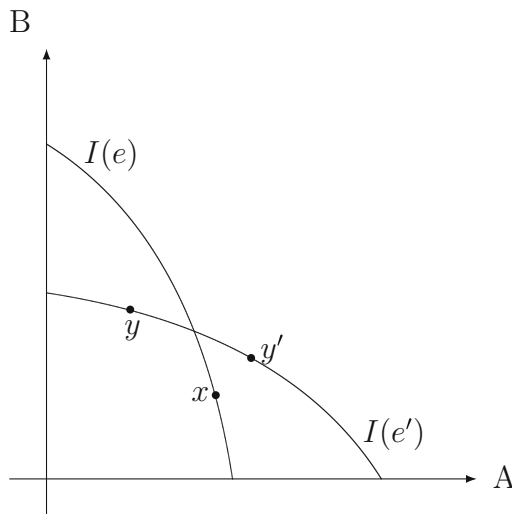


Fig. 2.3 Kaldor-improvement

Why should one get convinced by such unwarranted story of potential reallocation when he is, in fact, losing because of the change? If the reallocation is left undone such criterion is deceptive, and if the allocation is indeed done we just need the Pareto criterion and it is just redundant.

The other problem is that an allocation which Kaldor-improves upon another may be Kaldor-improved upon by the latter. See Fig. 2.4, in which (y_A, y_B) is a Kaldor-improvement of (x_A, x_B) through the potential reallocation to (y'_A, y'_B) , and (x_A, x_B) is a Kaldor-improvement of (y_A, y_B) through the potential reallocation to (x'_A, x'_B) . Hence, the Kaldor-criterion cannot rank properly between allocations in general.

One can explain this by using the utility possibility frontiers. See Fig. 2.5, in which two utility possibility frontiers are drawn, $I(e)$ obtained from e and $I(e')$ obtained from e' . Then y on $I(e')$ makes a Kaldor-improvement of x on $I(e)$ since we can pick y' on $I(e')$ which is in the upper-right of x . However, x makes a Kaldor improvement of y as well, since we can pick x' on $I(e)$ which is in the upper-right of y .

Let me introduce a criterion which is the “complement” of the Kaldor criterion. The so-called *Hicks criterion* says that a change should be accepted if we cannot make everybody better off by reallocating the allocation

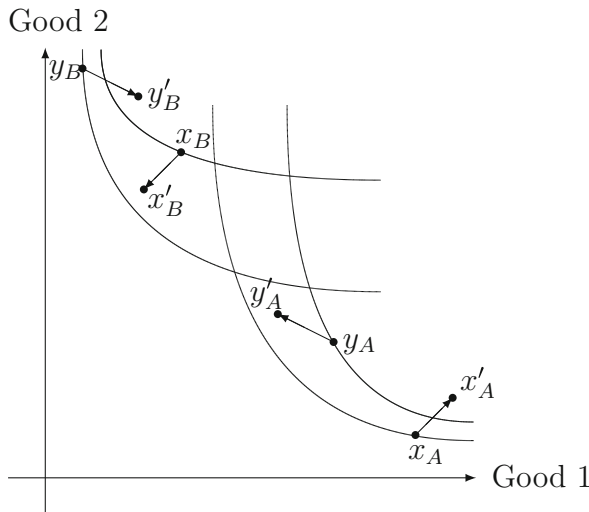


Fig. 2.4 Mutual Kaldor-improvements

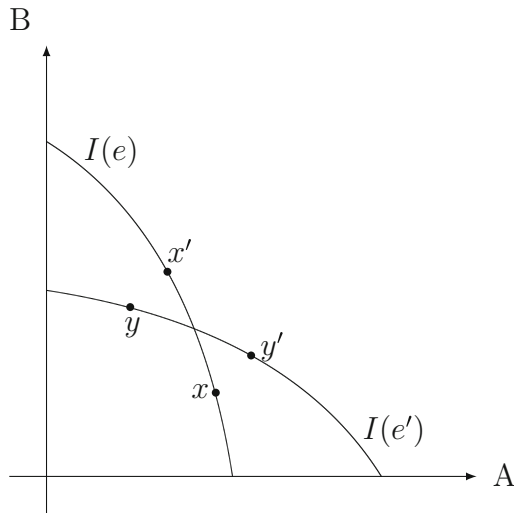


Fig. 2.5 Mutual Kaldor-improvements

before the change than in the allocation obtained by the change. In other words, one is a Hicks-improvement of another if the latter is not a Kaldor-improvement of the former.

Formally,

Definition 2.5 An allocation $y = (y_1, \dots, y_n)$ is a *Hicks-improvement* of $x = (x_1, \dots, x_n)$ if there exists no allocation $x' = (x'_1, \dots, x'_n)$ with

$$\sum_{i=1}^n x'_i = \sum_{i=1}^n x_i$$

such that it holds

$$x'_i \succsim_i y_i$$

for all i and

$$x'_i \succ_i y_i$$

for at least one i .

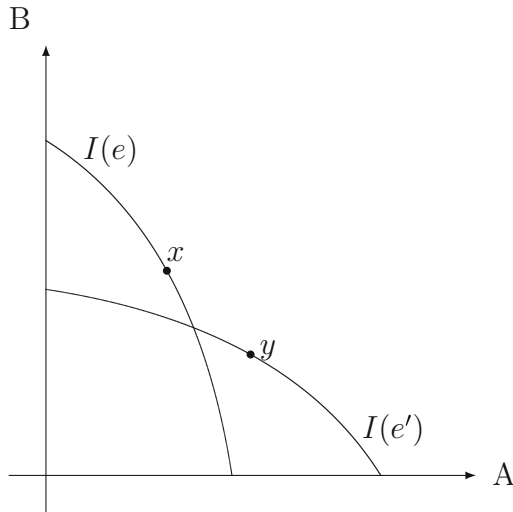


Fig. 2.6 Hicks improvement

Let me explain this using utility possibility frontiers. See Fig. 2.6. There, we can never go to the upper-right of y on $I(e')$ by reallocating x on $I(e)$. Hence y is a Hicks improvement of x .

The same comments as above apply to the Hicks criterion. Besides the ethical issue, an allocation which Hicks-improves upon another may be Hicks-improved upon by the latter. See Fig. 2.6 again. There, we can never go to the upper-right of y on $I(e')$ by reallocating x on $I(e)$. Hence y is a Hicks improvement of x . However, it is also the case that we can never go to the upper-right of x on $I(e)$ by reallocating y on $I(e')$. Hence x is a Hicks improvement of y .

Since the Kaldor criterion and Hicks criterion are the “complement” of each other, if we impose both we can avoid the problem that two allocations dominate each other under the Kaldor criterion alone and under the Hicks criterion alone, respectively. It is called Scitovsky criterion.

Definition 2.6 An allocation $y = (y_1, \dots, y_n)$ is a *Scitovsky-improvement* or of $x = (x_1, \dots, x_n)$ if y is both a Kaldor-improvement and a Hicks-improvement of x .

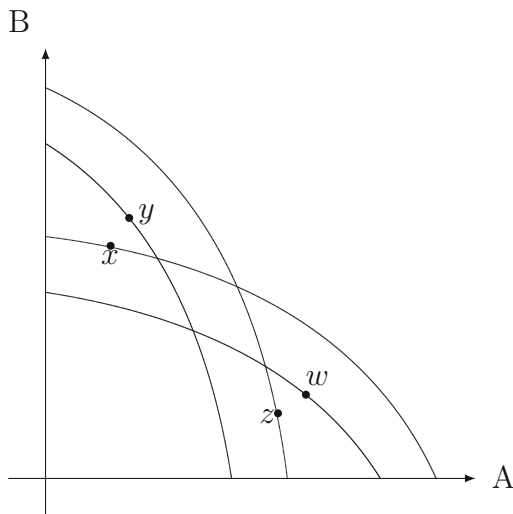


Fig. 2.7 Gorman paradox

If one is a Hicks-improvement of another it means the latter is not a Kaldor-improvement of the former. Hence it is always the case that if one is a Scitovsky-improvement of another the latter is not a Scitovsky-improvement of the former.

However, the ranking by the Scitovsky-improvement may be intransitive, that is, it may have a cycle. See Fig. 2.7. There y is a Scitovsky-improvement of x , z is a Scitovsky-improvement of y , w is a Scitovsky-improvement of z , but x is a Scitovsky-improvement of w . It is called the *Gorman paradox*. So the Scitovsky-improvement does not help, unfortunately.

Samuelson considered a weakening of the condition, stating that one allocation should be better than another when the entire utility possibility frontier given by the former is above the entire utility possibility frontier by the latter. Let us call this *Samuelson criterion*. This leads to a cycle again when it is combined with the Pareto criterion, however. The same example works. In Fig. 2.7, y is a Pareto-improvement of x , z is a Samuelson-improvement of y , w is a Pareto-improvement of z , but x is a Samuelson-improvement of w .

Now, we are pretty much in deadlock.

2.6 Social Welfare Function

2.6.1 Arrovian Social Welfare Function

We saw that the Pareto criterion alone is silent about which efficient allocation is socially desirable, and cannot rank between efficient allocation even including indifference.

Also, it is orthogonal to any notion of fairness. Consider, for example, between a slightly inefficient allocation and an efficient but extremely unfair (in some sense) allocation. Then, we will be required to give a quantitative judgment over several mutually orthogonal criteria. This motivates us to provide a complete *ranking* over allocations.

Let $\omega \in \mathbb{R}_+^n$ be the social endowment vector, and let

$$X = \left\{ x \in \mathbb{R}_{++}^{nI} : \sum_{i=1}^I x_i = \omega \right\}$$

be the set of feasible allocations in which everybody receives positive consumption.

Let \mathcal{R} be the set of complete, transitive, continuous convex, and strongly monotone preference relations over \mathbb{R}_+^n . Let \mathcal{R}_0 be the set of complete and transitive orderings over X .

An Arrovian social welfare functional is a mapping $R : \mathcal{R}^I \rightarrow \mathcal{R}_0$, where $R(\succsim)$ denotes the social ranking given a profile of individual preferences $\succsim = (\succsim_i)_{i=1}^I \in \mathcal{R}^I$, and let $P(\succsim)$ denote the corresponding strict ranking.

The following two axioms are natural to require.

Axiom 2.1 *Pareto*: For all $\succsim \in \mathcal{R}^I$ and for all $x, y \in X$, if

$$x \succ_i y$$

for all $i = 1, \dots, I$, then

$$x P(\succsim) y.$$

Axiom 2.2 *Nondictatorship*: There is no $i = 1, \dots, I$ such that for all $\succsim \in \mathcal{R}^I$ and for all $x, y \in X$, if

$$x \succ_i y$$

then

$$x P(\succsim)y.$$

Now the well-known independence of irrelevant alternatives axiom basically states that only the ordinal information about preferences should matter.

Axiom 2.3 *Independence of Irrelevant Alternatives*: For all $\succsim, \succsim' \in \mathcal{R}^I$ and for all $x, y \in X$, if

$$x \succsim_i y \iff x \succsim'_i y$$

for all $i = 1, \dots, I$, then

$$x R(\succsim)y \iff x R(\succsim')y.$$

Here is a version Arrow theorem stated for exchange economy, which is proven, for example, by Bordes et al. [5].

Theorem 2.4 *Let $n \geq 2$. Then, there is no social welfare functional $R : \mathcal{R}^I \rightarrow \mathcal{R}_0$ which satisfies Independence of Irrelevant Alternatives, Pareto, and Nondictatorship.*

2.6.2 Bergson–Samuelson Social Welfare Function

We saw that it is impossible to aggregate preferences so that the aggregation is independent of the choice of cardinalization of preference representation.

We may still accept the idea that evaluation of allocation should depend only on individuals' utilities, and exclude any "paternalistic" judgment which involves something beyond individual utility functions, while it has to involve a choice of cardinalization.

Let us take utility representation of each individual's preference *as given*, which are assumed to be monotone and concave, and consider aggregating them. Then, a Bergson–Samuelson social welfare function is given in the form

$$U(x) = W(u_1(x_1), \dots, u_I(x_I))$$

where $W : \prod_{i=1}^I u_i(\mathbb{R}_+^I) \rightarrow \mathbb{R}$ is an aggregator function.

Let us focus on the class of additive Bergson–Samuelson social welfare functions, given the form

$$U(x) = \sum_{i=1}^I \alpha_i u_i(x_i)$$

where $\alpha \in \mathbb{R}_+^I \setminus \{\mathbf{0}\}$ be a fixed welfare weight vector.

Then the following claims hold.

Theorem 2.5 *For any $\alpha \in \mathbb{R}_+^I \setminus \{\mathbf{0}\}$, the maximizer of*

$$U(x) = \sum_{i=1}^I \alpha_i u_i(x_i)$$

in the set of feasible allocations is Pareto-efficient.

Theorem 2.6 *Let x be any Pareto-efficient allocation in the set of feasible allocations. Then there is a welfare weight vector $\alpha \in \mathbb{R}_+^I \setminus \{\mathbf{0}\}$ such that x is maximizing*

$$U(x) = \sum_{i=1}^I \alpha_i u_i(x_i)$$

in the set of feasible allocations.

We should note that unless we have a strong belief or evidence that a particular way of cardinalization is the reasonable one among many, focusing on the additive aggregation like above is no more than for a mathematical convenience. For example, take an exponential transformation of the

original cardinalization, so that

$$v_i(x_i) = \exp u_i(x_i),$$

implying

$$u_i(x_i) = \log v_i(x_i)$$

Then we obtain

$$\begin{aligned} U(x) &= \sum_{i=1}^I \alpha_i u_i(x_i) \\ &= \sum_{i=1}^I \alpha_i \log v_i(x_i) \\ &= \log \prod_{i=1}^I v_i(x_i)^{\alpha_i} \end{aligned}$$

which is ordinally equivalent to another Bergson–Samuelson social welfare function

$$V(x) = \prod_{i=1}^I v_i(x_i)^{\alpha_i}$$

which is multiplicative. This point is the key point in the so-called Harsanyi–Sen debate on whether establishing an additive aggregation theorem indeed provides a formal foundation of Benthamite utilitarianism. To understand, see the comprehensive treatment by Weymark [6].

2.6.3 Negishi Approach

Negishi [7] showed that competitive market maximizes a weighted sum of individual utilities, where the weights are determined endogenously so that each individual's one is equal to the inverse of his marginal utility of income. Such weight vector is called *Negishi weights*.

Assume differentiable preference, and go back to the first-order condition for competitive equilibrium, where

$$Du_i(x_i) = \lambda_i p$$

for each $i = 1, \dots, I$.

Now, let $\alpha_i = \frac{1}{\lambda_i}$ for each i , and consider the weighted sum of utilities

$$\sum_{i=1}^I \alpha_i u_i(x_i)$$

Then competitive equilibrium allocation maximizes this weighted sum of utilities since it yields an extreme value for the Lagrangean in the form

$$L = \sum_{i=1}^I \alpha_i u_i(x_i) - \sum_{k=1}^n \mu_k \left(\sum_{i=1}^I x_{ik} - \sum_{i=1}^I e_{ik} \right)$$

as the Lagrange multiplier on Good k is taken to be $\mu_k = p_k$ for each $k = 1, \dots, n$ and $\alpha_i = \frac{1}{\lambda_i}$ for each i .

We should be careful, though, because such Negishi weight vector $\alpha = (\alpha_1, \dots, \alpha_I)$ is *endogenously* determined in equilibrium. This makes a critical difference from Bergson–Samuelson social welfare function in which welfare weights are *exogenously* chosen by the planner, which reflects his value judgment. For a *fixed* profile of initial endowment the Negishi “social welfare function” behaves as if it is a Bergson–Samuelson social welfare function, but such welfare weight changes as initial endowment changes, which is not the case in BS.

2.7 General Equilibrium Under Uncertainty

Aggregate expected consumer surplus is a prominent one as an efficiency measure in partial equilibrium welfare analysis under uncertainty. We will give a general equilibrium characterization of when the use of such

measure is justified. Here, we briefly review the general equilibrium models of resource allocation under uncertainty.

2.7.1 The Environment

Focus on the two-period setting, in which there is no consumption or earning taking place in Period 0.

There are S states of the world in Period 1. There are n goods at each state in Period 1. Hence, the consumption space is thus \mathbb{R}_+^{Sn} , where its element for individual $i = 1, \dots, I$ is denoted by $x_i = (x_{i1}, \dots, x_{iS})$.

Each individual i has

- Preference \succsim_i over \mathbb{R}_+^{Sn} .
— Typically, it is assumed to be represented in the expected utility form

$$u_i(x_i) = \sum_{s=1}^S \pi_s v_i(x_{is})$$

where the function v_i is called von-Neumann/Morgenstern index.

- Endowment $\omega_i \in \mathbb{R}_{++}^{Sn}$

Note that von-Neumann/Morgenstern index of the utility function, not a utility function, which forms a class of additive representations of preference. As far as we restrict attention to the class of additive representations of a given preference, which is a proper subset of the whole set of representation of the preference, such index has cardinal meaning, as its curvature explains the degree of risk aversion. Note, however, that *overall* representation is still ordinal. For example, take an exponential transformation of the expected utility form, then we have

$$e^{u_i(x_i)} = e^{\sum_{s=1}^S \pi_s v_i(x_{is})} = \prod_{s=1}^S \left(e^{v_i(x_{is})} \right)^{\pi_s}.$$

Now let $\tilde{u}_i(x_i) = e^{u_i(x_i)}$ and $\tilde{v}_i(z) = e^{v_i(z)}$. Then we obtain

$$\tilde{u}_i(x_i) = \prod_{s=1}^S (\tilde{v}_i(z))^{\pi_s}.$$

which is a geometric mean rather than arithmetic mean.

2.7.2 Arrow–Debreu Market

Let us first describe the case that there is a complete system of markets for all state-contingent consumptions.

Definition 2.7 An *Arrow–Debreu equilibrium* is a pair of price vector $p \in \mathbb{R}_+^{Sn} \setminus \{\mathbf{0}\}$ and an allocation x such that $p \cdot x_i \leq p \cdot \omega_i$ and it holds

$$p \cdot x'_i \leq p \cdot \omega_i \implies x_i \succsim x'_i$$

for all $x'_i \in \mathbb{R}_+^{Sn}$ for all $i = 1, \dots, I$, and

$$\sum_{i=1}^I x_i = \sum_{i=1}^I \omega_i$$

From the first welfare theorem, allocation in Arrow–Debreu equilibrium is Pareto-efficient according to the individuals' *ex-ante* preferences over state-contingent consumptions.

2.7.3 Sequential Trade

Now consider that there is not necessary a complete system of markets for state-contingent consumptions. Instead, let us consider a possibly incomplete system of asset markets.

Consider that there are K assets. Let R denote the return matrix, where R_{sk} denote Asset k 's gross return rate at State s . Asset price vector is denoted by $q \in \mathbb{R}^K$ and spot price vectors are denoted by $p_s \in \mathbb{R}_{++}^n$ for each $s = 1, \dots, S$.

Then individual $i = 1, \dots, I$ faces a sequence of budget constraints in the form

$$\sum_{k=1}^K q_k z_{ik} \leq 0$$

$$p_s x_{is} \leq \sum_{k=1}^K R_{sk} z_{ik} + p_s \omega_{is}, \quad s = 1, \dots, S$$

The natural restriction on asset price system is that it allows no arbitrage.

Definition 2.8 (q, R) admits an arbitrage if there exists z such that $qz \leq 0$ and $R_s z \geq 0$ for all s and $R_s z > 0$ for at least one s .

Here is the well-known necessary and sufficient condition for the no arbitrage condition.

Theorem 2.7 (q, R) admits no arbitrage if and only if there exists $\mu \in \mathbb{R}_+^S \setminus \{0\}$ such that $q = \mu R$.

Here is the definition of competitive equilibrium which corresponds to the current setting of incomplete asset markets.

Definition 2.9 A Radner equilibrium is a quadruple of asset price vector q , spot price vectors $(p_s)_{s=1, \dots, S}$, consumption allocation x and portfolio allocation z such that for each i the consumption-portfolio pair (x_i, z_i) is optimal for \succsim_i under the constraint

$$\sum_{k=1}^K q_k z_{ik} \leq 0$$

$$p_s x_{is} \leq \sum_{k=1}^K R_{sk} z_{ik} + p_s \omega_{is}, \quad s = 1, \dots, S$$

The following observations are straightforward.

Proposition 2.8 *If (q, p, x, z) constitutes a Radner equilibrium given R then (q, R) admits no arbitrage.*

Proposition 2.9 *If (q, p, x, z) constitutes a Radner equilibrium given R then there exists $\mu \in \mathbb{R}_+^S \setminus \{0\}$ such that $q = \mu R$.*

Let us verify that when the asset markets are in fact complete Radner equilibrium and Arrow–Debreu equilibrium are equivalent.

Definition 2.10 The asset markets are complete when $\text{rank } R = S$.

Theorem 2.8 *Suppose the asset markets are complete.*

(i) *If (p, x) forms an Arrow–Debreu equilibrium then there is an asset price vector q and portfolio allocation such that (q, p, x, z) forms a Radner equilibrium.*

(ii) *If (q, p, x, z) forms a Radner equilibrium, then there is a vector $\mu \in \mathbb{R}_+^S \setminus \{0\}$ such that the price vector defined by $(\mu_1 p_1, \dots, \mu_S p_S)$, and x form an Arrow–Debreu equilibrium.*

2.7.4 Market Incompleteness and Efficiency

Incompleteness of asset markets in general leads to (ex-ante) inefficiency of allocation, we cannot hedge all the uncertainties. What about the second-best property? Here, assume that $n = 1$, and Let

$$U_i^*(z_i) = U_i(R_1 z_i + \omega_{i1}, \dots, R_S z_i + \omega_{iS})$$

for each $i = 1, \dots, I$

Definition 2.11 Asset allocation $(z_1, \dots, z_I) \in \mathbb{R}^{IK}$ is constrained Pareto-efficient if it is feasible (that is, $\sum_{i=1}^I z_i \leq 0$) and if there is no other feasible asset allocation (z'_1, \dots, z'_I) such that

$$U_i^*(z'_i) \geq U_i^*(z_i)$$

for all i and

$$U_i^*(z'_i) > U_i^*(z_i)$$

for at least one i .

When there is only one consumption good at each state, Radner equilibrium satisfies the constrained Pareto-efficiency.

Theorem 2.9 *Assume two periods and $n = 1$. Then asset allocation in Radner equilibrium is constrained Pareto-efficient.*

This result is not true when there are two or more goods, or there are more than two periods, because you cannot even define the indirect utility function defined with portfolio choices alone.

A crude intuition might tell us that when we have more assets our (ex-ante) welfare improves, as we have more devices to hedge uncertainty. *This is wrong.* That a complete market leads to an efficient allocation and whether our welfare monotonically improves as the market becomes “more complete” are different questions. In fact, Hart [8] shows an example that introducing a new security to trade makes everybody worse-off. We will come back to this problem in the last chapter.

Notes

1. See, for example, Katzner [9] as well as Mas-Colell et al. [1].
2. Strong Monotonicity can be weakened to Local Nonsatiation, which says that at any point and its open neighborhood (relative to the consumption space) there is always a strictly better point in it.

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