CVA Computing by PDE Models

Iñigo Arregui(✉), Beatriz Salvador, and Carlos Vázquez

Department of Mathematics, University of A Coruña, A Coruña, Spain
{inigo.arregui,beatriz.salvador,carlos.vazquez.cendon}@udc.es

Abstract. In order to incorporate the credit value adjustment (CVA) in derivative contracts, we propose a set of numerical methods to solve a nonlinear partial differential equation [2] modelling the CVA. Additionally to adequate boundary conditions proposals, characteristics methods, fixed point techniques and finite elements methods are designed and implemented. A numerical test illustrates the behavior of the model and methods.

Keywords: CVA · Modelling · Numerical methods

1 Introduction

Since 2007 crisis, when important financial entities went bankrupt, the counterparty risk has become an important ingredient in all contracts. It can be described as the risk to each party of a contract that the counterparty will not live up to its contractual obligations. Thus, the neutral risk value of a derivative must take into account the following adjustments [6,7]:

- CVA: the credit value adjustment is the amount by which the value of a security is adjusted downward because of the counterparty credit risk
- DVA: the debit value adjustment corresponds to the CVA of the bank, viewed from the point of view of its counterparty
- FVA: the funding value adjustment represents the difference between a collateralized and uncollateralized trade. Moreover FVA = FBA − FCA, where FCA is the adjustment due to existence of funding costs by the issuer, and the FBA is the adjustment due to the liquidity produced by the evolution in this value.

Thus, including counterparty risk in the pricing of derivatives represents an important change in the existent risk-free pricing models. In particular, you can formulate nonlinear partial differential equation (PDE) models which have to be mathematically analyzed and solved by means of numerical methods.

Our goal is to calculate the value of derivatives, accounting for all the associated cash flows that come from the derivative itself, the act of hedging, and the management of default risk and funding cost. We will refer to all value adjustments as XVA, which is defined by:

$$XVA = DVA - CVA - FCA + FBA = DVA - CVA + FVA.$$

I. Dimov et al. (Eds.): NAA 2016, LNCS 10187, pp. 15–24, 2017.
DOI: 10.1007/978-3-319-57099-0_2

In this work, we first propose boundary conditions for a one dimensional model [2], considering constant default intensities. Then, we propose a set of numerical methods (characteristics methods, fixed point techniques and finite elements) to achieve approximations of the solutions and a numerical test is solved.

2 Mathematical Model

In a first step, following [2] we hedge the derivative with a self-financing portfolio which covers all underlying risk factors of the model. Let us assume a portfolio Π consisting of:

- $\Delta(t)$ units of the underlying S,
- $\alpha_B(t)$ units of P_B, a default risky, zero-recovery, zero-coupon bond of party B
- $\alpha_C(t)$ units of P_C, an analogous bond for the counterparty C
- γ units of cash, which is made up of a financing amount, cash needed to buy a position in C's bond and a REPO amount, such that the portfolio value at time t hedges out the value of the derivative contract to the seller.

Thus,

$$-\hat{V}_t = \Pi_t = \Delta S_t + \alpha_B P_{B_t} + \alpha_C P_{C_t} + \gamma.$$

Imposing the self-financing feature of the portfolio, we deduce:

$$d\Pi_t = \Delta dS_t + \alpha_B dP_{B_t} + \alpha_C dP_{C_t} + (r\gamma_F^+ + r_F\gamma_F^- - r\gamma_{P_C} - r_R\gamma_R)dt;$$

then, applying Ito's Lemma and eliminating all risks in the portfolio we obtain the PDE modelling the value of the derivative including the counterparty risk:

$$\begin{cases} \partial_t \hat{V} + \mathcal{A}_t \hat{V} - r\hat{V} = (\lambda_B + \lambda_C)\hat{V} + s_F M^+ \\ \qquad\qquad\qquad\qquad - \lambda_B(R_B M^- + M^+) - \lambda_C(R_C M^+ + M^-) \\ \hat{V}(T,S) = H(S), \end{cases}$$

where the parabolic differential operator $\mathcal{A}_t$ is given by:

$$\mathcal{A}_t V \equiv \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - r_R S \frac{\partial V}{\partial S},$$

and λ_B and λ_C are constant default intensities from the counterparties, R_B and R_C are the constant recovery rates of the two counterparties, s_F is the funding cost of the entity and M represents the Mark-to-Market value of $\hat{V}$ at default.

We consider two scenarios for the determination of the derivative Mark-to-Market value at default, namely that recovery is either on the total risky value or on the riskless value. Thus, according to the M value, two PDEs are obtained:

- if $M = \hat{V}$, we obtain the nonlinear PDE:

$$\begin{cases} \partial_t \hat{V} + \mathcal{A}_t \hat{V} - r\hat{V} = (1 - R_B)\lambda_B \hat{V}^- + (1 - R_C)\lambda_C \hat{V}^+ + s_F \hat{V}^+ \\ \hat{V}(T,S) = H(S), \end{cases}$$

– if $M = V$, the following linear PDE is obtained:

$$\begin{cases} \partial_t \hat{V} + \mathcal{A}_t \hat{V} - (r + \lambda_B + \lambda_C)\hat{V} \\ \qquad\qquad = -(R_B\lambda_B + \lambda_C)V^- - (R_C\lambda_C + \lambda_B)V^+ + s_F V^+ \\ \hat{V}(T,S) = H(S), \end{cases}$$

where $H(S)$ represents the pay–off of the derivative. European vanilla call and put options and forwards have been considered.

As we aim to compute the value of adjustment, risk derivative value is written as:

$$\hat{V} = V + U,$$

where U is the total value adjustment and the risk-free value, V, satisfies the classical Black–Scholes equation without counterparty risk:

$$\begin{cases} \partial_t V + \mathcal{A}_t V - rV = 0 \\ V(T,S) = H(S). \end{cases} \tag{1}$$

Thus, the total value adjustment PDE is obtained:

– if $M = \hat{V}$, we get a final value problem governed by a nonlinear PDE:

$$\begin{cases} \partial_t U + \mathcal{A}_t U - rU = (1 - R_B)\lambda_B (V + U)^- \\ \qquad\qquad + (1 - R_C)\lambda_C (V + U)^+ + s_F (V + U)^+ \\ U(T,S) = 0, \end{cases}$$

– if $M = V$, an analogous linear problem is deduced:

$$\begin{cases} \partial_t U + \mathcal{A}_t U - (r + \lambda_B + \lambda_C)U = (1 - R_B)\lambda_B V^- \\ \qquad\qquad + (1 - R_C)\lambda_C V^+ + s_F V^+ \\ U(T,S) = 0. \end{cases}$$

In both cases, variable S belongs to the unbounded domain $[0, +\infty)$, while t lies in $[0, T]$.

By using Feynman–Kac theorem, we obtain the total value adjustment (XVA) in terms of the expected value:

– if $M = \hat{V}$,

$$\begin{aligned} U(t,S) = &- (1 - R_B)\int_t^T \lambda_B(u) D_r(t,u) \mathbb{E}_t[(V(u,S(u)) + U(u,S(u)))^-]du \\ &- (1 - R_C)\int_t^T \lambda_C(u) D_r(t,u) \mathbb{E}_t[(V(u,S(u)) + U(u,S(u)))^+]du \\ &- \int_t^T s_F(u) D_r(t,u) \mathbb{E}_t[(V(u,S(u)) + U(u,S(u)))^+]du, \end{aligned} \tag{2}$$

– if $M = V$,

$$\begin{aligned} U(t,S) = & -(1-R_B)\int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t,u)\mathbb{E}_t[(V^-(u,S(u))]du \\ & -(1-R_C)\int_t^T \lambda_C(u) D_{r+\lambda_B+\lambda_C}(t,u)\mathbb{E}_t[(V^+(u,S(u))]du \\ & -\int_t^T s_F(u) D_{r+\lambda_B+\lambda_C}(t,u)\mathbb{E}_t[(V^+(u,S(u))]du. \end{aligned} \tag{3}$$

where the three lines of (2) (or (3)), from top to bottom, correspond to DVA, CVA and FCA, respectively.

3 Numerical Methods

In order to solve the previous models, different numerical methods are proposed. In particular, a finite elements method is used for spatial discretization; thus, the truncation to a spatial bounded domain is required and adequate boundary conditions have to be deduced.

We will focus on the nonlinear problem, although similar methods are used in the linear one.

In any case, the change of variable $\tau = T - t$ is considered in order to write the following initial condition problem:

$$\begin{cases} \dfrac{\partial U}{\partial \tau} - \dfrac{\sigma^2}{2} S^2 \dfrac{\partial^2 U}{\partial S^2} - r_R S \dfrac{\partial U}{\partial S} + rU = \\ \qquad\qquad = -(1-R_B)\lambda_B (V+U)^- - (1-R_C)\lambda_C (V+U)^+ - s_F (V+U)^+ \\ U(0,S) = 0. \end{cases} \tag{4}$$

3.1 Characteristics Method

Analogously to other advection–diffusion equations, we use a characteristics method [11] for time discretization. Thus, we consider the material derivative:

$$\frac{DU}{D\tau} = \frac{\partial U}{\partial \tau} + \frac{\partial U}{\partial S}\frac{\partial S}{\partial \tau}$$

so that we can write our equation as:

$$\begin{aligned} & \frac{DU}{D\tau} - \frac{\sigma^2}{2}\frac{\partial}{\partial S}\left(S^2 \frac{\partial U}{\partial S}\right) + rU = \\ & \qquad = -(1-R_B)\lambda_B (V+U)^- - (1-R_C)\lambda_C (V+U)^+ - s_F (V+U)^+. \end{aligned}$$

The use of characteristics for time discretizations leads to:

$$\begin{aligned} \frac{U^{n+1} - U^n \circ \chi^n}{\Delta\tau} - \frac{\sigma^2}{2}\frac{\partial}{\partial S}\left(S^2 \frac{\partial U^{n+1}}{\partial S}\right) + rU^{n+1} = -(1-R_B)\lambda_B (V+U^{n+1})^- \\ -(1-R_C)\lambda_C (V+U^{n+1})^+ - s_F (V+U^{n+1})^+ \end{aligned} \tag{5}$$

where $U^n(\cdot) \approx U(\tau^n, \cdot)$ and $\chi^n \equiv \chi(S, \tau^{n+1}; \tau^n)$ satisfies the final value problem:

$$\begin{cases} \dfrac{\partial \chi}{\partial \tau} = (\sigma^2 + \gamma_s - q_s)\chi(\tau) \\ \chi(\tau^{n+1}) = S. \end{cases}$$

The solution of this problem is

$$\chi(S, \tau^{n+1}; \tau^n) = S \exp((r_R - \sigma^2)\Delta\tau),$$

so that we can evaluate $U^n \circ \chi^n$ at each step of (5).

3.2 Fixed Point Scheme

In order to solve the of nonlinear Eq. (5) at each iteration of the characteristics method, we propose a fixed point algorithm. Thus, the global scheme can be written in the following way:

Let $N > 1$, $\varepsilon > 0$, U^0 given.
For $n = 0, 1, 2, \ldots$
  Let $U^{n+1,0} = U^n$
  For $k = 0, 1, 2, \ldots,$ $U^{n+1,k+1}$ is computed to satisfy:

$$\begin{aligned} (1 + r\Delta\tau)\, U^{n+1,k+1} - \frac{\sigma^2 \Delta\tau}{2} \frac{\partial}{\partial S}\left(S^2 \frac{\partial U^{n+1,k+1}}{\partial S}\right) = U^n \circ \chi^n \\ - \Delta\tau \big[(1 - R_B)\lambda_B(V^{n+1} + U^{n+1,k})^- \\ + (1 - R_C)\lambda_C(V^{n+1} + U^{n+1,k})^+ + s_F(V^{n+1} + U^{n+1,k})^+\big] \end{aligned} \tag{6}$$

  until $\|U^{n,k+1} - U^{n,k}\| \leq \varepsilon$.

3.3 Boundary Conditions

As previously indicated, we will use finite elements to discretize the previous equations. Thus, we need to truncate the unbounded domain $[0, +\infty)$ into a bounded one.

We will assume $S \in [0, S_\infty]$, where $S_\infty > 0$ is large enough. Let us introduce function f, defined by:

$$f(U, V) = (1 - R_B)\lambda_B(V + U)^- + (1 - R_C)\lambda_C(V + U)^+ + s_F(V + U)^+.$$

The left boundary condition is obtained just by replacing $S = 0$ in (4):

$$\partial_\tau U + rU = -f(U, V);$$

this equation is approximated by an implicit Euler method:

$$U^{n+1}(0) - U^n(0) + \Delta\tau\, r\, U^{n+1}(0) = -\Delta\tau\, f(U^{n+1}(0), V^{n+1}(0)),$$

so that the nonhomogeneous Dirichlet boundary condition is obtained for each step of the global algorithm:

$$U^{n+1,k+1}(0) = \frac{1}{1+r\Delta\tau}\left(U^n(0) - \Delta\tau\left[(1-R_B)\lambda_B(V^{n+1}(0)+U^{n+1,k}(0))^- \right.\right.$$
$$\left.\left. + (1-R_C)\lambda_C(V^{n+1}(0)+U^{n+1,k}(0))^+ + s_F(V^{n+1}(0)+U^{n+1,k}(0))^+\right]\right).$$

In order to deduce the boundary condition at $S = S_\infty$, we multiply Eq. (4) by S^{-2}; thus, when S tends to infinity the following condition is obtained:

$$\lim_{S\to\infty}\frac{\partial^2 U}{\partial S^2} = 0.$$

Then, following [5], we consider a solution of the form

$$U = H_0 + H_1 S,$$

where H_0 and H_1 are constant coefficients. By introducing this expression into each fixed point iteration, two simpler ODEs are obtained and the nonhomogeneous Dirichlet condition is posed:

$$\begin{aligned} U^{n+1,k+1}(S_\infty) = H_1^{n+1,k+1}S_\infty = \frac{1}{(r+\Delta\tau)}&\left((U^n\circ\chi^n)(S_\infty)\right. \\ &- \Delta\tau\left[(1-R_B)\lambda_B(V^{n+1}(S_\infty)+U^{n+1,k}(S_\infty))^-\right. \\ &\quad + (1-R_C)\lambda_C(V^{n+1}(S_\infty)+U^{n+1,k}(S_\infty))^+ \\ &\quad \left.\left. + s_F(V^{n+1}(S_\infty)+U^{n+1,k}(S_\infty))^+\right]\right). \end{aligned} \tag{7}$$

3.4 Finite Elements Method

We can now proceed with the spatial discretization. For this purpose, let us consider the functional spaces:

$$H^1(0,S_\infty) = \{\varphi \in L^2(0,S_\infty)/\frac{\partial\varphi}{\partial S} \in L^2(0,S_\infty)\}$$
$$W = H_0^1(0,S_\infty) = \{\varphi \in H^1(0,S_\infty)/\varphi(t,0)=0, \varphi(t,S_\infty)=0\}.$$

If we multiply both members of (6) by a function $\varphi \in V$ and integrate on $[0,S_\infty]$, the variational formulation consists in finding a function $U^{n+1} \in W$ such that:

$$(1+r\Delta\tau)\int_0^{S_\infty} U^{n+1}\varphi\,dS - \Delta\tau\int_0^{S_\infty}\frac{\partial}{\partial S}\left(\frac{\sigma^2}{2}S^2\frac{\partial U^{n+1}}{\partial S}\right)\varphi\,dS$$
$$= \int_0^{S_\infty}(U^n\circ\chi^n)(S)\varphi\,dS - \Delta\tau\int_0^{S_\infty} f(U^{n+1},V^{n+1})\varphi\,dS, \quad \forall\varphi\in W.$$

Since W is an infinite dimension functional space, a finite dimension subspace W_h is built. For this purpose, we consider a uniform finite element mesh. Let $M > 0$ such that

$$h = \frac{S_\infty - S_0}{M+1} > 0,$$

and $S_j = S_0 + jh$ for $j = 0, \ldots, M+1$.

Let us define the functional spaces

$$W_h = \{\varphi_h : (0, S_\infty) \to \mathbb{R} \,/\, \varphi_h \in \mathcal{C}(0, S_\infty),\ \varphi_h|_{[S_j, S_{j+1}]} \in \mathcal{P}_1\}$$
$$W_{h,0} = \{\varphi_h \in W_h \,/\, \varphi_h(0) = 0, \varphi_h(S_\infty) = 0\}.$$

The discrete problem consists in finding $U_h^{n+1} \in W_h$ such that:

$$(1 + r\Delta\tau) \int_0^{S_\infty} U_h^{n+1} \varphi_h \, dS - \Delta\tau \int_0^{S_\infty} \frac{\partial}{\partial S} \left(\frac{\sigma^2}{2} S^2 \frac{\partial U_h^{n+1}}{\partial S} \right) \varphi_h \, dS$$
$$= \int_0^{S_\infty} (U_h^n \circ \chi^n)(S) \varphi_h \, dS - \Delta\tau \int_0^{S_\infty} f(U_h^{n+1}, V^{n+1}) \varphi_h \, dS,$$

forall $\varphi_h \in W_{h,0}$. Applying Green's formula and classical properties of integrals, we obtain:

$$\sum_{j=0}^{M} (1 + r\Delta\tau) \int_{S_j}^{S_{j+1}} U_h^{n+1} \varphi_h dS + \sum_{j=0}^{M} \Delta\tau \int_{S_j}^{S_{j+1}} \left(\frac{\sigma^2}{2} S^2 \frac{\partial U_h^{n+1}}{\partial S} \right) \frac{\partial \varphi_h}{\partial S} dS$$
$$= \sum_{j=0}^{M} \int_{S_j}^{S_{j+1}} (U_h^n \circ \chi^n) \varphi_h dS - \sum_{j=0}^{M} \Delta\tau \int_{S_j}^{S_{j+1}} f(U_h^{n+1}, V^{n+1}) \varphi_h dS, \tag{8}$$

where U_h^{n+1} and φ_h^{n+1} are polynomials of degree less or equal than one on each interval $[S_j, S_{j+1}]$.

We have thus deduced a system of linear Eq. (8), that can be written as:

$$\left[(1 + r\Delta\tau) A_h^1 + \frac{\Delta\tau}{2} \sigma^2 A_h^2 \right] U_h = b_h^1 - \Delta\tau b_h^2,$$

where matrices A_h^1 and A_h^2 and vectors b_h^1 and b_h^2 are built by conveniently assembling the contributions of each element. Moreover, adequate quadrature formulae are used in order to approximate each integral. More precisely,

$$\left[A_h^j \right]^1 \approx \frac{h}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$
$$\left[A_h^j \right]^2 \approx \frac{1}{h_j} \sum_{i=0}^{2} \omega_i (S_j + h_j x_i)^2 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$
$$\left[b_h^j \right]^1 \approx \frac{S_{j+1} - S_j}{2} (U^n \circ \chi^n) \left(\frac{S_{j+1} + S_j}{2} \right) \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$\left[b_h^j \right]^2 \approx \frac{S_{j+1} - S_j}{2} \begin{pmatrix} f(U^{n+1}(S_j), V^{n+1}(S_j)) \\ f(U^{n+1}(S_{j+1}), V^{n+1}(S_{j+1})) \end{pmatrix},$$

where Simpson, three node Gaussian, midpoint and trapezoidal formulae have been respectively used.

4 Numerical Results

In this section, one problem is simulated and the behaviour of the XVA is analyzed. We first study the error and order of convergence of the applied numerical methods, for which we take advantage of the analytic solution of the XVA problem in particular cases. If $M = \hat{V}$ and considering funding cost, $s_F = (1-R_B)\lambda_B$, the analytical solution is:

$$U(t, S) = -(1 - \exp(-((1 - R_B)\lambda_B + (1 - R_C)\lambda_C)(T - t)))\, V(t, S).$$

As we can observe in Table 1, the order of convergence obtained with the discrete norm $L^\infty((0, T) \times L^2([0, S_\infty]))$ is one.

Table 1. Relative errors in norm $L^\infty((0, T) \times L^2([0, S_\infty]))$, convergence ratios and order. Example with finite elements scheme. The input parameters are $E = 15$, $S \in [0, 4E]$, $r = 0.03$, $r_R = 0.015$, $\sigma = 0.25$, $t \in [0, 5]$, $\lambda_B = 0.02$, $\lambda_C = 0.05$, $R_B = 0.4$ and $R_C = 0.4$

Time step	Space step	Error	R	Order
400	50	0.02232872	-	-
800	100	0.01192059	1.87312280	0.90544548
1600	200	0.00617545	1.93031711	0.94883787
3200	400	0.00315299	1.95860211	0.96982435
6400	800	0.00160323	1.96665313	0.97574253

We show in Figs. 1 and 2 the XVA value as a percentage of the risk-free value, V. We can see the relevance of the choice of the mark-to-market value at default (either V or $\hat{V}$), as well as the funding cost.

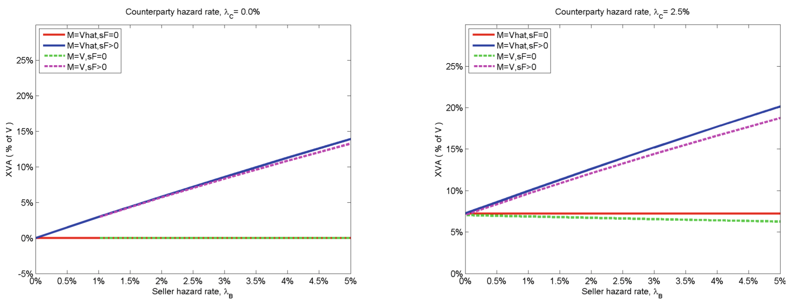

Fig. 1. CVA in the cases $M = \hat{V}$ and $M = V$ for $\lambda_C = 0\%$ and $\lambda_C = 2.5\%$ in $t = 0$

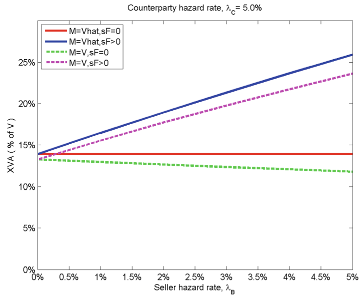

Fig. 2. CVA in the cases $M = \hat{V}$ and $M = V$ for $\lambda_C = 5\%$ in $t = 0$

5 Conclusions

In this work, we have assessed the derivative value taking into account different adjustments in the risk-free value. Models are posed when V depends on one stochastic factor, S.

To solve this problem, different numerical methods have been applied. We have analyzed one example where we can observe the relevance of considering different kinds of risks and funding cost regarding risk-free value, as well as the performance of the numerical methods.

Therefore, we can conclude the significance of taking into account different adjustments. As a result, a new valuation framework of derivatives is created, where both types of financing and counterparty risk must be considered.

References

1. Burgard, C., Kjaer, M.: In the balance. Risk **11**, 1–12 (2011)
2. Burgard, C., Kjaer, M.: PDE representations of derivatives with bilateral counterparty risk and funding costs. J. Credit Risk **7**(3), 1–19 (2011)
3. Burgard, C., Kjaer, M.: Funding costs, funding strategies. J. Credit Risk **11**, 1–17 (2013)
4. Calvo Garrido, M.C.: Mathematical analysis and numerical methods for pricing some pension plans and mortgages. Ph.D. thesis, Universidade da Coruña (2014)
5. Castillo, D., Ferreiro, A.M., García-Rodríguez, J.A., Vázquez, C.: Numerical methods to solve PDE models for pricing business companies in different regimes and implementation in GPUs. Appl. Math. Comput. **219**, 11233–11257 (2013)
6. Gregory, J.: Counterparty Credit Risk and Credit Value Adjustment. Wiley Finance, Hoboken (2012)
7. Hull, J., White, A.: Valuing derivatives: funding value adjustment and fair value. Financ. Anal. J. **3**, 1–27 (2014)
8. Huzinger, C.B.: Reviewing a framework to price a credit risky derivative post the credit crisis. Ph.D. thesis, Witwatersrand University (2014)
9. Mikosh, T.: Elementary Stochastic Calculus with Finance in View. World Scientific Press, Singapur (1998)

10. Ruiz, I.: A Complete XVA Valuation Framework. Why the "Law of One Price" is dead. iRuiz Consulting **12**, 1–23 (2014)
11. Vázquez, C.: An upwind numerical approach for an American and European option pricing model. Appl. Math. Comput. **97**, 273–286 (1998)
12. Wilmott, P., Howison, S., Dewynne, J.: The mathematics of financial derivatives. A students introduction. Cambridge University Press, Cambridge (1996)
13. Wilmott, P., Howison, S., Dewynne, J.: Option Pricing: Mathematical Models and Computation. Oxford Financial Press, Oxford (1996)

http://www.springer.com/978-3-319-57098-3

Numerical Analysis and Its Applications
6th International Conference, NAA 2016, Lozenetz,
Bulgaria, June 15-22, 2016, Revised Selected Papers
Dimov, I.; Faragó, I.; Vulkov, L. (Eds.)
2017, XIV, 785 p. 208 illus., Softcover
ISBN: 978-3-319-57098-3