Chapter 2
Time Encoding and Decoding in Bandlimited and Shift-Invariant Spaces

Abstract The IF neuron represents one of the most common models for describing the behaviour of the spiking neurons. The IF model belongs to the more general class of time encoding machines (TEM). Using mathematical tools from nonuniform sampling theory, it was proven that, the input of a TEM belonging to a bandlimited space can be perfectly reconstructed from the corresponding output sequence of spike times. This result was subsequently generalised for inputs belonging to the more general shift-invariant spaces. All the state-of-the-art reconstruction algorithms for TEMs are studied in a unifying manner with the reconstruction algorithms from nonuniform sampling theory. In this case, the sampling times are different for every input, and thus the algorithms become computationally demanding when processing a large number of inputs. This chapter reviews the nonuniform sampling theory and the state-of-the-art mathematical formulation describing the encoding and decoding for TEMs, for inputs belonging to bandlimited as well as shift-invariant spaces.

2.1 Introduction

A fundamental problem in information processing is representing a continuous function as a discrete sequence of values. This problem was originally addressed in uniform sampling theory, pioneered by Shannon (1949) and Kotelnikov (1933), and nonuniform sampling theory, in the works of Beurling and Malliavin (1967) and Landau (1967).

The sampling methods based on the theories above record the amplitude value of a function at predefined time locations. A dual method, sampling based on timing, records the time location when the amplitude of a function, or an operator applied to the function, exceeds a threshold value (Gontier and Vetterli 2014). The TEM, which performs sampling based on timing, has first been defined by Lazar and Tóth (2003) as a real-time asynchronous mechanism that encodes the amplitude information of a function into a time sequence.

The TEMs have been used as replacements for classical analog-to-digital converters in signal processing applications such as brain machine interface (BMI) prototypes (Bashirullah et al. 2007) for encoding neural recordings, and human area

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network (HAN) prototypes for encoding biomonitoring information (Káldy et al. 2007). In both cases it was shown that the TEMs represent encoding devices with lower power and higher resolution than the classical analog-to-digital converters.

In neuroscience, the TEMs have been used for describing neuron models. One example is the integrate-and-fire (IF) neuron, which represents one of the most common models of the spiking neuron (Lapicque 1907; Tuckwell 1988). Lazar and Pnevmatikakis (2008b) have developed conditions under which a bandlimited input of the IF neuron can be reconstructed perfectly. The result has been extended to other TEMs, such as populations of IF neurons (Lazar and Pnevmatikakis 2008b; Lazar and Slutskiy 2015), IF neurons with refractory period (Lazar 2004), leaky IF (LIF) neurons (Lazar 2005), Hodgkin-Huxley neurons (Lazar 2010) and asynchronous sigma-delta modulators (ASDMs) (Lazar and Tóth 2004a).

In many practical applications, the space of bandlimited functions is considered too restrictive. Under certain conditions, functions belonging to the more general shift-invariant spaces (SIS) can be perfectly reconstructed from uniform (Aldroubi et al. 1994) as well as nonuniform samples (Aldroubi and Feichtinger 1998; Aldroubi and Gröchenig 2001). Furthermore, Gontier and Vetterli (2014) have provided sufficient conditions for reconstructing the input of an IF neuron belonging to a SIS from the associated output sequence.

All the above reconstruction algorithms for TEMs share an important drawback. They are studied in a unifying manner with the reconstruction algorithms from nonuniform sampling theory and, in this case, the time locations of the corresponding nonuniform sampling times are input dependent, and thus different for every reconstruction. In practical applications, this causes significantly higher computational complexity than the reconstruction algorithms from uniform samples.

This chapter reviews nonuniform sampling theory in Sect. 2.1. The reconstruction algorithms for TEMs are presented in Sect. 2.2 for bandlimited spaces and in Sect. 2.3 for shift-invariant spaces. Conclusions are in Sect. 2.4.

2.2 Nonuniform Sampling and Reconstruction of Bandlimited Functions

Uniform sampling is studied in a unifying manner with the theory of harmonic Fourier series, as demonstrated by Shannon (1949). If the samples are not uniform in time, the reconstruction problem can be addressed using tools from frame theory (Christensen 2003) and theory of nonharmonic Fourier series (Young 1980). Feichtinger and Gröchenig (1994) proposed several iterative algorithms for reconstructing functions efficiently from their nonuniform samples.

In the classical sampling theory, a function f can be reconstructed from the sequence of uniform samples $\{f(kT)\}_{k\in\mathbb{Z}}$, if $f \in PW_\Omega$ and $T = \frac{\pi}{\Omega}$, where PW_Ω is the Paley-Wiener space of bandwidth Ω

$$PW_{\Omega} = \left\{u \in \mathscr{L}^2(\mathbb{R}) : \operatorname{supp}(\widehat{u}) \subseteq [-\Omega, \Omega]\right\}, \tag{2.1}$$

where $\widehat{u}$ denotes the Fourier transform of function u and $\operatorname{supp}(\widehat{u})$ denotes the support of $\widehat{u}$. The space PW_{Ω} is endowed with norm $\|\cdot\|_{L^2}$ and inner product $\langle\cdot\rangle_{L^2}$. Then f can be reconstructed as (Shannon 1949)

$$f(t) = \sum_{k\in\mathbb{Z}} f(kT) \cdot \operatorname{sinc}(\Omega(t - kT)).$$

The set of functions $\left\{\sqrt{\frac{\Omega}{\pi}} \cdot \operatorname{sinc}(\Omega(\cdot - kT))\right\}_{k\in\mathbb{Z}}$ forms an orthonormal basis on PW_{Ω}, which is equivalent to the fact that $\left\{\sqrt{\frac{\pi}{\Omega}} \cdot e^{-ikT\cdot}\right\}_{k\in\mathbb{Z}}$ forms an orthonormal basis on $L^2[-\Omega, \Omega]$ (Naylor and Sell 1982), also known as the harmonic Fourier basis (Aldroubi and Gröchenig 2001), where

$$L^2[-\Omega, \Omega] \triangleq \left\{f : [-\Omega, \Omega] \to \mathbb{C} : \int_{-\infty}^{\infty} |f(\omega)|^2 d\omega < \infty\right\}.$$

The following definition was presented in (Gröchenig and Razafinjatovo 1996) for characterising a nonuniform sequence of sampling times $\{x_k\}_{k\in\mathbb{Z}}$.

Definition 2.1 A sequence $\{x_k\}_{k\in\mathbb{Z}}$ is called a set of sampling for PW_{Ω} if there exist $A, B > 0$ such that

$$A\|f\|_{L^2}^2 \leq \sum_{k\in\mathbb{Z}} |f(x_k)|^2 \leq B\|f\|_{L^2}^2, \forall f \in PW_{\Omega}.$$

This is also equivalent to the fact that $\{g_{\Omega}(\cdot - x_k)\}_{k\in\mathbb{Z}}$ is a frame on PW_{Ω}, where $g_{\Omega} = \frac{sin(\Omega\cdot)}{\pi\cdot}$ is the reproducing kernel of PW_{Ω}. If this is true, then any function $f \in PW_{\Omega}$ can be reconstructed from its samples at points $\{x_k\}_{k\in\mathbb{Z}}$, which satisfy

$$f(x_k) = \langle f, g_{\Omega}(\cdot - x_k)\rangle_{L^2}, \forall k \in \mathbb{Z}.$$

Lazar and Pnevmatikakis (2008b) have proven that $\{g_{\Omega}(\cdot - x_k)\}_{k\in\mathbb{Z}}$ is a frame on PW_{Ω} if and only if $\left\{e^{-ix_k\cdot}\right\}_{k\in\mathbb{Z}}$ is a frame for $L^2[-\Omega, \Omega]$, i.e.,

$$A\|f\|_{L_{\Omega}^2}^2 \leq \sum_{k\in\mathbb{Z}} |\langle f, e^{-ix_k\cdot}\rangle_{L_{\Omega}^2}|^2 \leq B\|f\|_{L_{\Omega}^2}^2, \forall f \in L^2[-\Omega, \Omega], \tag{2.2}$$

where $\langle f_1, f_2\rangle_{L_{\Omega}^2} \triangleq \frac{1}{2\pi}\int_{-\Omega}^{\Omega} f_1(\omega) f_2^*(\omega) d\omega$, $\|f\|_{L_{\Omega}^2} \triangleq \sqrt{\langle f, f\rangle_{L_{\Omega}^2}}$ are the inner product and norm on $L^2[-\Omega, \Omega]$, respectively, and f_2^* denotes the conjugate of complex function f_2. By substituting f with f^* in (2.2) and using the properties of the inner product, it follows that this is also equivalent to the fact that $\left\{e^{ix_k\cdot}\right\}_{k\in\mathbb{Z}}$ is a frame on $L^2[-\Omega, \Omega]$. The latter is one of the main problems of interest in the theory of nonharmonic Fourier series (Young 1980). The main results that establish conditions for the existence of frames on $L^2[-\Omega, \Omega]$ are presented as follows. The review is limited to countable sequences of reals $\{x_k\}_{k\in\mathbb{Z}}$.

Definition 2.2 Sequence $\{x_k\}_{k\in\mathbb{Z}}$ is called relatively separated if $\exists\zeta > 0$ such that $|x_n - x_k| > \zeta, \forall n, k \in \mathbb{Z}, n \neq k$.

Definition 2.3 A relatively separated sequence $X = \{x_k\}_{k\in\mathbb{Z}}$ is uniformly dense with uniform density $d(X)$ if

$$\exists L > 0, \forall k \in \mathbb{Z}, \left| x_k - \frac{k}{d(X)} \right| \leq L.$$

The following theorem was proven by Duffin and Schaeffer (1952).

Theorem 2.1 (Duffin and Schaeffer) *Let* $X = \{x_k\}_{k\in\mathbb{Z}}$ *be a uniformly dense sequence with uniform density* $d(X)$*. Then* $\left\{e^{ix_k\cdot}\right\}_{k\in\mathbb{Z}}$ *is a frame for* $L^2[-\Omega, \Omega]$ *provided that* $0 < \frac{\Omega}{\pi} < d(X)$.

For relatively separated sequences that are not uniformly dense, Landau (1967) established the following condition for the existence of a frame on $L^2[-\Omega, \Omega]$.

Theorem 2.2 (Landau) *Let* $X = \{x_k\}_{k\in\mathbb{Z}}$ *be a relatively separated sequence. Then the lower uniform density of* X*, defined as*

$$D^-(X) \triangleq \lim_{r\to\infty} \frac{\inf_{x\in\mathbb{R}} \operatorname{card}\left(X \cap [x, x+r]\right)}{r},$$

always exists. Moreover, if $D^-(X) > \frac{\Omega}{\pi}$*, then* $\left\{e^{ix_k\cdot}\right\}_{k\in\mathbb{Z}}$ *is a frame for* $L^2[-\Omega, \Omega]$.

The lower density of a relatively separated sequence X that is uniformly dense is $D^-(X) = d(X)$ (Jaffard 1991). The following is an alternative definition for the density of a sequence.

Definition 2.4 Sequence $\{x_k\}_{k\in\mathbb{Z}}$ is called Δ- dense if $|x_{k+1} - x_k| \leq \Delta, \forall k \in \mathbb{Z}$.

A Δ-dense sequence $X = \{x_k\}_{k\in\mathbb{Z}}$ that is relatively separated has a lower density that is greater or equal to that of $Y = \{k\Delta\}_{k\in\mathbb{Z}}$, for which $D^-(Y) = \frac{1}{\Delta}$. It follows that Theorem 2.2 can be applied to sequence X provided that $\Delta < \frac{\pi}{\Omega}$, which was proven explicitly in the following theorem (Benedetto 1992).

Theorem 2.3 (Benedetto) *Let* $\{x_k\}_{k\in\mathbb{Z}}$ *be a strictly increasing, relatively separated,* Δ*-dense sequence. Then* $\left\{e^{ix_k\cdot}\right\}_{k\in\mathbb{Z}}$ *is a frame for* $L^2[-\Omega, \Omega]$ *provided that* $\Delta < \frac{\pi}{\Omega}$.

A function can be reconstructed from its nonuniform samples $\{x_k\}_{k\in\mathbb{Z}}$, provided that $\{g_\Omega(\cdot - x_k)\}_{k\in\mathbb{Z}}$ is a frame on PW_Ω, using the frame expansion or the dual frame expansion presented in Appendix A. In practice, this involves calculating the dual frame, which is a challenging task for generic nonuniform sequences.

Feichtinger and Gröchenig (1994) have developed an efficient iterative algorithm for reconstructing a function from its nonuniform samples with arbitrary accuracy.

Theorem 2.4 *Let $\{x_k\}_{k\in\mathbb{Z}}$ be a Δ-dense sequence and let*

$$\mathscr{V} : PW_\Omega \to PW_\Omega, \mathscr{V} f = \mathscr{P}_{PW_\Omega}\left(\sum_{k\in\mathbb{Z}} f(x_k)1_{[y_{k-1},y_k[}\right),$$

where $\mathscr{P}_{PW_\Omega} : L^2(\mathbb{R}) \to PW_\Omega$ is the orthogonal projection operator on PW_Ω, $y_k \triangleq \frac{x_k+x_{k+1}}{2}$, and $1_{[y_{k-1},y_k[}$ is the characteristic function of interval $[y_{k-1}, y_k[, \forall k \in \mathbb{Z}$. Provided that $\Delta < \frac{\pi}{\Omega}$, any function $f \in PW_\Omega$ can be reconstructed iteratively from its samples $\{f(x_k)\}_{k\in\mathbb{Z}}$ as follows

$$\begin{aligned} f_0 &= \mathscr{V} f \\ f_{n+1} &= f_n + \mathscr{V}(f - f_n), \forall n \geq 0. \end{aligned}$$

Then $\lim_{n\to\infty} f_n = f$ *and*

$$\|f - f_n\|_{L^2} \leq \left(\frac{\Delta\Omega}{\pi}\right)^{n+1} \|f\|_{L^2}, \forall n \in \mathbb{N}.$$

2.3 Time Encoding and Decoding in Bandlimited Spaces

The review in this chapter is focused on the time encoding machine (TEM), which is a mechanism performing sampling based on timing, the dual of nonuniform sampling. The TEM is defined mathematically as an operator $\mathscr{T} : PW_\Omega \to \mathbb{R}^\mathbb{Z}$ that maps a function $u \in PW_\Omega$ to a strictly increasing sequence of reals $\mathscr{T} u = \{t_k\}_{k\in\mathbb{Z}}$ satisfying $\lim_{k\to\pm\infty} t_k = \pm\infty$. If operator $\mathscr{T}$ is invertible, then $\mathscr{T}^{-1} : \mathbb{R}^\mathbb{Z} \to PW_\Omega$ is called a time decoding machine (TDM).

The TEMs arise in neuroscience as models of spiking neurons, called integrate-and-fire (IF) neurons (Dayan and Abbott 2003). Reconstruction algorithms have been designed for the bandlimited inputs of linear (Lazar and Pnevmatikakis 2008b) as well as nonlinear (Lazar and Slutskiy 2015) filter banks in cascade with ensembles of ideal IF neurons, ideal IF neurons with refractory period (Lazar 2004), leaky IF (LIF) neurons (Lazar 2005), populations of LIF neurons with random thresholds (Lazar et al. 2010), as well as populations of Hodgkin-Huxley neurons (Lazar 2007, 2010). Florescu and Coca (2015) have introduced a new reconstruction algorithm for ideal IF neurons that redefines IF time encoding as a uniform sampling problem.

The asynchronous sigma-delta modulator (ASDM) (Roza 1997), an alternative to the common analog to digital converters, is an example of a nonlinear circuit that can be modelled with a TEM. Lazar and Tóth (2004a) have developed an iterative algorithm that reconstructs the bandlimited input of an ASDM with arbitrary accuracy from its output sequence and demonstrated the strong relationship between TEMs and nonuniform sampling.

This section reviews the reconstruction algorithms developed for the above TEMs for bandlimited stimuli.

2.3.1 The Ideal IF Neuron

The ideal IF neuron, depicted in Fig. 2.1, consists of an adder and an ideal integrator. Each time the integrator output reaches a threshold value δ, the neuron fires and the integrator is reset.

Definition 2.5 An ideal IF neuron generates output spike sequence $\{t_k\}_{k\in\mathbb{Z}}$ when presented with input u, satisfying $|u| \le c < b$, such that

$$\int_{t_k}^{t_{k+1}} u(\tau)d\tau = C\delta - b(t_{k+1} - t_k), \forall k \in \mathbb{Z}, \tag{2.3}$$

where δ, C, and b represent the threshold, integration constant, and bias, respectively. Without reducing the generality, it is assumed that $t_1 = 0$.

Equation 2.3 is known as the t-transform of the ideal IF neuron. Lazar and Pnevmatikakis (2008b) have proven that for any input $u \in PW_\Omega$ the t- transform can be expressed as

$$\mathscr{L}_k^{\mathbb{T}_u} u = \left\langle u, \phi_k^{\mathbb{T}_u} \right\rangle_{L^2} = q_k, \forall k \in \mathbb{Z}, \tag{2.4}$$

where $\phi_k^{\mathbb{T}_u} \triangleq g_\Omega * 1_{[t_k, t_{k+1}[}$, and $1_{[t_k, t_{k+1}[}$ is the characteristic function of interval $[t_k, t_{k+1}[, \forall k \in \mathbb{Z}$.

Therefore, the t - transform represents the orthogonal projection of u on functions $\{\phi_k^{\mathbb{T}_u}\}_{k\in\mathbb{Z}}$. As in nonuniform sampling theory, where the samples at points $\{x_k\}_{k\in\mathbb{Z}}$ represent the projection of a continuous function on $\{g_\Omega(\cdot - x_k)\}_{k\in\mathbb{Z}}$, an important problem in encoding with ideal IF neurons is establishing conditions for which $\{\phi_k^{\mathbb{T}_u}\}_{k\in\mathbb{Z}}$ forms a frame on PW_Ω. The following lemma addresses this issue (Lazar and Pnevmatikakis 2008b).

Lemma 2.1 *Let* $\mathbb{T} = \{t_k\}_{k\in\mathbb{Z}}$ *be a relatively separated sequence of reals. Then sequences* $\{g_\Omega(\cdot - s_k)\}_{k\in\mathbb{Z}}$ *and* $\{\phi_k^{\mathbb{T}}\}_{k\in\mathbb{Z}}$ *are frames on* PW_Ω *provided that* $D^-(\mathbb{T}) >$

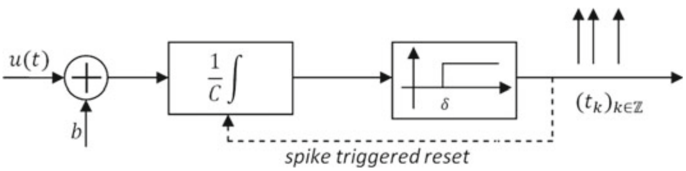

Fig. 2.1 The ideal IF neuron

$\frac{\Omega}{\pi}$, *where* $s_k \triangleq \frac{t_{k-1}+t_k}{2}$ *and* $D^-(\mathbb{T})$ *represents the lower density of sequence* $\mathbb{T}$

$$D^-(\mathbb{T}) \triangleq \lim_{r\to\infty} \frac{\inf_{x\in\mathbb{R}} \operatorname{card}(\mathbb{T} \cap [x, x+r])}{r}. \tag{2.5}$$

For a sequence $\mathbb{T}_u$, generated by an ideal IF neuron with parameters $\{b, \bar{\delta}\}$ when presented with input $u \in PW_\Omega$, Lazar and Pnevmatikakis (2008b) have shown that $D^-(\mathbb{T}_u) \geq \frac{b-c}{\bar{\delta}}$, and later on that $D^-(\mathbb{T}_u) = \frac{b}{\bar{\delta}}$ (Lazar and Pnevmatikakis 2011). Furthermore, $\mathbb{T}_u$ is relatively separated as it satisfies (Lazar 2004)

$$\frac{\bar{\delta}}{b+c} \leq t_{k+1} - t_k \leq \frac{\bar{\delta}}{b-c}, \forall k \in \mathbb{Z}. \tag{2.6}$$

If $\frac{b}{\bar{\delta}} > \frac{\Omega}{\pi}$, due to Lemma 2.1, input u can be represented as

$$u(t) = \sum_{k\in\mathbb{Z}} c_k g_\Omega(t - s_k), \forall t \in \mathbb{R}.$$

Coefficients $\{c_k\}_{k\in\mathbb{Z}}$ satisfy (Lazar and Pnevmatikakis 2008b)

$$\mathbf{c} = \mathbf{G}^+\mathbf{q}, \tag{2.7}$$

where $\mathbf{G}^+$ denotes the Moore-Penrose pseudoinverse of matrix $\mathbf{G}$, $[\mathbf{c}]_k = c_k$, $[\mathbf{q}]_k = \bar{\delta} - b(t_{k+1} - t_k)$, and $[\mathbf{G}]_{k,l} = \int_{t_k}^{t_{k+1}} g_\Omega(\tau - s_l)d\tau, \forall k, l \in \mathbb{Z}$.

The reconstruction of bandlimited inputs is only possible if the IF neuron parameters satisfy $\frac{b}{\bar{\delta}} > \frac{\Omega}{\pi}$. If this is not valid, Lazar and Pnevmatikakis (2008b) proved that a bandlimited input can still be recovered from a sufficiently large population of IF neurons. Moreover, they designed a model that arises in several sensory systems, consisting of a bank of filters in cascade with a population of IF neurons, depicted in Fig. 2.2. The filters, which model the processing taking place in the dendritic trees of biological neurons, are required to satisfy the following stability property.

Definition 2.6 A linear filter h is called bounded-input bounded-output (BIBO) stable if

$$\int_{-\infty}^{\infty} |h(\tau)|d\tau < \infty.$$

The t-transform of the population of IF neurons satisfies

$$\int_{t_k^j}^{t_{k+1}^j} (h^j * u)(\tau)d\tau = q_k^j, \forall k \in \mathbb{Z}, \forall j = 1, \ldots, N, \tag{2.8}$$

where $q_k^j = \bar{\delta}^j - b^j(t_{k+1}^j - t_k^j), \forall k \in \mathbb{Z}, \bar{\delta}^j = C^j\delta^j$ and C^j, δ^j denote the integration constant and threshold of neuron $j, \forall j = 1, \ldots, N$. To prevent the situation in which

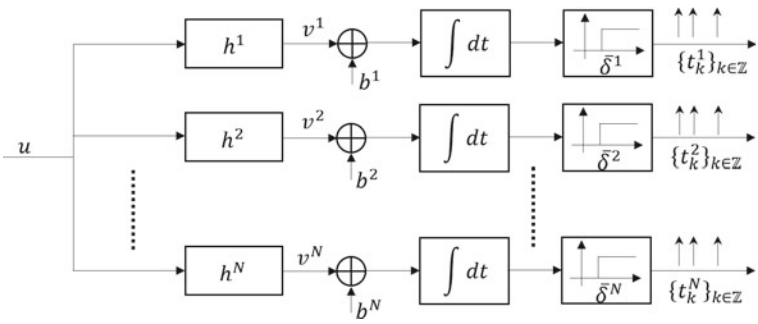

Fig. 2.2 Single-input multi-output population of N ideal IF neurons

different neurons trigger the same spike times, i.e., some neurons don't encode any new information, the filters are required to satisfy the following property (Lazar and Pnevmatikakis 2008b).

Definition 2.7 The filters $\{h^j\}_{j=1,\ldots,N}$ are called linearly independent if $\nexists a_j$, $j = 1, \ldots, N$, not all zero, and real numbers α^j, $j = 1, \ldots, N$, such that

$$\sum_{j=1}^{N} a_j (h_j * g_\Omega)(\cdot - \alpha_j) = 0 \; a.e.$$

Let $\mathbb{T}_u^j \triangleq \{t_k^j\}_{k\in\mathbb{Z}}, \forall j = 1, \ldots, N$. Then (2.8) is equivalent to (Lazar and Pnevmatikakis 2008b)

$$\langle u, \phi_{k,j}^{\mathbb{T}_u^j} \rangle_{L^2} = q_k^j, \forall k \in \mathbb{Z}, \forall j = 1, \ldots, N,$$

where $\phi_{k,j}^{\mathbb{T}_u^j} \triangleq \widetilde{h}^j * g_\Omega * 1_{[t_k^j, t_{k+1}^j[}$ and $\widetilde{h}^j \triangleq h^j(-\cdot)$, $\forall k \in \mathbb{Z}, \forall j = 1, \ldots, N$.

Let $\mathbb{T}_u \triangleq \bigcup_{j=1,\ldots,N} \mathbb{T}_u^j$ and $\psi_k^j \triangleq (\widetilde{h}^j * g_\Omega)(\cdot - s_k), \forall k \in \mathbb{Z}, \forall j = 1, \ldots, N$. Lazar and Pnevmatikakis (2008b) have proven that sequences $\{\psi_k^j\}_{k\in\mathbb{Z}, j=1,\ldots,N}$ and $\{\phi_{k,j}^{\mathbb{T}_u^j}\}_{k\in\mathbb{Z}, j=1,\ldots,N}$ are both frames on PW_Ω provided that

$$\sum_{j=1}^{N} \frac{1}{\bar{\delta}^j}(b^j - c\int_{-\infty}^{\infty} |h^j(\tau)| d\tau) > \frac{\Omega}{\pi}, \tag{2.9}$$

and thus input u can be recovered as

$$u = \sum_{j=1}^{N} \sum_{k\in\mathbb{Z}} c_k^j \psi_k^j.$$

Coefficients $\{c_k^j\}_{k\in\mathbb{Z},j=1,\dots,N}$ satisfy (Lazar and Pnevmatikakis 2008b)

$$\mathbf{c} = \mathbf{G}^{+}\mathbf{q},$$

where $\mathbf{c} = [\mathbf{c}^1, \dots, \mathbf{c}^N]^T$, $[\mathbf{c}^j]_k = c_k^j$, $\mathbf{q} = [\mathbf{q}^1, \dots, \mathbf{q}^N]^T$, $[\mathbf{q}^j]_k = q_k^j$, and

$$\mathbf{G} = \begin{pmatrix} \mathbf{G}^{11} & \dots & \mathbf{G}^{1N} \\ \dots & & \dots \\ \mathbf{G}^{N1} & \dots & \mathbf{G}^{NN} \end{pmatrix}, [\mathbf{G}^{nj}]_{k,l} = \int_{t_k^n}^{t_{k+1}^n} (h^n * \widetilde{h}^j * g)(\tau - s_l^j)d\tau.$$

The reconstruction methodology was further extended to three dimensional inputs, representing an analog monochromatic video stream (Lazar and Pnevmatikakis 2008a), and to multiple inputs of different dimensions (Lazar and Slutskiy 2013).

Lazar and Slutskiy (2015) have shown that the dendritic computations in biological neurons can be modelled more accurately with nonlinear filters. The paper developed an algorithm that reconstructs perfectly the inputs of a multi-input multi-output circuit consisting of a bank of nonlinear filters in cascade with a population of ideal IF neurons.

2.3.2 *The Ideal IF Neuron with Refractory Period*

The ideal IF neuron with refractory period represents an extension of the ideal IF neuron, incorporating the absolute refractory period of a biological neuron (Lazar 2004).

Definition 2.8 The ideal IF neuron with refractory period generates the output sequence of spike times $\{t_k\}_{k\in\mathbb{Z}}$ when presented with input u, satisfying $|u| \le c < b$, such that

$$\int_{t_k+\Delta_r}^{t_{k+1}} u(\tau)d\tau = \bar{\delta} - b(t_{k+1} - t_k - \Delta_r), \forall k \in \mathbb{Z},$$

where Δ_r represents the absolute refractory period and $\bar{\delta} = C\delta$.

Lazar (2004) has proven that Δ_r does not lead to information loss, and that any input $u \in PW_\Omega$ can be reconstructed perfectly from spike times $\{t_k\}_{k\in\mathbb{Z}}$ provided that

$$\frac{\bar{\delta}}{b-c} \cdot \frac{\Omega}{\pi} < \frac{1-\varepsilon}{1+\varepsilon},$$

where $\varepsilon = \sqrt{\frac{\Delta_r}{\bar{\delta}/(b+c)+\Delta_r}}$. Function u is recovered as

$$u = \mathbf{g}\,\mathbf{G}^{+}\mathbf{q},$$

where $[\mathbf{g}]_{1,l} = g_\Omega(\cdot - s_l)$, $[\mathbf{G}]_{k,l} = \int_{t_k+\Delta_r}^{t_{k+1}} g_\Omega(\tau - s_l)d\tau$, and $[\mathbf{q}]_{k,1} = \bar{\delta} - b(t_{k+1} - t_k - \Delta_r)$, $\forall l, k \in \mathbb{Z}$.

2.3.3 The Leaky IF Neuron

The leaky IF (LIF) neuron is a TEM that consists of an adder in cascade with a linear RC filter. The neuron triggers a spike each time the filter output y reaches a threshold value δ.

Definition 2.9 The LIF neuron generates the output sequence of spike times $\{t_k\}_{k\in\mathbb{Z}}$ when presented with input u, satisfying $|u| \le c < b$, such that

$$\int_{t_k}^{t_{k+1}} u(\tau)e^{-\frac{t_{k+1}-\tau}{RC}}d\tau = C(\delta - bR) + C\left(bR - y(t_0)\right)e^{-\frac{t_{k+1}-t_k}{RC}},$$

where b is the bias and $y(t_0)$ is the initial condition of the RC filter.

The inputs u that additionally satisfy $u \in PW_\Omega$ can be recovered from spike times $\{t_k\}_{k\in\mathbb{Z}}$ provided that (Lazar 2005)

$$RC \cdot \ln\left(1 - \frac{\delta - y(t_0)}{\delta - (b-c)R}\right)\frac{\Omega}{\pi} < \frac{(b-c)R - \delta}{(b-c)R + \delta - 2y(t_0)}.$$

Function u is recovered as

$$u = \mathbf{g}\,\mathbf{G}^{+}\mathbf{q},$$

where $[\mathbf{g}]_{1,l} = g_\Omega(\cdot - s_l)$, $[\mathbf{G}]_{k,l} = \int_{t_k}^{t_{k+1}} g_\Omega(\tau - s_l)e^{-\frac{t_{k+1}-\tau}{RC}}d\tau$, and $[\mathbf{q}]_{k,1} = C(\delta - bR) + C\left(bR - y(t_0)\right)e^{-\frac{t_{k+1}-t_k}{RC}}$.

Alternative reconstruction algorithms have been proposed for the input u of a LIF neuron on a finite time horizon (Lazar and Pnevmatikakis 2009), i.e., $u \in L^2[0, T]$ and $\mathbb{T}_u = \{t_k\}_{k=1,\dots,N}$. Instead of perfect reconstruction, the paper proposes an algorithm that is consistent, i.e., the reconstructed function generates exactly the same spike train $\mathbb{T}_u$ as the original input function. An extra constraint imposed on the reconstructed signal is to minimize the following cost function

$$\mathscr{C}u = \left[\int_0^T \left(\frac{d^2u}{dt^2}(\tau)\right)^2 d\tau\right]^{1/2}.$$

Function u is reconstructed using the following theorem, presented in (Lazar and Pnevmatikakis 2009).

Theorem 2.5 *The consistent reconstruction, optimal with respect to cost function $\mathscr{C}u$, is unique, and has the expression*

$$u_{opt}(t) = d_0 + d_1 x + \sum_{k=1}^{N-1} c_k \xi_k(x), \tag{2.10}$$

where

$$\xi_k(t) = \int_{t_k}^{t_{k+1}} |t-s|^3 e^{-\frac{t_{k+1}-s}{RC}} ds, \tag{2.11}$$

where $|\cdot|$ *denotes the absolute value. Coefficients* d_0, d_1 *and* c_k, $k = 1, 2, \ldots, N-1$, *are the solution of the following linear system:*

$$\begin{bmatrix} \boldsymbol{G} & \boldsymbol{p} & \boldsymbol{r} \\ \boldsymbol{p}^T & 0 & 0 \\ \boldsymbol{r}^T & 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{c} \\ d_0 \\ d_1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{q} \\ 0 \\ 0 \end{bmatrix}, \tag{2.12}$$

where $\boldsymbol{c} = [c_1, c_2, \ldots, c_{N-1}]^T$, *and* $\boldsymbol{q} = [q_1, q_2, \ldots, q_{N-1}]^T$. *Matrix* $[\boldsymbol{G}]_{(N-1)\times(N-1)}$ *and column vectors* $\boldsymbol{p}$ *and* $\boldsymbol{r}$ *have the following expressions:*

$$\begin{aligned} [\boldsymbol{p}]_k &= <\phi_k, 1>_{L_T^2}, \\ [\boldsymbol{r}]_k &= <\phi_k, t>_{L_T^2}, \\ [\boldsymbol{G}]_{kl} &= <\phi_k, \zeta_l>_{L_T^2}, \end{aligned} \tag{2.13}$$

where $<\cdot,\cdot>_{L_T^2}$ *is the standard inner product on* $L^2[0,T]$, $<f, g>_{L_T^2} = \int_0^T f(\tau) g^*(\tau) d\tau$, $\forall f, g \in L^2[0,T]$, *and* $\phi_k \triangleq e^{-\frac{t_{k+1}-\cdot}{RC}} \cdot 1_{[t_k, t_{k+1}]}$.

2.3.4 The Leaky IF Neuron with Random Threshold

The LIF neuron with random threshold has been proposed as a model that incorporates the variability in the biological spiking neurons (Lazar et al. 2010).

Definition 2.10 The LIF neuron with random threshold generates a spike time sequence $\{t_k\}_{k=1,\ldots,N}$ when presented with input u, satisfying $|u| \le c < b$, such that

$$\int_{t_k}^{t_{k+1}} u(\tau) e^{-\frac{t_{k+1}-\tau}{RC}} d\tau = q_k + \varepsilon_k, k = 1, \ldots, (N-1), \tag{2.14}$$

where $q_k \triangleq C\delta - bRC\left(1 - e^{-\frac{t_{k+1}-t_k}{RC}}\right)$, $\varepsilon_k \triangleq C(\delta_k - \delta)$ and δ_k is the random threshold drawn from the normal distribution with zero mean and variance $(C\sigma)^2$, for $k = 1, \ldots, (N-1)$.

Lazar et al. (2010) have considered the reconstruction problem for periodic inputs $u \in \mathbb{H}_\Omega^M$. They have proven that, in this case, (2.14) is equivalent to

$$\langle u, \chi_k \rangle = q_k + \varepsilon_k,$$

where $\{\chi_k\}_{k=1,\ldots,(N-1)}$ can be calculated using the orthonormal basis $\{e_m\}_{m=-M,\ldots,M}$, $e_m \triangleq \frac{1}{\sqrt{T}} e^{jm\frac{\Omega}{M}t}$, $m = -M, \ldots, M$ as

$$\chi_k = \sum_{-M}^{M} b_{m,k} e_m, \forall k = 1, \ldots, (N-1),$$

where

$$b_{m,k} \triangleq \frac{RC \cdot e_{-m}(t_{k+1}) + (y_k - RC) e_{-m}(t_k)}{\sqrt{T}(1 - jmRC \cdot \Omega/M)}, \forall m = -M, \ldots, M,$$
$$y_k \triangleq RC \left(1 - e^{-\frac{t_{k+1} - t_k}{RC}}\right), \forall k = 1, \ldots, (N-1).$$

The reconstruction u_{opt} is calculated as the solution to the minimization problem

$$u_{opt} = \underset{u \in \mathbb{H}_\Omega^M}{\operatorname{argmin}} \left(\sum_{k=1}^{N-1} (q_k - \langle u, \chi_k \rangle_{\mathbb{H}_\Omega^M}) + (N-1)\mu \|u\|^2_{\mathbb{H}_\Omega^M} \right),$$

where μ is a positive parameter that regulates the tradeoff between smoothness and the faithfulness of measurements.

Lazar et al. (2010) have proven that function u_{opt} satisfies

$$u_{opt} = \mathbf{e} \cdot \mathbf{c},$$

where $[\mathbf{e}]_1 = e_m, [\mathbf{c}]_m = c_m, \forall m = -M, \ldots, M$, are line and column vectors, respectively, and

$$\mathbf{c} = (\mathbf{G}^H \mathbf{G} + (N-1)\mu \mathbf{I})^{-1} \mathbf{G}^H \mathbf{q},$$

where $[\mathbf{G}]_{km} = b^*_{m,k}$, $[\mathbf{q}]_k = q_k$, $k = 1, \ldots, (N-1)$, $m = -M, \ldots, M$, and $\mathbf{I}$ is the identify matrix of dimension $(2M+1) \times (2M+1)$.

2.3.5 The Hodgkin-Huxley Neuron

The Hodgkin-Huxley neuron is one of the best known biophysically realistic models for the spiking neuron introduced by Hodgkin and Huxley (1952).

Definition 2.11 The Hodgkin-Huxley neuron generates output function V when presented with input I such that

$$
\begin{aligned}
C\frac{dV}{dt} &= -g_{Na}m^3h(V - E_{Na}) - g_K n^4(V - E_K) - g_L(V - E_L) + I \\
\frac{dm}{dt} &= \alpha_m(V)(1 - m) - \beta_m(V)m \\
\frac{dh}{dt} &= \alpha_H(V)(1 - h) - \beta_h(V)h \\
\frac{dn}{dt} &= \alpha_n(V)(1 - n) - \beta_n(V)n.
\end{aligned}
$$

In the definition above V represents the membrane voltage, i.e., the difference in electric potential between the interior and the exterior of the neuron and m, h, and n are the gating variables. The gating variables model the conductance ($resistance^{-1}$) of the corresponding ion channel, which is a membrane protein that establishes a resting membrane potential of the neuron.

The Hodgkin-Huxley equations can be expressed in matrix form as

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}), \tag{2.15}$$

where $\mathbf{x} = [V, m, h, n]^T$ and $\mathbf{f} : \mathbb{R}^4 \to \mathbb{R}^4$ has an expression given by the system of equations above.

A Hodgkin-Huxley neuron stimulated via multiplicative coupling by input function u is described by (Lazar 2007)

$$\frac{d\mathbf{y}}{dt} = (b + u(t))\mathbf{f}(\mathbf{y}), \tag{2.16}$$

where b is a constant satisfying $u(t) + b > 0, \forall t \in \mathbb{R}$, and $\mathbf{y} = [y_1, y_2, y_3, y_4]^T$. The solution to system (2.16) is $\mathbf{y}(t) = \mathbf{x}\left(bt + \int_0^t u(\tau)d\tau\right)$, where $\mathbf{x}$ is the solution to (2.15). The spiking times of the Hodgkin-Huxley neuron, denoted $\{\delta_k\}_{k\in\mathbb{Z}}$, are defined as the local maxima of function $[\mathbf{x}]_1 = V$. Similarly, the spike times of the Hodgkin-Huxley neuron with multiplicative coupling $\{t_k\}_{k\in\mathbb{Z}}$ are the local maxima of function $[\mathbf{y}]_1 = y_1$ (Lazar 2010).

The following lemma proves that a Hodgkin-Huxley neuron with multiplicative coupling and an ideal IF neuron with variable threshold sequence $\{\delta_{k+1} - \delta_k\}_{k\in\mathbb{Z}}$ are input-output equivalent, i.e., they both trigger the same spike sequence $\{t_k\}_{k\in\mathbb{Z}}$ when presented with the same input u (Lazar 2010).

Lemma 2.2 *The spike times* $\{\delta_k\}_{k\in\mathbb{Z}}$ *and* $\{t_k\}_{k\in\mathbb{Z}}$ *satisfy*

$$\int_{t_k}^{t_{k+1}} u(\tau)d\tau = \delta_{k+1} - \delta_k - b(t_{k+1} - t_k), \forall k \in \mathbb{Z}.$$

Lazar (2010) has shown that, under the assumption $\delta_{k+1} - \delta_k = \delta, \forall k \in \mathbb{Z}$, a bandlimited input $u \in PW_\Omega$ of a Hodgkin-Huxley neuron with multiplicative coupling can be perfectly reconstructed from the generated output spike sequence provided that $\delta/b < \pi/\Omega$.

The Hodgkin-Huxley neuron with multiplicative coupling belongs to a more general class of models, namely TEMs with multiplicative coupling, which are described by the following system of equations (Lazar 2006)

$$\frac{d\mathbf{y}}{dt} = (b + u(t))\mathbf{f}(\mathbf{y}), \tag{2.17}$$

where $\mathbf{y} = [y_1, \ldots, y_n]^T$, $f : \mathbb{R}^n \to \mathbb{R}^n$ is an arbitrary continuous non-linear function, and $u \le c < b$. The solution satisfies $\mathbf{y}(t) = \mathbf{x}\left(bt + \int_0^t u(\tau)d\tau\right)$, where $\mathbf{x} = [x_1, \ldots, x_n]^T$ and

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}). \tag{2.18}$$

The output of this TEM is defined as the sequence of zeros of function y_1, denoted $\{t_k\}_{k\in\mathbb{Z}}$. Lazar (2006) has shown that the TEM with multiplicative coupling is input-output equivalent with an ideal IF neuron with variable threshold sequence $\{\delta_{k+1} - \delta_k\}_{k\in\mathbb{Z}}$, where $\{\delta_k\}_{k\in\mathbb{Z}}$ denotes the sequence of zeros of function x_1.

2.3.6 *The Asynchronous Sigma-Delta Modulator*

The TEM has also been used as a model for non biological circuits. The ASDM, depicted in Fig. 2.3, represents an efficient replacement for the classical A/D converter (Lazar and Tóth 2004a, b). It consists of an adder, an ideal integrator, and a noninverting Schmitt trigger with parameters $\{\delta/2, 1\}$.

The input of the circuit is assumed to be bounded by $|u(t)| \le c < 1, \forall t \in \mathbb{R}$. Function z switches between -1 and 1 at times $\mathbb{T}_u = \{t_k\}_{k\in\mathbb{Z}}$, with initial value $z(t_0) = -1$. Function y is increasing or decreasing when z is negative or positive, respectively.

Definition 2.12 The ASDM circuit generates output sequence $\mathbb{T}_u = \{t_k\}_{k\in\mathbb{Z}}$ when presented with input u, satisfying $|u(t)| \le c < 1, \forall t \in \mathbb{R}$, such that

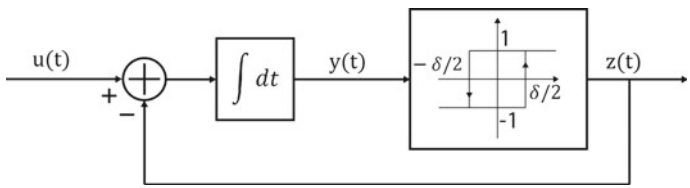

Fig. 2.3 The ASDM circuit

$$\int_{t_k}^{t_{k+1}} u(\tau)d\tau = (-1)^k[\delta - (t_{k+1} - t_k)], \forall k \in \mathbb{Z}.$$

Lazar and Tóth (2004a) have proposed the following theorem for the reconstruction of u with arbitrary accuracy.

Theorem 2.6 *Let $\mathbb{T}_u = \{t_k\}_{k\in\mathbb{Z}}$ be the time sequence generated by an ASDM when presented with bounded input $u \in PW_\Omega$, $|u(t)| \leq c < 1, \forall t \in \mathbb{R}$. Let $\{u_l\}_{l\in\mathbb{Z}}$ be a sequence of bandlimited functions satisfying the recursive equation*

$$u_{l+1} = u_l + \mathscr{Z}_\Omega(u - u_l), \forall l \in \mathbb{N},$$

where $u_0 = \mathscr{Z}_\Omega u$ and

$$\mathscr{Z}_\Omega : PW_\Omega \to PW_\Omega, \mathscr{Z}_\Omega u \triangleq \sum_{k\in\mathbb{Z}} \mathscr{L}_k^{\mathbb{T}_u} u \cdot g(\cdot - s_{k+1}).$$

If $r \triangleq \frac{\Omega}{\pi} \cdot \frac{\delta}{1-c} < 1$, then input u can be recovered with arbitrary precision from the associated sequence $\mathbb{T}_u$ as

$$u = \lim_{l\to\infty} u_l.$$

Moreover,

$$\|u - u_l\|_{L^2} \leq r^{l+1} \cdot \|u\|_{L^2}, \forall l \in \mathbb{N}.$$

Lazar and Tóth (2004a) have presented the following theorem to demonstrate the similarity between Theorem 2.6 and nonuniform sampling theory.

Theorem 2.7 *Let $\mathbb{T}_u = \{t_k\}_{k\in\mathbb{Z}}$ be the time sequence generated by an ASDM when presented with bounded input $u \in PW_\Omega$, $|u(t)| \leq c < 1, \forall t \in \mathbb{R}$. Let $\{u_l\}_{l\in\mathbb{Z}}$ be a sequence of bandlimited functions satisfying the recursive equation*

$$u_{l+1} = u_l + \mathscr{Z}_\Omega^*(u - u_l), \forall l \in \mathbb{N},$$

where $u_0 = \mathscr{Z}_\Omega^* u$ and

$$\mathscr{Z}_\Omega^* : PW_\Omega \to PW_\Omega, \mathscr{Z}_\Omega^* u = \sum_{k\in\mathbb{Z}} u(s_{k+1})\phi_k^{\mathbb{T}_u}, \forall u \in PW_\Omega.$$

where $\phi_k^{\mathbb{T}_u} = g * 1_{[t_k, t_{k+1}[}$. If $r = \frac{\Omega}{\pi} \cdot \frac{\delta}{1-c} < 1$, then u can be reconstructed from $\{u(s_k)\}_{k\in\mathbb{Z}}$ as

$$u = \lim_{l\to\infty} u_l.$$

Moreover,

$$\|u - u_l\|_{L^2} \leq r^{l+1} \cdot \|u\|_{L^2}, \forall l \in \mathbb{N}.$$

The algorithm in Theorem 2.6 has also been presented in matrix form (Lazar and Tóth 2004a), namely $u = \lim_{l\to\infty} u_l$, where

$$u_l = \mathbf{g}^T \mathbf{P}_l \mathbf{q},$$

where $\mathbf{g}^T$ denotes the transpose of vector $\mathbf{g}$, $[\mathbf{g}]_k = g_\Omega(\cdot - s_k)$, $s_k = \frac{t_{k-1}+t_k}{2}$, $\forall k \in \mathbb{Z}$, $\mathbf{P}_l = \sum_{j=0}^{l}(\mathbf{I}-\mathbf{G})^j$, $[\mathbf{G}]_{k,l} = \int_{t_k}^{t_{k+1}} g_\Omega(\tau - s_l)d\tau$ and $[\mathbf{q}]_k = (-1)^k[\delta - (t_{k+1} - t_k)]$, $\forall k, l \in \mathbb{Z}$. Moreover, the paper proves that

$$u = \lim_{l\to\infty} u_l = \lim_{l\to\infty} \mathbf{g}^T \mathbf{P}_l \mathbf{q} = \mathbf{g}^T \mathbf{G}^+ \mathbf{q}. \tag{2.19}$$

Reconstruction formula (2.19) can also be derived using the theory of frames. Sequence $\mathbb{T}_u$ satisfies (Lazar and Tóth 2004a)

$$\frac{\delta}{1+c} \leq t_{k+1} - t_k \leq \frac{\delta}{1-c}, \forall k \in \mathbb{Z}.$$

Then the following holds

$$\frac{\delta}{1+c} \leq s_{k+1} - s_k = \frac{t_{k+1} - t_{k-1}}{2} \leq \frac{\delta}{1-c}, \forall k \in \mathbb{Z}.$$

Therefore, $\{s_k\}_{k\in\mathbb{Z}}$ is relatively separated and $\frac{\delta}{1-c}$- dense. According to Theorem 2.3 it follows that, if $\frac{\delta}{1-c} < \frac{\pi}{\Omega}$, then $\{g_\Omega(\cdot - s_k)\}_{k\in\mathbb{Z}}$ is a frame on PW_Ω and thus (2.19) holds true.

From a computational point of view, the main disadvantage of the reconstruction approach in (2.7) and (2.19) is that a new set of functions $\{g_\Omega(\cdot - s_k)\}_{k=2,\dots,N}$, matrix $\boldsymbol{G}$, and its pseudoinverse $\boldsymbol{G}^+$ have to be calculated for every sequence $\mathbb{T}_u = \{t_k\}_{k=1,\dots,N}$ of spike times. Alternatively, functions $\{g_\Omega(\cdot - s_k)\}_{k=2,\dots,N}$ and sequence $\{\mathscr{L}_k^{\mathbb{T}_u} u_l\}_{l<L}$ are calculated with Theorem 2.6 for every sequence $\mathbb{T}_u$ with $L \in \mathbb{N}$ an arbitrarily large number.

2.4 Time Encoding and Decoding in Shift-Invariant Spaces

The space PW_Ω is spanned by a basis $\{\text{sinc}\,(\Omega(\cdot - k\pi/\Omega))\}_{k\in\mathbb{Z}}$ of functions that have infinite time support and slow decay, which often creates complexity issues during numerical implementations (Aldroubi and Gröchenig 2001). The more general SIS is spanned by a set of functions $\{\lambda(\cdot - kT)\}_{k\in\mathbb{Z}}$, which are required to form a frame (Christensen 2003).

Gontier and Vetterli (2014) extended the results of Lazar and Tóth (2003) to SIS using the non-uniform sampling framework developed by Aldroubi and Gröchenig (2001), Aldroubi and Feichtinger (1998), Feichtinger et al. (1995), Gröchenig (1992, 1993), and Gröchenig and Schwab (2003). The TEMs considered by Gontier and Vetterli (2014) are the C-TEM and the IF-TEM, which is a generalization of the ideal IF neuron. The paper designed two algorithms that reconstruct the inputs of the C-TEM and IF-TEM belonging to a SIS and proved the close relationship between the two.

This section introduces the theory of SIS and presents the reconstruction algorithms for the IF-TEM and C-TEM.

The shift invariant space of order 2 generated by function λ is defined by (Unser 2000)

$$V_T^2(\lambda) = \left\{ u(t) = \sum_{k\in\mathbb{Z}} c_k \lambda(t-kT), (c_k)_{k\in\mathbb{Z}} \in l^2(\mathbb{R}) \right\}. \tag{2.20}$$

If $\lambda(t) = \sin(\Omega t)/(\Omega t)$ and $T = \frac{\pi}{\Omega}$ then u is bandlimited and $V_T^2(\lambda) = PW_\Omega$.

An important problem in the theory of SIS is establishing conditions for which the set of functions $\{\lambda(t-kT)\}_{k\in\mathbb{Z}}$ forms a frame on $V_T^2(\lambda)$. To address this problem, the following periodic function is defined

$$G_\lambda^T(\omega) \triangleq \left(\sum_{k\in\mathbb{Z}} \left| \widehat{\lambda}\left(\frac{\omega + 2k\pi}{T} \right) \right|^2 \right)^{1/2}, \forall\omega \in [0, 2\pi].$$

The following theorem, presented in (Christensen 2003), proves several important properties of function G_λ^T.

Theorem 2.8 *Let $\lambda \in L^2(\mathbb{R})$. Then $\{\lambda(\cdot - kT)\}_{k\in\mathbb{Z}}$ is a frame sequence for $V_T^2(\lambda)$ with bounds $A, B > 0$ if and only if*

$$A \leq \frac{G_\lambda^T(\omega)}{T} \leq B, a.e.\ \omega \in [0, 2\pi]\backslash N_0, \tag{2.21}$$

where $N_0 = \{\omega \in [0, 2\pi] : G_\lambda^T(\omega) = 0\}$. Moreover, $\{\lambda'(\cdot - kT)\}_{k\in\mathbb{Z}}$ is a Bessel sequence for $V_T^2(\lambda)$ with bound $B' > 0$ if and only if

$$\frac{G_{\lambda'}^T(\omega)}{T} \leq B', a.e.\ \omega \in [0, 2\pi]. \tag{2.22}$$

Gontier and Vetterli (2014) consider shift invariant spaces with integer shifts $V^2(\lambda) = V_1^2(\lambda)$, for which the inner product is defined as

$$\langle f, g \rangle_{V_1^2} = \frac{1}{2\pi} \int_0^{2\pi} \widehat{c}(\omega)\widehat{d}(\omega)^* G_\lambda(\omega)^2 d\omega, \tag{2.23}$$

where $G_\lambda = G_\lambda^1$, $\widehat{c}(\omega) = \sum_{k\in\mathbb{Z}} c_k e^{-ik\omega}$, $\widehat{d}(\omega) = \sum_{k\in\mathbb{Z}} d_k e^{-ik\omega}$, and c_k, d_k are the coefficients in $V^2(\lambda)$ of f and g, respectively. The dual frame of $\{\lambda(\cdot - k)\}_{k\in\mathbb{Z}}$, denoted $\{\widetilde{\lambda}(\cdot - k)\}_{k\in\mathbb{Z}}$, satisfies (Gontier and Vetterli 2014)

$$\widetilde{\lambda}(t) = \mathscr{F}^{-1}\left(\frac{\widehat{\lambda}(\omega)}{(G_\lambda(\omega))^2}\right), \forall\omega \in \mathbb{R}.$$

The dual frame is useful for calculating the coefficients corresponding to the expansion of function u in space $V^2(\lambda)$. More generally, it can be used to calculate the coefficients of the orthogonal projection $\mathscr{P}_{V^2}$ of an arbitrary function $f \in L^2(\mathbb{R})$ onto $V^2(\lambda)$ as $\langle f, \widetilde{\lambda}(\cdot - k)\rangle_{V^2}$, where

$$\mathscr{P}_{V^2} f = \sum_{k\in\mathbb{Z}} \langle f, \widetilde{\lambda}(\cdot - k)\rangle_{V^2} \cdot \lambda(\cdot - k).$$

Gontier and Vetterli (2014) restrict function λ to space $H^1(\mathbb{R})$, i.e., a Sobolev space defined by

$$H^1(\mathbb{R}) \triangleq \{f \in L^2(\mathbb{R}) : \|f\|_{H^1} < \infty\}, \|f\|^2_{H^1} = \|f\|^2_{L^2} + \|f\,'\|^2_{L^2},$$

where $f\,'$ is a weak derivative of f, which satisfies

$$\int_{-\infty}^{\infty} f\,'(\tau)v(\tau)d\tau = -\int_{-\infty}^{\infty} f(\tau)v\,'(\tau)d\tau, \forall v \in C_0^\infty(\mathbb{R}),$$

where $C_0^\infty(\mathbb{R})$ is the class of smooth functions on $\mathbb{R}$ with compact support.

Let $u \in V_T^2(\lambda)$, $\lambda \in H^1(\mathbb{R})$. Then its weak derivative $u\,'$ is bounded by (Gontier and Vetterli 2014)

$$\|u\,'\|_{L^2} \leq \frac{1}{\rho}\|u\|_{L^2}, \tag{2.24}$$

where $\rho \triangleq \inf_{\omega\in[0,2\pi[} \frac{G_\lambda(\omega)}{G'_\lambda(\omega)}$. To ensure that this bound is finite and different than 0, λ is restricted to satisfy conditions (2.21) and (2.22) for $\forall\omega \in [0, 2\pi]$, namely $\lambda \in W$, where

$$W = \{\lambda \in H^1(\mathbb{R}) : \exists A, B, B' > 0, A \leq G_\lambda(\omega) \leq B, G_{\lambda'}(\omega) \leq B', \forall\omega \in [0, 2\pi]\}. \tag{2.25}$$

Function λ is required to belong to Sobolev space $H_1(\mathbb{R})$ such that the assumptions are satisfied in the next lemma, which establishes two properties for space $V^2(\lambda)$ (Gontier and Vetterli 2014).

Lemma 2.3 *Let $\lambda \in W$. Then $V^2(\lambda)$ is a RKHS and $V^2(\lambda) \hookrightarrow C(\mathbb{R})$, where $C(\mathbb{R})$ denotes the class of continuous functions on $\mathbb{R}$.*

The reproducing kernel on $V^2(\lambda)$ is denoted by $K : \mathbb{R}^2 \to \mathbb{R}$ and has the expression (Gontier and Vetterli 2014)

$$K(x, t) = \sum_{k\in\mathbb{Z}} \lambda(x - k)\widetilde{\lambda}(t - k) = \sum_{k\in\mathbb{Z}} \widetilde{\lambda}(x - k)\lambda(t - k).$$

Definition 2.13 A crossing TEM (C-TEM) with continuous test functions $\{\Phi_k\}$ generates a sequence $\mathbb{CT}_u = \{t_k\}_{k\in\mathbb{Z}}$, when presented with input $u \in V^2(\lambda)$, such that

A) The value of Φ_k at a given time is $\Phi_k(t) = \Phi\left(t, \{t_j, j \leq k\}\right)$, where $\Phi : \mathbb{R} \times \{t_j, j \leq k\} \to \mathbb{R}$.
B) $u(t_{k+1}) = \Phi_k(t_{k+1})$;
C) $u(t) \neq \Phi_k(t), \forall t \in \left]t_k, t_{k+1}\right[$.

The problem of reconstructing u from the output of a C-TEM is the same as the one of reconstructing u from its nonuniform samples at times $\mathbb{CT}_u$. The following operator is required to design a reconstruction algorithm.

$$\mathscr{V} : V^2(\lambda) \to V^2(\lambda), \mathscr{V}u \triangleq \sum_{k\in\mathbb{Z}} u(t_k)1_{[s_k, s_{k+1}[},$$

where $s_k = \frac{t_{k-1}+t_k}{2}, \forall k \in \mathbb{Z}$.

The following theorem was proven in (Gontier and Vetterli 2014) and generalizes Theorem 2.4 proven by Feichtinger and Gröchenig (1994) for bandlimited spaces.

Theorem 2.9 *Let $\mathbb{CT}_u = \{t_k\}_{k\in\mathbb{Z}}$ be the sequence generated by a C-TEM when presented with input $u \in V^2(\lambda), \lambda \in W$. Then u can be reconstructed iteratively from $\mathbb{CT}_u$ provided that there exists $\Delta > 0$ such that $\mathbb{CT}_u$ is Δ-dense and*

$$\Delta < \pi\rho. \tag{2.26}$$

where $\rho = \inf_{\omega\in[0,2\pi[} \frac{G_\lambda(\omega)}{G_{\lambda'}(\omega)}$. The reconstruction is performed with

$$\begin{aligned} u_1 &= \mathscr{P}_{V^2}\mathscr{V}u \\ u_{n+1} &= u_1 + (\mathscr{I} - \mathscr{P}_{V^2}\mathscr{V})u_n, \end{aligned} \tag{2.27}$$

where $\mathscr{I}$ is the identity operator. The functions u_n satisfy

$$\|u - u_n\|_{L^2} \leq \left(\frac{\Delta}{\pi\rho}\right)^n \|u\|_{L^2}, \forall n \in \mathbb{N}^*.$$

For the particular case $V^2(\lambda) = PW_\Omega, \rho = \frac{1}{\Omega}$ and the requirement (4.6) is $\Delta < \frac{\pi}{\Omega}$, which is in line with the result for bandlimited spaces in Theorem 2.4.

Definition 2.14 An IF-TEM with test functions $\{\Phi_k\}$ generates a sampling sequence $\mathbb{IT}_u = \{t_k\}_{k\in\mathbb{Z}}$, when presented with input $u \in V^2(\lambda)$, such that

A) The value of Φ_k at a given time is $\Phi_k(t) = \Phi\left(t, \{t_j, j \leq k\}\right)$, where $\Phi : \mathbb{R} \times \{t_j, j \leq k\} \to \mathbb{R}$.
B) $\mathscr{L}_k^{\mathbb{IT}_u}u = \Phi_k(t_{k+1})$;
C) $\int_{t_k}^{t} u(\tau)d\tau \neq \Phi_k(t), \forall t \in \left]t_k, t_{k+1}\right[$, where $\mathscr{L}_k^{\mathbb{IT}_u}$ is an operator mapping function u onto the real axis, i.e., $\mathscr{L}_k^{\mathbb{IT}_u} : V^2(\lambda) \to \mathbb{R}, \mathscr{L}_k^{\mathbb{IT}_u}u \triangleq \int_{t_k}^{t_{k+1}} u(t)dt$.

The ideal IF neuron is an IF-TEM with test functions $\Phi_k(t) = \bar{\delta} - b(t - t_k), \forall k \in \mathbb{Z}$, where $\bar{\delta} \triangleq C\delta$, and δ, C, and b are the threshold, integration constant, and bias, respectively. Moreover, the ASDM circuit can be modelled as an IF-TEM with test functions $\Phi_k(t) = (-1)^k[\delta - (t - t_k)], \forall k \in \mathbb{Z}$.

The following operator is used to reconstruct the input of an IF-TEM

$$\mathscr{Z} : V^2(\lambda) \to V^2(\lambda), (\mathscr{Z}u)(t) \triangleq \sum_{n\in\mathbb{Z}} \mathscr{L}_n^{\mathbb{IT}_u} u \cdot K(s_{n+1}, t), \forall u \in V^2(\lambda),$$

where K is the reproducing kernel in $V^2(\lambda)$. To show the relationship between the IF-TEM and the C-TEM, operator V' is defined as

$$\mathscr{V}' : V^2(\lambda) \to V^2(\lambda), \mathscr{V}'u \triangleq \sum_{k\in\mathbb{Z}} u(s_{k+1}) 1_{[t_k, t_{k+1}[}.$$

Gontier and Vetterli (2014) have proven that operator $\mathscr{P}_{V^2}\mathscr{V}'$ is the adjoint of $\mathscr{P}_{V^2}\mathscr{Z}$, i.e.

$$\langle f, \mathscr{P}_{V^2}\mathscr{V}'g\rangle_{V^2} = \langle \mathscr{P}_{V^2}\mathscr{Z} f, g\rangle_{V^2}, \forall f, g \in V^2(\lambda). \tag{2.28}$$

The operators $\mathscr{V}$ and $\mathscr{V}'$ are very similar, and due to (2.28) the IF-TEM is called the quasi-adjoint of the C-TEM (Gontier and Vetterli 2014).

Theorem 2.10 *Let $\mathbb{IT}_u = \{t_k\}_{k\in\mathbb{Z}}$ be the sequence generated by an IF-TEM when presented with input $u \in V^2(\lambda), \lambda \in W$. Then u can be reconstructed iteratively from $\mathbb{IT}_u$ provided that there exists $\Delta > 0$ such that $\mathbb{IT}_u$ is Δ-dense and*

$$\Delta < \pi\rho. \tag{2.29}$$

where $\rho = \inf_{\omega\in[0,2\pi[} \frac{G_\lambda(\omega)}{G'_\lambda(\omega)}$. The reconstruction is performed with

$$\begin{aligned} u_1 &= \mathscr{P}_{V^2}\mathscr{Z}u \\ u_{n+1} &= u_1 + (\mathscr{I} - \mathscr{P}_{V^2}\mathscr{Z})u_n, \end{aligned} \tag{2.30}$$

where $\mathscr{I}$ is the identity operator. The functions u_n satisfy

$$\|u - u_n\|_{L^2} \le \left(\frac{\Delta}{\pi\rho}\right)^n \|u\|_{L^2}, \forall n \in \mathbb{N}^*.$$

Theorem 2.6 is the particular case of Theorem 2.10 for $u \in PW_\Omega$, where $\rho = \frac{1}{\Omega}$ and the requirement (4.7) is $\Delta < \frac{\pi}{\Omega}$.

From a computational point of view, the main disadvantage of the reconstruction approach in Theorem 2.10 is that functions $\{K(s_k, \cdot)\}_{k=1,\ldots,N}$ and values $\{\mathscr{L}_k^{\mathbb{IT}_u} u_l\}_{l<L}$ are calculated for every sequence $\mathbb{IT}_u$, where $L \in \mathbb{N}$ is an arbitrarily large number.

2.5 Conclusions

This chapter presented two dual sampling methods, namely nonuniform sampling and sampling based on timing. As a consequence of this duality, the corresponding two algorithms for reconstructing a function from its samples are studied in a unifying manner.

An important class of models that performs sampling based on timing is the TEM. Several models of the biological spiking neuron have been represented as TEMs, including the ideal IF model, the LIF neuron, the IF model with variable threshold and the IF model with absolute refractory period. The TEM has also been proposed as a model for the ASDM, an efficient encoding circuit representing a suitable replacement for the classical A/D converter.

The existent algorithms for reconstructing the input of a TEM belonging to bandlimited or shift-invariant spaces exploit the classical formulation of time encoding, where the stimulus is projected onto a set of input dependent frame functions. As a consequence, input u^j is reconstructed from sequence $\mathbb{T}_{u^j}$ in a space spanned by a new set of functions for every j. This process becomes computationally demanding for a large number of reconstructions.

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