Preface

I could not have written this book if it was not for the invitation of Professor Tania Maria Campos Mendonça to present a seminar for the doctoral students of the mathematics education graduate program of the Universidade Bandeirante de São Paulo (UNIBAN). The seminar focused on the teaching of algebra to all elementary school students, analyzing its difficulties and exploring other introduction strategies. Therefore, the main purpose was not to present the registers of semiotic representation, but to understand how they allow analyzing the cognitive processes that we should encourage and develop in students, so they understand and use the basic tools that algebra provides. Thus, the presentations and materials for the working sessions were prepared for this purpose.

Very quickly, the request of the doctoral students and the program's professors, who participated in the seminar, was for the representation registers. A large proportion of the participants knew the distinction between the kinds of transformations of semiotic representations: conversions and treatments. Some participants had seen the relevancy of this distinction to identify the students' understanding difficulties and to analyze the cognitive processes that underlie understanding and non-understanding phenomena during the mathematics learning process. But that raised several important questions for the seminar participants, such as when and how to use such a distinction in the research work related to mathematics education. So, the seminar focused on the following issues:

- How to situate the representation registers in relation to other semiotic "theories"?
- Why use a semio-cognitive analysis of mathematical activity for teaching mathematics?
- How to distinguish the different kinds of registers?
- How to organize tasks and learning activities that take into account the registers as variables?
- How to analyze students' productions in terms of registers?

The theoretical aspect was less important than the analysis tool and work method. The questions and discussions that arose, in a way, determined the plan of this book.

This book is based on the analyses developed in *Sémiosis et Pensée Humaine* (1995), but the main ideas are presented from another perspective.

In *Sémiosis et Pensée Humaine*, analyses focused on the cognitive processes of mathematical reasoning, in which the natural language is explicitly used, even when symbolic formulations are used. The purpose was to analyze the specific cognitive process of reasoning that underlies proofs in geometry. And that led us to consider that the language and its use are not only constituted by the words used but also by the discursive operations employed when speaking and reasoning whether in a deductive, argumentative, or purely semantic way. It is only when these discursive operations are taken into account that language appears as a true representation register and the relationship between language and thought can be systematically examined.

In this book, on the contrary, we are more interested in the other registers, especially those that allow visualizing in mathematics. Figures in geometry, graphs, and the various types of tables used in statistics or in other areas raise at least as many difficulties as reasoning in geometry. But we pay less attention to them because we believe that in this case, since we "see," the recognition cognitive processes would be the same as in any iconic representation, such as images, diagrams, maps, and photographs. However, this is clearly wrong.

The various kinds of representation used to visualize in mathematics become a source of misunderstanding, all the more important that they have an increasing place in the teaching of mathematics for two reasons. First, the emphasis is placed on practical activities that involve using a lot of varied, iconic representations, such as figures, "curves," and tables, among others. Second, the use of a computer for everything that concerns mathematical visualization, both in geometry and in analysis, and geometrical or graphical software opens considerable possibilities of creation and visual exploration. But does software suffice to develop in the students the ability to anticipate the different possible transformations of a given figure into others completely different? Does it make students aware of the one-to-one mapping between graphic visual values and the terms of the equations they represent?

Another issue has also become more explicit after the publication of *Semiosis et Pensée Humaine*. It is about the particular epistemological status of mathematical knowledge in relation to the other domains of scientific knowledge. Certainly, didactics is concerned about epistemology, but it is only an intra-mathematical epistemology, in which we follow, historically, the formation steps of the concepts or mathematical objects, each with its own motivations and obstacles. This intra-mathematical epistemology cannot be considered an epistemology of scientific knowledge; insofar it does not take into account the formation of concepts in chemistry, geology, botany, or paleontology. The ways to get access to these objects of knowledge and the kinds of proving are not at all the same as in mathematics. This fact raises a serious question regarding the learning process of students who must switch between maths and other scientific disciplines and, therefore, different ways of working on the same day or the same week. Does teaching really help them to realize how the way of thinking and working in mathematics is different from that in other sciences?

In fact, the specific understanding problems that students face when learning mathematics are rooted in the particular epistemological status of mathematical knowledge, not only on questions of the pedagogical organization of activities. Indeed, the way we get access to mathematical objects is radically different from the way we do for the objects of other scientific disciplines. This is the crucial point for learning mathematics, and it is also the first challenge in the teaching of mathematics. It is necessary to develop a kind of cognitive funtioning wholly different from the one mobilized in the practice of the other sciences so that students understand how learning mathematics contributes to their global intellectual development. This goes against the general direction of most research on mathematics teaching, where it is assumed that the model to acquire and build knowledge would be the same for all areas of knowledge, mathematical and non-mathematical. In order to see and to teach mathematics differently, we must, instead, be aware of the specific cognitive processes that mathematical thinking requires and develop them with the students, even if by doing this, teachers have the impression "of not teaching (momentarily) math"!

And algebra? In the seminar, we then turned to the issue of learning to use equations as a problem-solving tool. This acquisition is generally the main objective of teaching algebra in elementary school. But where to start? By introducing letters and literal calculation to solve numerical problems? But this produces insurmountable obstacles for many students. Cognitive analysis leads to another approach. First, it highlights the specific discursive designation operation of any object that putting data into equation requires as opposed to the designation operations in the practice of ordinary language. Second, it emphasizes the diversity of conversions underlying the problem statements. Therefore, other initial goals and tasks that consider the registers used for producing a problem statement are required. Thus, about these two crucial points, the introduction of letters and problem-solving, it is necessary to:

- Dissociate the introduction of letters from any problem-solving activity.
- Associate *the first use of the letters to the functional designation of numbers.*
- To teach how to make up problems that can be solved mathematically so that students become able to solve any problem.
- Practice all the various conversions of representation that can be involved in any verbal description of non-mathematical situations. Conversions required to solve a problem are not the same when starting from the description of a real situation or the equations that allow solving the problem of this situation.

What matters in introducing algebra at middle school is that students develop an awareness of specific cognitive operations required to put data into equations and to be able to recognize whenever equations can be used to solve real problems.

This analysis was developed in the theoretical framework of registers of semiotic representation. And it provides an application of this theory to an area of mathematics, where, unlike geometry, cognitive activity seemed reduced to pure mathematical operations! In a way, the two parts of the seminar form a whole. Nevertheless, we felt it important to publish them in two volumes.

This book can be read or used in four different ways.

We can do a more or less complete *linear reading* as with any other work. Just following the text, issues such as the justifications of the concepts and explanations of the distinctions necessary to analyze the cognitive functioning of mathematical thinking are introduced gradually. This reading allows us to understand the coherence of the entire line of thinking and to learn the internal relationships that form the theoretical framework of analysis. However, this reading retains only an overall impression that is often of little use, unless we do several local readings.

A cross and synoptic reading is also possible. We do not start from the text any longer, but from the terms that crystallize the strengths of the theoretical framework or analysis tool. Therefore, we enter into the text via the index terms. But, here, we must avoid the trap of the contents of dictionaries, hit list, and keywords. These indices isolate one term from another as if their meanings were not linked to each other. We should not consider a single term, but several and, if possible, a group of terms. The terms and pairs of terms are regrouped under a word, which condense them, to facilitate the cross reading. We only grasp the reality and the complexity of semiotic phenomena when we understand all the distinctions that characterize them.

This point is essential because it touches on a very frequent confusion between two types of conceptualization. One seeks primarily to have definitions, and even definitions presented with the help of some uppercase or lowercase letters, which we then apply to what we observe or even allow us to build a general model. In didactic research, these definitions relate almost always to crystallizer words and are limited to them. The other type describes the phenomena so that we will be able to identify the variables and the underlying processes that make their complexity. But we do not define a priori the relevant variables. To identify them, it is necessary to isolate them from systematic observations. And is it necessary to remember that the relevant variables we seek to identify are about the understanding/misunderstanding processes in mathematics learning? In this way of conceptualizing, the condensed words do not matter as much as the set of variations and distinctions they regroup in a descriptive network.

A practical reading is another possibility; it starts from examples and shows the way to analyze them. To facilitate this reading, we made a list of representations of objects and mathematical operations that are briefly presented here, but whose explanation can be found in other publications. The choice of the examples highlighted during the seminar has been guided by our experience of working with teachers and sometimes with teachers from different disciplines looking for an interdisciplinary collaboration. However, in this practical reading, we must never limit ourselves to one example, but consider at least two very different examples. A theoretical approach should allow researchers and teachers to analyze the understanding process in algebra as well as in geometry or calculus. In all these areas, the cognitive way of thinking and working is the same, and it has nothing in common with the one mobilized in geology, chemistry, or botany.

Finally, *we can look at it as a cartoon*. In this case, we start from all the figures that are part of the text. So we can start by looking at the two pictures that open Chaps. 1 and 2, respectively. They give an insight of the epistemological and cognitive issues raised by the analysis of knowledge and its formation: the necessary distinction between representation and object, the various kinds of representation, recognition of the represented objects, the conditions to get access to the objects, etc. Similarly, we can start by looking at eight small sequences of figures in Chap. 3 to become aware of the heterogeneity of figural operations underlying visualization in geometry. Here, we have a first spectral decomposition of the different ways of "seeing" required in any geometric activity. How do we expect to advance in the research about the teaching of geometry if we do not take into account all the figural operations that constitute the specific cognitive variables for geometric visualization? How can teachers understand the difficulties of their students if they are not aware of this cognitive complexity and, instead, adhere to a syncretic notion of an overall figure opposite to any drawing always particular?

There is obviously no difference among these different types of reading. We start as we want. But all of them are needed to see how the complexity of the studied phenomena leads to the development of a set of distinctions corresponding to experimentally isolable factors and to master the cognitive way of analyzing mathematical activity.

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http://www.springer.com/978-3-319-56909-3

Understanding the Mathematical Way of Thinking – The Registers of Semiotic Representations
Duval, R.
2017, XXIV, 117 p. 50 illus., 9 illus. in color., Hardcover
ISBN: 978-3-319-56909-3