Chapter 2
Mathematical Activity and the Transformations of Semiotic Representations

The different notions of signs and representations are misleading because they seem to reduce them to simple phenomena of cross-reference to the objects, which signs and representations stand for. The signs and representations would mainly fulfill this function: "evoke what is absent" or "communicate" a thought that is not obvious to others. And, then, we would be dealing with objects rather than with the signs or representations. *However, in mathematics, this function is secondary*. What matters with semiotic representations is first their intrinsic potential for their transformation into other semiotic representations new and equivalent. This is because the power of calculation in development, control of reasoning and the creativity of the mathematical visualization depend on the potential of semiotic systems developed in mathematics and not on the represented objects. In fact, the necessity of semiotic representations in mathematical knowledge covers two very different problems, the non-perceptive access to the mathematical objects and the transformation of semiotic representations into others new but keeping the same denotation.

The first problem is epistemological. Would we have a concrete perception of numbers, functions, etc. or, on the contrary, would the access to them go through, necessarily and immediately, the mobilization of semiotic representations? To support the first hypothesis, the examples of integers and simple 2D geometrical shapes with their topological properties are often put forward. Then, the use of semiotic representations would only become necessary when the magnitude or complexity of these mathematical objects were beyond our limited capacity for intuition or immediate memory, or when we leave the field of the finite and discrete. To support the second hypothesis, there is the fact that this distinction between what would be directly intuitive and what exceeds the limited capacity of our mind is at least fluent and fictitious. In fact, this is based on extremely partial observations.

The second problem is cognitive. It concerns the nature of the mathematical work and, more deeply, the way in which the mathematical thinking functions. Does

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it consist of the mobilization of "concepts" and the use of common reasoning abilities or, on the contrary, it depends on mobilized semiotic representation systems and the stages of specific thought of mathematics? This issue is cognitive and not mathematical. The difference appears clearly in the analysis of what is called very generally "problem-solving". From a mathematical viewpoint, what we analyze is always the resolution of a given problem and therefore, *we start from its solution* to explicit the mathematical properties that lead to its solution. The analysis is, therefore, retroactive and proper the particular problem that is given or selected, but we can only assume that students will be able to solve all the others problems that are similar. From the cognitive viewpoint, what we analyse is the process that allow us to recognize the mathematical knowledge to be used in the context of the given problem, whatever it may be. For there is no point in explaining the solution to students if each of them cannot see how he/she could have recognized by his/her own the properties to be used. In other words, the cognitive issue is about the intellectual gestures specific to the mathematical work, *even before we have any idea of the solution.*

The question about the nature of mathematical work is not just a cognitive issue. It is also a methodological issue. What kind of observations need to be made and data to be collected for analyzing the cognitive processes of mathematical activity?

– the way students understand or misunderstand the mathematical concepts to be used for solving the given problem? But this remains within the range of mental representations. We cannot read in the mind of the others. We only interpret verbal, graphical or gestural expressions, which are very often more allusive than explicit and their interpretation and cannot be checked.
– the semiotic representations, considered under the only aspect or their "reference to an object"? But, this leads to subordinate them to hypothetical mental representations, which would be the important ones.

We will examine separately the two problems that the mathematical activity raises, the mode of access to mathematical objects and the transformations of semiotic representations. To the epistemological problem, we will present a test about the basic requirement of never confusing the objects with their representations. Is it as clear in mathematics as in the perception of real things? To the cognitive problem, we will see why and how the mathematical way of working must be analyzed in terms of transformation of semiotic representations. *Semiosis* is at the center of the cognitive processes of mathematical thinking through two kinds of transformations of semiotic representation. There is no *noesis without semiosis*, no mathematical thinking without transformation of semiotic representations whatever they are. That is the answer to the true question for everybody who is not a mathematician: "To do maths, what is it"?

2.1 Two Epistemological Situations, One Irreducible to the Other, in the Access to Objects of Knowledge

It is generally claimed that the way of accessing objects of knowledge would be fundamentally the same for all fields of knowledge: first, the experience with the material objects themselves and their iconic representations, followed by the development of their first mental representations and conceptualization. And from this viewpoint, the access to objects in mathematics learning and understanding would be the same than for learning and understanding in botany, chemistry, astronomy, etc. All general cognitive models and some semiotic approaches were built on this assumption.

To check the validity of this assumption we can make a juxtaposition test. It enables to answer two very simple questions.

(Q.1) Can we juxtapose the object itself and its representations?

This seems natural since it amounts to comparing a representation with the object it represents.

(Q.2) When we juxtapose different representations can we recognize whether they are representations of the same object or not?

This means presenting different representations at the same time or in parallel, as we see in encyclopedias, magazines, textbooks, and web pages that sometimes practice this up to kaleidoscopic vertigo.

We can do this test first with material objects and elementary mathematical objects. Would the results be the same to the first question in both cases?

2.1.1 The Juxtaposition Test with a Material Object: The Photo Montage of Kosuth

Look at this photograph taken by Kosuth, in 1965 (Fig. 2.1).

It is the result of two successive pictures. The first picture is that of a chair on which we can sit. This picture is then fixed to the wall, next to the chair and, on the other side, a post explaining the word "chair" is fixed on the wall. The second photo shows this montage. It creates an effect of placing in abyss (*mise en abyme*) the iconic representation of the chair.

The paradox of this photo is that, somehow, it erases any distinction between representation and object, placing the object and its various representations on the same plane, as it is indicated in the caption of this photo: "One and three chairs". This paradox goes against the example considered by Plato to highlight the epistemological requirement on which all knowledge is based on: do not confuse the object and its representation. For trees by a river and their reflections in the water remain separated and quite different as two pages of an open book and, unless we

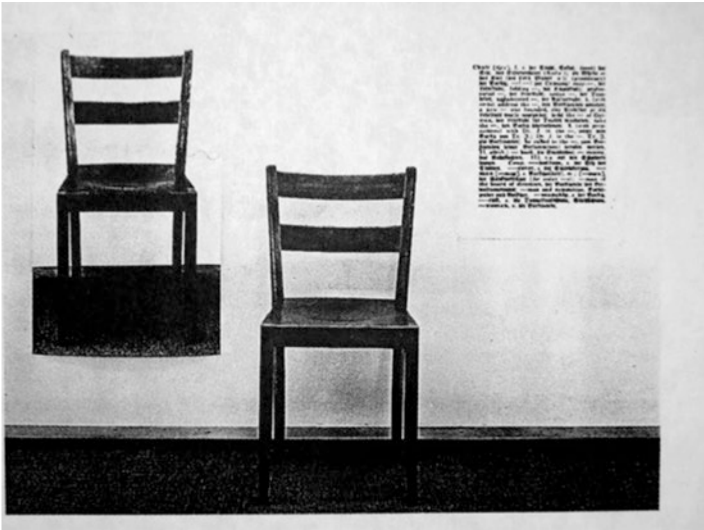

Fig. 2.1 "One and three chairs": the juxtaposition of the real chair and its representations

are not fully aware, we do not confuse any real thing with its images. However, on the photo of Kosuth the real chair does not overlap completely with its representations because it is in the center, and it is the first element of a setting that falls into the abyss.

The main interest in this photo resides in the fact that it juxtaposes a non-semiotic with a semiotic representation, i.e., a photo and text describing what a "chair" is. We could indeed join other possible semiotic representations in this setting. For example, the mounting template of free chair kits, the arrows to be drawn to connect the different representations between them and the material chair placed against the wall. We may have then "one and *n* chairs" (Fig. 2.2).

First, Kosuth's photo montage illustrates perfectly the two characteristics of the representations (P1) and (P2) we saw in the previous chapter. There are many possible representations of the same object, and the diversity of representations depends on the systems that allow their production.

It is observed that for material objects, there is a direct and immediate access to the objects themselves, and we can juxtapose them with their different possible representations.

The classic cognitive models, as well as the didactic sequence organizations, start from this empirical epistemological situation. They explain *in which order the transition activities between the different types of representation must be organized from direct experience of the objects themselves, in order to make students "construct scientific concepts".*

JUXTAPOSED ELEMENTS	EPISTEMOLOGICAL VALUE of the juxtaposed elements	
1. A **chair** against a wall.	(1) The **object itself** to which we have access independent of its representations.	
	and its different representations	
	NON SEMIOTIC or	SEMIOTIC
2. A **photograph** of this chair.	(2) An **image** produced physically by a photographic camera.	
3. A **text** from the dictionary explaining the word "chair."		(3) A **verbal description**
4. The **drawings are** showing how to assemble the different parts of the chair.		(4) **Another type of image,** *produced by drawing traces.*
5. The **arrows** on the wall are establishing the connections between {1. 2. 3. 4.}.		(5) **A scheme** (*of the "conceptual network" type?*).

Fig. 2.2 The juxtaposition test for a material object

Free arrangements of unit marks into spatial configurations: MATERIALS OR DRAWN ITEMS (PSEUDO-OBJECTS)	Verbal denomination of numbers in mother tongue: ORAL (MENTAL) PRODUCTION OF WORDS	Double internal organization: position and base: invention of "0". WRITTEN FORM OF EXPRESSIONS COMBINING DIGITS
IIII *ou* II II *doigts, allumettes*  *la tâche piagetienne* ((((I) I) I) I) *inclusions lineaires* *configuration polygonale*	**"four »** Lexicon varies considerably according mother tongues **"0"** is not pronouced in the verbal denomination of numbers	in the decimal system : **4** in the binary system**: 100** in a fraction form of the decimal system: **64/16**

Fig. 2.3 Juxtaposition of "presentations" of an integer

2.1.2 *The Juxtaposition Test with the Natural Numbers*

Let us now consider a natural number. The question here is whether the first natural are perceptually accessible, unlike larger ones that are only be accessible by using semiotic number systems. Let us taken again the example of integer representation given in the preceding chapter (Chap. 1, Fig. 1.7). We can add the mother tongue words designating integers to the units marks and decimal system. So get the following justaposition (Fig. 2.3).

In this juxtaposition of eight "presentations" of the same number, which one constitutes the perception of the number itself and which presentations would be only representations?

Husserl and Wittgenstein consider that clusters of unit marks are the first perception or intuition of numbers or at least their "proper" representations.[1] Nevertheless, it is difficult to consider the clusters of unit marks as numbers. They only represent numbers for those who can perform *a counting operation*. This operation is complex. It requires some specific conditions:

- *discerning separately and successively* each element of the cluster,
- matching each of all the elements with one word of a sequence of words *always spoken in the same order*,
- attributing to the cluster of marks *the value of the last term enumerated*, which means, according the counting context, either the ordinal or the cardinal.

The unit marks, which can be stones or fingers, *take the place* of any material object. In other words, the counting operation mobilizes THE COORDINATION OF TWO DIFFERENT REPRESENTATIONS: for any cluster of marks units a set of verbal denominations and/or written expressions combining digits. The unit marks fulfill only two functions. They constitute a kind of external memory of the counting, and they provides an immediate synthetic apprehension of the collection of counted material units.

In summary, the first natural numbers are given as objects only in a counting activity of unit marks, which requires an explicit or implicit semiotic production. None of these kinds of representation alone can be considered the natural numbers themselves. The next chapter shows that the same thing happens with the simplest geometric objects. They are not given perceptually but require specific operations, which, to be carried out, go against the intuitive operation of the perceptual Gestalt recognition.

The answer to the first question of the juxtaposition test is problematic. Because the access to the numbers is not direct, but must go through extremely varied semiotic representations, ranging from the most rudimentary verbal descriptions to more complex semiotic systems. This epistemological situation of empirical inaccessibility of mathematical objects is radically different from the epistemological situation of access to all the other objects of scientific knowledge. There is no surprise here, because mathematics begins when we do not limit ourselves to what is given concretely or physically any longer, but when we put it in the framework of what we can conceive as possible.

The second question concerns the recognition of the same object in very different representations (Q.2). Does this recognition depend on the same processes in both epistemological situations? In other words, can we still apply the empirical analysis models of knowledge acquisition to the mathematical objects, whose accessibility is semiotic and not empirical? This question is not theoretical. Any organization of a sequence of activities for student's learning involves, implicitly or explicitly, an answer to that question.

[1] Husserl, E. (1891). *La philosophie de l'arithmétique*. Trad. J. English. Paris: PUF, 1972. Wittgenstein, L. (1983). *Remarques sur les fondements des mathématiques.* Paris: Gallimard, 1983, p. 138–145.

2.1.3 How to Recognize the Same Object in Different Representations?

This question translates to the cognitive level the fundamental epistemological requirement of never confusing the representation of the object with the represented object. Two different semiotic representations of the same object can always be taken for representations of two different objects because their respective contents are quite different or, on the contrary, two representations of two differents objects for representations of the same object because their contents are similar. How can we know, then, when we are facing two representations whether they are representations of two separate things or of one and same thing?

The cognitive difficulty results from the fact that two different representations do not have or do not explicitly present the same features of the object they represent. Because even for images, the representation can have been produced as if the object was seen from the front, profile, back, etc. And we know that in the schematics produced for scientific observation purposes, for example, in anatomy or geology, the drawings of the same object can appear to have nothing in common. Instead of images, Frege used as an example the completely different symbolic expressions of the same number.[2] In other words, when we talk about representations, we speak of this complex relationship in which the content of the representation depends on both the kind of the representation used and the represented object used:

{{**CONTENT** of representation, **SYSTEM** of semiotic representation} represented **OBJECT**}.[3]

In the empirical accessibility situation, this cognitive difficulty is easily overcome because the object itself is always accessible outside its representations. Therefore, we can juxtapose it, i.e., associate it with each of the representations that we can produce. Thus, in anatomy and geology, we can always present a specimen of what is represented, i.e., a material object to associate to the different images (Fig. 2.4, left column). There is no more difficulty to recognize it in the different schemes or cuts. The association between the representations and the object itself, the words and the designated things, a work and its model etc., appear

[2] We quote two very clear passages from Frege. The first is taken from the article 'Fonction et concept', published in 1891. "The difference of the designations is not a different reason to be different from the designated … This tendency to not recognize as object what is not perceived by the meaning has as consequence to mistake the signs of the numbers for the numbers themselves, for the true objects of research, in which case 7 and 5 + 2 would be different." The second is taken from the article "Sens et denotation ', published in 1892." We would only know how to distinguish a = a and a = b if the difference of signs corresponded to a difference in the way the designated object is given" (Frege, G. (1971). *Ecrits logiques et philosophiques.* 1 Trad. Imbert. Paris: Seuil, 1971, p. 81, 103).

[3] Duval, R. (2008). Eight problems for a semiotic approach in Mathematics Education. In: Radford, L.; Schubring, G.; Seeger, F. (Eds.). *Semiotics in Mathematics Education; epistemology, history, classroom and culture*. Sense Publishers, p. 39–61.

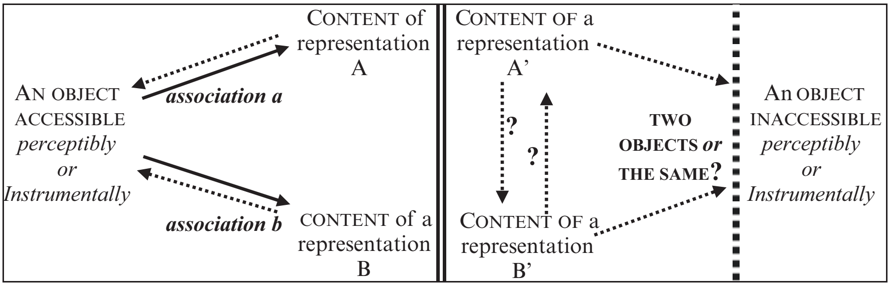

Fig. 2.4 Recognition of the same object in two epistemologically opposite situations

as the fundamental cognitive process to "make sense" and to verify, and hence, acquire new knowledge.

This cognitive operation is not possible in the special epistemological situation of non-empirical access to mathematical objects. We can only juxtapose representations, but never an object and its representation, because the objects of knowledge are not accessible outside the semiotic representations. Also, we need to have a second representation whose content is different from the one of the first, so we can no longer confuse the mathematical object and its representation. But, the question that arises is how can we recognize the same object in two representations whose contents have nothing in common (Fig. 2.4, right column)?

There is, evidently, a local response based, for example, on an operation. To find out whether "3/4" is the same number as "0.75" it is sufficient to divide 3 by 4. But this may be less obvious while finding the corresponding fractional writing to "0.76". However, everything changes when there is no more calculation operation to switch from one representation to another. This is the case when two representations are semiotically heterogeneous, when their respective contents mobilize units of meaning of different kinds (words, symbols, 1D and 2D shapes) and/or when the organization of these units of meanings are of different natures (based either on syntagmatic combinations, or spatial relations and positions). We shall see the importance of these two criteria in the classification of representation registers (Chap. 4). In this situation, the recognition of a single object represented by two representations, A and B is based on a one-to-one mapping between the respective meaning units of the two representations.

Take for example a graduated straight line. It has at least three types of visual units: two kinds of marks corresponding to two scales of magnitude—the first is the division into units, the second is the first division of each unit—and the intervals between the marks for each division scale. We can thus realize an one-to-one mapping between the end of each interval and the sequence of digits 1, 2, 3, … or 0.1, 0.2, 0.3 … We shall see that realizing such an one-to-one mapping between the "number line" and the relative numbers is more complex than is generally assumed. In other words, realizing a one-to-one mapping between the meaning units, which constitute the respective contents of two different representations, is the cognitive condition to be able to recognize the same object in these two representations.

Two questions appear cognitively crucial so that the students understand and acquire the mathematical way of thinking.

(Q.3) How to make one learn to DISCRIMINATE THE UNITS OF MEANING RELEVANT IN THE DIVERSITY OF SEMIOTIC REPRESENTATIONS that are mobilized in mathematics?

(Q.4) How make one become aware of the central role of the one-to-one mapping operation between the meaning units discriminated in two different representations?

Contrary to what has been always postulated in mathematics education, discrimination of the relevant units of meaning in different representations does not result from the acquisition of concepts, but it is the prerequisite for this acquisition. Similarly, the search for the "right" representation or even the juxtaposition of multiple representations are only a misleading help. The "right" representations cannot be associated with the mathematical objects they represent because these are not directly or empirically accessible. The only possible means to access empirically inaccessible objects is to realize the one-to-one mapping cognitive operation between meaning units of the semiotic representations used, whatever they may be.

2.1.4 A Fundamental Cognitive Operation in Mathematics: One-to-One Mapping

We don't have paid sufficient attention to the central role played by the one-to-one mapping elements between meaning units of two different semiotic representations. This operation is the only one that is crucial from both the mathematical and cognitive viewpoints.

The importance of this operation in mathematics appeared with the semiotic revolution, especially with the development of the Analysis of the notion of function. But to illustrate this two-sided operation, we shall consider the simplest and most spectacular example, the historically famous question: are there more natural than even numbers or as many even as many natural numbers? This question seems absurd to most educated people, the inclusion of the even numbers seems so conceptually evident.

The mathematical response consists of one-to-one mapping between natural and even numbers, by paralleling them over two lines to remove the obstacle of the inclusion of even numbers. But this raises the visual obstacle of a twofold occurrence of some objects: any even number occurs in both sequences! It was when they became aware of COUNTING ONLY THE COLUMNS AS MEANING UNITS (indicated by the arrows) *that the students (12–13 years old) had the insight of the*

Paralleling of two sequences of numbers to create new meaning units: the vertical associations of two numbers	*VISUAL ONE-TO-ONE MAPPING between the units of each sequence*	**1** ↕ **2**	**2** ↕ **4**	**3** ↕ **6**	4 ↕ **8**	…. ↕ ….
oral or mute verbalization	*DISCURSIVE ONE-TO-ONE MAPPING* by counting the new meaning units	↑ One	↑ two	↑ three	↑ four	↑ …..

Fig. 2.5 Twofold one-to-one mapping between two kinds of semiotic representation

one-to-one mapping.[4] We can analyze this discovery as a twofold one-to-one mapping between two kinds of semiotic representation:

- a mathematical one, which leads to the notion of infinity as a set equipotent with one of its own parts,
- a cognitive one, which leaves no trace because it is made orally or even just "mentally" (Fig. 2.5)

Of course, from a mathematical point of view, the first correspondence is the only thing that matters because it breaks the inclusion linked to the representation of the sequence of natural numbers. But, from the cognitive point of view, the second is the crucial one. Without an explicit or implicit counting of the new meaning units shown by the arrows, the mathematical one-to-one mapping between one natural and one even number cannot be understood.

The only cognitive operation for making out new properties, or giving access to mathematical objects is the *one-to-one mapping meaning units from two semiotic representations differing from each other by their respective contents.* The central character of this operation is always ignored because the classical explanation of the formation of concepts *based on the perception of material objects*, and therefore on the differences or similarities of sense data, seems the most obvious acquisition processes for common knowledge. Piaget's analysis of the genesis of the number for the child was the seminal pattern. Although it resorted to this cognitive operation to prepare the famous proof test of the acquisition of number notion, this reduced its importance in two ways. First, the operation is limited to the one-to-one mapping of two rows of pearls, whose spacing of one can vary noticeably (Fig. 2.3, column 1). The invariance of the answers is criterion for the acquisition of the cardinal number notion. And above all, one-to-one mapping has no role in this acquisition, which is explained by two other operations: classification and seriation.

[4] Duval, R. (1983). L'obstacle du dédoublement des objets mathématiques. *Educational Studies in Mathematics*, 14, p. 385–414.

The mathematical and cognitive operations are related to the elements of the respective contents of two semiotic representations. But, the cognitive operation diverges from the mathematical operation in that it cannot objectively defined once and for all, because there are multiple ways to discriminate the meaning units, which make up the content of the semiotic representations. *Its outcome is the recognition of the object represented by two different representations*.

2.2 The Transformation of Semiotic Representations at the Heart of the Mathematical Way of Working

In general, the answer to the question—what does it mean "to do math?" is: "solve problems". Solving problems is, therefore, placed in the forefront of organizing classroom activities. However, this response is in reality vague. It does not say anything about the *mathematical way of working* that should enable anybody to solve problems. Thus, the didactic analysis of solving problems is local and mostly retrospective. It starts from the mathematical solution of a problem posed to explain all the properties that must be discovered and used during the research phase. But, the steps to be performed during this stage are still a black box for many students. To understand the solution when it is explained by the teacher or another student does not allow us to grasp how we should have handled the problem in order to solve it by ourselves. Why be surprised then that many students find themselves back in the same situation of incomprehension or mental block when they face a problem previously explained, but whose the context or one of its conditions has been changed?

The key feature of mathematical of the mathematical work consists of TRANSFORMING THE SEMIOTIC REPRESENTATIONS, given or obtained in the context of a proposed problem, into other semiotic representations. This is where mathematical activity differs from other sciences such as physics, astronomy, biology or geology, etc. This explains why, in mathematics, *a semiotic representation is only interesting insofar as it can transformed into another representation, and not first because the object it represents*. This key feature of the mathematical work involves a complete reversal of the common cognitive viewpoint about the representations and particularly about the semiotic representations. Semiotic representations are not only useful for working with or about the objects. If we want to describe, from a cognitive point of view, the mathematical way of working in mathematics, we must focus on the transformations of semiotic representations and analyze the different kinds of transformation.

2.2.1 Description of an Elementary Mathematical Activity: The Development of Polygonal Unit Marks Configuration

The representations produced only with only unit marks offer no restrictions to any spatial arrangements we want to organize (Fig. 2.3, column 1). Starting with a simple rule, one can with a token, for example, add other tokens to mark the vertices of a square and reiterate … We immediately see that there are two possible mathematical descriptions of polygonal configurations we have produced: one that is the number of elements of each configuration and another, the evolution of the configuration sequences thus generated. Of course, using a literal notation, we can generalize the successive numerical descriptions into a formula.

Here we will limit ourselves to the analysis of the transformations of representations that are involved in a single numerical description activity of polygonal configuration. We will modify slightly an elementary classical situation, by changing the development procedure first, and, then, the shape of the polygonal configuration (Fig. 2.6).

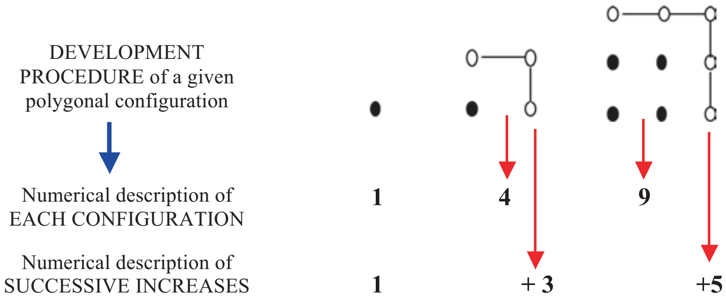

Fig. 2.6 Double numerical description of the development of a polygonal configuration

In the first description, the counted units are the unit marks of each figure. In the second, the counted units are those we need to join in the penultimate figure to obtain the last, i.e., the difference of the unit marks between two successive figures (the white tokens on the figures above). So we have two numerical descriptions possible according to whether one-to-one mapping is made with the square or only two sides of the new square obtained.

We can make a first observation regarding the description of the successive increases. *To carry out this activity with a material, we need to have tokens of two different colors*, one color for the already placed tokens and another for those placed later. If the tokens are of the same color, we must distinguish the tokens added from the previously placed in each successive configurations. The tokens of the same color hide the successive increases rather than show them. We need, therefore, to keep in mind the visual memories of the previous configuration in order to compare it with the new configuration now perceived on the desk, i.e. to shift the focus of

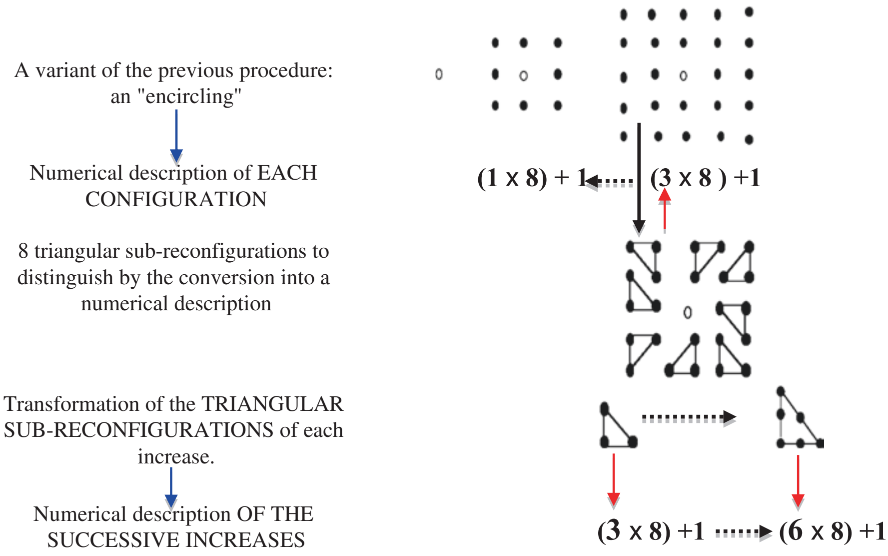

Fig. 2.7 First variant of the development of the polygonal configurations

attention from any configuration to the successive increases! This forces us to wonder about the value of the help that many of the proposed material handlings contribute.

We change now very slightly the transformation procedure. Will the description of the successive increases also be simple? And is it still the same type of problem? (Fig. 2.7).

The counting activity requires to focus on the figural units, i.e., the triangular sub-configurations of tokens instead of the only tokens. But, here, a perceptual conflict arises regarding the recognition of the shape of the figural units to be counted. We need to use as figural units the triangular sub-configurations while the dominant overall shape is a square. We therefore cannot count the two sides of the new square as in Fig. 2.6.

This first variation of the task allows an important observation. Each polygonal configuration is suitable for the distinction of QUITE DIFFERENT FIGURAL UNITS POSSIBLE, but recognizing some of them excludes the correlative recognition of others. The question is, then, to know which will allow to choose from among the many possible figural units, those that are relevant to the task. It is clear that the knowledge of the geometrical properties of the square and triangles cannot help to discern and choose.

Let us apply now this transformation procedure by "successive encircling" by using regular hexagons instead of tokens. As this new task is very complex to carry out without using software, we replace the hexagons by points. We then have a

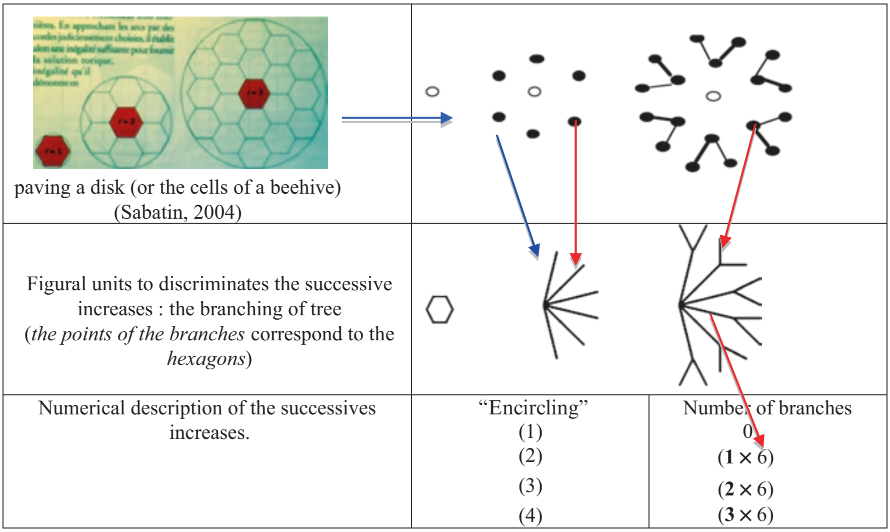

Fig. 2.8 Second variant of the development situation of polygonal configurations

variant of the previous task (Fig. 2.8). From a cognitive point of view, is this still the same task?[5]

Here, the cognitive activity required by this task presents an important difference compared with the preceding variant. First, it is necessary to introduce a transitional auxiliary representation, that of a tree to see how one-to-one mapping between the numerical description and the successive encircling procedure by regular hexagons is possible. It is no longer a “geometric” reconfiguration, because it is not the polygonal configuration and sub-configuration shapes that matter now, but *the points of branching and the number of branches of each point* instead.

The comparison of these three development situations of polygonal configurations shows that a very slight modification of a mathematical task at hand leads to important differences in the cognitive activity required to carry out the task. In the three situations the task is based on the same operation of one-to-mapping (red arrows) between tokens or figural units and numerical expressions. However *two basic kinds of semiotic transformation* are required in the variant situations. The first is the need of an internal shape transformation of the geometrical configuration (Fig. 2.7, dark dotted arrows) before performing the one-to-one mapping (red arrows). The second is the need of introduce quite different representations (Fig. 2.8, blue arrows). Thus, in both variants, two operations of one-to-one mapping, one internal shape transformation (Fig. 2.7) are required to find out the numerical

[5] Sabatin, A. (2004). L’âme de géomètre des abeilles. Les formes de la vie. *Dossier pour la Science*, 44, 72–77

description of the given polygonal configuration development. The initial task (Fig. 2.6.) is more simple because none of these two kinds of semiotic transformation (dark dotted arrows and blue arrows) is required, even if the numerical description asked requires an one-to-one mapping with real objects or marks units (Figs. 2.3 and 2.6).

Whether blue, dark or red (change of the kind of representation, internal transformation of representation, one-to-one mapping between meaning units from two representation contents), all the arrows mark *the cognitive activity that a student must use, either to be able to succeed when solving a mathematical task or to understand the solution.* The first question we ask is whether students facing semiotic representations, whatever they may be, can discriminate the different units of meaning that form the contents of each representation, *in order to recognize the different possible one-to-one mapping with other quite different representations* and whether when looking at figures, diagrams, they can see other spatial organizations besides those imposed by the given configuration. Such transformations of semiotic representations are only required in mathematics and can be truly practiced only in mathematics.

The mathematical way of working can be analyzed here like the transformations, in parallel, of at least two kinds of semiotic representations of numbers, each fulfilling a role different from the other. Some have the heuristic role of exploitation, or intuitive role, in relation to the other, the latter having a description role. But in reality, to make these transformations, *we must implicitly or explicitly go back and forth constantly* between the two kinds of representations. Therefore, they can all perform locally a function of anticipation or control, without the possibility of attributing these cognitive functions, respectively, to configurations of token units or numerical descriptions. At later stages of the mathematical work, it is the numerical description that will be privileged and will allow another kind of transformation: using letters to condense a sequence of local numerical descriptions into a general description.

Is it possible to generalize this analysis based on the variation of a mathematical activities? The two kinds of representations required by the mathematical tasks of polygonal configuration development may seem too narrow. Suppose we no longer practice spatial arrangements about unit marks, but about the signs of decimal numbers. We then have the famous Triangle of Pascal or the Gauss' solution for the sum of natural numbers, with a figural unit count activity that allow a more direct passage to the formulas. And, in the field of combinatorics, we find the same semio-cognitive gestures of the mathematical work. Of course, there is a limitation, we are in the discrete and countable field. As we move to the magnitudes, measurements or the mathematical continuity, do we find the same semio-cognitive gestures? Mathematics mobilizes many other types of semiotic representations. Before studying them, we must revisit the question of the relationship between the semiotic transformations and mathematical activity from another angle.

2.2.2 *Representational Transformations Specific to each Kind of Semiotic Representation: The Case of Representation of Numbers*

We just saw that the mathematical activity is the transformation of semiotic representations. But, we also found that the mathematical activity can mobilize very different semiotic representations to represent the same objects. We can then ask ourselves if, *in fact, we perform the same operations with the different semiotic representations or, on the contrary, we perform different operations.* The question arises when we switch the kind of semiotic representation (Figs. 2.6, 2.7, and 2.8, blue arrows).

To answer this question, take the example of the representation of numbers, because there is no notion of number without notion of operation that can be carried out on the numbers. Do we make the same calculations when we switch the representation of numbers? This question is only interesting in the elementary situations where the answer is not obvious: the first natural numbers that we can represent either by the unit marks or decimal number system, and the operations with the relative integers.

2.2.2.1 Operations with Small Natural Numbers

There is a fundamental difference between the representations using unit marks and those produced with either the decimal or any other base *n* system.

Representations by unit marks are only a support *for operations that are external to the unit marks:* regroup them or separate them into clusters, arrange them according to the polygonal configuration (Fig. 2.3), or order them at regular intervals over

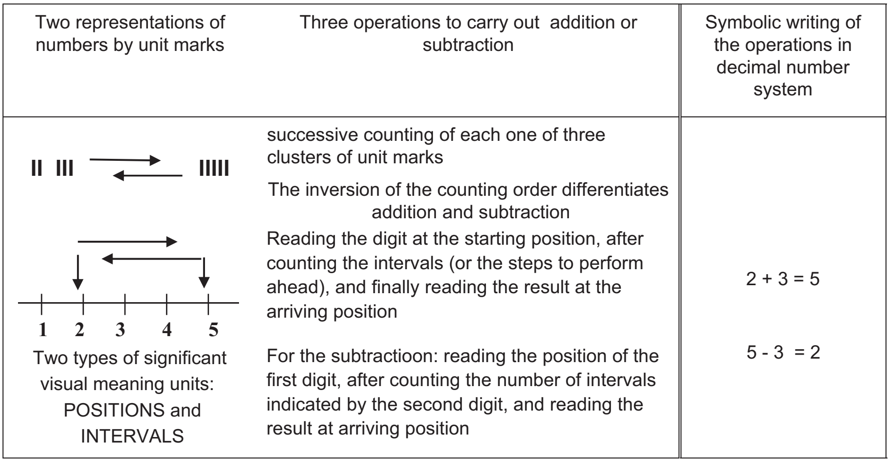

Fig. 2.9 The additive operations with unit marks

a straight line and encode them, that is, to fix a counting (Fig. 2.9). As either unit marks or digits are used, we do not perform the same external operations to add or subtract two natural numbers from one another.

To add and subtract with representations using unit marks, there are three operations, which are not the same whether the unit marks are associated with digits or not (Fig. 2.9 column 1). And in the case of the number line, the numbers are represented by two kinds of visual meaning units. But in both cases the calculation is not based on any change in the representation of numbers.

However, with the representation of the numbers in the decimal system, *the operations become intrinsic to the representation system* and are carried out by transformations of the digital expressions of numbers. This appears when we consider the numbers that exceed the base system and, therefore, require two digits or more. The limit of the base is marked by this sign that does not refer to a number: "0". Calculation operations rest on the transformation of two digital expressions that take into account the limits of the base and the position of digits in order to get a third digital expression: 13 + 18 = 31. More generally, a semiotic system of representation of numbers is characterized by its calculation power. The calculation algorithms are related to the operations intrinsic to each system of representations of numbers.

From the representations of numbers by unit marks to the true use of the decimal system, there is the counter-intuitive semiotic threshold of 'zero'. An always hidden threshold and whose difficulty is underestimated. It resurges when it is asked to multiplying or dividing a number by a factor of 10 or a factor of 100 etc. It is useful to remember that multiplication, division or square root operations cannot be made with the unit marks, but require a representation system involving a double organization (Fig. 2.3).

Therefore, there is a small overlapping zone in which the operations appear to be the same, whether we use a representation of the numbers by the unit marks or a decimal representation of numbers. It concerns the natural numbers that can be represented above the threshold of 'zero', i.e., whenever it is not needed to mobilize the calculation possibilities provided by of the decimal system. As if, for example, we would drive a Ferrari only in first gear and just touch the accelerator. For all the numbers in that zone, there is not only a perfect congruence, i.e., complete transparency between the different representations of numbers, but the additive operations may be the same whatever the representation is used. Can we then speak of the intuitive or concrete character of the first numbers and oppose it to the symbolic nature of the knowledge of other numbers, as was done by Leibniz and Husserl, or as it is still done in some didactic approaches? To think so is to forget two essential things. The first concerns the semiotic representation of numbers. The zero threshold varies according to the adopted system. Why, then, a base ten would be more intuitive than a base two, a base seven or a base twelve? The second concerns what cognitively we consider intuition. Simultaneous visual perception of unit marks? But then, can the number vary considerably depending on its disposition or the distance that separates them? The visual perception of a successive series of unit marks? We fall back on the issue of the capacity of short memory, essential for the

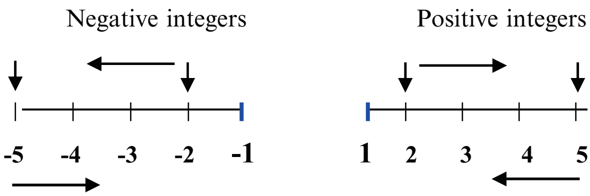

Fig. 2.10 The negative integers as a reverse mirror of positive integers

perception of successive meaning units when listening to a sentence or a phone number for example. There is nothing more uncertain and more variable than the dividing line between what is intuitive and what is not, between what is concrete and what is not.

2.2.2.2 Operations with Relative Integers

The extension of the set of natural numbers to the relative integers requires to encode the decimal notation. It consists of using the operation symbols "−" and "+" for each natural number. This encoding introduces *a new dimension to the value of the opposing sign numbers.* For example, "+1" is not just the opposite of the nine other digits that can be used instead of it, but gets a supplementary value, totally different, that of the opposite to "−1". This opposition is equivalent to reversing the order of unit marks on a straight line, as an image in a mirror (Fig. 2.10).

Of course, this dual representation of the relative integers may seem incomplete. The "0" is missing, which brings together the two representations to constitue what is called the "number line". So the symbols "+" or "−" mean respectively the position value "to the right" or "to the left" of *the origin "0"*. But, would it be that simple? We cannot forget that the number representations are only interesting insofar as they allow calculation. The question is whether the calculation process is the same with each of these two representations, the number line and the encoded digits of the decimal system, or not.

The operations of addition and subtraction of positive integers remain the same as those described above in Fig. 2.9: they corresponds respectively to "steps forward" and "steps backward" (Fig. 2.10, horizontal arrows). It is the reverse for the addition and substraction of negative integers. *But, in both cases we do not get over the barrier of "zero"*. Also, introducing here the "0″ origin to merge the two representations into the "number line" is a cognitive jump that complicates the understanding of these operations.

The numerical line is necessary only when we want to add or subtract a positive integer and a negative integer since it is necessary to overcome the barrier or limit of "zero". But here everything gets complicated because we fall into complete semiotic ambiguity, as we can see it comparing the below calculations (Fig. 2.11).

The semiotic ambiguity is between the position value and the interval value for the numbers, and between the encoding of the decimal numbers and the operations

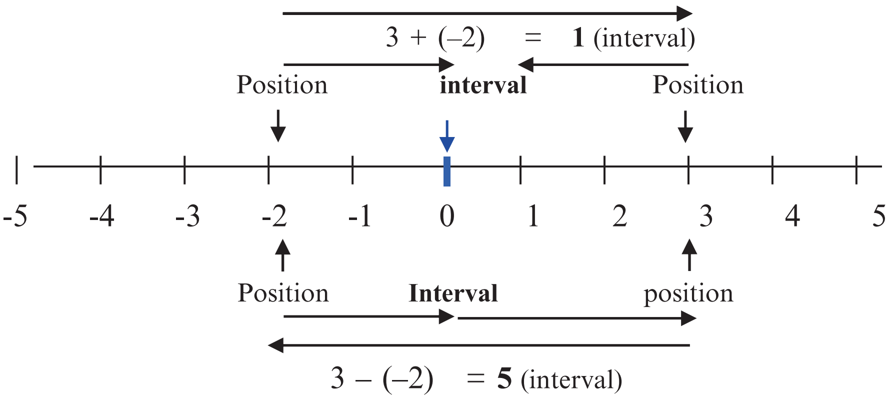

Fig. 2.11 Elementary, my dear student!

steps forwards or backwards for the two symbols "+" and "–" . To add, it is necessary *to start from the position of the positive number* and moving as many steps backwards as indicated by the negative number. The result is the number of the interval between the respective end positions of the two opposite arrows. To subtract, it is necessary *to start from the position of the negative number* and moving as many steps forwards as indicated by this negative number. The result is number of intervals between the positions of the negative number and positive number. It is also necessary to take into account the fact that intervals are oriented. So, if we had considered the operations

$$(-3)+2=\ldots, \quad \textit{and} \quad (-3) \quad (+2)=\ldots$$

the orientation of the interval is translated as positive or negative number. For addition, it depends on its position on the right or on the left side of the origin "0". For subtraction operation, the value of the interval is always a positive number. Thus, we see the double semiotic ambiguity that the number line arises compared to all other representations.

Let us now look at the symbolic writing of operations. We see, immediately, an increase and diversification *of the possible numerical expressions* for these two operations with the relative integers. Two factors explain them. First, the operation (addition or subtraction) and second, the fact that the operation concerns two integers, both positive or both negative or one positive and one negative. We therefore get eight different numerical expressions possible, that we can represent in a table. To construct such a table it is necessary not to change the absolute value of the numbers and keep them in the eight possible numerical expressions (here |3| and |2|). It is the condition to compare the different possible numerical expressions of the operations and their results (Fig. 2.12).

Two observations are obvious on this table.

First, we obtain the same result in two different ways for each line (A_1 and C_1, etc.) whether performing an addition or subtraction. *The comparison of two numerical*

Operations convertible into steps whether on the positive half line, or on the negative	Operations convertible into steps on the numerical line by overcoming the barrier of "zero"
A_1 3 + 2 = **5** A_2 3 – 2 = **1**	C_1 3 – (–2) = **5** C_2 3 + (–2) = **1**
B_1 (–3) + (–2) = **(-5)** B_2 (–3) – (–2) = **(-1)**	C_3 (–3) – 2 = **(-5)** C_4 (–3) + 2 = **(-1)**

Fig. 2.12 The variations in writing to represent the addition operations with relative integers

expressions on every line of the table shows that subtracting a number is the same as adding its opposite, thus (C_1 and A_1, A_2 and C_2, etc.). It is, of course, *AN IMPORTANT POINT FOR LEARNING ALGEBRA*, since it is this transformation of expression that allows changing the term of an equation from one side to the other. We may also add a third column to the table in which the two numerical equalities of each line are changed into a third. For example, for the first line and the second line:

$$3 + 2 = 3 - (-2)$$

$$3 - 2 = 3 + (-2)$$

So, we have here an example of the transformation mechanism of semiotic representations described by Frege as characteristic of mathematical thinking. *Two expressions whose content meanings are different can be substituted one for another.* On each line of the table, the numerical expressions are obviously different, since they do not use the same symbols of operation and nor does the same relative integers, but they both refer to the same number. We can replace one for the other in any equation in which one of the two expressions occurs.

Second, the two columns correspond to the cognitive jump from the visual representation of the half-lines to the number line. There is no observable difference between the numerical expressions of the columns. But if we look at their respective visual representation support we observe that the visualization of the additive operations with the relative integers becomes very complex. We face a *semiotic overload* of arrows above the numerical line to indicate the position of numbers, the direction of the steps to be done that is always the same as direction indicated par the symbol of the operation to be done, the interval value (Fig. 2.11). This semiotic overload is too often ignored in the didactical use of the number line. Didactic use of number line is so simplified that most arrows needed for indicating all the cognitive operations involved in its support use are missing. Beyond this specific case it raises an important question about the teaching and use of didactic or pedagogical tools: can we use different kinds of representations without blurring the means of visualization or expression transformation specific to each kind, and without short-circuiting the necessary realization of the one-to-one mapping between the meaning units from two representation contents?

2.2.2.3 The Operations with Rational Numbers

There is an important semiocognitive gap between operations performed with the means of transformation of numerical expressions, which decimal number system provides and the ones provided by a graduated line, i.e., by mark units ordered on a straight line from the origin "0". This gap increases as we move from operations with relative integers to operations with rational numbers.

R. Adjiage showed that, to carry out operations with rational numbers, *one must use TWO GRADUATED STRAIGHT LINES* with *different scales of graduation*: the one-to-one mapping between the position of the unit marks on one straight line and their positions on the other depends on the ratio of intervals of the two graduations.[6] What is essential is that the two graduations will be chosen independently of each other, without keeping the same ratio, as in the recurrent division of an interval. Changing the ratio is here the key point for the organization of learning situations, because fractions are the appropriate numerical expressions for the one-to-one mapping between the position of the unit marks on one straight line and their positions on the other. Thus, for a fixed graduation on the first straight line we may have several other lines each with different ratios and, for the position of an unit mark on this first line, we get different fractions indicating its corresponding position on the other lines 1/2, 3/5 or 5/7 according to the ratio of the graduations. So these different fractions can be compared without calculation, only by relating them to the first graduated straight line taken as a reference. Then we can see *whether a fraction is greater or smaller than the unit interval of the first straight line, i.e., greater or smaller than "1"* and also any other fraction. That does not only define the additive operations, but also the multiplicative ones, which are different from those that can be carried out with the fractional or the decimal writing.

2.3 Conclusion: The Cognitive Analysis of the Mathematical Activity and the Functioning of the Mathematical Thinking

The cognitive analysis of mathematical activity focuses on the problems and the processes of mathematical understanding. But, the comprehension criteria are not exactly the same from the cognitive view point as from the mathematical one. From the mathematical viewpoint, comprehension begins with what is called

[6] Adjiage, R.; Pluvinage, F. (2000). Un registre géométrique unidimensionnel pour l'expression des rationnels. *Recherches en Didactique des Mathématiques*, 20(1), 41–88.

"validation", "justification", "validation", "demonstration", according to the level of requirement. From the cognitive viewpoint, two essential conditions are necessary so that we can speak of comprehension. First, FAST RECOGNITION of the objects themselves through their multiple representations possible, and second, SELF-CONFIDENCE to begin exploring one one's own possible ways in any new task and check their relevance. As long as these two cognitive conditions are not met, whatever is done or explained in mathematics remains for the students a bit like "dark matter". There are therefore two key issues for the cognitive analysis of mathematical activity and the functioning of the mathematical thinking.

The first concerns the access mode to mathematical objects. In this chapter, we took the example of the numbers, but we could have also used the simpler geometric objects such as the elementary figures of Euclidean geometry, or functions etc. This issue is, first of all, epistemological. And, on this point, there is a misunderstanding. When we speak of epistemology, we think *of intra-mathematical* epistemology, essentially focused on the historical stages of the discovery and development of mathematical objects. There are thus epistemological studies of the different kind of numbers, functions, vectors, etc. But, here, it is not what this is about. *The epistemological issue is not intra-mathematical but scientific,* i.e., about the scientific knowledge in the heterogeneous range of its areas and methods. It is whether we have access to mathematical objects, whatever they are, in the same way as phenomena and objects studied in physics, chemistry, geology, astronomy, or biology. In other words, it involves the implicit or explicit choice of a cognitive model of the functioning of thought. Can we use general cognitive models based on an empirical, direct or instrumental, access to the objects of knowledge to analyze the questions regarding mathematics understanding and learning, whereas access to mathematical objects is not empirical, but only semiotic? As the search for examples and counterexamples shows, *the criterion of reality in mathematics is not what is empirically given, but all possible cases that can semiotically represented or constructed.* This issue has nothing theoretical. We have only to observe a student who should successively move from one 50 min session of geography to another of mathematics and then to another of geology or physics, etc. all this within the same day! Not only the objects of study are different, but also and more importantly, the ways of thinking and working.

The second key issue concerns the nature of mathematical activity, whatever the objects, areas or mathematical frameworks. What are we doing when we do math? Obviously to say "we are solving problems" does not answer the question, because this really means "solving *mathematically* problems", which are mathematical problems even if they are presented as real-life problems! In fact, the crucial point is whether an objective description of mathematical activity is possible or not, regardless any claim based on the unverifiable obviousness of introspection. We can formulate it in two ways. What characteristics are observable? Or, what are the intellectual gestures that make you able to work mathematically?

We have seen that the transformation of semiotic representations is the process we find in all forms of mathematical activity. Whether to explore situations, solve problems or demonstrate conjectures, this drives the mathematical activity. The development of mathematical activity depends on two factors: the variety of semiotic representations that can be used, and the need to produce and consider, alternately or in parallel, explicitly or implicitly two quite different representations of the same object.

Taking as an example the most immediately accessible natural numbers, we have highlighted the need to take into account, as a priority, these two factors in order to analyze the cognitive processes underlying mathematical activity.

- the one-to one mapping between meaning units from two semiotic representations differing from each other by their respective contents ist the cognitive prerequisite condition to recognize whether two semiotic representations represent the same object or not.
- We cannot isolate directly the meanings units which make up the content of semiotic representations, and therefore, there are different ways to discriminate them. This depends on the organizational level on which it is focused.
- Some semiotic representations are mixed representations. They result from the superposition or fusion of two kinds of representations, the straight line and the unit marks for a numerical encoding of only some points on the line. We obtain thus many mixed representations: the number line, the graduated straight line (to measure), the real line. In this example, the mixed representations lead to confuse contiguity (adjacency), consecution, visual continuum and mathematical continuum.
- In any semiotic transformation, it is necessary to distinguish the starting representation and new representation produced, i.e. the arrival representation. This raises the question of whether *the inverse transformation is cognitively equivalent to the direct transformation,* that is, whether semiotic transformations are reversible or not.
- There are two kinds of cognitive tasks in mathematical activity. Two semiotic representations differing each other by their respective contents are given or directly juxtaposed. Then you have to RECOGNIZE whether they are two representations of the same object, or not. On the contrary, the semiotic representations that are given do not differ in kind: verbal statements or symbolic expressions, or geometrical figures, etc. Then you have to PRODUCE NEW representations of the same object in another kind of representation. Solving problem always involves, explicitly or implicitly this cognitive task more or less complex.

These key points raise, of course, three questions. What kinds of semiotic representations are used or can be used in mathematics? Do the two factors of semiotic transformations that are at the core of mathematical activity correspond to central processes of the cognitive functioning of thought? What method of analysis (both for the organization of observations and the interpretations of data collected) allows to study the cognitive phenomena related to mathematical understanding and learning? These three questions will be addressed in the following chapters.

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