Simplicial Topological Coding and Homology of Spin Networks

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Abstract We study the commutation of the stabilizer generators embedded in the q-representation of higher dimensional simplicial complex. The specific geometric structure and topological characteristics of 1-simplex connectivity are generalized to higher dimensional structure of spin networks encoded in ordered complex via combinatorial optimization of a closed compact space. Obtained results of a consistent homology-chain basis are used to define connectivity and dynamical self organization of spin network system via continuous sequences of simplicial maps.

Keywords Spin network · Simplicial complex · Graph state · Combinatorial optimization · Quantum code

1 Introduction

Spin networks [1–6] can be presented by purely combinatorial structures: one-dimensional simplicial complexes with edges labeled by numbers $j = 0$, $1/2$, 1, $3/2$, etc. These numbers stand for total angular momentum or "spin". The imposed condition is that three edges meet at each vertex, with the corresponding spins: j_1, j_2, j_3, adding up to an even integer and satisfying the triangle inequality. These rules are motivated by the quantum properties of angular momentum: if we combine a component with spin j_1 and a component with spin j_2, the spin j_3 of the unit system satisfies exactly latter constraint. In such setting, given that $\mathbb{F}$ is a general field, a spin network represents quantum states of $\mathbb{F}$-geometry on $d = 3 + 1$ dimensional space defined by tensor product states

$$H_{j_1 j_2 j_3 j_4} \equiv \bigotimes_{i=1}^{4} H_{ji}, \;\; H_{\perp} \equiv \bigoplus_{\{J\}} H^0_{\{J\}}. \tag{1}$$

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I.S. Kotsireas and E. Martínez-Moro (eds.), *Applications of Computer Algebra*, Springer Proceedings in Mathematics & Statistics 198,
DOI 10.1007/978-3-319-56932-1_2

where $\{J\}$ runs over the set of ordered 4-tuples of integers or half-integers such that $H^0_{\{J\}}$ is nonempty complex obtained from the n-skeleton H^n, constructed from H^{n-1} by attaching n-simplexes via maps $\phi : K^{n-1} \to H^{n-1}$. In the PR model [1] a partition function is defined for a given three-dimensional simplicial complex [by deforming SU(2) to a quantum group [4], where the partition function depends only on the topology of the manifold which is triangulated by the simplicial complex] by means of the following: to each edge of the complex is associated a spin [i.e., an irreducible unitary SU(2) representation, determined only by its dimension $d \equiv 2j + 1$].

In particular case, we are interested in the homomorphisms of the simplicial q-th homology group which represents the free abelian group generated by the q-cycles, and their induced mapping on the stabilizer group (S_G) basis. Assuming that Γ and S_G are free abelian groups with bases $g_1, \ldots, g_n$ and $g'_1, \ldots, g'_m$, respectively, if $f : \Gamma \to S_G$ is a homomorphism, then $f\left(g_j\right) = \sum_{i=1}^{m} (-1)^i \lambda_{ij} g'_i$ for unique integers λ_{ij}, where the parity of any transposition is -1. More general, giving that K is a simplicial complex, and S_G is an abelian group, then for non-negative integer q, to each $(q+1)$-tuple $(x_0, x_1, \ldots, x_q)$ of vertices spanning a simplex $\sigma_q(K)$, there corresponds an element $\alpha(x_0, x_1, \ldots, x_q)$ of S_G defining a homomorphism $\alpha : C_q(K) \to S_G$, where $C_q(K)$ denotes the corresponding chain group, i.e., finitely generated abelian group on the oriented simplices.

This paper is organized as follows. After introducing basic concepts, in Sect. 3 we present a realization of the spin networks in terms of simplicial manifolds, associated with the properties of the fundamental groups. A distinctive feature of these groups is that they are topological invariant, i.e., topological spaces of the same homotopy description have the same fundamental group, and a loop differentiable property [7]. Details of the stabilizer formalism with the implementation to spin network unit on graph state are discussed in Sect. 4.

2 Preliminaries

2.1 *Simplicial Complexes*

Let $x_0, \ldots, x_q$ be points geometrically independent in $\mathbb{R}^m$ where $m \geqslant q$. The q-simplex $\sigma_q = \left\langle x_0, \ldots, x_q \right\rangle$ is a compact (bounded and closed) subset of $\mathbb{R}^m$, given by

$$\sigma_q = \left\{ v \in \mathbb{R}^m \,|\, v = \sum_{i=0}^{q} c_i x_i,\, c_i \geqslant 0,\, \sum_{i=0}^{q} c_i = 1 \right\}. \tag{2}$$

For an integer n such that $0 \leqslant n \leqslant q$, $n+1$ points define a n-simplex $\sigma_n = \left\langle x_{i_0}, \ldots, x_{i_n} \right\rangle$ denoted as n-face of σ_q. In particular, $\mathcal{K}$ represent a set of finite number of simplexes in $\mathbb{R}^m$ called simplicial complex [8, 9] if

- $\sigma \in \mathcal{K}$ and $\sigma' \leqslant \sigma$, then $\sigma' \in \mathcal{K}$.

- $\sigma, \sigma' \in \mathcal{K}$, then the intersection $\sigma \cap \sigma'$ is either empty set or a common face of σ and σ', i.e., either $\sigma \cap \sigma' = \emptyset$ or $\sigma \cap \sigma' \leqslant \sigma$, and $\sigma \cap \sigma' \leqslant \sigma'$.

Let $\sigma_q = \left[x_0, \ldots, x_q\right]$ $(q > 0)$ denote an oriented q-simplex, then the boundary $\partial_q \sigma_q$ of σ_q is an $(q-1)$-chain where ∂_q, called boundary operator, defines a homomorphism map $\partial_q : C_q(K) \to C_{q-1}(K)$. For K representing the n-dimensional simplicial complex, there exists a sequence of free Abelian groups and homomorphisms, called chain complex [10, 11]:

$0 \xrightarrow{i} C_n(K) \xrightarrow{\partial_n} C_{n-1}(K) \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} C_1(K) \xrightarrow{\partial_1} C_0(K) \xrightarrow{\partial_0} 0,$ where $i : 0 \to C_n(K)$.

Let $\mathcal{K}$ be a finite simplicial complex, where $|\mathcal{K}|$ represents the union of all the simplices $\sigma \in \mathcal{K}$. A topological space X which is homemeomorphic to $|\mathcal{K}|$ represents a polyhedron where $\mathcal{K}$ is a triangulation of X. If X and $K_q \subseteq \mathcal{K}$ are simplicial complexes, a morphism $\phi : \mathcal{K} \to K_q$ is a function $\phi : \mathcal{K}(\sigma_0) \to K_q(\sigma_0)$, where σ_0 denotes 0-simplexes or vertices, such that if $\sigma_q \in \mathcal{K}$ is a q-simplex spanned by the affinely independent set $x_0, \ldots, x_q$ of $(q+1)$ points, then the elements of the set $\phi(x_0), \ldots, \phi(x_q)$ form an affinely independent set of points spanning a simplex $\phi_\sigma \in K_q$, where $\dim \phi_\sigma \leq \dim \sigma$. In particular, a morphism $\phi : \mathcal{K} \to K_q$ of simplicial complexes for distinct elements $x_i \to \phi(x_i)$ determines a unique map of the simplex σ to ϕ_σ, by generating a piecewise-affine map of spaces $|\phi| : |\mathcal{K}| \to \left|K_q\right|$, where $|\cdot|$ is a functor from the category K of simplicial complexes to the category TOP of topological spaces. Considering X as a topological space and assigning a base point [12] $*p \in X$, a loop established at p is a path $\alpha : [0, 1] \to X$ with $\alpha(0) = \alpha(1) = p$. Then a map: $P : [0, 1]^2 \to X$ with $P(t, 0) = \alpha(t)$, $P(t, 1) = \beta(t)$ and $P(0, \tau) = P(1, \tau) = p$, $\forall (t, \tau) \in [0, 1]$ determines two homotopic loops α, β which can be deformed one from other via other loops on the set of common paths, defining an equivalence relation. The homotopy class of α loop is denoted as $[\alpha]$. In particular, two loops α, β are denoted with the homotopy classes $[\alpha][\beta] = \alpha * \beta$ for the path

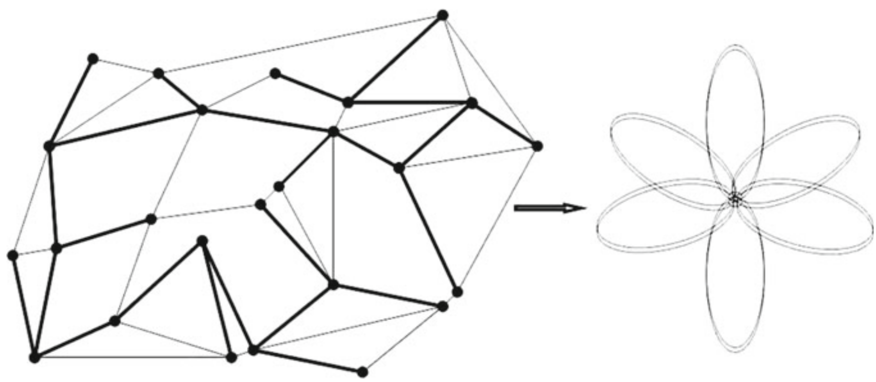

Fig. 1 Spin network represented via subgraph $X \in G$, is a maximal tree which is homotopy equivalent to a wedge of *circles*

that goes twice around α, then around β, such that $\alpha * \beta(t) = \alpha(2t), t \leqslant 1/2$ and $\beta(2t-1), t \leqslant 1/2$, see Fig. 1. right, which illustrates the wedge of six circles generated by gluing together a collection of spaces at a base point twice around loops α, β.

3 Homotopy of Spin Networks Embeded in Simplicial Complex

Let V be the vertex set and $e_1, e_2, \ldots, e_n$ be the sequence of edges on $V \times V$, connected along a path from a point a to a point b on the surface S, given by: $e_i = P_i P_{i+1}$, $P_1 = a$, $P_{n+1} = b$, where distinct edges possess orientation which coincides with the path direction. Then the path can be associated to the 1-chain: $e_1 + e_1 \cdots + e_n$. A linear transformation of the 1-chain module is associated by each group element action $g \in \Gamma$ which permutes the edges in either the successive mirror or the dual tiling, defining: $\alpha_1 e_1 + \alpha_2 e_2 + \cdots \alpha_n e_n \to \alpha_1 g e_1 + \alpha_2 g e_2 + \cdots \alpha_n g e_n$. In general, the Γ-action commutes with the boundary operator, i.e., $\partial g n = g \partial n$ for every chain, where

$$gZ_n(S;\mathbb{R}) = Z_n(S;\mathbb{R}) = \ker\partial_n : C_n(S;\mathbb{R}) \to C_{n-1}(S;\mathbb{R}),$$

$$gB_n(S;\mathbb{R}) = B_n(S;\mathbb{R}) = \mathrm{im}\partial_n : C_{n+1}(S;\mathbb{R}) \to C_n(S;\mathbb{R}),$$

resulting that distinct elements of Γ map homology classes to homology classes, yielding a linear action of Γ on $H_n(S;\mathbb{R}) = \frac{Z_n(S;\mathbb{R})}{B_n(S;\mathbb{R})}$. Then, a corresponding vertex set V represents a submodule for $V \subseteq H_n(S;\mathbb{R})$ which is Γ-invariant or a Γ-submodule if $gV = V,\ \forall g \in \Gamma$. Such action of group Γ on the homology chain is known as the homology representation.

Let Γ be a group, where $S \subseteq \Gamma$ is a generator subset. Let $\bar{S}$ be a set of inverses of S with $A = S \sqcup \bar{S}$. Then, an underlying graph [12, 13] of spin network $G = G(\Gamma, S)$ is

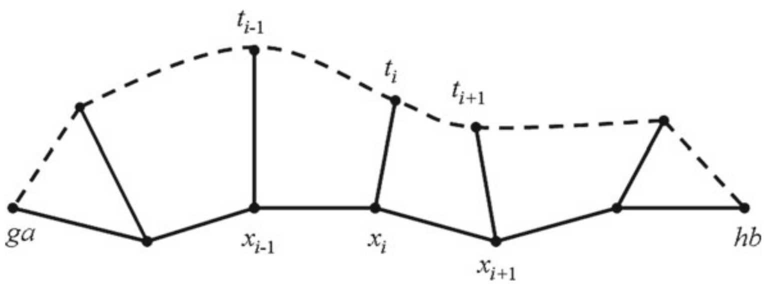

Fig. 2 Construction of a spin network by the union of the set of flat connections which can be defined over the multiply connected manifolds [14], given by unit intervals of a finite set of *curves* crosshatching only at their endpoints of the metric space [15]

established by connecting vertices $g, h \in V_G$, where the set $V_G \subset \Gamma$, by establishing edge in A under condition

$$(g, h \in V_G) = \begin{cases} 1, & if\ g^{-1}h \in A \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

That is, for distinct $g \in \Gamma$ and $a \in A$ there is an edge relating g to ga. In particular, the directed edge from g to ga is defined as the element a.

Given any $g, h \subseteq \Gamma$, let $\alpha \subseteq V_G$ be a geodesic connecting ga to a point hb, by selecting a sequence of points, $ga = x_0, x_1, \ldots, x_n = hb$, see Fig. 2, along α, such that $d(x_i, x_{i+1}) \leqslant 1, \forall i$. For each i, $g_i \in V_G$ are selected so that $\alpha : [a, b] \to [0, 1]$.

Definition 1 Given a metric space (M, d), let $I \subseteq \mathrm{R}$ be an interval. A path which denotes unit interval (geodesic) is
$\gamma : I \to \{M \mid d(\gamma(t), \gamma(\tau)) = |t - \tau|, \forall(t, \tau) \in I\}$.

Assuming $\gamma : [a, b] \to M$ is an arbitrary path, its length is represented as

$$\sup \left\{ \sum_{i=1}^{n} d\ (\gamma(t_{i-1}), \gamma(t_i)) |\, a = t_0 < t_1 < \cdots < t_n = b \right\}. \tag{4}$$

Theorem 1 *Let X_σ and $X_{\sigma'}$ be subspaces of X such that the covering dimension of simplexes σ, σ' is maximal covering of X. Let $\gamma_i : X_{\sigma\sigma'} \to X_\gamma$ and $\gamma_j : X_\gamma \to X$ be the inclusions, resulting that $h_\gamma : \Pi\left(X_\gamma\right) \to \Lambda$ are functors into a groupoid defining commutativity relation $h_{\sigma'}\Pi(i_{\sigma'}) = h_\sigma \Pi(i_\sigma)$, i.e., a different path γ gives the same result, where a unique functor $\lambda : \Pi(X) \to \Lambda$ is defined such that $h_{\sigma'} = \lambda\Pi(j_{\sigma'}),\ h_\sigma = \lambda\Pi(j_\sigma)$ as*

$$\begin{array}{ccc} \Pi(X_{\sigma\sigma'}) & \xrightarrow{\Pi(i_\sigma)} & \Pi(X_\sigma) \\ \Pi(i_{\sigma'}) \Big\downarrow & & \Big\downarrow \Pi(j_\sigma) \\ \Pi(X_{\sigma'}) & \xrightarrow{\Pi\left(j_{\sigma'}\right)} & \Pi(X) \end{array} \tag{5}$$

is a pushout in groupoid of the inclusions $X_\sigma \supset X_{\sigma\sigma'} \subset X_{\sigma'}$.

Proof In particular, a path $\gamma : [a, b] \to X$ represents a morphism $[\gamma]$ in $\Pi\left(X_\gamma\right)$ from $\gamma(a)$ to $\gamma(b)$ if we arrange it with an increasing homeomorphism $\alpha : [a, b] \to [0, 1]$. If $a = t_0 < t_1 < \cdots < t_n = b$ then γ establishes the composition of the morphisms $\left[\gamma \left| \left[t_i, t_{i+1}\right]\right.\right]$. Let $\gamma : I \to X$ be a path and let $\sigma : \{0, \ldots, n\} \to \{0, 1\}$ $|\gamma\left(\left[t_i, t_{i+1}\right]\right) \subset X(\sigma_i)$, then there exists a decomposition in affine space: $0 = t_0 < t_1 < \cdots < t_{n+1} = 1$ such that: $\left[\gamma \left| \left[t_i, t_{i+1}\right]\right.\right] \subset X(\sigma_i),\ i = 0, \ldots, n$. The construction $\left[\gamma \left| \left[t_i, t_{i+1}\right]\right.\right]$ as path γ_i in X_{γ_i}, produces composition

$$[\gamma] = \Pi\left(\sigma_{\gamma(n)}\right)[\gamma_n] \circ \cdots \circ \Pi\left(\sigma_{\gamma(0)}\right)[\gamma_0]. \tag{6}$$

If subdivision λ exists, then $\lambda[\gamma] = h_{\gamma(n)}[\gamma_n] \circ \cdots \circ h_{\gamma(0)}[\gamma_0]$ is inclined by the homotopy composition of the path.
Let $h : \mathcal{K} \to X$ be a homotopy of paths from a to b. We consider edge-paths in the subsets of 3-simplex ($\mathcal{K}_3$) which path-connect coordinates $a = (000)$ and $b = (111)$, see Fig. 3. These paths differ from $h_{\gamma(0)}$ and $h_{\gamma(1)}$ by composition with a constant interval. h generates two paths in $\mathcal{K}$, which give the same result since they differ by a homotopy on subinterval which belongs to the subsets $\sigma_i \in \mathcal{K}_3,\ i = 1, \ldots, 4$. $\square$

Given a topological space X representing the union of subsets $X_\sigma, X_{\sigma'}$, general properties of X encompassed from those of $X_\sigma, X_{\sigma'}$, and $X_{\sigma\sigma'} = X_\sigma \cap X_{\sigma'}$ can be inferred from the Theorem 2.

Theorem 2 [15, 16]. *Let K_0 and K_1 be subspaces of simplicial complex $\mathcal{K}$ such that the maximal dimension simplexes $\sigma_0 \in K_0,\ \sigma_1 \in K_1$, represent covering of X. Considering $\gamma_i : K_{01} = K_0 \cap K_1 \to K_\gamma$ and $\gamma_j : K_\gamma \to X$ as inclusions, in particular, let K_0, K_1, K_{01} be path connected with base $* \in K_{01}$. Then Eq. (7)*

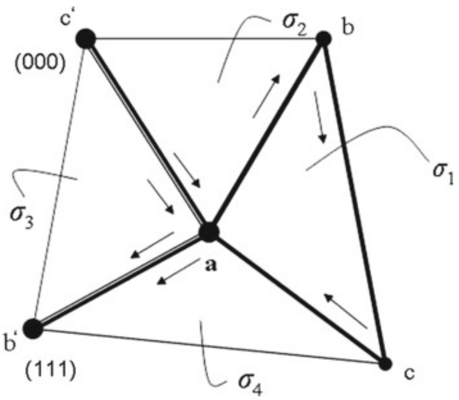

Fig. 3 Two different paths along arrows (marked by *thin* and *thick black lines*) induce the following stabilizer generator sets on a base (a face) which belongs to incident simplexes (see Theorem 2 and Sect. 4):

$$\begin{aligned}
\sigma_1 \cap \sigma_2 &= \{\{a, b\}\} \to \{|000\rangle, |001\rangle, |110\rangle, -|111\rangle\},\\
\sigma_4 \cap \sigma_1 &= \{\{c, a\}\} \to \{|000\rangle, |101\rangle, |010\rangle, -|111\rangle\},\\
\sigma_3 \cap \sigma_4 &= \left\{\left\{a, b'\right\}\right\} \to \{|000\rangle, |110\rangle, |001\rangle, -|111\rangle\},\\
\sigma_2 \cap \sigma_3 &= \left\{\left\{c', a\right\}\right\} \to \{|000\rangle, |010\rangle, |101\rangle, -|111\rangle\}.
\end{aligned}$$

$$\begin{array}{ccc} \pi_1(K_{01},*) & \xrightarrow{\pi(i_{1*})} & \pi_1(K_1,*) \\ \pi_1(i_{0*})\Big\downarrow & & \Big\downarrow\pi_1(j_{1*}) \\ \pi_1(K_0,*) & \xrightarrow{\pi_1(j_{0*})} & \pi_1(X,*) \end{array} \tag{7}$$

is a pushout in topological space of the inclusions $K_0 \supset K_{01} \subset K_1$, representing a fundamental group.

Proof Assuming that simplicial complex $\mathcal{K}$ is path connected and $z \in \mathcal{K}$, where $z = *$, then $r : \Pi(\mathcal{K}) \to \pi_1(\mathcal{K}, z)$ induces morphism compositions over the full subset z. For each $z \in \mathcal{K}$ exists a morphism such that $u_z = \mathrm{id}$, $u_y \alpha u_x^{-1}$ where α: $x \to y$, represented by:

$$\begin{array}{ccccc} \Pi(K_0) & \longleftarrow & \Pi(K_{01}) & \longrightarrow & \Pi(K_1) \\ \Big\downarrow r_1 & & \Big\downarrow r_{01} & & \Big\downarrow r_0 \\ \pi_1(K_0,*) & \leftarrow & \pi_1(K_{01},*) & \rightarrow & \pi_1(K_1,*). \end{array} \tag{8}$$

Precisely, restriction of $\mathcal{K}$ to subcomplexes: K_{01}, K_0, K_1, and X with a base point $z = *$, yields a commutative relation where morphisms in $\Pi(X)$ are respectively assigned by the composition of morphisms in $\Pi(K_0)$ and $\Pi(K_1)$, likewise, the group $\pi_1(X,*)$ is formed by the images of j_0* and j_1*. □

4 Application to Graph State and Spin Network

Graph state is represented in scope of the stabilizer formalism [17, 18] via tensor products of Pauli operators σ_X and σ_Z, whose composition and structure are based on the complexity of the underlying graph which can be seen as one-dimensional simplicial complex. The stabilizers establish a group (S_G) under multiplication, formed from n generators g_i, associated to a number of vertices x_i of the graph [19]. In particular, stabilizer generators are induced on the vertex set V_G of a graph G by the bijective mapping $(\Gamma(V_G), A) \to (S_G, \cdot)$, see Sect. 3, Eq. (3). Graph state is obtained by relating each vertex $x_i \in V_G$ with a stabilizer generator $g_i = \sigma_X^i \sigma_Z^{ij}$, where $g_i \left|G\right\rangle = \left|G\right\rangle$, $\forall i = 1, \ldots, n$. The stabilizer generators [20–22] g_i for n graph state generate the complete Abelian stabilizer group S_G of $\left|G\right\rangle$ with multiplication. The group S_G consists of 2^n elements which uniquely represent a graph state

$$\left|G\right\rangle = \left\{ \sum_{i=1}^{2^n} \alpha_i \left|x_i\right\rangle = \sum_i \alpha_i S_G^i \left|x_i\right\rangle, \sum_i \left|\alpha_i\right|^2 = 1 \right\}. \tag{9}$$

The stabilizer group S_G is formed from a set of $n-k$ generators $g_1, \ldots, g_{n-k}$, which: (a) commute; (b) are unitary and Hermitian; and (c) $g_i^2 = I$. Each element of the stabilizer group S_G can be expressed as a product of the generators as $S_k = g_1^{\alpha_1} \cdots g_{n-k}^{\alpha_{n-k}},\ S_k \in S_G,\ \alpha_i \in \{0, 1\}, i = 1, \ldots, n-k$, where $S \subseteq G_n$ with G_n denoting a corresponding Pauli group for n qubit state.

Definition 2 Stabilizer code of length n is represented by the fixed point set [23] $S_k = \{I, X, Y, Z\}$ of Pauli operators:

$$I = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ X \equiv \sigma_X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ Y \equiv \sigma_Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ Z \equiv \sigma_Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$k = 1 \ldots, n$ such that $S_1, S_2, \ldots, S_k$ are acting over n qubits (i.e., over $\left(\mathbb{C}^2\right)^{\otimes n}$).

When stabilizers S are composed of elements $\{\sigma_i\}_{i=X,Y,Z}$ of $\{I, X\}^{\otimes n}$ and $\{\sigma_i\}_{i=X,Y,Z}$ of $\{I, Z\}^{\otimes n}$, it can be seen that $\left[\sigma_i \sigma_j\right] = 2i\varepsilon_{ijk}\sigma_k$ and $\left\{\sigma_i \sigma_j\right\} = 2\delta_{ij}$. Precisely, I represents the identity matrix of size 2, X denotes the Pauli matrix encoding the bit flip error and Z denotes the Pauli matrix describing the phase error. The isomorphisms between $\{I, X\}, \{I, Z\}$ and the vector space $\mathbb{F}_2$ makes possible establishing a connection between classical and quantum codes. On the basis of these isomorphisms, the stabilizers relate to binary vectors and the commutation relation corresponds to the orthogonality relation in $\mathbb{F}_2^n$.

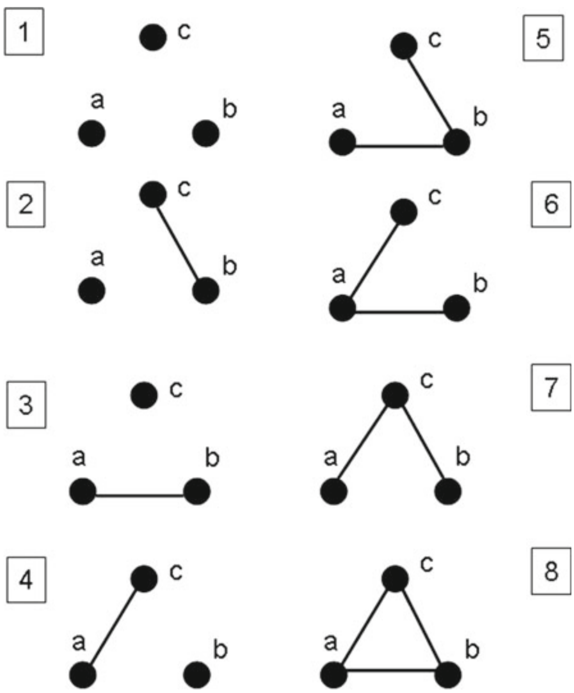

Fig. 4 Stabilizer generators for three-partite graph states representing elementary segment of spin network, see Eqs. (10, 11)

In particular, stabilizers of the graph state, given in Fig. 4, are represented by each row of the binary matrix [24, 25]

$$\left(\begin{array}{ccc|ccc} 0&0&0&0&0&0 \\ 1&0&0&0&0&1 \\ 0&1&0&0&0&1 \\ 0&0&1&1&1&0 \\ 1&1&0&0&0&0 \\ 1&0&1&1&1&1 \\ 0&1&1&1&1&1 \\ 1&1&1&1&1&0 \end{array}\right) \tag{10}$$

where nodes of element of the spin network $\{a, b, c\}$, where: $a + b \geqslant c = 2k,\ a + c \geqslant b = 2k,\ b + c \geqslant a = 2k$ are encoded in the graph state ($n = 3$) establishing incidence relations via following generators:

$$\begin{aligned}
&(1)\ \{\{a\}, \{b\}, \{c\}\} \rightarrow \{|000\rangle\},\\
&(2)\ \{\{a\}\} \rightarrow \{|000\rangle, |001\rangle, |010\rangle, -|011\rangle\},\\
&(3)\ \{\{b\}\} \rightarrow \{|000\rangle, |100\rangle, |001\rangle, -|101\rangle\},\\
&(4)\ \{\{c\}\} \rightarrow \{|000\rangle, |010\rangle, |100\rangle, -|110\rangle\},\\
&(5)\ \{\{a, c\}\} \rightarrow \{|000\rangle, |010\rangle, |101\rangle, |111\rangle\},\\
&(6)\ \{\{b, c\}\} \rightarrow \{|000\rangle, |100\rangle, |011\rangle, |111\rangle\},\\
&(7)\ \{\{a, b\}\} \rightarrow \{|000\rangle, |001\rangle, |110\rangle, |111\rangle\},\\
&(8)\ \{\{a, b, c\}\} \rightarrow \{|100\rangle, |010\rangle, |001\rangle, -|111\rangle\},
\end{aligned} \tag{11}$$

where (5–7) represent standard three-qubit flip code on the code subspace: $V_S = \{|000\rangle, |111\rangle\}$ for stabilizer set $S = \{I, Z_1Z_2, Z_2Z_3, Z_1Z_3\}, I = (Z_1Z_2)^2$.

5 Conclusion

We have analyzed and demonstrated implementation of graph states in composing the spin networks architectures. The characterization of graph states is utilized via the underlying graph construction defined in terms of affine simplexes with respect to path-connection induced homeomorphisms and polytope construction herein. Future outlook is implementation of higher dimensional homologies in order to establish a self-correcting memory which allows secure data processing without continual active error correction via stabilizer measurement.

Acknowledgements Author acknowledges the sponsorship provided through FP7 EU Commission framework.

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Applications of Computer Algebra
Kalamata, Greece, July 20–23 2015
Kotsireas, I.S.; Martínez-Moro, E. (Eds.)
2017, XIV, 515 p. 126 illus., 73 illus. in color., Hardcover
ISBN: 978-3-319-56930-7