Chapter 2
Parametrically Excited Dynamic Systems

We turn now to the statistical analysis of stochastic dynamic systems related to random parametric excitation in space and time. Such systems, appearing in many branches of physics, can be described by ordinary differential equations, as well as by partial differential equations. Stochastic structure formation for such systems in random media in the form of *clustering* is related to the parametric excitation of various physical fields in these media. *Clustering of a particular field implies the appearance of compact regions with large field values against the background of surrounding areas with relatively low field values*. Statistical averaging, expectedly, destroys all the information on clusters. Such challenges occur in fluid dynamics (*a passive scalar tracer in a turbulent flow*), in magnetohydrodynamics (*a passive vector tracer—magnetic field in a turbulent flow*), and in the propagation of waves of various origins (acoustic and radio waves, light and laser radiation) in random media. All these issues are commonly considered in the kinematic approximation and share the following two most important traits.

1. At fixed points in space, the field realizations in time are random processes which possess a specific character: they have the shape of peaks that appear at random instants of time. The intervals between them are characterized by low intensity and long duration. Such a realization of a random process in time for any location in space stems from the lognormal one-time distribution of probabilities, which has a slightly sloping 'tail'. The large but rare outliers (fluctuations) come from these tails. The main statistical characteristics of the processes being considered are the one-time probability density, one-time moment functions, typical realization curve characterizing the key features in the behavior of realizations of random processes, and Lyapunov exponent. In one-dimensional tasks described by ordinary differential equations with initial or boundary conditions, only such physical phenomena as *dynamic localization* can be observed in a number of cases (see Chap. 5).

2. The structure formation itself of a stochastic field takes place in physical space and is described through a related statistical analysis based on the ideas of statistical topography of a stochastic field. In the simplest problem formulation, under

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V.I. Klyatskin, *Fundamentals of Stochastic Nature Sciences*,
Understanding Complex Systems, DOI 10.1007/978-3-319-56922-2_2

statistical homogeneity in space, all one-point statistical characteristics of a random field are independent of spatial locations. Accordingly, the equation for the one-point probability density of a random field coincides in form with the equation for the probability density of a random process at each point in space, although the sense of these equations is substantially different. Accordingly, the statistical analysis of these equations should also be completely different.

A detailed discussion of these questions can be found in monographs [28, 29] and articles [30–33, 35].

First of all, a question arises as to whether or not such physical phenomena as localization and clustering appear in individual realizations of the processes and fields being considered, and if yes, then over which characteristic time (or on which spatial scales).

2.1 Lognormal Random Process

The phenomenon of structure formation in stochastic, parametrically excited dynamic systems on its own is well known in physics. For example, solutions of one-dimensional problems on parametric excitation, described by ordinary differential equations, are random processes.

The simplest dynamic system of that kind defines a lognormal random process $y(t;\alpha)$ described by a first-order ordinary stochastic differential equation:

$$\frac{d}{dt}y(t;\alpha) = \{-\alpha + z(t)\}y(t;\alpha), \;\; y(0;\alpha) = 1, \tag{2.1}$$

where $z(t)$ is a Gaussian random process of *white noise* with the parameters

$$\langle z(t)\rangle = 0, \quad B_z(t-t') = \left\langle z(t)z(t')\right\rangle = 2D\delta(t-t').$$

The solution to Eq. (2.1) takes the form

$$y(t;\alpha) = \exp\left\{-\alpha t + \int\limits_0^t d\tau z(\tau)\right\}. \tag{2.2}$$

It should be noted that the change in the sign of parameter α in Eq. (2.2) is statistically equivalent to passing to the process $1/y(t)$ [36].

Figure 2.1 presents realizations of the lognormal random process $y(t;\alpha)$ given by formula (2.2) for positive and negative values of the parameter α and $|\alpha|/D = 1$ (the dashed lines correspond to the functions $\exp\{-Dt\}$ for $\alpha > 0$, and $\exp\{Dt\}$ for $\alpha < 0$). The presence of rare, but strong, spikes (fluctuations) with respect to the dashed curves in the directions of both large and small values can be seen in Fig. 2.1.

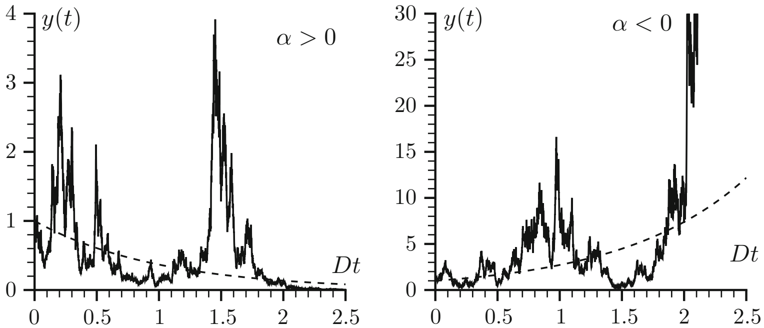

Fig. 2.1 Realizations of the lognormal process $y(t)$ for $\alpha > 0$, $\alpha < 0$ and $|\alpha|/D = 1$

This property of random processes is called *intermittency*; it was intensively studied in the 1980s (see, for example, Refs. [37, 38]). The curve with respect to which we identify the outliers (fluctuations) will be referred to as the *typical realization curve*.

2.2 Uncorrected Error of the Past Unavoidably Results in Errors of the Present and Future

Once you have missed the first buttonhole,
you will never manage to button up.

Johann Wolfgang von Goete

The authors of practically every one of the numerous articles exploring the properties of intermittency cite paper [38] when turning to the notion of 'intermittency'. The term *intermittency* on its own emerged in studies of the velocity field and temperature spots in turbulent media [39, 40] (as, for instance, stated in Ref. [38]). However, even at that time it was already well known that one-point distributions of velocity fields and temperature fluctuations are close to the Gaussian ones (see, for example, Ref. [41]). The term *intermittency* is certainly a telling one, and it characterizes temporal variability of a random field at a fixed location in space, i.e., the variability of a random process with respect to its mean value.

At that time, certainly, it was also known that stochastic instability (parametric excitation) could occur in dynamic systems as a consequence of fluctuations in the internal parameters of the system. However, for a long time, up to the 1980s, nobody took interest in these questions. The merit of the authors of Ref. [38] is that they, in all probability, were the first to draw attention to the possibility of stochastic structure formation as a consequence of such parametric excitation, which had been known at that time from various kinds of observations.

The abstract to paper [38] states, "The processes of instability in random media are characterized by formation of specific structures in which a growing quantity reaches record-high values. Despite the rareness of such concentrations namely they confine the main part of integral characteristics of a growing quantity (the mean value, the mean squared value, etc.). The appearance of such structures is referred to as the phenomenon of *intermittency*."

Further, on page 2 of this paper, one can read, 'Structures appeared in a random medium show peculiar features; they have the form of spikes (outliers) appeared at random locations and at random times. The inter-spike intervals are characterized by low intensity and long extent. The general name of such a pattern is *intermittency*'.

Thus, in Ref. [38] strong rare outliers (fluctuations) are termed *specific structures*, while the process proper whereby these structures (outliers or fluctuations) are formed is called the phenomenon of *intermittency*.

In our understanding, *intermittency* is a general property of all random processes, independent of the amplitude of possible fluctuations, while *structure formation constitutes a certain type of evolution of stochastic dynamic systems in space and time*.

Reference [38] treated these notions as identical. At present, for example, some scientists call large rare outliers (fluctuations) characteristic of both the stochastic linear and nonlinear Leontovich equations (see below) rogue (freak) waves (see, for example, the lectures by Zakharov [42], paper [43] and monograph [44]). A rogue wave is undoubtedly a phenomenon of spatio-temporal watermass clustering, and it should be considered on the basis of an appropriate statistical analysis of the evolution of random fields.

At the same time, p. 5 of paper [38] has the following text: 'The main sign of intermittency is namely an anomalous relationship between successive statistical moments (in comparison with the Gaussian one)'. Thus, this paper opposes the Gaussian random processes and fields to the parametrically excited ones. However, as we have seen in Sect. 1.1, stochastic structure formation can occur even in the Gaussian fields.

It should be noted that the statistical theory of stationary extremal statistical processes is an independent branch of probability theory (see, for example, review [45]). However, in our opinion, this area has nothing to do with stochastic structure formation in space and time.

A fundamental feature of stochastic dynamic systems described by partial differential equations is that their solutions comprise random fields in space and time. The difficulty in explaining processes of structure formation in these systems is related to two factors. First, at any fixed point in space, the random field constitutes a random process in time. Second, for any fixed instant of time, the random field represents a random process over its spatial coordinates. Intermittency (i.e., variability) occurs namely for random processes (with respect to time or spatial coordinate); it is a general property of any random process irrespective of the *nature of its origin*.

In this study, the intermittency of a random process is understood as a more or less uninterrupted alteration of outliers (fluctuations) of this process toward larger as well as lower values with respect to the deterministic curve—*the curve of typical* *realization*, which is the median of the integral probability distribution function (see Sect. 5.1). In this case, a lognormal, parametrically excited random process can exponentially decay with time in individual realizations (certainly, with some fluctuations), which corresponds to the phenomenon of *dynamic localization*. An exponential growth of a random process with time is also possible, which corresponds to the absence of dynamic localization. A peculiarity of a lognormal random process is the presence of rare anomalously high spikes (fluctuations) on the curves of the process (see Fig. 2.1), related to the long sloped 'tail' of the probability density (see Chap. 5). All traditional statistical characteristics, such as moment and correlation functions of arbitrary order, result from these fluctuations.

By introducing the notion of the typical realization curve for a random process, we return to the historical sense of the concept of intermittency, which is general for all random processes and has a strict probabilistic definition and a transparent physical meaning.

2.3 Oscillator with Randomly Varying Frequency (Stochastic Parametric Resonance)

A more complicated problem that cannot be solved in analytical form is the problem on *stochastic parametric resonance*. Such a system is described by the second-order equation

$$\begin{gathered}\frac{d^2}{dt^2}x(t)+\omega_0^2[1+z(t)]x(t)=0,\\ x(0)=x_0,\quad \frac{d}{dt}x(0)=v_0,\end{gathered}\tag{2.3}$$

where $z(t)$ is the random function of time. This equation is characteristic of almost all fields of physics. It is physically obvious that dynamic system (2.3) is capable of parametric excitation, because random process $z(t)$ has harmonic components of all frequencies, including frequencies $2\omega_0/n (n = 1, 2, \ldots)$ that exactly correspond to the frequencies of parametric resonance in the system with periodic function $z(t)$, as, for example, in the case of the *Mathieu equation*.

This problem will be statistically analyzed in Chap. 6.

http://www.springer.com/978-3-319-56921-5

Fundamentals of Stochastic Nature Sciences
Klyatskin, V.I.
2017, XII, 190 p. 62 illus., 11 illus. in color., Hardcover
ISBN: 978-3-319-56921-5