CHAPTER 2

Methods of constructing topological vector spaces

In this chapter we consider projective limits (in particular, products) of families of topological vector spaces, inductive limits (in particular, topological direct sums) of families of locally convex spaces, including strict inductive limits and inductive limits with compact embeddings, tensor products of locally convex spaces, and nuclear spaces.

Throughout this chapter the symbol $\mathbb{K}$ denotes (if it is not explicitly stated otherwise) the field of complex or the field of real numbers; it is assumed that all considered vector and topological vector spaces are spaces over $\mathbb{K}$.

2.1. Projective topologies

We define projective limits for topological vector spaces, but the construction applies to general topological spaces.

2.1.1. Definition. *Let E be a vector space and let $\mathfrak{A}$ be some index set. Suppose that for every $\alpha \in \mathfrak{A}$ we are given a topological vector space E_α and a linear mapping $g_\alpha\colon\ E \to E_\alpha$. The projective topology of the family of spaces $\{E_\alpha\}$ with respect to the family of mappings $\{g_\alpha\}$ is the weakest topology in E with respect to which all mappings $\{g_\alpha\}$ are continuous. The projective limit of the family $\{E_\alpha\}$ with respect to the family of mappings $\{g_\alpha\}$ is the vector space E equipped with this topology.*

We verify that the topology in this definition is well-defined, i.e., we prove that the projective topology of the family of spaces $\{E_\alpha\}$ with respect to the family $\{g_\alpha\}$ of mappings exists. The proof is an explicit description of this topology. For every index $\alpha \in \mathfrak{A}$ let $\mathcal{W}_\alpha$ be the class of all sets of the form ${g_\alpha}^{-1}(V)$, where V is an open subset in E_α. Set $\mathcal{W} = \bigcup_\alpha \mathcal{W}_\alpha$. Then the collection $\mathcal{P}$ of the intersections of all possible finite families of sets from $\mathcal{W}$ forms a base of a topology τ in E possessing the required properties, i.e., the weakest topology t in E for which all mappings $g_\alpha\colon\ (E,t) \to E_\alpha$ are continuous.

In order to see this, we have to verify the following assertions:

(1) $\mathcal{P}$ is a base of some topology τ in E; (2) all mappings $g_\alpha\colon\ (E,\tau) \to E_\alpha$ are continuous; (3) the topology τ is majorized by every topology t in E for which all mappings $g_\alpha\colon\ (E,t) \to E_\alpha$ are continuous.

The validity of (1) follows from the fact that the intersection of every finite collection of subsets from $\mathcal{P}$ belongs to $\mathcal{P}$, so that the collection of all subsets

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of the space E each of which is the union of some family of sets in $\mathcal{P}$ forms a topology. It will be denoted by τ. It is clear that $\mathcal{P}$ is a base of this topology (the family $\mathcal{W}$ is a prebase).

If now $\alpha \in \mathfrak{A}$ and V is an open subset in E_α, then $g_\alpha^{-1}(V) \in \mathcal{W} \subset \mathcal{P} \subset \tau$ by the definition of $\mathcal{W}$, $\mathcal{P}$ and τ, so that the mappings $g_\alpha\colon (E, \tau) \to E_\alpha$ are continuous, i.e., (2) holds.

Let us verify (3). Let t be a topology in E such that for every $\alpha \in \mathfrak{A}$ the mapping $g_\alpha\colon (E, t) \to E_\alpha$ is continuous. Then every set in $\mathcal{W}$ is open in t, i.e., we have $\mathcal{W} \subset t$. Hence $\tau \subset t$ by the definition of τ.

2.1.2. Remark. So far we have never used that for every $\alpha \in \mathfrak{A}$ the topology of the space E_α agrees with the structure of a vector space and that all mappings g_α are linear, as well as that E and E_α are vector spaces. Thus, the definition of the projective limit is meaningful and remains correct without these assumptions (i.e., in the case where E is an arbitrary set, $\{E_\alpha\}$ is an arbitrary family of topological spaces and $\{g_\alpha\}$ is an arbitrary family of mappings from E to E_α).

2.1.3. Proposition. *Under the assumptions of Definition* 2.1.1 *the projective topology τ agrees with the vector structure of the space E.*

PROOF. This follows from Corollary 1.2.9. Indeed, all E_α are topological vector spaces, all mappings g_α are linear. Hence every family of sets $\mathcal{W}_\alpha$ is invariant with respect to translations; then the families $\mathcal{W}$ and $\mathcal{P}$ are also invariant with respect to translations, hence the topology τ is invariant as well.

Further, by the very definition of the topology τ the set $\mathcal{O}$ of all subsets of the space E each of which is the intersection of some finite family of sets of the form $g_\alpha^{-1}(V)$, where V is an open circled neighborhood of zero in E_α, is a base of neighborhoods of zero in the topology τ, moreover, this base possesses properties (1) and (2) from Proposition 1.2.2. Hence by Corollary 1.2.9 the topology τ agrees with the structure of a vector space. □

The term the "projective limit" is often used for a more special construction (which will be discussed below); on the other hand, projective topologies in our sense are also called "initial".

2.1.4. Proposition. *If in Proposition* 2.1.1 *all topological vector spaces E_α are locally convex, then the topology τ is locally convex as well.*

PROOF. The justification is similar to the proof of Proposition 2.1.3; we have only replace the words "an open circled neighborhood" by the words "an open convex circled neighborhood". □

2.1.5. Proposition. *Let E be a topological vector space that is the projective limit of a family $\{E_\alpha\colon \alpha \in \mathfrak{A}\}$ of topological vector spaces with respect to linear mappings $\{g_\alpha\colon \alpha \in \mathfrak{A}\}$. A mapping f of an arbitrary topological space G to the topological vector space E is continuous at a point $x \in G$ precisely when for every $\alpha \in \mathfrak{A}$ the mapping $g_\alpha \circ f\colon G \to E_\alpha$ is continuous at this point.*

PROOF. Since the composition of continuous (at the corresponding points) mappings is continuous, it suffices to prove that the continuity of the mappings

$g_\alpha \circ f$ ($\alpha \in \mathfrak{A}$) implies the continuity of f. Let V be an open neighborhood of the point $f(x)$. We have to show that there exists a neighborhood W of the point x such that $f(W) \subset V$. By the definition of the projective topology there exist indices $\alpha_1, \ldots, \alpha_n \in \mathfrak{A}$ and open subsets $V_1, \ldots, V_n$ of the spaces $E_{\alpha_1}, \ldots, E_{\alpha_n}$ for which $f(x) \in \bigcap_{i=1}^n g_{\alpha_i}^{-1}(V_i) \subset V$. Since every mapping $g_{\alpha_i} \circ f$ is continuous, there exists an open neighborhood W_i of the point x such that $(g_{\alpha_i} \circ f)(W_i) \subset V_i$ for every index $i \in \{1, 2, \ldots, n\}$. This means that $f(W_i) \subset g_{\alpha_i}^{-1}(V_i)$ for every such i. Hence $f(\cap_i W_i) \subset g_{\alpha_i}^{-1}(V_i)$ for all i. Therefore, if $W = \bigcap_{i=1}^n W_i$, then we have $f(W) \subset \bigcap_{i=1}^n g_{\alpha_i}^{-1}(V_i)$. □

If E is a locally convex space and p is a continuous seminorm on E, then the symbol E_p or (E_p, p) denotes the normed space defined as follows: the vector space E_p is the vector factor-space of the vector space E by its subspace $p^{-1}(0)$; in addition, p is the norm on E_p defined as follows: if $x \in E_p$ and x_1 is a representative of the class x, then $p(x) = p(x_1)$.

Note that the canonical mapping $g_p \colon E \to E_p$ is continuous as the composition of two continuous mappings: the identity mapping of the space E equipped with the original topology to E equipped with the topology defined by the seminorm p (the latter is denoted by the symbol (E, p)) and the canonical mapping of (E, p) to the quotient E_p (actually this is the same mapping g_p, but considered as a mapping from (E, p) to the space E_p).

A general projective limit is a very universal object.

2.1.6. Proposition. *Every locally convex space E is the projective limit of the family $\{E_p \colon p \in \mathcal{P}\}$ of normed spaces with respect to the canonical mappings $\{g_p \colon p \in \mathcal{P}\}$, where $\mathcal{P}$ is the set of all continuous seminorms on E.*

PROOF. This follows from the fact that every locally convex topology is defined by the set of all continuous seminorms on this space. □

2.1.7. Remark. A similar proposition is valid for arbitrary topological vector spaces (i.e., spaces that are not locally convex). It suffices to replace the word "seminorm" by the word "quasi-norm" and the word "normed" by the word "metrizable" in the formulation of the proposition above.

2.1.8. Remark. For every seminorm p on E we denote by $\mathcal{E}_p$ the Banach space serving as the completion of the normed space E_p. It follows from the previous proposition that every locally convex space is the projective limit of the family of Banach spaces $\{\mathcal{E}_p\}$ with respect to the corresponding canonical mappings. A similar proposition is valid for arbitrary topological vector spaces, i.e., not necessarily locally convex (in this case the role of Banach spaces is played by complete metrizable topological vector spaces).

2.1.9. Remark. If τ is the projective topology in a vector space E with respect to a certain family $\{E_\alpha \colon \alpha \in \mathfrak{A}\}$ of topological vector spaces and linear mappings $\{g_\alpha \in \mathcal{L}(E, E_\alpha) \colon \alpha \in \mathfrak{A}\}$, then in order that τ be separated, it is necessary and sufficient that for every nonzero element $x \in E$ one could find $\alpha \in \mathfrak{A}$ and a neighborhood of zero V_α in E_α such that $g_\alpha(x) \notin V_\alpha$. In particular, if all topological vector spaces E_α are separated, then in order that E be also separated, it is necessary and sufficient that for every nonzero element $x \in E$ one could find an index $\alpha \in \mathfrak{A}$ such that $g_\alpha(x) \neq 0$. Both assertions follow directly from the definition.

2.2. Examples of projective limits

Let us consider some examples of projective limits.

2.2.1. Example. (*The least upper bound of a family of topologies in a vector space.*) Let E be a vector space and let $\mathfrak{A}$ be an index set such that for every $\alpha \in \mathfrak{A}$ we are given a topology τ_α in E compatible with the structure of a vector space. Then there exists a topology τ in E that is the least upper bound of the set $\{\tau_\alpha\colon\ \alpha \in \mathfrak{A}\}$ in the set of all topologies in E, i.e., the weakest among topologies in E each of which majorizes every topology τ_α. The topology τ agrees with the structure of a vector space and is locally convex if so are all τ_α.

Indeed, the required property holds for the topology of the projective limit of the family of topological vector spaces $\{E_\alpha\colon\ \alpha \in \mathfrak{A}\}$ with respect to the family of mappings $\{g_\alpha\colon\ \alpha \in \mathfrak{A}\}$, where for every $\alpha \in \mathfrak{A}$ we take $E_\alpha = (E, \tau_\alpha)$, and g_α is the identity mapping of the space E. The locally convex case follows from Proposition 2.1.4.

2.2.2. Example. (*Subspaces*, see Example 1.3.10). Let E be a topological vector space, let E_1 be its topological vector subspace (i.e., a vector subspace equipped with the induced topology), and let $g\colon E_1 \to E$ be the canonical embedding. Then E_1 is the projective limit of the one-element family of topological vector spaces $\{E\}$ with respect to the one-element family of mappings $\{g\}$.

2.2.3. Example. (*The product of topological vector spaces.*) Let $\mathfrak{A}$ be a nonempty set and let (E_α, τ_α) be a topological vector space, $\alpha \in \mathfrak{A}$. Let E be the vector space that is the product of the family of vector spaces $\{E_\alpha\colon\ \alpha \in \mathfrak{A}\}$. Thus, the set of elements of E is the set of all functions f on the set $\mathfrak{A}$ with values in the set $\bigcup_{\alpha\in\mathfrak{A}} E_\alpha$ such that $f(\alpha) \in E_\alpha$ for every $\alpha \in \mathfrak{A}$; the structure of a vector space in E is introduced by the relations

$$(\lambda_1 f_1 + \lambda_2 f_2)(\alpha) = \lambda_1 f_1(\alpha) + \lambda_2 f_2(\alpha), \quad \lambda_1, \lambda_2 \in \mathbb{K},\ f_1, f_2 \in E.$$

For every $\alpha \in \mathfrak{A}$ let pr_α denote the projection of E onto E_α defined as follows: if $f \in E$, then $\mathrm{pr}_\alpha(f) = f(\alpha)$. The topology (in E) of the projective limit of the family $\{E_\alpha\colon\ \alpha \in \mathfrak{A}\}$ of topological vector spaces with respect to the family of mappings $\{\mathrm{pr}_\alpha\colon\ \alpha \in \mathfrak{A}\}$ coincides with the topology of Tychonoff's product; this follows from their definitions. Throughout the product of a family of topological vector spaces will be understood as their product of vector spaces equipped with the Tychonoff product topology; if $\{G_\alpha\}$ is the corresponding family of topological vector spaces, then the symbol $\prod_{\alpha\in\mathfrak{A}} G_\alpha$ will denote their product.

2.2.4. Proposition. *Suppose that E is the projective limit of a certain family $\{E_\alpha\colon\ \alpha \in \mathfrak{A}\}$ of topological vector spaces with respect to a certain family of mappings $\{g_\alpha \in \mathcal{L}(E, E_\alpha)\colon\ \alpha \in \mathfrak{A}\}$; assume that its topology is separated. Then*

E is isomorphic — as a topological vector space — to some topological vector subspace in the product G of a family topological vector spaces $\{E_\alpha\colon\ \alpha \in \mathfrak{A}\}$.

PROOF. The mapping $\Psi\colon E \to G$ defined by: $\Psi(x)(\alpha) = g_\alpha(x)$ is linear by the linearity of all g_α. This mapping is injective, since, due to the assumption that E is separated, for every $x \in E$ there exists $\alpha \in \mathfrak{A}$ such that $g_\alpha(x) \neq 0$. Further, for every $\alpha \in \mathfrak{A}$ the composition $\mathrm{pr}_\alpha \circ \Psi$ coincides with the mapping g_α, hence is continuous. By Proposition 2.1.5 the mapping Ψ is continuous. For completing the proof it remains to verify that the mapping $\Psi^{-1}\colon\ \Psi(E) \to E$ is also continuous (assuming that the vector subspace $\Psi(E)$ of the space G is equipped with the topology induced by the topology of the space G). For every index $\alpha \in \mathfrak{A}$ the composition of the mapping Ψ^{-1} taking the element $g(x)$ (i.e., the function $\alpha \mapsto g_\alpha(x)$) of the space $\Psi(E)$ to the element $x \in E$ and the mapping g_α coincides with the restriction to $\Psi(E)$ of the projection mapping $\mathrm{pr}_\alpha\colon\ g(x) \mapsto g_\alpha(x)$ and hence is continuous by the definition of the product topology. Hence — again by Proposition 2.1.5 — the mapping Ψ^{-1} is also continuous. Thus, the mapping Ψ is a linear homeomorphism of E onto the topological vector subspace $\Psi(E)$ of the space $\prod_\alpha E_\alpha$. □

2.2.5. Corollary. *Every Hausdorff locally convex space E is isomorphic to a topological vector subspace of the product $\prod_{p\in\mathcal{P}} E_p$, where $\mathcal{P}$ is the set of all continuous seminorms on E.*

This fact follows from Propositions 2.1.6 and 2.2.4.

2.2.6. Example. (*Weak topologies*, Example 1.3.23). Let E be a vector space, let G be a vector subspace in E^*, and, for every $g \in G$, let E_g be a copy of the field $\mathbb{K}$ considered as a one-dimensional topological vector space (over the field $\mathbb{K}$). Then the topology of the projective limit of the family of topological vector spaces $\{E_g\colon\ g \in G\}$ with respect to the family of mappings $\{g\colon\ g \in G\}$ is the weak topology in E defined by the elements of the set G.

2.2.7. Example. (*Limits of inverse spectra of topological vector spaces.*) Let $\mathfrak{A}$ be a directed set. A family $\{E_\alpha\colon\ \alpha \in \mathfrak{A}\}$ of topological vector spaces is called the *inverse spectre of topological vector spaces* E_α if, for every pair of indices $\alpha, \beta \in \mathfrak{A}$ with $\alpha \leqslant \beta$, a continuous linear mapping $\psi_{\alpha\beta}\colon\ E_\beta \to E_\alpha$ is given. The limit of such inverse spectre is the topological vector subspace in the product $\prod_\alpha E_\alpha$ denoted by the symbol $\varprojlim E_\alpha$ and consisting of elements $g \in \prod_\alpha E_\alpha$ such that $g(\alpha) = \psi_{\alpha\beta} g(\beta)$ whenever $\alpha, \beta \in \mathfrak{A}$, $\alpha \leqslant \beta$. For example, if $E_\beta \subset E_\alpha$ whenever $\alpha < \beta$ and the natural embedding $E_\beta \to E_\alpha$ is continuous, then $\bigcap_\alpha E_\alpha$ is the limit of the inverse spectre of the spaces E_α.

Every topological vector subspace of the product of an arbitrary family of topological vector spaces is the projective limit of this family with respect to the family of mappings of the regarded space to the factors that are the restrictions of the projection mappings of the product to these factors. Therefore, in particular, the space $\varprojlim E_\alpha$ is the projective limit of the family of topological vector spaces

$\{E_\alpha\colon \alpha \in \mathfrak{A}\}$ with respect to the family of mappings that are the restrictions of the corresponding projections.

Note also that by the continuity of the mappings $\psi_{\alpha\beta}$ the topological vector space $\varprojlim E_\alpha$ is a closed subspace in the product $\prod_\alpha E_\alpha$ (verify this). Since, in addition, any closed subset in a complete topological vector space is complete, and the product of an arbitrary family of complete topological vector spaces is a complete topological vector space, the limit of an inverse spectre of complete topological vector spaces is a complete topological vector space. An analogous assertion is valid also for quasi-complete topological vector spaces.

Let us introduce one more interesting class of spaces.

2.2.8. Example. (*Countably normed spaces.*) This term is used for locally convex spaces that are limits of inverse spectra of Banach spaces possessing the following properties: (a) the index set is the set of natural numbers with its usual order; (b) all mappings ψ_{nj} (defined for $j \geqslant n$) are injective.

The notion of a countably normed space was introduced by Gelfand and Shilov [**193**] by means of another definition: in their book a topological vector space is called countably normed if it is locally convex, metrizable, complete, and its topology can be defined by a countable family of compatible norms. Here two norms on a vector space E are called *compatible* if every sequence of elements in this space that is Cauchy with respect to both norms and converges to zero with respect to one of them must converge to zero with respect to the other.

As an example of incompatible norms on an infinite-dimensional Banach space X we can take the original norm $\|\cdot\|$ on X and the norm $x \mapsto \|x\| + |l(x)|$, where l is any discontinuous linear functional on X. Then one can always find vectors $x_n \in X$ such that $\|x_n\| \to 0$ and $l(x_n) = 1$.

We shall call our definition of countably normed spaces Definition I, and the definition from the book [**193**] will be called Definition II.

We now show that these definitions are equivalent. First we show that the requirements of Definition I imply the requirements of Definition II. Thus, let us consider the inverse spectre $\{E_n\colon n \in \mathbb{N}\}$ of Banach spaces satisfying the conditions in Definition I.

For every $j \in \mathbb{N}$, the norm of E_j will be denoted by the symbol $\|\cdot\|_j$. Since all mappings $\psi_{nj}\colon E_j \to E_n$ are injective, every space E_n can be identified as a vector space with a subspace of each space E_i $(i < n)$ with smaller indices. Thus, replacing the words "is a vector subspace" by the symbol $\subset\subset$, we obtain the following chain of relations:

$$\cdots \subset\subset E_n \subset\subset E_{n-1} \subset\subset \cdots \subset\subset E_1.$$

Here the mappings ψ_{jr} will coincide with the corresponding (identical) embeddings. We emphasize that in general the spaces E_r are not topological vector subspaces of the spaces E_j with smaller indices j, they are merely vector subspaces.

Therefore, the space $\varprojlim E_n$ can be identified as a vector space with the intersection $\bigcap_n E_n$ of the spaces E_n. Indeed, $g \in \varprojlim E_n$ precisely when for all $j \geqslant n$ we have the equality $\psi_{nj}\big(g(j)\big) = g(n)$. Since ψ_{nj} is an embedding, these equalities actually mean that $g(1) = g(2) = \cdots = g(n) = \cdots$, so that a natural identification of $\varprojlim E_n$ and $\bigcap_n E_n$ can be defined as follows: $g \in \varprojlim E_n \Longleftrightarrow g(1) \in \bigcap_n E_n$.

Thus, we have shown that $\varprojlim E_n$ coincides with $\bigcap_n E_n$ as a vector space. As we have observed before Example 2.2.8, the topology of the space $\varprojlim E_n$ is the topology of the projective limit of the family of topological vector spaces $\{E_j\colon j \in \mathbb{N}\}$ with respect to the family of mappings that are the restrictions to $\varprojlim E_n$ of the mappings $\mathrm{pr}_j\colon \prod_n E_n \to E_j$. Identifying $\varprojlim E_n$ and $\bigcap_n E_n$, we can assume that the mappings pr_j are defined on $\bigcap_n E_n$; each of these mappings is an embedding into the respective space; say, pr_j coincides on $\bigcap_n E_n$ with the identical embedding $\mathrm{in}_j\colon \bigcap_n E_n \to E_j$. Hence if we equip $\bigcap_n E_n$ with the topology τ of the projective limit of the family of Banach spaces $\{E_n\colon n \in \mathbb{N}\}$ with respect to the mappings in_j, then the identification of the spaces $\varprojlim E_n$ and $\bigcap_n E_n$ described above will be their identification as topological vector spaces.

It follows from Definition I that the topology of the projective limit in $\bigcap_n E_n$ defined above is given by the family of norms $\|\cdot\|_n$ (more precisely, by the restrictions of these norms to $\bigcap_n E_n$). We show that these norms are compatible. Suppose that a sequence $\{x_k\} \subset \bigcap_n E_n$ is Cauchy in both norms $\|\cdot\|_j$, $\|\cdot\|_n$ and $\|x_k\|_n \to 0$. If $n > j$, then the relation $\|x_k\|_j \to 0$ follows from the continuity of the embedding $\psi_{jn}\colon E_n \to E_j$. If $n < j$, then $\{x_k\}$ converges to some x in E_j by the completeness of E_j, which gives the equality $x = 0$ by the continuity and the injectivity of ψ_{nj}. Thus, it is shown that the space $(\bigcap_n E_n, \tau)$, hence also the space $\varprojlim E_n$ that coincides with it, is a countably normed space in the sense of Definition II.

Suppose now that the space E satisfies the conditions of Definition II and let $\{p_j\}$ be compatible norms defining the topology of this space. Replacing, if necessary, the norms p_j by the norms $p_j' = \sum_{n=1}^{j} p_n$ (which define the same topology) and keeping the previous notation we can assume that for all $x \in E$ and n we have $p_n(x) \leqslant p_{n+1}(x)$; the consistence of norms is preserved.

Further, let E_n be the completion of E with respect to the norm p_n, $n \in \mathbb{N}$. For each pair $n, j \in \mathbb{N}$ we define a continuous linear mapping $\psi_{n,n+j}\colon E_{n+j} \to E_n$ as the extension by the continuity of the identity mapping (such an extension exists according to Proposition 1.7.14).

The consistency of our norms yields the injectivity of these mappings, i.e., the equality $\psi_{n,n+j}(x) = 0$, where $x \in E_{n+j}$, yields that $x = 0$. Indeed, there exists a sequence $\{x_i\} \subset E$ converging in E_{n+j} to x. This sequence is fundamental in the norm p_{n+j}, moreover, $p_n(x_i) \to 0$, since $\psi_{n,n+j}(x) = 0$. By the consistency of these norms $p_{n+j}(x_i) \to 0$, whence $x = 0$.

Thus, the family $\{E_n\colon n \in \mathbb{N}\}$ of Banach spaces forms a spectre with respect to the injective mappings $\{\psi_{ns}\colon n \leqslant s\}$ (ψ_{nn} is the identity mapping of E_n to E_n, which, as above, is assumed to be an embedding of vector spaces).

As we have shown above, the space $\varprojlim E_n$ can be identified with the vector space $\bigcap_n E_n$ equipped with the topology defined by the family of norms p_n. Thus, for the proof of the fact that E is a countably normed space in the sense of Definition I, it remains to verify that the set $\bigcap_n E_n$ coincides as a vector space with E.

Since the inclusion $E \subset \bigcap_n E_n$ is true by definition (all spaces E_n are completions of E), we have to prove the opposite inclusion. Let $x \in \bigcap_n E_n$. This means that for every $n \in \mathbb{N}$ there is a sequence $\{x_j^n\} \subset E_n$ converging to x in E_n, i.e., in the norm p_n. Then we can choose a "quasi-diagonal" sequence $\{x_{j(n)}^n\}$ converging in every space E_n, hence fundamental in every norm p_n. Since they define the topology in E, this means that the sequence $\{x_{j(n)}^n\}$ is Cauchy in E and by the completeness of E it converges to some element $z \in E$. This sequence converges to z also in the space E_1 (being converging in E, it converges in every norm p_n). However, $\{x_{j(n)}^n\}$ is a subsequence of the sequence $\{x_j^1\}$ converging to x in E_1 and hence also converges in this space to x. Thus, $x = z$, i.e., $x \in E$. This completes the proof of the equivalence of both definitions.

2.2.9. Remark. Let us note the following fact established in Example 2.2.8. Suppose we are given a sequence of embedded Banach spaces $E_{n+1} \subset\subset E_n$ (as above, the symbol $\subset\subset$ means that the left space is a vector subspace in the right space), where all embeddings are continuous. Such a family can be regarded as the inverse spectre of these spaces the index set of which is the set of natural numbers and the role of the mappings ψ_{nj} is played by the embeddings.

Then the limit of such inverse spectre is (as a vector space) the intersection of all spaces E_n and its topology is defined by means of the restrictions to $\bigcap_n E_n$ of the norms p_n of the Banach spaces E_n.

2.2.10. Example. Certainly, not every Fréchet space is countably normed. For example, the countable product of the real lines $\mathbb{R}^\infty$ is not, since on it there are no continuous norms at all (every neighborhood of zero in this space contains some infinite-dimensional vector subspace).

2.2.11. Example. More interesting is the fact that even a Fréchet space whose topology is defined by a countable collection of norms need not be countably normed (observe by the way that in order the topology of a locally convex space, not necessarily metrizable, could be defined by a family of norms it is sufficient that this space had at least one continuous norm).

A Fréchet space in question can be defined as follows. Let E be the space of all continuously differentiable real functions f on the real line with the following property: $|f(t)| + |f'(t)| \to 0$ as $|t| \to \infty$. For every natural number n denote by p_n the norm on E defined by the equality

$$p_n(f) = \max_{t\in\mathbb{R}^1} |f(t)| + \max_{t\in[-n,n]} |f'(t)| \\ + \max\Bigl\{\bigl|f'\bigl(r + 1/(2k)\bigr)\bigr| \colon\ r \in \mathbb{Z};\ |r| > n;\ k = 1, 2, \ldots, n\Bigr\}.$$

Let τ be the topology in the space E defined by the countable family of norms $\mathcal{P} = \{p_n\colon\ n \in \mathbb{N}\}$; then (E, τ) is a Fréchet space.

We show that on this space there exists no countable family of compatible norms defining its topology. First of all, no two norms in the family $\mathcal{P}$ are compatible (verify this!).

Suppose now that E possesses a family $\mathcal{P}_1$ of compatible norms defining the topology of E. We can assume that these norms are increasing, so that the set of the corresponding balls $\{x \in E\colon\ q_j(x) < \varepsilon\}$ is a base (not only a prebase) of neighborhoods of zero. Let $q \in \mathcal{P}_1$. Then there exist two distinct norms $p_{j_1}, p_{j_2} \in \mathcal{P}$ such that $q(x) \leqslant Cp_{j_1}(x) \leqslant Cp_{j_2}(x)$ for every $x \in E$, where $C > 0$ is some number. We can assume that $C = 1$. Certainly, the norms p_{j_1} and p_{j_2} are not compatible . We can also find a norm $q' \in \mathcal{P}_1$ and a norm $p_{j_3} \in \mathcal{P}$ for which $q' \leqslant C' p_{j_3}$, so that without loss of generality we can assume that

$$q(x) \leqslant p_{j_1}(x) \leqslant p_{j_2}(x) \leqslant q'(x) \leqslant p_{j_3}(x) \quad \text{for all } x \in E.$$

Certainly, the norms p_{j_1} and p_{j_3} are not compatible as well.

We now show that the norms q and q' are not compatible. Let $\{a_n\}$ be a sequence of elements in E fundamental in the norm p_{j_3}, hence also in the norms p_{j_1} and p_{j_2}, and converging to zero in the norm p_{j_1}, but not converging to zero in the norm p_{j_2} (hence in the norm p_{j_3}). The fact that the topology τ cannot be defined by a family of pairwise compatible norms is implied by the property of $\mathcal{P}$ that, for every three norms p_{j_1}, p_{j_2}, p_{j_3} with $p_{j_1} \leqslant p_{j_2} \leqslant p_{j_3}$, there is a sequence converging to zero in the first norm, fundamental in all the three norms, but not converging to zero in the last two norms. Since $p_{j_1}(a_n) \geqslant q(a_n)$, we have $q(a_n) \to 0$. By the inequality $q' \leqslant p_{j_3}$ the sequence $\{a_n\}$ is fundamental in the norm q'. Finally, the inequality $p_{j_2} \leqslant q'$ yields that it cannot converge to zero in the norm q', since otherwise it would converge to zero in the norm p_{j_2}.

Note also that the relation $|f(t)| + |f'(t)| \to 0$ as $|t| \to \infty$, which is valid for every function $f \in E$, yields that the space (E, τ) is separable.

2.3. Inductive topologies

The concept of an inductive topology is dual to that of a projective topology, but the corresponding results related to one or the other topology are not completely symmetric; the nature of this asymmetry will be clear below.

2.3.1. Definition. *Let E be a vector space and let $\mathfrak{A}$ be a nonempty index set such that for every $\alpha \in \mathfrak{A}$ we are given a locally convex space E_α and a linear mapping $g_\alpha\colon\ E_\alpha \to E$. The inductive topology of the family of spaces $\{E_\alpha\}$ with respect to the family of mappings $\{g_\alpha\}$ (more precisely, the inductive topology in the category of locally convex topologies) is the strongest locally convex topology in E with respect to which all mappings g_α are continuous. The inductive limit of the family $\{E_\alpha\}$ with respect to the mappings $\{g_\alpha\}$ is the vector space E equipped with this topology. Notation*: $E = \mathrm{ind}_\alpha E_\alpha$.

2.3.2. Remark. (i) Let $\mathcal{V}$ be the set of all convex circled absorbent sets V in the space E such that $g_\alpha^{-1}(V)$ is a neighborhood of zero in E_α for each $\alpha \in \mathfrak{A}$.

Then $\mathcal{V}$ is a base of neighborhoods of zero in the inductive topology of the family $\{E_\alpha\}$ with respect to the family of mappings $\{g_\alpha\}$. Indeed, according to Proposition 1.2.11, the set $\mathcal{V}$ is a base of neighborhoods of zero of some locally convex topology τ in E. It follows from the definition of $\mathcal{V}$ that all mappings $g_\alpha\colon E_\alpha \to (E,\tau)$ are continuous. If τ_1 is an arbitrary locally convex topology in E for which all mappings $g_\alpha\colon E_\alpha \to (E,\tau_1)$ are continuous and $\mathcal{V}_1$ is a base of its neighborhoods of zero consisting of circled convex sets (certainly, they are all absorbing), then for every $\alpha \in \mathfrak{A}$ and every $V \in \mathcal{V}_1$ the set $g_\alpha^{-1}(V)$ is a neighborhood of zero in E_α, which by the definition of $\mathcal{V}$ yields that $\mathcal{V}_1 \subset \mathcal{V}$, hence we have $\tau_1 \subset \tau$.

Thus, if E_α is a locally convex space with a basis of absolutely convex neighborhoods of zero $\mathcal{V}_\alpha$, then the set $\mathcal{V}$ of absolutely convex envelopes of all possible sets of the form $\bigcup_\alpha g_\alpha(V_\alpha)$, where $V_\alpha \in \mathcal{V}_\alpha$, is a basis of neighborhoods of zero in the space E.

(ii) If E_α are general topological vector spaces, then the inductive topology on the space E in the category of topological vector spaces is introduced as the strongest vector topology for which all mappings g_α are continuous. This topology also exists. According to Example 2.2.1, the set T of all vector topologies in E for which all maps g_α are continuous is nonempty, moreover, it has the weakest topology. One can verify that T has the required strongest topology, but one can also define explicitly the corresponding base of neighborhoods of zero (see Jarchow [**237**, § 4.1]). For simplicity we assume that E is the union of $g_\alpha(E_\alpha)$. In every space E_α we take a basis $\mathcal{U}_\alpha$ of circled neighborhoods of zero and introduce in E a base of zero $\mathcal{U}$ consisting of the sets of the form

$$U = \bigcup_{n=1}^{\infty} \sum_{k=1}^{n} \bigcup_{\alpha\in\mathfrak{A}} g_\alpha(U_{\alpha,k}), \quad U_{\alpha,k} \in \mathcal{U}_\alpha,$$

where $\sum$ denotes a vector sum. For a countable collection of E_n we can take the sets $U = \bigcup_{K\in\mathcal{K}} \sum_{k\in K} g_k(U_k)$, where $U_k \in \mathcal{U}_k$ and $\mathcal{K}$ is the set of all finite subsets of $\mathbb{N}$. It is clear that U is circled and absorbent and that there exists $V \in \mathcal{U}$ with $V+V \subset U$ (one can take $V_{\alpha,k} \subset U_{\alpha,2k} \cap U_{\alpha,2k-1}$ in $\mathcal{U}_\alpha$). By Corollary 1.2.9 the class $\mathcal{U}$ is a base of neighborhoods of zero of a vector topology τ_2 in E. The continuity of $g_\alpha\colon E_\alpha \to (E,\tau)$ is obvious. The maximality of τ_2 is seen from the fact that if all g_α are continuous when E is equipped with a vector topology τ_0 and W_0 is a balanced neighborhood of zero in it, then we can find $U \in \mathcal{U}$ with $U \subset W$ by taking $W_k \in \tau_0$, $W_k + W_k \subset W_{k-1}$, $U_{\alpha,k} \subset g_\alpha^{-1}(W_k)$. If E_n are locally convex, then so is the topology τ_2, but for uncountable $\mathfrak{A}$ this is false (see an example below).

In addition to inductive topologies in the categories of locally convex spaces and topological vector spaces, the space E can be equipped with the strongest topology (not necessarily vector) with the property that all mappings g_α are continuous. It turns out that even if all E_α are locally convex, these three topologies can be distinct.

2.3.3. Example. Let E be a real vector space with an algebraic basis of cardinality of the continuum, let $\mathcal{F}$ be the set of all its finite-dimensional vector subspaces each of which is equipped with the standard topology, and let g_F be the canonical embedding of F into E for every $F \in \mathcal{F}$ (so here the index set is $\mathcal{F}$ itself). Let τ_1 be the strongest locally convex topology in E for which all g_F are continuous (i.e., the locally convex inductive topology), let τ_2 be the strongest vector topology in which all g_F are continuous, finally, let τ_3 be the strongest topology in E (not necessarily vector) for which the same mappings are continuous.

Then, obviously, $\tau_3 \supset \tau_2 \supset \tau_1$. We show that $\tau_3 \neq \tau_2 \neq \tau_1$.

For the proof of the inequality $\tau_2 \neq \tau_1$ we observe that the topology τ_1 can be defined by the set of all seminorms E and the topology τ_2 can be defined by the set of all quasi-norms on E. Let $\mathcal{P}$ be a Hamel basis in E and let q be the quasi-norm on E defined by the equality

$$q\Bigl(\sum_{e\in\mathcal{P}} c_e \cdot e\Bigr) = \Bigl(\sum_{e\in\mathcal{P}} \sqrt{|c_e|}\Bigr)^2.$$

Certainly, the set of nonzero coefficients in the sum $\sum_{e\in\mathcal{P}} c_e \cdot e$ is finite.

We show that on E there is no seminorm p such that

$$\{x \in E\colon\ p(x) \leqslant 1\} \subset \{x \in E\colon\ q(x) \leqslant 1\}.$$

Indeed, if p is such a seminorm on E, then for every $a \in E$ with $p(a) \leqslant 1$ we have the inequality $q(a) \leqslant 1$. On the other hand, for some $C > 0$ the set $\mathcal{P}_C = \{e \in \mathcal{P}\colon\ p(e) \leqslant C\}$ is infinite, since if such $C > 0$ does not exist, then the set $\mathcal{P}$ is at most countable, which is false by assumption. Thus, for every n we can find n different elements $e_1, \ldots, e_n \in \mathcal{P}_C$. Then for $\eta_j = C^{-1}e_j$ we have the estimate $p\bigl(n^{-1}\sum_{j=1}^n \eta_j\bigr) \leqslant 1$, while $q\bigl(n^{-1}\sum_{j=1}^n \eta_j\bigr) = C^{-1}n$. Hence for a sufficiently large n we obtain the inequality $q\bigl(n^{-1}\sum_{j=1}^n \eta_j\bigr) > 1$ contradicting the aforementioned estimate. Therefore, $\tau_2 \neq \tau_1$. A simpler assertion is proven.

Let us proceed to the proof of the inequality $\tau_3 \neq \tau_2$. Let $\|\cdot\|$ be the norm on E defined by the equality $\|\sum_{e\in\mathcal{P}} c_e \cdot e\| = \max_{e\in\mathcal{P}} |c_e|$, where we use the previous notation. For every positive function φ on the set $\mathcal{P}\times\mathcal{P}$ we choose a neighborhood of zero V_φ in the topology τ_3 with the following property: if $\|\lambda(e_1 + e_2)\| \geqslant \varphi(e_1, e_2)$, then $\lambda(e_1 + e_2)$ is not contained in V_φ. Such a neighborhood of zero in τ_3 exists, since a set is open in the topology τ_3 precisely when its intersection with each finite-dimensional subspace in E is open in the standard topology of this finite-dimensional subspace; for the neighborhood V_φ we can take $E \setminus \bigcup_{e,b\in\mathcal{P}}\bigl\{\lambda(e+b)\colon\ \|\lambda(e+b)\| \geqslant \varphi(e,b)\bigr\}$.

In order to prove that the topology τ_3 is stronger than τ_2, it suffices to show that there exists a function $\varphi\colon\ \mathcal{P}\times\mathcal{P} \to \mathbb{R}^+$ for which there is no quasi-norm q on E such that if $W = \{x \in E\colon\ q(x) \leqslant 1\}$, then $W + W \subset V_\varphi$. We prove a formally stronger assertion: for some function $\varphi\colon\ \mathcal{P}\times\mathcal{P} \to \mathbb{R}^+$ there is no neighborhood of zero W in the topology τ_3 such that $W + W \subset V_\varphi$. For every neighborhood of zero W in the topology τ_3 we choose some positive function ψ_W on $\mathcal{P}$ with the following property: for every $e \in \mathcal{P}$ one has $\{\lambda e\colon\ \|\lambda e\| < \psi_W(e)\} \subset W$. Since the inclusion $W + W \subset V_\varphi$ yields that $\min\bigl(\psi_W(e_1), \psi_W(e_2)\bigr) \leqslant \varphi(e_1, e_2)$ for all $e_1, e_2 \in \mathcal{P}$ (by our choice of V_φ), it suffices to find a function $\varphi\colon\ \mathcal{P}\times\mathcal{P} \to \mathbb{R}^+$ for which the inequality

$$\min\bigl(\psi(e), \psi(b)\bigr) \leqslant \varphi(e, b)$$

cannot hold for all $e, b \in \mathcal{P}$ at once, whatever a function $\psi\colon\ \mathcal{P} \to \mathbb{R}^+$ be.

A function φ with this property can be constructed in the following way. Let $\{e_j\}$ be some countable subset of the set $\mathcal{P}$, let S be the set of all sequences of positive real numbers, and let $\mathcal{P}_0$ be some subset of $\mathcal{P}$ of cardinality of the continuum. Let g be a one-to-one mapping of $\mathcal{P}_0$ onto S. For $e \in \mathcal{P}_0$ we set $\varphi(e, e_j) = g(e)_j$, where $\bigl(g(e)\bigr)_{j=1}^{\infty}$ is the sequence in S that is the image of the element e under the mapping g. For other pairs $(e, b) \in \mathcal{P}\times\mathcal{P}$ we define $\varphi(e, b)$ in an arbitrary way. Then, given an arbitrary a positive function ψ on $\mathcal{P}$, we obtain that $\varphi(e, b) \leqslant \min\bigl(\psi(e), \psi(b)\bigr)$ for some e, b. This follows from the fact that the set $g(\mathcal{P}_0)$ contains all sequences of positive numbers, in particular, the sequence of the numbers $\alpha_j = \psi(e_j)/j$. It is clear that if $g(e)_j = \alpha_j$, then we have $\varphi(e, e_j) \leqslant \min\bigl(\psi(e), \psi(e_j)\bigr)$ for all sufficiently large j (for arbitrary $\psi(e)$).

Hille and Phillips [**222**] introduced the "finitely open" topology in a vector space in which a set is open precisely when its intersection with every finite-dimensional subspace is open in the standard topology of this subspace (see Remark 1.10.4); thus, this topology coincides with our topology τ_3 (as already noted above). It was mentioned in [**222**] that the authors were unaware whether this topology agrees with the vector structure; a negative answer was given later in Kakutani, Klee [**254**].

The proof of the noncoincidence of the topologies τ_3 and τ_2 given above is actually a proof of the fact that if the algebraic dimension of a vector space is at least of cardinality of the continuum, then the finitely open topology in it is not compatible with the vector structure.

2.3.4. Remark. The term "an inductive limit" is often used for a more special construction (which will be described below). On the other hand, if E is an arbitrary set, $\{E_\alpha\colon\ \alpha \in \mathfrak{A}\}$ is a family of topological spaces such that for every $\alpha \in \mathfrak{A}$ a mapping $g_\alpha\colon\ E_\alpha \to E$ is given, then in E there exists the strongest topology for which all these mapping are continuous. This topology is called the inductive topology (in the class of all topologies) of the family of topological spaces $\{E_\alpha\}$ with respect to the family of mappings $\{g_\alpha\}$; sometimes this topology is also called final. Throughout, however, the terms an inductive limit and an inductive topology are used only in the sense of our definition.

The following simple result is frequently used.

2.3.5. Proposition. *Let a locally convex space E be the inductive limit of some family $\{E_\alpha\colon\ \alpha \in \mathfrak{A}\}$ of locally convex vector spaces with respect to linear mappings $g_\alpha\colon\ E_\alpha \to E$. A linear mapping f from the space E to a locally convex space G is continuous precisely when for every $\alpha \in \mathfrak{A}$ the mapping $f \circ g_\alpha$ is continuous. Therefore, sequentially continuous linear mappings of inductive limits of metrizable spaces are continuous.*

PROOF. As in Proposition 2.1.5, it suffices to show that the continuity of the mappings $f \circ g_\alpha$ implies the continuity of the mapping f. To this end we have to show that if V is a convex circled neighborhood of zero in G, then $f^{-1}(V)$ is a neighborhood of zero in E. The set $f^{-1}(V)$ is convex, circled and absorbent by the linearity of f. Finally, the continuity of all mappings g_α implies that all sets $g_\alpha^{-1}\bigl(f^{-1}(V)\bigr)$ are neighborhoods of zero in the respective spaces. Now it remains to use Remark 2.3.2. □

This proposition is not completely symmetric to Proposition 2.1.5, dual to which it is: here we assume (unlike Proposition 2.1.5) that f is linear and that G is locally convex. Neither of these assumptions can be dropped (the circumstance that now we consider the continuity everywhere and not at a point is not essential, since for linear mappings this is the same). Note also that this proposition does not extend to closed linear subspaces of inductive limits even for linear functionals (see Exercise 2.10.63).

2.4. Examples of inductive limits

We consider some more examples of inductive topologies.

2.4.1. Example. (*The least lower bound of a set of locally convex topologies in a vector space.*) Let E be a vector space such that for every α from some index set $\mathfrak{A}$ we have a locally convex topology τ_α in E. Then among all locally convex topologies in E majorized by the topologies of the family $\{\tau_\alpha\}$ there exists the strongest one — the inductive topology τ of the family of locally convex spaces (E,τ_α) with respect to the family of the "canonical" mappings $(E,\tau_\alpha) \to E$ (each of which is the identical mapping of the vector space E into itself). If $\mathcal{V}_\alpha$ is the class of all neighborhoods of zero in the topology τ_α, then $\bigcap_\alpha \mathcal{V}_\alpha$ is the class of all neighborhoods of zero in the topology τ (Remark 2.3.2).

2.4.2. Example. (*Quotients.*) Let E be a locally convex space and let E_1 be its vector subspace. Then the topology in the quotient E/E_1 making it a topological quotient is the inductive topology of the (one-element) family of locally convex spaces $\{E\}$ with respect to the (one-element) family of mappings from E to E/E_1 whose unique element is the canonical mapping of E onto E/E_1 (see Example 1.3.13).

2.4.3. Example. (*Topological direct sums of locally convex spaces.*) Let (E,τ) be a locally convex space such that the vector space E is a direct sum of a family $\{E_\alpha\}$ of its vector subspaces and every E_α is equipped with the locally convex topology τ_α induced by the topology τ. The space (E,τ) is called the *topological direct sum* of the family of its topological vector subspaces (E_α,τ_α) if the topology τ is the inductive limit of the topological vector spaces (E_α,τ_α) with respect to the canonical embeddings $E_\alpha \to E$.

In the just described situation we assume in advance that the topological vector space (E,τ) is given; so we consider the topological direct sum of a family of topological vector subspaces.

Suppose now that we are given a family $\{(E_\alpha, \tau_\alpha)\}$ of locally convex spaces about which we do not assume in advance that they are subspaces of some vector space.

In this case the topological direct sum of the family $\{(E_\alpha, \tau_\alpha)\}$ of locally convex spaces is the locally convex space (E, τ) defined as follows. The vector space E is the vector subspace in the product $\prod_\alpha E_\alpha$ of the family $\{E_\alpha\}$ of vector spaces consisting of all functions $f \in \prod_\alpha E_\alpha$ (defined on the index set $\mathfrak{A}$) each of which does not vanish in at most finitely many points; τ is the inductive topology of the family of locally convex spaces $\{(E_\alpha, \tau_\alpha)\}$ with respect to the "canonical" embeddings $g_\alpha\colon\ (E_\alpha, \tau_\alpha) \to E$, where $g_\alpha(x) = f \in E$ for all $\alpha \in \mathfrak{A}$ and $x \in E_\alpha$, and f is defined by $f(\alpha) = x$, $f(\beta) = 0$, whenever $\beta \neq \alpha$.

The topological direct sum of the family of locally convex spaces (E_α, τ_α) is denoted by $\bigl(\bigoplus_\alpha E_\alpha, \bigoplus_\alpha \tau_\alpha\bigr)$. Thus, $\bigl(\bigoplus_\alpha E_\alpha, \bigoplus_\alpha \tau_\alpha\bigr)$ is the topological direct sum of its topological vector subspaces $g_\alpha(E_\alpha)$.

Throughout we identify every locally convex space E_α with its image $g_\alpha(E_\alpha)$ in the sum $\bigoplus_\alpha E_\alpha$, assuming that the spaces E_α are topological vector subspaces in their topological direct sum.

It follows that the topological direct sum of a finite family of locally convex spaces and their product are the same object. For infinite families of topological vector spaces that are not locally convex, topological direct sums are considered very seldom (however, see Jarchow [**237**, § 4.3]).

2.4.4. Definition. *The topological vector space E equal the direct sum of its topological vector subspaces $E_1, \ldots, E_n$ is called the topological direct sum of the family $\{E_1, \ldots, E_n\}$ of its subspaces if E is canonically isomorphic to the product $\prod_{j=1}^n E_j$ of the topological vector spaces E_j.*

This means that the topology of E is the strongest one among all vector topologies in E for which all canonical embeddings $E_j \to E$ are continuous.

2.4.5. Proposition. *Let E_γ, where $\gamma \in \Gamma$, be a collection of locally convex spaces. Then the dual to their direct sum is the product of the dual spaces and the dual to their product is the direct sum of the dual spaces, i.e.,*

$$\Bigl(\bigoplus_\gamma E_\gamma\Bigr)' = \prod_\gamma E'_\gamma, \quad \Bigl(\prod_\gamma E_\gamma\Bigr)' = \bigoplus_\gamma E'_\gamma.$$

Moreover, a set is bounded in $\bigoplus_\gamma E_\gamma$ precisely when it is contained and is bounded in the sum of finitely many spaces E_γ; a set is bounded in $\prod_\gamma E_\gamma$ precisely when it is contained in the product of bounded sets in the factors.

In particular, $(\mathbb{R}^T)'$ is the direct sum of T copies of the real line.

PROOF. The first equality for the dual spaces is obvious. For verifying the second one we observe that if a linear function f on the product of the spaces E_γ is bounded on a basis neighborhood of zero of the form $U \times \prod_{\gamma \notin \Gamma_0}$, where Γ_0 is finite, then $f \in \bigoplus_{\gamma \in \Gamma_0} E'_\gamma$.

Let $A \subset \bigoplus_\gamma E_\gamma$ be bounded. If there are infinitely many indices γ_n for which the projections of A to E_γ contain nonzero vectors v_n, then, taking $f_n \in E'_{\gamma_n}$ such that $f_n(v_n) = n$, we obtain a continuous linear functional unbounded on A. The last assertion of the proposition is obvious. □

2.4.6. Proposition. *Let E be the inductive limit of a family $\{E_\alpha\colon\ \alpha \in \mathfrak{A}\}$ of locally convex spaces with respect to a family $\{g_\alpha \in \mathcal{L}(E_\alpha, E)\colon\ \alpha \in \mathfrak{A}\}$ of linear mappings such that the linear span of the set $\bigcup_\alpha g_\alpha(E_\alpha)$ coincides with E. Then E is isomorphic as a topological vector space to some topological vector quotient of the topological direct sum $\bigl(\bigoplus_\alpha E_\alpha, \bigoplus_\alpha \tau_\alpha\bigr)$.*

PROOF. Let $G = \bigl(\bigoplus_\alpha E_\alpha, \bigoplus_\alpha \tau_\alpha\bigr)$ and let $\Phi\colon\ G \to E$ be the linear mapping defined by the formula $\Phi(g) = \sum_\alpha g(\alpha)$; in the latter sum the number of nonzero terms is finite. The surjectivity of Φ follows from the coincidence of the linear span of $\bigcup_\alpha g_\alpha(E_\alpha)$ with E. The mapping Φ is continuous according to Proposition 2.3.5, since $g_\alpha = \Phi \circ \mathrm{im}_\alpha$, where im_α is the canonical embedding of E_α to $\bigoplus_\alpha E_\alpha$ defined in Example 2.4.3, where it is denoted by g_α. Let Ψ_1 denote the canonical mapping of the space G onto $G/\mathrm{Ker}\,\Phi$ and let Ψ_2 be the linear one-to-one mapping of the space $G/\mathrm{Ker}\,\Phi$ onto E defined by the equality $\Psi_2 \circ \Psi_1 = \Phi$. We show that Ψ_2 and Ψ_2^{-1} are continuous; this will prove that E is isomorphic (as a topological vector space) to the space $G/\mathrm{Ker}\,\Phi$.

The continuity of Ψ_2 follows from Proposition 2.3.5 and the continuity of Ψ_1 and Φ; here the role of the inductive limit mentioned in Proposition 2.3.5 is played by the quotient $G/\mathrm{Ker}\,\Phi$ (it is the inductive limit of the family of spaces $\{E_\alpha\}$ with respect to the family of mappings $\{\Psi_1 \circ g_\alpha\}$). The continuity of Ψ_2^{-1} follows, again according to Proposition 2.3.5, by the fact that $\Psi_2^{-1} \circ g_\alpha = \Psi_1 \circ \mathrm{im}_\alpha$, where all mappings $\Psi_1 \circ \mathrm{im}_\alpha$ are continuous (now the role of the inductive limit mentioned in Proposition 2.3.5 is played by E itself). □

In Chapter 3 we shall consider the duality between inductive and projective limits under various topologies on the dual space.

2.4.7. Example. (*Limits of direct spectra of locally convex spaces.*) Let $\mathfrak{A}$ be a directed set. A *direct spectre of locally convex spaces* with the index set $\mathfrak{A}$ is a family $\{(E_\alpha, \tau_\alpha)\colon\ \alpha \in \mathfrak{A}\}$ of locally convex spaces (with this index set) provided that to every pair $\alpha, \beta \in \mathfrak{A}$ of indices with $\alpha \leqslant \beta$ a continuous mapping $A_{\beta\alpha}\colon\ E_\alpha \to E_\beta$ is associated.

The limit of such direct spectre is the topological vector quotient of the topological direct sum $\bigl(\bigoplus_\alpha E_\alpha, \bigoplus_\alpha \tau_\alpha\bigr)$ by its vector subspace generated by the set G defined as follows: $f \in G$ if there exist $\alpha, \beta \in \mathfrak{A}$ such that $\alpha \leqslant \beta$, $f(\beta) = A_{\beta\alpha} f(\alpha)$ and $f(\gamma) = 0$ whenever $\gamma \notin \{\alpha, \beta\}$.

Every topological vector quotient of the topological direct sum of an arbitrary family of locally convex spaces is the inductive limit of this family with respect to the family of mappings that are the compositions of the canonical embeddings of the spaces in the family into their sum and the canonical mapping of the latter onto its quotient (this fact was used also in the proof of Proposition 2.4.6). Hence the limit of a direct spectre of locally convex spaces — we shall denote it by the symbol $\varinjlim E_\alpha$ — is the inductive limit of the family $\{(E_\alpha, \tau_\alpha)\colon\ \alpha \in \mathfrak{A}\}$ of locally convex spaces with respect to the just described family of mappings. Note also that here, unlike the case of limits of inverse spectra, in general one cannot assert that the space $\varinjlim E_\alpha$ is complete if all spaces (E_α, τ_α) are complete (see Exercise 2.10.23).

In Schaefer's book [**436**], the object called above "the limit of a direct spectre" is called "an inductive limit"; thus, the meaning of the term an "inductive limit" is broader here than in [**436**] (where it corresponds to an "inductive topology").

Let us now consider one special – but the most important for applications – class of direct spectra.

2.4.8. Example. Suppose that the index set $\mathfrak{A}$ is the set of natural numbers $\mathbb{N}$ with its usual order and that all mappings $A_{ij}\colon E_j \to E_i$ (defined for $j \leqslant i$) are injective. Due to the latter assumption we can assume that every locally convex space E_i is a vector subspace in E_{i+n} (notation: $E_i \subset\subset E_{i+n}$). By the continuity of the mappings A_{ij} the topology induced on E_i by the topology τ_{i+n} of the space E_{i+n} is majorized by the original topology τ_i of the space E_i.

Thus, we assume that $E_1 \subset\subset E_2 \subset\subset \ldots \subset\subset E_n \subset\subset \ldots$ and $E = \bigcup_{j=1}^\infty E_j$; the set E has a naturally defined structure of a vector space: if $x_1, x_2 \in E$, then $x_1, x_2 \in E_n$ for some $n \in \mathbb{N}$, and $\lambda_1 x_1 + \lambda_2 x_2$ in E is defined as the corresponding linear combination in E_n; since $E_n \subset\subset E_{n+j}$, this definition does not depend on our choice of E_n.

Under the stated conditions the vector space E equipped with the inductive topology of the family $\{E_n\colon n \in \mathbb{N}\}$ with respect to the embeddings $E_n \to E$ is "canonically isomorphic" to the space $\varinjlim E_n$.

This isomorphism $\Psi\colon E \to \varinjlim E_n$ can be described as follows. Let $x \in E$, i.e., $x \in E_n$ for some $n \in \mathbb{N}$. Then $\Psi(x)$ is an element of the space $\varinjlim E_n$ (that is a quotient of the space $\bigoplus_n E_n$) which is the image under the canonical mapping $\Phi\colon \bigoplus_n E_n \to \varinjlim E_n$ of the element $g_x^n \in \bigoplus_n E_n$ defined as follows: $g_x^n(j) = 0$ if $j \neq n$; $g_x^n(n) = x$. This definition is independent of our choice of n, since if $j \geqslant n$, then $A_{jn}(x) = x$ and hence $\Phi(g_x^n) = \Phi(g_x^j)$. The mapping Ψ is surjective: if $a \in \varinjlim R_n$, $g \in \Phi^{-1}(a)$, then $\Psi\bigl(\sum_n g(n)\bigr) = a$ (the set $\{n\colon g(n) \neq 0\}$ is finite).

Finally, the mappings Ψ and Ψ^{-1} are continuous; the proof of this is similar to the proof of Proposition 2.4.6.

Thus, the space E, equipped with the inductive topology defined above is a realization of a direct spectre. This realization will be denoted by the symbols $\operatorname{ind} E_n$ or $\operatorname{ind}_n E_n$; the locally convex space $\operatorname{ind} E_n$ will be called the *inductive limit of the increasing sequence of locally convex spaces* E_n.

Note that it often happens that it is not E that is constructed by means of a priori defined spaces E_n, but the spaces E_n are defined as suitable vector subspaces (with locally convex topologies) of an a priori given space E (without topology) that is their set-theoretic union. the term the "inductive limit of an increasing sequence of locally convex spaces E_n" means precisely the same as the term the "inductive limit of a family $\{E_n\}$ of locally convex spaces with respect to embeddings $E_n \to E$". Though, once the space $\bigcup_{n=1}^{\infty} E_n$ is introduced, the difference between the latter and the former situations disappears. One should have in mind that even the inductive limit of a sequence of Banach spaces can be non-separated (Exercise 2.10.24). The inductive limit of a sequence of Fréchet spaces is called an *LF-space*.

2.4.9. Example. Let $\mathcal{D}$ be the vector space of all infinitely differentiable (real) functions on $\mathbb{R}^1$ with compact support. For every $n \in \mathbb{N}$ let $\mathcal{D}_n := \mathcal{D}[-n,n]$ denote the subspace consisting of all functions vanishing outside of $[-n,n]$ equipped with the topology defined by the norms

$$p_i(\varphi) = \max_{j\in\{0,1,\ldots,i\}} \max_t |\varphi^{(j)}(t)|.$$

Then $\mathcal{D}[-1,1] \subset\subset \mathcal{D}[-2,2] \subset\subset \cdots \subset\subset \mathcal{D}[-n,n] \subset\subset \cdots$ and $\bigcup_n \mathcal{D}[-n,n] = \mathcal{D}$, so that we are in the situation described in the second part of the previous example.

The inductive topology in $\mathcal{D}$ of the family of locally convex spaces $\mathcal{D}[-n,n]$, where $n \in \mathbb{N}$, with respect to the family of embeddings $\mathcal{D}[-n,n] \to \mathcal{D}$ coincides with the topology in $\mathcal{D}$ introduced in Example 1.3.21. Thus, $\mathcal{D} = \varinjlim \mathcal{D}[-n,n]$. This topology in $\mathcal{D}$ is regarded as the standard one; if it is not stated otherwise, it is assumed (in this book and also in other books) that $\mathcal{D}$ is equipped with this topology.

The space $\mathcal{D}[-n,n]$ can be naturally identified with the space of all infinitely differentiable functions on the interval $[-n,n]$ vanishing with all derivatives at the points $\pm n$; to every function in the first space we associate its restriction to the interval $[-n,n]$, and every function in the second space is associated with its extension by zero from $[-n,n]$ to $\mathbb{R}^1$. For the second space we use the same notation $\mathcal{D}[-n,n]$ and equip it with the topology defined by the same norms $p'_j(\varphi) = \max_{i\leqslant j} \max_t |\varphi^{(i)}(t)|$. Although now $\mathcal{D}[-n,n]$ is not formally a subspace in $\mathcal{D}[-n-j,n+j]$, we obtain the same inductive limit.

2.4.10. Remark. (i) For every $j \in \{0,1,2,\ldots\}$ and every natural n let $\mathcal{K}^j_{[-n,n]}$ denote the vector space of all j-fold continuously differentiable real functions on $\mathbb{R}^1$ (for $j=0$ we obtain just continuous functions) vanishing outside the interval $[-n,n]$ and equipped with the topology generated by the norms q_j defined by the equality

$$q_j(\varphi) = \max\bigl\{\max_t |\varphi^{(i)}(t)|\colon\ i = 0,1,\ldots,j\bigr\}.$$

Then, for every $n \in \mathbb{N}$, the family $\{\mathcal{K}^j_{[-n,n]}\colon\ j = 0,1,\ldots\}$ is the inverse spectre with respect to the embeddings $\mathcal{K}^{j+1}_{[-n,n]} \to \mathcal{K}^j_{[-n,n]}$, moreover, the spaces $\varprojlim_j \mathcal{K}^j_{[-n,n]}$ and $\mathcal{D}[-n,n]$ coincide as locally convex spaces (Example 2.4.9). At the same time, for every $j \geqslant 0$ the family $\{\mathcal{K}^j_{[-n,n]}\colon\ n \in \mathbb{N}\}$ forms a direct spectre with respect to the embeddings $\mathcal{K}^j_{[-n,n]} \to \mathcal{K}^j_{[-n-1,n+1]}$. For every $j \geqslant 0$ the set $\mathcal{K}^j = \operatorname*{ind}_n \mathcal{K}^j_{[-n,n]}$ (in place of the symbol $\mathcal{K}^0$ we shall usually use the symbol $\mathcal{K}$). The locally convex spaces $\{\mathcal{K}^j\colon\ j \geqslant 0\}$ form an inverse spectre with respect to the embeddings $\mathcal{K}^{j+1} \to \mathcal{K}^j$, and the locally convex space $\varprojlim \mathcal{K}^j$ can be identified as a vector space with the space $\bigcap_{j=1}^{\infty} \mathcal{K}^j$, which coincides as a vector space with $\mathcal{D}$. However, as locally convex spaces the space $\mathcal{D}$ with the standard topology and the space $\varprojlim \mathcal{K}^j$ are not isomorphic. When we identify them as vector spaces the topology of the space $\mathcal{D}$ turns out to be strictly stronger than the topology of the space $\varprojlim \mathcal{K}^j$. Note that no linear mapping of one of these spaces onto the other can be continuous in both directions (i.e., is not an isomorphism of topological vector spaces) at least for the reason that the locally convex space $\mathcal{D}$ with the standard topology is barrelled, but the locally convex space $\varprojlim \mathcal{K}^j$ is not (the definition of a barrelled space is given in § 3.5, the proof of the fact that $\varprojlim \mathcal{K}^j$ is not barrelled is delegated to Exercise 3.12.55).

Taking into account the definition of the standard topology in $\mathcal{D}$, the observations made above can be summarized as the inequality

$$\varprojlim{}_j \bigl(\operatorname*{ind}_n \mathcal{K}^j_{[-n,n]}\bigr) \neq \operatorname*{ind}_n \bigl(\varprojlim{}_j \mathcal{K}^j_{[-n,n]}\bigr).$$

Thus, the operations of forming direct and inverse spectra do not commute.

(ii) So far we have considered spaces of functions with compact support on $\mathbb{R}^1$ (for simplicity we had in mind real functions, but nothing changes if we consider complex spaces of complex functions). Completely analogous constructions apply to spaces of functions on $\mathbb{R}^n$ and even on domains in $\mathbb{R}^n$.

Namely, let Ω be an open set in $\mathbb{R}^n$. A function on Ω has compact support if it vanishes outside a compact set in Ω (we consider real or complex functions). Let $K_1 \subset K_2 \subset K_3 \subset \cdots$ be an increasing sequence of compact subsets of Ω such that $\bigcup_{m=1}^{\infty} K_m = \Omega$. If in the previous discussion of Example 2.4.9, the symbols $[-n,n]$ and $\mathbb{R}^1$ are replaced with K_m and Ω, the words "compact interval" by the word "compact", and the symbols $\mathcal{K}^j$ and $\mathcal{D}$ with the symbols $\mathcal{K}^j(\Omega)$ and $\mathcal{D}(\Omega)$ denoting the spaces of all j-fold differentiable and all infinitely differentiable functions with compact support in Ω, respectively, and assume that the symbols $|\varphi^{(j)}(x)|$ denote (certain) norms on the spaces to which belong the elements $\varphi^{(j)}(x)$, say, the sums of all mixed partial derivatives of order j, then all our definitions remain meaningful and all our assertions will be valid. For example, $\mathcal{D}(\mathbb{R}^n)$ is the inductive limit of the spaces $\mathcal{D}_m(\mathbb{R}^n)$ of smooth functions with support in the ball $K_m = \{x\colon\ |x| \leqslant m\}$.

2.4.11. Example. (*The strongest locally convex topology in a vector space.*) Let E be a vector space. Then the inductive topology in E of the empty family of locally convex spaces with respect to the empty family of mappings is the strongest locally convex topology in E; a base of neighborhoods of zero is, for example, the set of all convex circled absorbent subsets of the space E. This topology can be defined by the set of all seminorms on E. Note that the strongest locally convex topology in a vector space has already been considered in Examples 1.3.18 and 2.4.1. An interesting situation arises if we take for E the space $\mathbb{R}_0^{\infty}$ of all finite sequences, i.e., the union of $\mathbb{R}^n$. On this space the strongest locally convex topology coincides with the topology of the inductive limit of $\mathbb{R}^n$ (this is clear from Proposition 2.3.5).

2.4.12. Example. (*Spaces of germs of continuous, infinitely differentiable and analytic functions at a fixed point.*) We consider only the case of analytic functions, other cases are similar.

Let $z \in \mathbb{C}$ and let E_1 be the set of all complex functions each of which is defined in some neighborhood of the point z and is analytic in this neighborhood and let $\sim$ be the equivalence relation in E_1 defined as follows: $f \sim g$ if and only if there exists a neighborhood V of the point z such that f and g are defined in V and $f(x) = g(x)$ for all $x \in V$.

Finally, let E be the quotient of E_1 with respect to this equivalence relation. Then E is naturally equipped with a vector structure; its elements are called germs of analytic functions at the point z.

For every $n \in \mathbb{N}$ we take the space E^n of all complex functions continuous on the disc $S_n = \{x \in \mathbb{C}\colon\ |x - z| \leqslant 1/n\}$ and analytic inside S_n and equip it with the topology defined by the norm

$$p(\varphi) = \max\{|\varphi(x)|\colon\ |x - z| \leqslant 1/n\}.$$

Let g_n be the embedding of E^n into E associating to every function $\varphi \in E^n$ the germ containing it (note that in the remaining three cases mentioned above, similar mappings are not embeddings). We define similarly an embedding $E^n \to E^{n+1}$: to every function in E^n we associate its restriction to S_{n+1}.

Thus, $E^1 \subset\subset E^2 \subset\subset \cdots \subset\subset E^n \subset\subset \cdots$, $E^n \subset\subset E$, and $E = \bigcup_{n=1}^{\infty} E^n$. Let us equip E with the inductive topology of the family of locally convex spaces E^n with respect to the embeddings g_n. This makes E the inductive limit of the increasing sequence of locally convex spaces E_n.

Finally, note that spaces like $\mathcal{D}$ can be defined on smooth manifolds, and spaces of germs of holomorphic functions can be considered on complex manifolds.

In § 2.6 and § 2.7 we describe two classes of inductive limits of increasing sequences of locally convex spaces most often encountered in applications: strict inductive limits and inductive limits of increasing sequences of locally convex spaces with compact embeddings.

2.5. Grothendieck's construction

In this section we consider a method of constructing normed spaces associated with absolutely convex sets in topological vector spaces. This method, which became popular after Grothendieck's works and found numerous applications in the theory of locally convex spaces, the theory of Banach space, the operator theory, and measure theory, consists in the following procedure.

Let E be a Hausdorff topological vector space and let B be its bounded absolutely convex subset. Denote by E_B the vector subspace $\bigcup_{n=1}^{\infty} nB$ (the fact that this is indeed a vector space follows from the absolute convexity of the set B) equipped with the norm p_B that is the Minkowski functional of the set B. The space E_B is also equipped with the topology generated by this norm. The fact that the Minkowski functional is a seminorm on E_B follows from the condition that B is an absolutely convex absorbent subset of the vector space E_B; the condition that B is a bounded subset of a Hausdorff topological vector space yields that the regarded Minkowski functional is a norm.

The set E_B is a vector subspace of the vector space E. However, in the general case, it is not a topological vector subspace in E, i.e., the topology induced on E_B by the topology of the space E does not coincide with the just defined topology of the normed space E_B. The boundedness of the set B implies that the canonical embedding of E_B into E is continuous; this is equivalent to saying that the topology induced in E_B from E is weaker than the norm topology in E_B. See also § 2.10(iii).

2.5.1. Proposition. *If B is an absolutely convex bounded sequentially complete subset of a Hausdorff topological vector space E, then the normed space E_B is complete, i.e., is a Banach space.*

PROOF. Let a sequence $\{a_n\} \subset E_B$ be Cauchy in the norm p_B. This means that for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $p_B(a_k - a_j) < \varepsilon$ for all $k, j > n_0$. Since the topology induced in E_B by the topology of the space E is majorized by the topology defined by the norm p_B, the sequence $\{a_n\}$ is Cauchy also in E. Hence for every fixed $k \in \mathbb{N}$ the sequence $\{a_k - a_j\}$ is Cauchy in the topology of the space E. Since the inequality $p_B(a_k - a_j) < \varepsilon$ yields that $a_k - a_j \in \varepsilon B$, it follows that $a_k - a_j \in \varepsilon B$ whenever $k, j > n_0$. The set εB is sequentially complete in E, hence the Cauchy sequence $\{a_k - a_j\}_{j=1}^{\infty}$ of its elements converges in the topology of the space E to some element $b_k \in B$. Hence the sequence $\{a_n\}$ converges in the topology of E, moreover, if a is its limit, then $a_k - a = b_k \in \varepsilon B \subset E_B$ for all $k > n_0$.

Since $a_k \in E_B$, we have $a \in E_B$. In addition, the inclusion $a_k - a \in \varepsilon B$ (fulfilled for all $k > n_0$) yields that for such k we have $p_B(a_k - a) \leqslant \varepsilon$. This means that $a_k \to a$ in E_B, because ε was arbitrary. $\square$

Similarly one can prove that if E is a vector space and τ_1 and τ_2 are two Hausdorff topologies in E compatible with the vector structure and τ_1 is stronger than τ_2 and possesses a base of neighborhoods of zero sequentially complete (or complete) in the topology τ_2, then the topological vector space (E, τ_1) is sequentially complete (respectively, complete).

2.5.2. Definition. *A Banach disc in a topological vector space E is an absolutely convex bounded subset B of the space E for which the space E_B is complete.*

The proposition above yields the following fact:
in order that an absolutely convex bounded subset B of a topological vector space be a Banach disc it is sufficient that it be sequentially complete (*which holds, in particular, if B is compact*); *moreover, it is sufficient that B be sequentially complete in a topology in E that agrees with the duality between E and E'*; *in particular, it is sufficient that this set be compact in some of such topologies.*

Neither of the aforementioned conditions is necessary.

2.5.3. Example. (i) The subset

$$\big\{x = (x_n) \in c_0\colon\ (nx_n) \in c_0,\ |x_n| \leqslant 1/n\ \ \forall n\big\}$$

of the space c_0 is a Banach disc, although it is not compact in any topology that agrees with the duality between c_0 and $(c_0)' = l^1$.

(ii) the closed unit ball U of $C[0,1]$ is a Banach disc in $L^2[0,1]$ which is not closed and not precompact. The closure of U in $L^2[0,1]$ is also a Banach disc for which the associated Banach space is $L^\infty[0,1]$, where $C[0,1]$ has infinite codimension.

(iii) The space $E = l^1$ is naturally embedded into l^2; let A be its closed unit ball. Let us take the functional $f \in l^\infty$ defined by the sequence $(1, 1, \ldots)$. Set $B = A \cap f^{-1}(0)$ and obtain a Banach disc. Certainly, it is easy to describe B explicitly: this is the set of sequences $x = (x_n)$ such that $\sum_{n=1}^\infty |x_n| \leqslant 1$ and $\sum_{n=1}^\infty x_n = 0$. The set B is the closed unit ball in the Banach space E_B, that is a closed hyperplane in l^1. But in l^2 the set B is not closed. For example, the vector $h = (2^{-1}, 0, 0, \ldots) \notin B$ is the limit in l^2 of the sequence of vectors $h_k \in B$ of the form $\big(2^{-1}, -(2k)^{-1}, \ldots, -(2k)^{-1}, 0, 0, \ldots\big)$, where the number of components $-(2k)^{-1}$ equals k; here $\|h - h_k\|_{l^2}^2 = (4k)^{-1}$.

A similar property holds for the non-closed precompact set

$$\{x = (x_n) \in l^2\colon\ |x_n| \leqslant 2^{-n},\ \lim_{n\to\infty} 2^n x_n = 0\}.$$

In fact, such examples are quite common.

2.5.4. Example. In every infinite-dimensional Fréchet space F there is a non-closed Banach disc B that serves as the closed unit ball of the Banach space E_B. In addition, we can take B totally bounded in F.

PROOF. The assertion reduces to the case $F = l^2$, since one can find an infinite-dimensional separable Hilbert space continuously embedded into F (then the previous example can be used). To this end it suffices to take in F a bounded sequence $\{v_n\}$ of linearly independent vectors and set $Tx = \sum_{n=1}^\infty 2^{-n} x_n v_n$, $x = (x_n) \in l^2$. The orthogonal complement to the kernel of T is injectively mapped to F and the image of the obtained mapping contains $\{v_n\}$. We observe that the constructed embedding is compact. $\square$

Clearly, a similar example exists in every locally convex space to which one can continuously embed an infinite-dimensional Hilbert space.

It is rather surprising that the closure of a Banach disc is not always a Banach disc (see Pérez Carreras, Bonet [**385**, Remark 8.3.21]).

Let us give a simple application of the notion of a Banach disc.

2.5.5. Example. Let B be a Banach disc in a topological vector space E and let $S \subset E$ be an absolutely convex closed set such that its linear span contains B. Then there exists $t > 0$ such that $tB \subset S$.

Indeed, the set $S \cap E_B$ is closed in the Banach space E_B with norm p_B and the latter is the union of the closed sets $n(S \cap E_B)$. By the classical Baire category theorem (see Bogachev, Smolyanov [**72**, § 1.5] or Rudin [**425**, p. 43]) for some n the set $n(S \cap E_B)$ has inner points (in the topology of the norm p_B). Hence this is true also for the set $S \cap E_B$, which by its absolute convexity proves our assertion.

Let us introduce one more important concept.

2.5.6. Definition. *A barrel in a locally convex space E is a closed absolutely convex absorbent set.*

The next simple but useful fact following from the previous example is called the Banach–Mackey theorem.

2.5.7. Theorem. *In every locally convex space every barrel absorbs every Banach disc.*

2.5.8. Proposition. *Let S be a convex subset of a locally convex space E such that every infinite sequence of its elements has a limit point in E. Then S is contained in some Banach disc. In particular, S is absorbed by every barrel.*

PROOF. In analogy with the usual l^1 we introduce the Banach space $l^1(S)$ consisting of real functions ξ on S non-vanishing in at most countably many points and having a finite norm $\|\xi\| = \sum_s |\xi(s)|$.

Let $\widetilde{E}$ denote the completion of E. Consider the mapping $T\colon\ l^1(S) \to \widetilde{E}$, $T\xi = \sum_{s\in S} \xi(s)s$; for every $x \in l^1(S)$ there are only countably many points s_n for which $\xi(s_n) \neq 0$, hence the series $\sum_{n=1}^{\infty} \xi(s_n)s_n$ converges in $\widetilde{E}$ by the completeness of $\widetilde{E}$ and boundedness of S following from our assumptions. It is also seen from this that T is continuous. The desired disc will be the image of the closed unit ball U in $l^1(S)$ if we verify that $S \subset T(U)$ and $T(U) \subset E$. The former is obvious, since $s = T(e_s)$, where e_s is the indicator function of the point s. For proving the latter it suffices to show that $T\xi \in E$ for all $\xi \in U$ such that $\xi(s_n) > 0$ for all points in the support of ξ. In this case $v_N = M_N^{-1} \sum_{n=1}^{N} \xi(s_n)s_n \in S$ by convexity, where $M_N := \sum_{n=1}^{N} \xi(s_n)$. By our assumption, the sequence $\{v_N\}$ has a limit point $v \in E$, but $M_N \to \|\xi\|$ and $M_N v_N \to T\xi$ in $\widetilde{E}$, whence it follows that $\|\xi\| v = T\xi$. Hence $T\xi \in E$. □

2.5.9. Theorem. *Let F be a metrizable locally convex space and let $A \subset F$ be a bounded set. Then there exists a bounded closed absolutely convex set $B \subset F$ such that $A \subset B$ and the normed space E_B induces on A the same topology as F.*

If A is precompact or compact in F, then it will retain this property in E_B.

In addition, for every sequence of bounded sets $A_n \subset F$ there is a bounded closed absolutely convex set D such that all sets A_n are bounded in E_D.

PROOF. Passing to the absolutely convex hull of A, we can assume that A is absolutely convex. In this case it suffices to find a bounded closed absolutely convex set $B \subset F$ containing A for which E_B and F induce the same neighborhoods of zero in A, which reduces to showing that, given a base $\{V_n\}$ of absolutely convex closed neighborhoods of zero in F, for every $\lambda > 0$ we can find n with $A \cap V_n \subset \lambda B$. If this is done, then for every point $a \in A$ and every $\lambda > 0$ we can take V_m such that $A \cap (a + V_m) \subset a + \lambda B$. Indeed, let

V_m be such that $A \cap V_m \subset 2^{-1}\lambda B$. If $a_1 = a + v \in A$, where $v \in V_m$, then $v/2 = (a_1 - a)/2 \in A \cap V_m \subset 2^{-1}\lambda B$ and hence $a_1 - a \in \lambda B$, i.e., $a_1 \in a + \lambda B$.

By the boundedness of A there are numbers $\lambda_n > 0$ for which $A \subset \bigcap_{n=1}^{\infty} \lambda_n V_n$. Clearly, we can assume that $\lambda_n \to +\infty$. Let us set $B := \bigcap_{n=1}^{\infty} \lambda_n^2 V_n$. Then B is absolutely convex, closed and bounded (since we have $\lambda_n^{-2} B \subset V_n$), $A \subset B$.

If $\lambda > 0$, then there exists n_0 such that $\lambda\lambda_n \geqslant 1$ for all $n \geqslant n_0$. Then $A \subset \lambda\lambda_n^2 V_n$ for all $n \geqslant n_0$. Since there exists m such that $V_m \subset \bigcap_{k \leqslant n_0} \lambda\lambda_k^2 V_k$, we have $A \cap V_m \subset \lambda\lambda_n^2 V_n$ for all n, whence we obtain $A \cap V_m \subset \lambda B$, as required.

Let us prove the last assertion. Let $\{V_n\}$ be a basis of closed neighborhoods of zero consisting of absolutely convex sets such that $V_{n+1} \subset V_n$. For every n there exists $c_n > 0$ such that $c_n A_n \subset V_n$. For D we can take the closed absolutely convex hull of the union of $c_n A_n$. □

Note that the spaces E_B and F induce on B also the same uniform structure (see § 1.12(i)).

Even if V is a convex balanced compact set in a Hilbert space, the Banach space (E_V, p_V) is not always separable. For example, this happens with the convex compact set $V = \{(x_n)\colon\ \sup_n |x_n| \leqslant 1\}$ in the weighted Hilbert space

$$E = \Bigl\{x = (x_n)\colon\ \|x\|^2 := \sum_{n=1}^{\infty} n^{-2} x_n^2 < \infty\Bigr\}.$$

In this case E_V coincides with l^∞. However, the following assertion is valid.

2.5.10. Theorem. *Let E be a complete metrizable locally convex space and let K be a compact set in E. Then there exists a convex balanced compact set V containing K such that the Banach space (E_V, p_V) is separable and K is compact with respect to the norm p_V.*

PROOF. Due to the completeness of E and two previous theorems, there exists a balanced convex compact set W_0 containing K such that K is compact as a subset of the Banach space E_{W_0}. Then the linear span of K in E_{W_0} is separable with respect to the norm p_{W_0}. The closure E_0 of this linear span with respect to the norm p_{W_0} in E_{W_0} gives the desired separable Banach space E_W, whose unit ball is the set $W := W_0 \cap E_0$. □

One can go further and obtain the following useful assertion — the Davies–Figiel–Johnson–Pełczyński theorem, a proof of which and of its corollary can be found in Diestel [**120**, Chapter V, § 4], Bogachev, Smolyanov [**72**, Theorem 8.6.24]. On reflexivity, see § 3.7.

2.5.11. Theorem. *Let X be a Banach space and let K be a convex balanced weakly compact set in X. Then there exists a bounded closed balanced convex set W containing K such that the Banach space (E_W, p_W) is reflexive. If K is compact, then W can be chosen compact and E_W can be made separable.*

2.5.12. Corollary. *In the situation of Theorem* 2.5.10 *the Banach space (E_V, p_V) can be chosen separable reflexive.*

As shown in Fonf, Johnson, Pisier, Preiss [**173**], it is not always possible to take for the space E_W a space with a Schauder basis (see the definition in Chapter 3).

In the case of compact V the topology of the space E_V is much stronger than the original topology of E. Nevertheless, as the following result shows, on the Banach space E_V there are sufficiently many linear functionals continuous with respect to this original topology.

2.5.13. Proposition. *Let V be a compact, convex and balanced set in a locally convex space E and let B^* be the unit ball in the dual to the Banach space E_V. Then the set of all functionals in B^* continuous with respect to the topology induced from E is dense in B^* in the topology of uniform convergence on compact sets in E_V.*

For a proof, see Bogachev, Smolyanov [**72**, Proposition 8.6.26].

2.5.14. Proposition. *Let $\{x_n\}$ be a sequence in a locally convex space converging to zero. Its closed absolutely convex hull K is compact precisely when the space E_K is Banach. In this case the compact K is metrizable.*

PROOF. If K is compact, then E_K is Banach. Conversely, let E_K be complete. Consider the linear mapping $T\colon\ l^1 \to E_K$, $T(y_n) = \sum_{n=1}^{\infty} y_n x_n$, which is defined and continuous, since the vectors x_n belong to the unit ball of E_K. Let U be the closed unit ball in l^1. Since $l^1 = c_0^*$, the set U is a metrizable compact in the topology $\sigma(l^1, c_0)$, see Theorem 3.1.4.

We show that T is continuous on U with this topology (on all of l^1 this can be false). If vectors $v^i = (v_n^i) \in U$ converge to $v = (v_n) \in U$ coordinate-wise, then for every continuous seminorm p on E and every $\varepsilon > 0$ we find a number m with the property that $p(x_n) < \varepsilon$ for all $n \geqslant m$, and then take a number M such that $|v_n^i - v_n| < \varepsilon\bigl(p(x_1) + \cdots + p(x_m) + 1\bigr)^{-1}$ for all $n = 1, \ldots, m$ and $i \geqslant M$. Then whenever $i \geqslant M$ we obtain the estimate

$$p(Tv^i - Tv) \leqslant \sum_{n=1}^{m} |v_n^i - v_n| p(x_n) + \varepsilon \sum_{n=1}^{\infty} \bigl(|v_n^i| + |v_n|\bigr) \leqslant 3\varepsilon,$$

which proves the continuity of T on U in the topology $\sigma(l^1, c_0)$. Therefore, $T(U)$ is a metrizable compact set in E. This yields at once also the compactness and metrizability of K, since $\{x_n\} \subset T(U)$, but actually we have $K = T(U)$, which is easily verified by observing that the absolutely convex hull of $\{x_n\}$ is dense in $T(U)$. □

2.5.15. Theorem. *In every locally convex space whose algebraic basis is at most of cardinality of the continuum there exists a hyperplane not containing Banach discs with an infinite-dimensional linear span.*

For a proof, see Pérez Carreras, Bonet [**385**, Theorem 6.3.11].

A compact set K in a Banach space X is called s-compact if it is contained in the closed absolutely convex hull of some sequence $\{x_n\}$ in X with the following property: for every $p \in \mathbb{N}$ the sequence $\{n^p x_n\}$ converges to zero.

2.5.16. Theorem. *For every s-compact set K in a Banach space X there exists an s-compact Banach disc C such that E_C is a separable Hilbert space, the embedding $E_C \to X$ is a nuclear operator* (*see the definition in* § 2.9), *and K remains s-compact also in E_C.*

For a proof, see [**385**, Theorem 6.5.4].

2.6. Strict inductive limits

Let $\{E_n\}$ be an increasing sequence of locally convex spaces:

$$E_1 \subset\subset E_2 \subset\subset \ldots \subset\subset E_n \subset\subset \ldots$$

such that for each index n the embedding $E_n \subset E_{n+1}$ is continuous, where $\subset\subset$ means "a linear subspace", as above. The union $E = \bigcup_{n=1}^{\infty} E_n$ is equipped with the strongest locally convex topology in which the embeddings $E_n \to E$ are continuous; for a base of neighborhoods of zero in it we can take convex sets $V \subset E$ such that every intersection $V \cap E_n$ is a neighborhood of zero in E_n. The space E is called the inductive limit of the sequence $\{E_n\}$ and denoted by the symbol $\mathrm{ind}_n E_n$ (see § 2.4). We shall see in § 2.10(i) that such inductive limit cannot be an inductive limit in the category of topological spaces, i.e., $\mathrm{ind}_n E_n$ can contain non-closed sets (even convex) whose intersections with all E_n are closed.

Two increasing sequences of locally convex spaces E_n and F_n are called equivalent if for every m there exist p and q such that $E_m \subset F_p$, $F_m \subset E_q$, and these embeddings are continuous.

It is readily seen that $\mathrm{ind}_n E_n = \mathrm{ind}_n F_n$ for two such sequences.

2.6.1. Definition. *If for all n the topology in E_n coincides with the topology induced from E_{n+1} and $E_n \neq E_{n+1}$, then the inductive limit of $\{E_n\}$ is called strict.*

The canonical example is the space $\mathcal{D}$ (see Example 2.4.9) represented as the union of the closed subspaces $\mathcal{D}_n$.

Let us show that the topology in E_n is induced by the topology in E.

2.6.2. Lemma. *If $E = \mathrm{ind}_n E_n$ is a strict inductive limit, then the topology in E_n is induced by the topology in E.*

If all spaces E_n are separated, then E is Hausdorff as well.

PROOF. It is clear that the topology in E_n is not weaker than the induced topology. Conversely, let U be an absolutely convex neighborhood of zero in E_n. Lemma 1.3.12 gives an absolutely convex neighborhood of zero U_{n+1} in E_{n+1} such that $U_{n+1} \cap E_n = U$. By induction we obtain increasing absolutely convex neighborhoods of zero $U_k \subset E_k$ with $k > n$ such that $U_{k+1} \cap E_k = U_k$. Then $V = \bigcup_{k=n+1}^{\infty} U_k$ is a neighborhood of zero in E and $V \cap E_n = U$. It is also seen from this that if all spaces E_n are separated, then so is E. $\square$

2.6.3. Proposition. *Let E be the strict inductive limit of separated locally convex spaces E_n such that every E_n is closed in E_{n+1}.*

(i) *A set A is bounded in E if and only if A is contained in some space E_n and is bounded in it.*

(ii) *Every compact set in E is a compact subset of some E_n, and similarly for totally bounded sets. In particular, every countable sequence converging in E is contained and converges in some of the spaces E_n.*

(iii) *All subspaces E_n are closed in E.*

PROOF. (i) It is clear that if A is bounded in E_n, then it is bounded in E. Conversely, let $A \subset E$ be bounded. If A is contained in no E_n, then there exist increasing numbers k_n and points $a_n \in (E_{k_n} \cap A) \backslash E_{k_{n-1}}$. By using Lemma 1.3.12 and induction we find absolutely convex neighborhoods of zero $V_n \subset E_{k_n}$ such that $n^{-1}a_n \notin V_{n+1}$ and $V_{n+1} \cap E_{k_n} = V_n$ for all n. The set $V = \bigcup_{n=1}^{\infty} V_n$ is a neighborhood of zero in E, but $n^{-1}a_n \notin V$, which contradicts the boundedness of A. Thus, A must belong to one of the sets E_n, and then it is bounded in it due to the coincidence of the topology in E_n with the induced one. Assertion (ii) follows at once by (i). Finally, (iii) follows from the previous lemma, since if a net of elements $x_\alpha \in E_n$ converges to a point $x \in E$, then $x \in E_m$ for some $m \geqslant n$, but E_n is closed in E_m by the closedness of E_n in E_{n+1}. □

Note that for general inductive limits these assertions are false (see Exercise 2.10.25). In addition, one should bear in mind that not every subspace $L \subset E$ is the inductive limit of $L \cap E_n$ (see Proposition 2.10.7 and Exercise 2.10.63).

2.6.4. Corollary. *In the situation of the previous proposition the space E is not metrizable.*

PROOF. If E is metrizable, then, taking a basis of decreasing neighborhoods of zero U_n in E and choosing a vector $a_n \in U_n \backslash E_n$ for each n, we obtain a sequence converging to zero, which contradicts the proven proposition. □

2.6.5. Proposition. *The strict inductive limit E of complete separated locally convex spaces E_n is complete.*

PROOF. Suppose that there is a point $z \notin E$ in the completion $\widetilde{E}$ of the space E. We observe that E_n is closed in $\widetilde{E}$ by the completeness of E_n. Hence for each n there exists an absolutely convex neighborhood of zero W_n such that $(z + W_n) \cap E_n = \varnothing$. These neighborhoods can be chosen in such a way that $W_{n+1} \subset W_n$ for all n.

The convex hull of the set $\bigcup_{n=1}^{\infty}(2^{-1}W_n \cap E_n)$ will be denoted by U. Then U is a neighborhood of zero in E, therefore, its closure $\overline{U}$ in $\widetilde{E}$ is a closed neighborhood of zero in $\widetilde{E}$. Since E is dense in $\widetilde{E}$, the set $(z + \overline{U}) \cap E$ is nonempty; hence for some n there exists $v \in (z + \overline{U}) \cap E_n$, i.e., $v = z + u \in E_n$, where $u \in \overline{U}$. We show that $\overline{U} \subset W_n + E_n$, whence it will follow that $u = w_n + y$, where $w_n \in W_n$, $y \in E_n$, but this will give the inclusion $z + w_n = v - y \in E_n$ contradicting our choice of W_n.

Thus, let $x \in \overline{U}$. Then $x \in U + 2^{-1}W_n$, so $x = u + w_n/2$, where $u \in U$, $w_n \in W_n$. Hence $x = \lambda_1 x_1 + \cdots + \lambda_k x_k + w_n/2$, where $x_j \in 2^{-1}W_j \cap E_j$, $|\lambda_1| + \cdots + |\lambda_k| = 1$. We can assume that $k \geqslant n$. We have

$$\lambda_1 x_1 + \cdots + \lambda_n x_n \in E_n, \quad \lambda_{n+1} x_{n+1} + \cdots + \lambda_k x_k \in 2^{-1}W_n,$$

since $W_j \subset W_n$ for $j > n$ and the neighborhood W_n is absolutely convex. Thus $\lambda_1 x_1 + \cdots + \lambda_k x_k \in E_n + 2^{-1}W_n$, whence we obtain that $x \in E_n + W_n$, as required. □

For example, $\mathcal{D}(\mathbb{R}^n)$ is complete. Some additional interesting information can be found in Pérez Carreras, Bonet [**385**, Chapter 8].

2.7. Inductive limits with compact embeddings

We now turn to another important case where the topology of E_n is a priori strictly stronger than the induced topology of E_{n+1}.

We shall say that an increasing sequence of Hausdorff locally convex spaces E_n is *regular* if, for each n, there is a closed neighborhood of zero in E_n with compact closure in E_{n+1}. If one can find a closed neighborhood of zero in E_n compact in E_{n+1}, then we shall say that we have a *strongly regular* sequence.

It is easy to see that a regular sequence is not always strongly regular.

Recall that the equivalence of sequences of embedded spaces are defined in § 2.6.

2.7.1. Lemma. *Every regular sequence $\{E_n\}$ is equivalent to some strongly regular sequence of separable reflexive Banach spaces.*

PROOF. For every n we find in E_n a closed absolutely convex neighborhood of zero V_n whose closure K_n is compact in E_{n+1} and set $X_n := E_{K_n}$ (see § 2.5). Then X_n is a Banach space, $E_n \subset X_n \subset E_{n+1} \subset X_{n+1}$, and these embeddings are continuous. Hence the closed unit ball of X_n (i.e., K_n) is compact in X_{n+1}. Thus, the sequences $\{E_n\}$ and $\{X_n\}$ are equivalent and $\{X_n\}$ is strongly regular.

Now with the aid of Corollary 2.5.12 we can obtain an equivalent sequence of separable reflexive Banach spaces. □

A sequence of absolutely convex compacts K_n in a locally convex space E will be called regular if $E = \bigcup_{n=1}^{\infty} K_n$, every K_n is contained in the algebraic kernel of K_{n+1} (see § 1.10) and is compact in the Banach space $E_{K_{n+1}}$.

2.7.2. Lemma. *Let $\{K_n\}$ be a regular sequence of absolutely convex compact sets. Then for each m and each scalar λ there exists a number n such that λK_m is contained in the algebraic kernel of K_n.*

PROOF. For $n > m$ let G_n be the algebraic kernel of $K_n \cap E_{K_{m+1}}$. Zero belongs to the topological interior of the set $K_n \cap E_{K_{m+1}}$ in the Banach space $E_{K_{m+1}}$. Hence the algebraic kernel of this set coincides with the topological one in the space $E_{K_{m+1}}$ (Exercise 1.12.82). Thus, the sets G_n are open in $E_{K_{m+1}}$. They cover λK_m, since the algebraic kernels of K_n cover E (since K_n is contained in the algebraic kernel of K_{n+1}), moreover, K_m is compact in this space. Hence λK_m is contained in some G_n and so belongs to the algebraic kernel of K_n. □

2.7.3. Lemma. *Let $\{K_n\}$ be a regular sequence of absolutely convex compact sets. Then each absolutely convex compact set $K \subset E$ is contained in some K_n.*

PROOF. By the continuity of the embedding $E_K \to E$ we obtain that the absolutely convex set set $K_n \cap E_K$ is closed in the Banach space E_K. Hence some intersection $K_m \cap E_K$ contains a neighborhood of zero in E_K. Therefore, there exists $\lambda > 0$ such that $K \subset \lambda K_m$. By the previous lemma K belongs to some K_n. □

2.7.4. Lemma. *In every regular inductive limit of a sequence of locally convex spaces there is a regular sequence of absolutely convex compact sets.*

PROOF. We represent the given space in the form of a regular limit of some sequence of Banach spaces X_n in which increasing closed unit balls U_n are compact in X_{n+1}. Then we can take $K_n = nU_n$. □

2.7.5. Lemma. *Let $E = \mathrm{ind}_n E_n$ be a regular inductive limit of some sequence of locally convex spaces E_n, let $\{K_n\}$ be an arbitrary regular sequence of absolutely convex compacts in E. Then E is the inductive limit of the sequence of Banach spaces E_{K_n}.*

PROOF. Let us take the same X_n and U_n as in the previous proof. Then the sequences $\{X_n\}$ and $\{E_{K_n}\}$ are equivalent by Lemma 2.7.3. □

We shall say that a topological space X is a *free union* of a sequence of its subspaces X_n if $X = \bigcup_{n=1}^{\infty} X_n$ and the closed sets in X are exactly those sets whose intersection which each X_n is closed in X_n in the induced topology. Thus, the space X is the inductive limit of the subspaces X_n in the category of general topological spaces. It is important to emphasize that if X_n are increasing locally convex spaces continuously embedded into embracing spaces, then the topology of the inductive limit in the category of general topological spaces can be strictly stronger than the topology of the inductive limit in the category of locally convex spaces. This happens in the case of the space $\mathcal{D}(\mathbb{R}^1)$: as we shall see below, it *is not* a free union of its closed subspaces $\mathcal{D}[-n,n]$. There is a principal difference between the next theorem and the case of strict inductive limits.

2.7.6. Theorem. *The inductive limit $E = \mathrm{ind}_n E_n$ of a regular sequence of locally convex spaces is a free union of any regular sequence of its absolutely convex compact subsets.*

PROOF. Let $\{K_n\}$ be a regular sequence of absolutely convex compacts in E and let A be a set such that all intersections $A \cap K_n$ are closed. We show that A is closed. Let $x_0 \notin A$. By assumption there exists a number p for which x_0 is contained in K_{p-1}, hence also in the algebraic kernel of K_p. By induction we find an increasing sequence of sets V_n, $n \geqslant p$, with the following properties: 1) V_n is a closed absolutely convex neighborhood of zero in E_{K_n} and is compact in $E_{K_{n+1}}$, 2) $x_0 + V_n \subset K_n$, 3) $(x_0 + V_n) \cap A = \varnothing$.

For the set V_p we take a closed ball of a sufficiently small radius in E_{K_p}; this is possible, since $A \cap (A \cap E_{K_p})$ is closed in E_{K_p}, $x_0 \notin A \cap (A \cap E_{K_p})$ and x_0 is contained in the algebraic kernel of K_p. If increasing absolutely convex compact sets $V_p, \ldots, V_n$ with the desired properties are already picked in the respective spaces up to E_{K_n}, then V_{n+1} is constructed as follows.

The set $A \cap K_{n+1}$ is closed in $E_{K_{n+1}}$ by the closedness in E and continuity of the embedding $E_{K_{n+1}} \to E$. The set $x_0 + V_n$ is disjoint with $A \cap K_{n+1}$ and with the unit sphere S_{n+1} in $E_{K_{n+1}}$ (since along with K_n it is contained in the algebraic kernel of K_{n+1}). Since $x_0 + V_n$ is compact in $E_{K_{n+1}}$ and the set $(A \cap K_{n+1}) \cup S_{n+1}$ is closed in the Banach space $E_{K_{n+1}}$, the distance d between them in the norm of $E_{K_{n+1}}$ is positive.

Therefore, setting $V_{n+1} = V_n + 2^{-1}dK_{n+1}$, we obtain an absolutely convex set compact in $E_{K_{n+2}}$ due to the compactness of V_n and K_{n+1} in $E_{K_{n+2}}$. It is clear that $V_n \subset V_{n+1}$ for all n and V_{n+1} is a closed neighborhood of zero in the space $E_{K_{n+1}}$. Finally, $x_0 + V_{n+1} \subset K_{n+1}$ and $(x_0 + V_{n+1}) \cap A = \varnothing$ due to our choice of d.

Thus, the sets V_n are constructed. Then the set $V = \bigcup_{n=p}^{\infty} V_n$ is a closed absolutely convex neighborhood of zero in E and $(x_0 + V) \cap A = \varnothing$. Therefore, $E\backslash A$ is open in E, i.e., A is closed. □

2.7.7. Corollary. *A set A in a regular inductive limit $\mathrm{ind}_n E_n$ is closed precisely when every intersection $A \cap E_n$ is closed in E_n.*

PROOF. Let $A \cap E_n$ be closed in E_n for all n. Taking a regular sequence of absolutely convex compact sets K_n, we obtain that $A \cap K_n$ is closed, which gives the closedness of A. The converse is obvious. □

2.7.8. Corollary. *A set A in a regular inductive limit $\mathrm{ind}_n E_n$ is closed precisely when it is sequentially closed, i.e., contains the limits of all its convergent countable sequences.*

PROOF. Let A be sequentially closed in E. We represent E as a regular inductive limit of a sequence of Banach spaces X_n. Then $A \cap X_n$ is sequentially closed in each Banach space X_n, which implies that A is closed in E. The converse is obvious. □

We observe that the previous two corollaries are not true for strict inductive limits (considered in § 2.6), but the next corollary is true for them.

2.7.9. Corollary. *In the situation of the theorem above, every compact set in the space E is contained in some subspace E_n. In addition, every convergent sequence in E is contained and converges in some E_n.*

Moreover, every bounded set in E is contained in some E_n and has compact closure there.

PROOF. It is clear that we can choose a regular sequence of absolutely convex compact sets K_n such that $2K_n \subset K_{n+1}$ for all n. Suppose that for every n there is a point a_n of a given bounded set A not belonging to K_{2n}. Then $2^{-n}a_n \notin K_n$, since $K_n \subset 2^{-n}K_{2n}$. The countable set $\{2^{-n}a_n\}$ has a finite intersection with each K_n and according to the results above turns out to be closed in E. However, this is impossible, since by the boundedness of the set A the sequence $\{2^{-n}a_n\}$ converges to zero that does not belong to this sequence. Thus, the set A is contained in some K_m, but we already know that the set K_m is contained and compact in some E_n. □

If the spaces E_n are distinct, then we see on account of the obtained results that the corresponding regular inductive limit is not metrizable. The proof of the following fact is delegated to Exercise 2.10.28.

2.7.10. Corollary. *Any regular inductive limit is complete.*

2.7.11. Example. Let E be the inductive limit of an increasing sequence of reflexive Banach spaces X_n with compact embeddings. Then E is a regular inductive limit. In particular, this is true if all spaces X_n are Hilbert. If all X_n are finite-dimensional, then this is true also in the case of continuous embeddings.

Indeed, any closed ball in X_n is compact in X_{n+1} by the weak compactness (which follows from the reflexivity) and totally bounded (which follows from the compactness of the embedding).

Here is yet another property of regular inductive limits that is not possessed by strict inductive limits (see Exercise 2.10.63).

2.7.12. Proposition. *Any closed linear subspace of a regular inductive limit* $\mathrm{ind}_n E_n$ *equipped with the induced topology is again a regular inductive limit.*

PROOF. Suppose that F is a closed linear subspace in a regular inductive limit $E = \mathrm{ind}_n E_n$. We can assume that E_n are reflexive Banach spaces such that the closed unit ball K_n of E_n is contained and is compact in the unit ball K_{n+1} of E_{n+1}. We show that $Q_n = K_n \cap F$ is a regular sequence of compact sets in F. It is clear that Q_n is an absolutely convex compact set in F, closed in $E_{Q_{n+1}}$. In addition, the set Q_n is totally bounded in $E_{Q_{n+1}}$. Indeed, it is compact in $E_{K_{n+1}}$, so for every $\varepsilon > 0$ there exist points $x_1, \ldots, x_m \in Q_n$ for which $Q_n \subset \bigcup_{i=1}^m (x_i + \varepsilon K_{n+1})$, whence $Q_n \subset \bigcup_{i=1}^m (x_i + \varepsilon Q_{n+1})$. Therefore, $\{Q_n\}$ is a regular sequence of compact sets in the subspace F. We now verify that F with the induced topology coincides with the regular inductive limit $\mathrm{ind}_n E_{Q_n}$. Let $A \subset F$ be such that all $A \cap Q_n$ are closed. Then $A \cap K_n$ is closed as well, whence it follows that A is closed in E, hence also in F with the induced topology. The converse is also true: if A is closed in F with the induced topology, then all sets $A \cap Q_n$ are closed. Thus, F with the induced topology is a free union of Q_n and hence coincides with $\mathrm{ind}_n E_{Q_n}$. $\square$

2.8. Tensor products

Let E_1 and E_2 be two locally convex spaces. The algebraic tensor product $E_1 \otimes E_2$ is usually defined as the natural quotient of the linear space formally generated by expressions of the form $x \otimes y$, where $x \in E_1$, $y \in E_2$, but here it will be convenient to define $E_1 \otimes E_2$ from the very beginning as the linear subspace in the algebraic dual to the space $B(E_1, E_2)$ of bilinear functions on $E_1 \times E_2$ generated by the elements

$$x \otimes y\colon\ b \mapsto b(x, y), \quad b \in B(E_1, E_2).$$

When using this embedding the necessary factorization is done automatically.

The space $E_1 \otimes E_2$ can be equipped with different locally convex topologies. The *projective topology* τ_π is defined as the strongest locally convex topology in which the canonical bilinear mapping $E_1 \times E_2 \to E_1 \otimes E_2$ is continuous.

It is readily verified (see Schaefer [**436**, p. 93]) that the dual to the space $(E_1 \otimes E_2, \tau_\pi)$ can be identified with the space of all continuous bilinear functions on $E_1 \times E_2$ in such a way that equicontinuous subsets in $(E_1 \otimes E_2, \tau_\pi)'$ will correspond to equicontinuous sets of bilinear functions on $E_1 \times E_2$. In order to define τ_π by means of seminorms the following construction of the tensor product of seminorms p and q on E_1 and E_2 is used. For $w \in E_1 \otimes E_2$ set

$$p \otimes q(w) = \inf\Bigl\{\sum_i p(x_i) q(y_i) \colon \ w = \sum_i x_i \otimes y_i\Bigr\},$$

where inf is taken over all representations of w in the indicated form. It is readily verified that $p \otimes q(x \otimes y) = p(x) q(y)$.

Any locally convex topology can be always defined by a directed family of seminorms $\mathcal{P}$, i.e., in such a way that for any seminorms $p_1, p_2 \in \mathcal{P}$ there exists $p_3 \in \mathcal{P}$ with $p_1 \leqslant p_3$, $p_2 \leqslant p_3$ (for example, one can take all possible finite sums of the original seminorms and their products by positive numbers). Let us take directed families of seminorms $\mathcal{P}_1$ and $\mathcal{P}_2$ on E_1 and E_2 defining the topologies. Then the collection of all seminorms $p_1 \otimes p_2$, where $p_1 \in \mathcal{P}_1$, $p_2 \in \mathcal{P}_2$, defines the *projective topology* τ_π. The completion of the space $(E_1 \otimes E_2, \tau_\pi)$ will be denoted by $E_1 \widetilde{\otimes}_\pi E_2$. The following result was obtained by Grothendieck.

2.8.1. Theorem. *If E_1 and E_2 are metrizable locally convex spaces, then every element $w \in E_1 \widetilde{\otimes}_\pi E_2$ can be represented in the form*

$$w = \sum_{i=1}^{\infty} \lambda_i x_i \otimes y_i, \quad \textit{where} \quad \sum_{i=1}^{\infty} |\lambda_i| < \infty,$$

where $x_i \to 0$ and $y_i \to 0$ in E_1 and E_2, respectively.

PROOF. Let us take in E_1 and E_2 increasing sequences of seminorms p_n and q_n generating the topologies. Set $r_n = p_n \otimes q_n$ and extend r_n to $E_1 \widetilde{\otimes}_\pi E_2$. There exist $w_n \in E_1 \otimes E_2$ such that $r_n(w - w_n) \leqslant n^{-2} 2^{-n-1}$ and $w_1 = 0$. Let $w_n = \sum_{i=1}^{i_n} \lambda_i x_i \otimes y_i$ and $v_n = w_{n+1} - w_n$. Then

$$\begin{aligned} r_n(v_n) &\leqslant r_n(w - w_n) + r_n(w - w_{n+1}) \\ &\leqslant r_n(w - w_n) + r_{n+1}(w - w_{n+1}) \leqslant n^{-2} 2^{-n}. \end{aligned}$$

It follows from the definition of r_n that there exist increasing numbers i_n, $n \in \mathbb{N}$ and representations $v_n = \sum_{i=i_n+1}^{i_{n+1}} \lambda_i \lambda_i x_i \otimes y_i$ such that $\sum_{i=i_n+1}^{i_{n+1}} |\lambda_i| \leqslant 2^{-n}$, $p_n(x_i) \leqslant n^{-1}$, $q_n(y_i) \leqslant n^{-1}$ for all $i_n < i \leqslant i_{n+1}$. Indeed, take a representation of the form $v_n = \sum_{i=i_n+1}^{i_{n+1}} u_i \otimes z_i$ with $\sum_{i=i_n+1}^{i_{n+1}} p_n(u_i) q_n(z_i) \leqslant n^{-2} 2^{-n}$, where $p_n(u_i) > 0$ and $q_n(z_i) > 0$. Now we can take $x_i = u_i \bigl(n p_n(u_i)\bigr)^{-1}$, $y_i = z_i \bigl(n q_n(z_i)\bigr)^{-1}$, $\lambda_i = n^2 p_n(u_i) q_n(z_i)$. Thus, the required representation is given by $w = \sum_{n=1}^{\infty} v_n = \sum_{i=1}^{\infty} \lambda_i x_i \otimes y_i$. $\square$

Taking absolutely convex neighborhoods of zero $U \subset E_1$ and $V \subset E_2$, we can define the following seminorm on $E_1 \otimes E_2$:

$$p_{U,V}(w) = \sup\Bigl\{\sum_i f(x_i)g(y_i) \colon\ w = \sum_i x_i \otimes y_i, f \in U^\circ, g \in V^\circ\Bigr\},$$

where sup is taken over all representations of w in the indicated form. It is readily verified that $p_{U,V}(w) \leqslant p_U \otimes p_V$. Hence the topology τ_ε generated by such seminorms is weaker than τ_π. This weaker topology is called the *topology of equicontinuous convergence*.

The completion of $(E_1 \otimes E_2, \tau_\varepsilon)$ will be denoted by the symbol $E_1 \widetilde{\otimes}_\varepsilon E_2$.

We shall see in the next section that for a given locally convex space E the equality $E \widetilde{\otimes}_\varepsilon F = E \widetilde{\otimes}_\pi F$ holds for all locally convex spaces F precisely when E is nuclear (which is also defined in the next section). Actually, it suffices to have this equality for all Banach spaces. Since infinite-dimensional Banach spaces are not nuclear, it is easy to give examples where the equality fails. Below we describe these two topologies explicitly in case of Hilbert spaces.

Yet another projective topology on the tensor product $E_1 \otimes E_2$ is called *inductive*. It is denoted by the symbol τ_i and is defined as the strongest locally convex topology on $E_1 \otimes E_2$ such that the canonical bilinear mapping $E_1 \times E_2 \to E_1 \otimes E_2$ is separately continuous. Hence the projective and inductive topologies coincide if all separately continuous bilinear mappings on $E_1 \times E_2$ are continuous. For example, the projective tensor topology on $\mathcal{D}(\mathbb{R}) \otimes \mathcal{D}(\mathbb{R})$ is strictly weaker than the inductive topology, since there is a discontinuous separately continuous bilinear form on $\mathcal{D}(\mathbb{R})$, see Exercise 2.10.49.

It is instructive to examine the case of Hilbert spaces E_1 and E_2. The projective tensor topology is generated by the so-called nuclear norm

$$\|w\|_{\mathcal{N}} = \inf\Bigl\{\sum_i \|x_i\|\,\|y_i\| \colon\ w = \sum_i x_i \otimes y_i\Bigr\},$$

the completion with respect to which in case $E_1 = E_2 = H$ gives the space $\mathcal{N}(H)$ of nuclear (or trace class) operators on H, i.e., $H \widetilde{\otimes}_\pi H = \mathcal{N}(H)$. The inductive topology gives the same. The topology τ_ε is generated by the norm

$$\|w\|_\infty = \sup\Bigl\{\sum_i f(x_i)g(y_i) \colon\ w = \sum_i x_i \otimes y_i,\ \|f\|_{E_1'} \leqslant 1,\ \|g\|_{E_2'} \leqslant 1\Bigr\},$$

which is the operator norm, and in case $E_1 = E_2 = H$ the completion of $H \widetilde{\otimes}_\varepsilon H$ is the space $\mathcal{K}(H)$ of compact operators on H. Actually, both constructions apply to any Banach spaces (where the elements $x \otimes y$ are regarded as operators from E_1' to E_2 acting by the formula $l \mapsto l(x)y$). However, in the Hilbert case the space $E_1 \otimes E_2$ can be equipped with the Hilbert–Schmidt norm (when we identify elements of $E_1 \otimes E_2$ with operators from E_1 to E_2), which in case $E_1 = E_2 = H$ after completing leads to the space $\mathcal{H}(H)$ of Hilbert–Schmidt operators (p. 144). This latter tensor product is called the Hilbert tensor product and is denoted by $E_1 \otimes_2 E_2$. Note that in these examples the regarded norms on elements $x \otimes y$ equal $\|x\|\,\|y\|$; such norms on $E_1 \otimes_2 E_2$ are called cross-norms. The operator norm and nuclear norm are extreme cross-norms. Using other norms on $E_1 \otimes_2 E_2$ we obtain different tensor products as completions.

Tensor products can be introduced in the language of categories, and they are connected with the so-called representable functors. We make brief remarks in this direction, although categories are never used in this book. In the theory of categories (see Mac Lane [**326**]), it is customary to consider along with usual sets certain "very large sets", called "classes" to distinguish them from sets, in order to use expressions like "the class of all locally convex spaces" (we recall that the concept "the set of all sets" is contradictory). A category is a class $\mathfrak{C}$ whose elements are called objects of the category, and it is assumed that for each pair A, B of objects we are given a set $\hom(A, B)$ whose elements are called morphisms from A to B, provided that they satisfy certain natural axioms.

For example, in the category of sets morphisms are arbitrary mappings. Another relevant example is the category $\mathcal{LCS}$ of real locally convex spaces; morphisms are continuous linear mappings. Certainly, one can also consider the category of complex locally convex spaces. In the category of topological spaces morphisms are continuous mappings.

We say that there is a covariant functor from a category $\mathfrak{C}_1$ to a category $\mathfrak{C}_2$ if to every object A of $\mathfrak{C}_1$ an object $F(A)$ of $\mathfrak{C}_2$ is defined and for every morphism $\varphi \in \hom(A, B)$, where $A, B \in \mathfrak{C}_1$, a morphism $F(\varphi) \in \hom\bigl(F(A), F(B)\bigr)$ is defined such that the identity mappings are taken to identity mappings and F respects compositions (provided they are defined). Contravariant functors are defined similarly, but $F(\varphi) \in \hom\bigl(F(B), F(A)\bigr)$.

For example, let $\mathfrak{S}$ be the category of sets and let $\mathfrak{C}$ be some category. The mapping $h^A\colon C \to \hom(A, C)$, where $A, C \in \mathfrak{C}$, along with the class of naturally defined mappings of sets $\hom(A_1, C)$ to sets $\hom(A_2, C)$ is a covariant functor from $\mathfrak{C}$ to $\mathfrak{S}$.

Isomorphisms of functors are defined in a natural way. A covariant functor F from a category $\mathfrak{C}$ to the category of sets $\mathfrak{S}$ is called representable if, for some object $A \in \mathfrak{C}$, there is an isomorphism of functors $f\colon h^A \to F$; then we say that the object A represents the functor F.

Let us return to the category $\mathcal{LCS}$ of real locally convex spaces. Suppose that we are given $E, G \in \mathcal{LCS}$. For each space $K \in \mathcal{LCS}$ let $B(E \times G, K)$ be the set of all continuous bilinear mappings from $E \times G$ to K. The category $\mathcal{T}$ will be defined as follows: its objects are pairs (K, B), where $K \in \mathcal{LCS}$, $B \in B(E \times G, K)$, and morphisms between objects (K_1, B_1) and (K_2, B_2) are linear operators $\psi\colon K_1 \to K_2$ such that $\psi \circ B_1 = B_2$. Another category is obtained by using the set $B_s(E \times G, K)$ of all separately continuous bilinear mappings.

One can verify that in both cases we obtain a representable covariant functor from the category $\mathcal{LCS}$ to the category of sets $\mathfrak{S}$. In the first case (continuous bilinear mappings) the representing object is the completion of the tensor product $E_1 \otimes E_2$ with the projective topology. The functor isomorphism $f\colon h^{E \otimes G} \to F$ takes the set of continuous linear operators from $E \otimes G$ to K to the set of their compositions with the canonical bilinear mapping from $E \times G$ to $E \otimes G$, which is continuous in this case.

In the second case (separately continuous bilinear mappings) the representing object is the completion of the tensor product $E_1 \otimes E_2$ with the inductive topology. The functor isomorphism is described similarly

2.9. Nuclear spaces

Let X and Y be two normed spaces. An operator $T \in \mathcal{L}(X,Y)$ is called *nuclear* if it can be represented in the form

$$Tx = \sum_{i=1}^{\infty} u_i(x)v_i, \quad \text{where } u_i \in X',\ v_i \in Y,\ \sum_{i=1}^{\infty} \|u_i\|\,\|v_i\| < \infty.$$

The infimum of the sums $\sum_{i=1}^{\infty} \|u_i\|\,\|v_i\|$ over all possible representations of T is called the *nuclear norm* of T and is denoted by the symbol $\|T\|_{\mathcal{N}}$. It is obvious that any nuclear operator between Banach spaces is compact (see the definition in § 3.10).

The concept of nuclear operator is naturally extended to locally convex spaces.

If V is an absolutely convex neighborhood of zero in a locally convex space E, then the Minkowski functional p_V of the set V is a seminorm and generates a norm on the quotient $E/p_V^{-1}(0)$. The completion of this normed space will be denoted by $\widetilde{E}_V$. The norm in $\widetilde{E}_V$ will be also denoted by the symbol p_V. If $p_V^{-1}(0) = 0$, then $\widetilde{E}_V$ is the completion of the usual space (E_V, p_V). We recall that the latter is complete if V is bounded and sequentially complete. However, in typical cases the set V is not bounded and $p_V^{-1}(0) \neq 0$. The natural mapping $j_V\colon E \to \widetilde{E}_V$ is continuous. Note also that if V contains an absolutely convex neighborhood of zero W, then the operator $\pi_{V,W}\colon \widetilde{E}_W \to \widetilde{E}_V$ generated by the natural inclusion $p_W^{-1}(0) \subset p_V^{-1}(0)$ and the surjection $E/p_W^{-1}(0) \to E/p_V^{-1}(0)$, is continuous.

Let E and F be two locally convex spaces and let $T\colon E \to F$ be a linear mapping such that for some absolutely convex neighborhood of zero $V \subset E$ the set $T(V)$ is contained in some Banach disc $B \subset F$. Let $\psi_B\colon F_B \to F$ be the natural embedding. Since $Tx = 0$ if $p_V(x) = 0$ (by the boundedness of $T(V)$), the operator T can be written in the form $T = \psi_B \circ \widetilde{T} \circ j_V$, where $\widetilde{T}\colon \widetilde{E}_V \to F_B$ is a continuous linear operator. If we can choose V and B in such a way that the operator $\widetilde{T}$ between the indicated Banach spaces becomes nuclear, then T is called *nuclear*. This yields at once that any nuclear operator is compact also in the case of locally convex spaces (see § 3.10). For normed spaces this gives the previous concept.

One can verify that nuclear mappings are characterized in the following way (see Schaefer [**436**, Theorem 7.1]).

2.9.1. Theorem. *An operator $T \in \mathcal{L}(E,F)$ is nuclear precisely when it has the form*

$$Tx = \sum_{n=1}^{\infty} \lambda_n l_n(x) y_n,$$

where $\sum_{n=1}^{\infty} |\lambda_n| < \infty$, $\{l_n\}$ is an equicontinuous sequence in E' and $\{y_n\} \subset F$ is a sequence contained in some Banach disc.

2.9.2. Definition. *A locally convex space E is called nuclear if it possesses a basis $\mathcal{V}$ of absolutely convex neighborhoods of zero with the property that for every neighborhood $V \in \mathcal{V}$ the canonical mapping $j_V\colon\ E \to \widetilde{E}_V$ is nuclear.*

The previous theorem implies that the canonical mappings j_V must have the form

$$j_V x = \sum_{n=1}^{\infty} \lambda_n l_n(x) y_n,$$

where $\sum_{n=1}^{\infty} |\lambda_n| < \infty$, $\{l_n\}$ is an equicontinuous sequence in E' and $\{y_n\} \subset V$.

It is readily verified that a locally convex space is nuclear precisely when its completion is nuclear.

As an example of an infinite-dimensional nuclear space one can take an infinite power $\mathbb{R}^T$ of $\mathbb{R}$; for a basis of neighborhoods of zero we can take the sets $V = \{x\colon\ |x(t_i)| < \varepsilon, i = 1, \ldots, n\}$, so $p_V^{-1}(0) = \{x\colon\ x(t_i) = 0, i = 1, \ldots, n\}$, hence the quotient $\mathbb{R}^T/p_V^{-1}(0)$ is finite-dimensional.

2.9.3. Theorem. *The following properties of a locally convex space E are equivalent*:

(i) *E is nuclear*;

(ii) *for every Banach space X, every operator $T \in \mathcal{L}(E, X)$ is nuclear*;

(iii) *every absolutely convex neighborhood of zero V in the space E contains an absolutely convex neighborhood of zero U such that the associated operator $\pi_{V,U}\colon\ \widetilde{E}_U \to \widetilde{E}_V$ is nuclear.*

PROOF. If E is nuclear and X is a Banach space with the open unit ball U, then $V = T^{-1}(U)$ is a neighborhood of zero in E, so the mapping $j_V\colon\ E \to \widetilde{E}_V$ is nuclear. Hence it has the form $\sum_{n=1}^{\infty} \lambda_n l_n(x) y_n$, where $\{l_n\}$ is an equicontinuous sequence in E', $\{\lambda_n\} \in l^1$, $\{y_n\}$ is contained in a Banach disc in $\widetilde{E}_V$. We observe that $Tx = \sum_{n=1}^{\infty} \lambda_n l_n(x) T y_n$, where $\{Ty_n\}$ is contained in a Banach disc in X.

It is clear that (iii) implies (i). Let (ii) hold and let V be an absolutely convex neighborhood of zero in E. Then the operator $j_V\colon\ E \to \widetilde{E}_V$ has the form indicated above. By the equicontinuity of $\{l_n\}$ there exists an absolutely convex neighborhood of zero W for which $|l_n(w)| \leqslant 1$ for all $w \in W$, $n \geqslant 1$. Let us take $U = W \cap V$. It is straightforward to verify that the operator $\pi_{V,U}$ is nuclear, since it has the form $\pi_{V,U} z = \sum_{n=1}^{\infty} \widetilde{l}_n(z) y_n$, where the functional $\widetilde{l}_n \in \widetilde{E}'_U$ generated by the functional l_n (the estimate $|l_n(w)| \leqslant 1$ on W means that the norm of $\widetilde{l}_n$ does not exceed 1) and the sequence $\{y_n\}$ is bounded in $\widetilde{E}_V$. □

Property (iii) and the compactness of nuclear operators yield the following important fact.

2.9.4. Corollary. *Every bounded set in any nuclear space is precompact.*

It is obvious from Property (ii) that a normed space is nuclear only when it is finite-dimensional. Nevertheless, as we shall now see, nuclear spaces are closely connected with Hilbert spaces.

2.9.5. Theorem. *Let U be a neighborhood of zero in a nuclear space E. Then there exist an absolutely convex neighborhood of zero $V \subset U$, $W \subset V$ such that the Banach spaces $\widetilde{E}_V$ and $\widetilde{E}_W$ are linearly isometric to l^2 or $\mathbb{R}^n$ and the operator $\pi_{V,W}\colon \widetilde{E}_W \to \widetilde{E}_V$ is nuclear. Hence the topology of E is generated by a family of seminorms defined by nonnegative definite Hermite forms.*

PROOF. We shall show that there exists an operator $A \in \mathcal{L}(E, l^2)$ such that $V = A^{-1}(B) \subset U$, where B is the open unit ball in l^2. We can assume that U is absolutely convex. Then $j_U\colon E \to \widetilde{E}_U$ has the form $j_U x = \sum_{n=1}^{\infty} \lambda_n l_n(x) y_n$, where $\lambda_n > 0$, $\sum_{n=1}^{\infty} \lambda_n = 1$, $\|y_n\| = 1$ in $\widetilde{E}_U$ and $\{l_n\}$ is equicontinuous in E'. Set $Ax = \bigl\{\sqrt{\lambda_n} l_n(x)\bigr\}$. The equicontinuity of $\{l_n\}$ yields that $A(U)$ is bounded in l^2. We have

$$p_U(j_U x) \leqslant \sum_{n=1}^{\infty} \lambda_n |l_n(x)| \leqslant \Bigl(\sum_{n=1}^{\infty} \lambda_n |l_n(x)|^2\Bigr)^{1/2} = \|Ax\|_{l^2}.$$

Thus, $V := A^{-1}(B) \subset U$. Finally, the quotient $E/p_V^{-1}(0)$ with norm p_V is linearly isometric to the Euclidean space $A(E)$, which gives a linear isometry between $\widetilde{E}_V$ and the closure of $A(V)$ in l^2. It remains to apply the previous theorem. □

Note that in place of the Hilbert space l^2 in this theorem one can take any space l^p, $1 \leqslant p \leqslant +\infty$. A justification is similar.

Let Λ be a set that has the cardinality equal to the minimal possible cardinality of a base of neighborhoods of zero in the space E.

2.9.6. Corollary. *If E is nuclear and $\{H_\lambda\}_{\lambda\in\Lambda}$ is a family of infinite-dimensional Hilbert spaces, then there exist linear mappings $T_\lambda\colon E \to H_\lambda$ such that the topology of E is the weakest one in which all T_λ are continuous.*

2.9.7. Corollary. *If a nuclear space E is complete, then it is isomorphic to a projective limit of a family of cardinality Λ of Hilbert spaces.*

A Fréchet space is nuclear precisely when it can be represented in the form of the projective limit $E = \varprojlim H_n$ of a sequence of separable Hilbert spaces H_n with nuclear mappings $\psi_{mn}\colon H_m \to H_n$ for $m < n$. In this case it is separable.

PROOF. The first assertion follows from what we have proved above. Let now E be a nuclear Fréchet space. We already know that E has a base of absolutely convex neighborhoods of zero V_n for which the spaces $H_n = \widetilde{E}_{V_n}$ are Hilbert. We can assume that $V_{n+1} \subset V_n$ and that all canonical mappings $\psi_{n,n+1}\colon \widetilde{E}_{V_{n+1}} \to \widetilde{E}_{V_n}$ are nuclear. This gives the required representation. Conversely, let E have the indicated form. Then E is the subspace in $\prod_{n=1}^{\infty} H_n$ determined by the conditions $\psi_{mn} x_m = x_n$, $m < n$, and its base of neighborhoods of zero is formed by the products $V = \prod_{i=1}^{m} B_i \times \prod_{j=m+1}^{\infty} H_j$, where B_i is a ball centered at the origin in H_i. Hence the canonical mapping $E \to \widetilde{E}_V$ can be identified with the projection of E to $\prod_{i=1}^{m} H_i$. This projection has the form $p = (p_1, \ldots, p_m)$, where p_i is the projection of E to H_i. Since $p_i = \psi_{in} \circ p_n$ for $n > m$, the mapping p is nuclear. □

2.9.8. Theorem. (i) *Any vector subspace and any Hausdorff quotient space of a nuclear space are nuclear.*

(ii) *The product of any family of nuclear spaces and a locally convex direct sum of a countable collection of nuclear spaces are nuclear.*

(iii) *The projective limits of arbitrary families of nuclear spaces and the inductive limits of countable collections of nuclear spaces are nuclear.*

(iv) *The projective tensor product of two nuclear spaces is nuclear.*

For a proof, see Schaefer [**436**, Chapter III, Theorem 7.4 and Theorem 7.5, p. 103, 105]. In terms of projective topologies the following characterization is known (see [**436**, p. 172, 184]).

2.9.9. Theorem. *A locally convex space E is nuclear precisely when we have $E\widetilde{\otimes}_\pi F = E\widetilde{\otimes}_\varepsilon F$ for every locally convex space F. Moreover, it suffices to have this equality for all Banach spaces F.*

Let us consider some examples of nuclear spaces.

2.9.10. Example. The following spaces are nuclear.

(i) Any vector subspace in any product $\mathbb{R}^T$ with the induced topology, in particular, any locally convex space with the weak topology.

(ii) The space Σ of rapidly decreasing sequences (Example 1.3.19).

(iii) The space $C_0^\infty(U)$ of smooth functions vanishing outside of a ball U with the topology of uniform convergence of all derivatives.

(iv) The spaces $\mathcal{D}(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n)$.

(v) The space $C^\infty(\mathbb{R}^n)$ of smooth functions on $\mathbb{R}^n$ with the topology of uniform convergence of all derivatives on compact sets. The space $C^\infty(U)$ is also nuclear for any open set U.

(vi) The space $H(U)$ of functions holomorphic in an open set $U \subset \mathbb{C}$ with the topology of uniform convergence on compact sets in U.

PROOF. Assertion (i) follows from what has been said above. The nuclearity of Σ is seen from the fact that the same topology is generated by Euclidean norms q_k (see Example 1.3.19) for which the corresponding operators between Hilbert spaces are nuclear. In (iii) we consider for simplicity the case $n = 1$. For norms on $C_0^\infty[0,1]$ we take the L^2-norms of the even order derivatives. It is straightforward to verify that the corresponding operators are nuclear. It would be also possible to employ an isomorphism with Σ (Theorem 1.12.15). The nuclearity of $\mathcal{D}(\mathbb{R}^n)$ follows from the previous theorem and (iii). The space $\mathcal{S}(\mathbb{R}^n)$ is nuclear, since is isomorphic to the space Σ (Theorem 1.12.15). Finally, $\mathcal{H}(U)$ is a closed subspace in the complex space $C^\infty(U)$. □

Let us mentioned some other classical results on nuclear spaces. Their proofs can be found, e.g., in Jarchow [**237**], where additional information is given.

The following important fact was established by B.S. Mityagin [**350**].

2.9.11. Theorem. *The spaces $C^\infty(\mathbb{R}^n)$ and Σ^∞ are isomorphic.*

The indicated spaces are universal in the following sense.

2.9.12. Theorem. (*Kōmura, Kōmura* [**287**]) *A locally convex space is nuclear precisely when it is isomorphic to a vector subspace with the induced topology in the space* Σ^A *for some set* A.

A Fréchet space is nuclear precisely when it is isomorphic to a subspace of the space Σ^∞ (*and also to a subspace of the space* $C^\infty(\mathrm{IR}^1)$).

B.S. Mityagin [**350**] obtained a criterion of nuclearity of a Fréchet space in terms of the ε-entropy of compact sets in it.

2.9.13. Theorem. *A Fréchet space* E *is nuclear precisely when for every compact set and every neighborhood of zero in it one has the equality*

$$\limsup_{\varepsilon\to 0} \log\log N(K,\varepsilon U)/\log\frac{1}{\varepsilon} = 0,$$

where $N(K,\varepsilon U) = \inf\Big\{N\colon\ K\subset \bigcup_{i=1}^N (x_i+\varepsilon U),\ x_i\in E\Big\}$.

There is also an interesting criterion of nuclearity of a Fréchet space in terms of *unconditionally convergent series*, i.e., series converging for all permutations of its elements, and absolutely convergent series, i.e., series with a general term x_n such that $\sum_{n=1}^\infty p(x_n) < \infty$ for every continuous seminorm p.

2.9.14. Theorem. *A Fréchet space is nuclear precisely when every unconditionally series in it is absolutely converging.*

The next result is known as the Dvoretzky–Rogers theorem.

2.9.15. Corollary. *A Banach space in which every unconditionally series converges absolutely is finite-dimensional.*

The next so-called *Schwartz kernel theorem* gives a characterization of nuclear spaces in terms of bilinear forms.

2.9.16. Theorem. *A locally convex space* E *is nuclear precisely when for every locally convex space* F *every continuous bilinear function* B *on the space* $E\times F$ *is nuclear, i.e., admits a representation*

$$B(x,y) = \sum_{n=1}^\infty l_n(x) f_n(y),$$

where $l_n\in E'$, $f_n\in F'$ *and there exist absolutely convex neighborhoods of zero* $U\subset E$ *and* $V\subset F$ *such that* $\sum_{n=1}^\infty p_{U^\circ}(l_n)p_{V^\circ}(f_n) < \infty$.

There is another "kernel theorem" also due to Laurent Schwartz for the concrete space $\mathcal{D}(\mathrm{IR}^n)$; see Hörmander [**227**, Theorem 5.2.1] for a proof.

2.9.17. Theorem. *A bilinear form* B *on* $\mathcal{D}(\mathrm{IR}^n)$ *is separately continuous if and only if it admits the representation*

$$B(\varphi,\psi) = F(\varphi\otimes\psi)$$

with $F\in\mathcal{D}'(\mathrm{IR}^{2n})$, *where* $(\varphi\otimes\psi)(x,y) = \varphi(x)\psi(y)$.

In § 3.7 we present some other facts related to the nuclearity of dual spaces. For connections with measure theory, see § 5.10.

2.10. Complements and exercises

(i) Properties of the spaces $\mathcal{D}$ and $\mathcal{D}'$ (139). (ii) Absolutely summing operators (143). (iii) Local completeness (145). Exercises (147).

2.10(i). Properties of the spaces $\mathcal{D}$ and $\mathcal{D}'$

Here we discuss a number of rather exotic properties of the spaces $\mathcal{D}$ and $\mathcal{D}'$. At the beginning of the 1950s it was even unknown whether there exist spaces with such properties at all, then an impressive ingenuity was recruited for constructing some artificial examples, and, finally, it was realized that the classical spaces $\mathcal{D}$ and $\mathcal{D}'$ serve as examples. Though, a verification of this turns out highly non-trivial.

One of the first questions arising at a closer look at the topology of the space $\mathcal{D}$ is this: why do not we introduce in $\mathcal{D}$ the topology $\tau_{\rm top}$ in which closed sets are exactly the sets having closed intersections with all $\mathcal{D}_n$? This topology (of the inductive topological limit) is not weaker than the one we introduced. However, it is different: the indicated stronger topology makes continuous all those functions whose restrictions to all $\mathcal{D}_n$ are continuous. The standard topology of $\mathcal{D}$ has no this property: the quadratic form

$$F(\varphi) = \sum_{n=1}^{\infty} \varphi(n)\varphi^{(n)}(0)$$

is discontinuous in this topology (Exercise 2.10.49), but is obviously continuous on all subspaces $\mathcal{D}_n$. Though, both topologies have equal supplies of continuous linear functions. However, a decisive drawback of the topology $\tau_{\rm top}$ is that it does not make $\mathcal{D}$ a topological vector space: the addition operation is discontinuous. Otherwise it would follow by Remark 2.3.2(ii) that $\tau_{\rm top}$ coincides with the standard topology (since the base of zero mentioned in that remark for countable collections of spaces with bases of absolutely convex sets consists of convex sets). The proofs of the results stated below can be found in Smolyanov [**480**], [**484**], [**486**], [**492**], [**493**], [**494**].

A set is called *sequentially closed* if it contains the limits of all its convergent sequences. A *Fréchet–Uryson space* is a space in which for every set A every point in its closure is the limit of a sequence of points in A (not necessarily distinct). Such spaces belong to the broader class of *sequential spaces* in which the sequential closedness is equivalent to the usual closedness (see Engelking [**154**]). Let $[A]$ denote the closure of A (denoted in other places by $\overline{A}$, but here another notation will be more convenient), let $[A]_s$ denote the sequential closure (the smallest sequentially closed set containing A), and let $[A]_{ss}$ denote the set of all limits of countable sequences of elements in A. For example, every metric space is a Fréchet–Uryson space, but the product of a continuum of real lines is not. As an example of a nonmetrizable Fréchet–Uryson space take any weakly compact set in a Banach space that is not metrizable in the weak topology, e.g., the closed ball in a non-separable Hilbert space (Theorem 3.4.11 and Exercise 3.12.50). Any regular inductive limit (see § 2.7) is a nonmetrizable sequential locally convex space, but this limit is not a Fréchet–Uryson space. A subset S of a locally convex space E will be called a *standard countable sequentially closed non-closed set* if

(i) $0 \notin S, 0 \in [S]$;

(ii) 0 is a unique non-isolated limit point of S (so that, in particular, all points of S are islolated);

(iii) every convergent sequence of elements of the set S is stationary (this yields the sequential closedness of S).

2.10.1. Proposition. *Let*

$$S_1 = \{k^{-1}\delta^{(n)}(\cdot) + \delta^{(k)}(\cdot - n)\colon\ k, n \in \mathbb{N}\} \subset \mathcal{D}'.$$

Then S_1 is a standard countable sequentially closed non-closed subset of the space $\mathcal{D}'$ with the topology of uniform convergence on bounded sets in $\mathcal{D}$, i.e., $\beta(\mathcal{D}', \mathcal{D})$.

PROOF. It is clear that $0 \notin S_1$. Let V be a neighborhood of zero in $\mathcal{D}'$. This means that there exist a bounded subset B in $\mathcal{D}$ and $\varepsilon > 0$ such that

$$V \supset V_1 = \{g \in \mathcal{D}'\colon\ |\langle g, \varphi\rangle| < \varepsilon\ \forall \varphi \in B\}.$$

Since B is bounded, there exists $\alpha > 0$ such that the interval $[-\alpha, \alpha]$ contains the supports of all functions in B. Let $n \in \mathbb{N}$, $n > \alpha$ and let $k \in \mathbb{N}$ be such that $k^{-1}\delta^{(n)} \in V_1$ (such k exists because V_1 is a neighborhood of zero). Then for every $\varphi \in B$ we have

$$(k^{-1}\delta^{(n)}, \varphi) + (\delta^{(k)}(\cdot + n), \varphi) = (k^{-1}\delta^{(n)}, \varphi) < \varepsilon,$$

so that $k^{-1}\delta^{(n)}(\cdot) + \delta^{(k)}(\cdot - n) \in V_1 \subset V$, i.e., $0 \in [S]$. We now verify that 0 is a unique limit point of the set S_1. Let $g \in [S_1]$, $g \notin S_1$, $g \neq 0$. Then

$$g = \sum_{j=1}^{n} \alpha_j \delta^{(j)} + \sum_{n=1}^{\infty} \sum_{r=1}^{r(n)} \beta_{rn} \delta^{(r)}(\cdot - n),$$

since the support of the limit point of the set S_1 must belong to the union of supports of the elements of this set. If $\beta_{rn} \neq 0$ for some r, n, then by the inclusion $g \in [S_1]$ we have $\beta_{rn} = 1$ and $\beta_{ij} = 0$ for every pair (i, j) that differs from the pair (r, n); it follows (again by the inclusion $g \in [S_1]$) that $\alpha_n = 1/r$ and $\alpha_j = 0$ if $j \neq n$. Thus, $g \in S_1$, which contradictions our assumption. For completing the proof of property (ii) it remains to observe that all points of the set S_1 are isolated. Let us now prove property (iii). Suppose that for some j we have the equality

$$a_j = k_j^{-1}\delta^{(n_j)} + \delta^{(k_j)}(\cdot - n_j),$$

and suppose that the sequence $\{a_j\}$ converges in $\mathcal{D}'$. Then we have $\sup_j n_j < \infty$, therefore, $\sup_j k_j < \infty$. Hence the set of different elements in the sequence $\{a_n\}$ is finite. Since this sequence converges, there exists $n \in \mathbb{N}$ such that $a_{j_1} = a_{j_2}$ whenever $j_1, j_2 > n$, i.e., the sequence $\{a_j\}$ is stationary. □

Similarly one can show that also the set

$$S_{1'} = \{k^{-1}\delta^{(n)}(\cdot) + k\delta(\cdot - n)\colon\ k, n \in \mathbb{N}\}$$

possesses the properties analogous to those of S_1. Moreover, both the convex hull of the set S_1 and the convex hull of $S_{1'}$ are sequentially closed, but not closed.

Note also (this will be used below) that the set S_1 (hence its convex span) is contained in a subspace of the space $\mathcal{D}'$ isomorphic to the product of $\mathbb{R}^\infty$ (the countable power of $\mathbb{R}^1$) and the topological direct sum $\mathbb{R}^{(\infty)}$ of a countable family of real lines. Since the space $\mathbb{R}^\infty$ is metrizable and $\mathbb{R}^{(\infty)}$ is sequential, we see that the product of two locally convex spaces, one of which is a Fréchet space and the other one is sequential, need not be a sequential space.

2.10.2. Proposition. *Let $a_{kn}(t) = g(t)(k+1)^{-n}\sin(kt)$, where $k, n \in \mathbb{N}$, $g \in \mathcal{D}_{[-1,1]}$, $g \neq 0$, and $b^1_{nk}(t) = k^{-1}p(t-n)$, where $p \in \mathcal{D}$ and* $\operatorname{supp} p \subset [1, \infty)$. *Then the set $S_2 = \{a_{kn} + b^1_{nk}\colon\ n, k \in \mathbb{N}\}$ is a standard sequentially closed closed subset of the space $\mathcal{D}$.*

2.10.3. Proposition. *Let G be the closed vector subspace of the space $\mathcal{D}'$ consisting of all generalized functions concentrated at zero, let $E = \mathcal{D}_{[-1,1]}$, and let $k, n \in \mathbb{N}$, $b^2_{nk} = k^{-1}\delta^{(n)} \in G$, $a_{kn} \in E$ be the same as in the previous proposition. Then the set $S_3 = \{a_{kn} + b^2_{nk}\colon\ n, k \in \mathbb{N}\}$ is a standard sequentially closed non-closed subset in the topological direct sum $E \oplus G$.*

2.10.4. Remark. The topological direct sum of a countable collection of real lines is not a space Fréchet–Uryson space, i.e., it contains sets some limit points of which are not limits of convergent sequences of its elements. Among such sets there exist even countable sets.

Indeed, let us realize such topological direct sum as the space G from the previous proposition. Set

$$S_4 = \{n^{-1}\delta + k^{-1}\delta^{(n)}\colon\ n, k \in \mathbb{N}\}.$$

Then $0 \notin S_4, 0 \in [S_4]$, but there is no sequence of elements in the set S_4 converging to zero. Certainly, S_4 is not sequentially closed, since $[S_4] \neq S_4$. Hence G is not sequential. Note also that $[S_4]_{ss} \neq [S_4]_s = [S_4]$. Similarly one can show that the infinite-dimensional Hilbert space with the weak topology is not sequential. Indeed, let $\{e_n\}$ be a countable orthonormal family in this space and let

$$A = \{k^{-1}e_n + ke_{n+1}\colon\ k, n \in \mathbb{N}\}.$$

Then $0 \in [A]$, $0 \notin A$ (verify this!), but no sequence of elements of the set A converges to zero. This example goes back to von Neumann.

Below for a subset A of a locally convex space the symbol $M(A)$ denotes the affine variety generated by A and the symbol $L(A)$ denotes the vector subspace generated by A.

2.10.5. Proposition. *For $k, n, m \in \mathbb{N}$ set*

$$b^3_{nk} = (k+1)^{-1}\delta^{(n)}(\cdot - k^{-1}),\ a^0_{kn} = \delta^{(k)}(\cdot - n),$$
$$C_m = \{a^0_{kn} + b^3_{nk}\colon\ k \in \mathbb{N}, n = 1, 2, \ldots, m\} \subset \mathcal{D}',$$
$$M = \bigcup_{m=1}^{\infty} M(C_m).$$

Then $0 \notin M, 0 \in [M], [M] = M \cup \{0\}$ *and* $[M]_s = M$*, so that* M *is a sequentially closed non-closed affine subvariety in the space* $\mathcal{D}'$*. Moreover, the vector subspace* $L(M)$ *of the space* $\mathcal{D}'$ *is closed and the linear functional* $f\colon\ L(M) \to \mathbb{R}$ *defined by the equality* $f(M) = 1$ *is sequentially continuous, but not continuous in the induced topology.*

2.10.6. Proposition. *For* $k, n \in \mathbb{N}$ *set*

$$b^4_{nk} = (k+1)^{-1}\delta^{(n)}(\cdot - k^{-1}) \in \mathcal{D}'_{[-1,1]}$$

and take elements a^1_{kn} *of the space* $\mathcal{D}_{[-1,1]}$ *defined by*

$$a^1_{kn}(t) = (k+1)^{-1}\sin(l_{kn}t)f(t),$$

where $f \in \mathcal{D}_{[-1,1]}$*,* $f \neq 0$*,* $l_{kn} \in \mathbb{N}$*, and* $l_{k_1n_1} = l_{k_2n_2}$ *precisely when* $k_1 = k_2$ *and* $n_1 = n_2$*. For every* $m \in \mathbb{N}$ *let*

$$C_m = \{a^1_{kn} + b^4_{nk}\colon\ k \in \mathbb{N},\ n = 1, 2, \ldots, m\}$$

and $M_1 = \bigcup_{m=1}^{\infty}[M(C_m)]$*. Then* $0 \notin M_1$*,* $0 \in [M_1]$*,* $[M_1] = M_1 \cup \{0\}$*,* $[M_1]_s = M_1$*. Therefore,* M_1 *is a sequentially closed non-closed affine subvariety in the space* $\mathcal{D}_{[-1,1]} \times \mathcal{D}'_{[-1,1]}$ *and the linear functional* f *on* $L(M_1)$ *defined by the equality* $f(M_1) = 1$ *is discontinuous, but sequentially continuous.*

2.10.7. Proposition. *Let* a^1_{kn} *be the elements of* $\mathcal{D}$ *defined by the equality from the previous proposition and let* $b^5_{nk} \in \mathcal{D}$ *be defined by* $b^5_{nk}(t) = 2^{(-2^k)}g\bigl(2^k(t - n)\bigr)$*, where* $g \in \mathcal{D}$*,* $\operatorname{supp} g \in [1, \infty), g \neq 0$*. For every* $m \in \mathbb{N}$ *let*

$$C^1_m = \{a^1_{kn} + b^5_{nk}\colon\ k \in \mathbb{N}, n = 1, 2, \ldots, m\}.$$

Then $M_2 = \bigcup_{m=1}^{\infty}[M(C^1_m)]$ *is a sequentially closed non-closed affine variety in* $\mathcal{D}$ *and the linear functional* f *on* $L(M_2)$ *such that* $f(M_2) = 1$ *is discontinuous, but sequentially continuous.*

It follows from this proposition that the subspace $L(M_2)$ is not the inductive limit of the intersections $L(M_2) \cap \mathcal{D}_n$.

2.10.8. Proposition. *Let* $\{r_j\}$ *be the sequence of all rational numbers, let* $(p, j, s) \mapsto n(p, j, s)$ *be a one-to-one mapping of* $\mathbb{N}^3$ *onto* $(2, 3, \ldots)$*, and let* $F_{\mathcal{D}'}$ *be the subvariety in the space* $\mathcal{D}'$ *generated by the set*

$$\Bigl\{r_p\delta(\cdot - r_j) + k^{-1}\delta^{(n(p,j,s))}(\cdot - k^{-1}) + \delta^{(k+1)}\bigl(\cdot - n(p, j, s)\bigr)\colon\ p, k, j, s \in \mathbb{N}\Bigr\}.$$

Then the subspace $F_{\mathcal{D}'}$ *is sequentially closed and dense in* $\mathcal{D}'$*, but is not closed (one has* $0 \in [F_{\mathcal{D}'}], 0 \notin F_{\mathcal{D}'}$*), so that* $F_{\mathcal{D}'} - a$*, where* $a \in F_{\mathcal{D}'}$*, is a sequentially closed non-closed everywhere dense vector subspace in* $\mathcal{D}'$*.*

2.10.9. Proposition. *Let* $(k, n) \mapsto l(k, n)$ *and* $(j, s) \mapsto n(j, s)$ *be two bijections of* $\mathbb{N}^2$ *onto* $\mathbb{N}$ *and let* $\{f_j\}$ *be a sequence of elements of* $\mathcal{D}$ *such that*

(1) *if* $\min_s n(j, s) = a(j)$ *and* $\min_{k,s} l\bigl(k, n(j, s)\bigr) = b(j)$*, then* $f_j^{(a(j))}(x) = 0$ *for* $x \in \bigl(0, 2\pi/b(j)\bigr)$*;*

(2) $\operatorname{supp} f_j \subset \bigl(-\infty, a(j)\bigr)$*;* (3) $[\{f_j\}] = \mathcal{D}$*.*

The existence of such a sequence follows from the relations $a(j) \to \infty, b(j) \to \infty$.

For each triple $k, j, s \in \mathbb{N}$ *consider the function*

$$\varphi_{k,j,s}(t) = f_j(t) + f(t)(k+1)^{n(j,s)} \sin\Bigl(l\bigl(k, n(j,s)\bigr)t\Bigr) + 2^{-2^k} g\Bigl(2^k\bigl(t - n(j,s)\bigr)\Bigr),$$

where $g \in \mathcal{D}$, $\operatorname{supp} g \subset [1, \infty)$, $f \in \mathcal{D}$, $\operatorname{supp} f \subset (-\infty, 1]$, $f(t) = 1$ *for all points* $t \in (0, 1/2)$. *Then the sequentially closed affine subvariety* $F_{\mathcal{D}}$ *of the space* $\mathcal{D}$ *generated by the set* $\{\varphi_{k,j,s}\colon\ k, j, s \in \mathbb{N}\}$ *is not closed and is everywhere dense, so that* $F_{\mathcal{D}} - a$, *where* $a \in F_{\mathcal{D}}$, *is a non-closed everywhere dense sequentially closed vector subspace in* $\mathcal{D}$.

2.10.10. Remark. A similar method can be applied for constructing a sequentially closed non-closed everywhere dense vector subspace in $\mathcal{D}_{[-1,1]} \oplus \mathcal{D}'_{[-1,1]}$.

2.10(ii). Absolutely summing operators

A series $\sum_{n=1}^{\infty} x_n$ in a Hausdorff locally convex space X is called *unconditionally convergent* if it converges for all permutations of indices. For a scalar series this is equivalent to absolute convergence; thus, the sum does not depend on a permutation. Hence if s is the sum of this series, then for every continuous seminorm q on X and every $\varepsilon > 0$ there exists N such that for every finite set of indices M containing $\{1, \ldots, N\}$ we have $q\bigl(s - \sum_{i \in M} x_i\bigr) < \varepsilon$.

More generally, a family of vectors $x_\gamma \in X$ indexed by some set Γ is called *summable to a vector* $s \in X$ if, for every continuous seminorm q on X and every $\varepsilon > 0$, there exists a finite subfamily $\Gamma_0 \subset \Gamma$ such that $q\bigl(s - \sum_{\gamma \in \Gamma_1} x_\gamma\bigr) < \varepsilon$ for every finite family Γ_1 containing Γ_0.

If $\sum_{\gamma \in \Gamma}^{\infty} q(x_\gamma) < \infty$ for every continuous seminorm q, then the family $\{x_\gamma\}_{\gamma \in \Gamma}$ is called *absolutely summable.* For $\Gamma = \mathbb{N}$ we say of an absolutely convergent series.

In the finite-dimensional space unconditional convergence of a series is equivalent to its absolute convergence. For example, the sign alternating series with a general term $(-1)^n n^{-1}$ converges, but not unconditionally. In the Hilbert space l^2 the series of vectors $x_n = n^{-1} e_n$, where $\{e_n\}$ is the standard basis, does not converges absolutely, but converges unconditionally, since whenever $k_1, \ldots, k_m > n$ we have $\|x_{k_1} + \cdots + x_{k_m}\|^2 < \sum_{k=n}^{\infty} k^{-2}$.

Let $p \in [1, +\infty)$. A family of vectors $\{x_\gamma\}_{\gamma \in \Gamma}$ is called *weakly* p*-summable* if $\sum_{\gamma \in \Gamma} |l(x_\gamma)|^p < \infty$ for all $l \in X'$.

If $\sum_{\gamma \in \Gamma} q(x_\gamma)^p < \infty$ for every continuous seminorm q on X, then $\{x_\gamma\}_{\gamma \in \Gamma}$ is called *absolutely* p*-summable.*

2.10.11. Lemma. *Suppose that a sequence* $\{x_n\}$ *in a normed space* X *is weakly* p*-summable. Then there exists* $C > 0$ *such that*

$$\sup_{l \in X',\, \|l\| \leqslant 1} \sum_{n=1}^{\infty} |l(x_n)|^p \leqslant C.$$

PROOF. Let us consider the linear mapping $S\colon X' \to l^p$, $Sl = \bigl(l(x_n)\bigr)_{n=1}^{\infty}$. We have to show its boundedness. Since X' and l^p are Banach spaces, it suffices to verify that the graph of S is closed (see § 3.9). Let $l_j \to l$ in X' and $Sl_j \to v$ in l^p, where $v = (v_n)$. Then $l_j(x_n) \to l(x_n)$ for every fixed n. On the other hand, we have $l_j(x_n) \to v_n$, whence $l(x_n) = v_n$, i.e., $v = Sl$, as required. □

2.10.12. Definition. *Let X and Y be two locally convex spaces. An operator $T \in \mathcal{L}(X, Y)$ is called absolutely p-summing if it takes weakly p-summing families to absolutely p-summing families.*

An operator T is called absolutely summing if it takes summing families to absolutely summing families.

It is readily seen that in these definitions it suffices to consider countable collections of vectors.

If X and Y are normed spaces, then $T \in \mathcal{L}(X, Y)$ is absolutely p-summing if the condition $\sum_{n=1}^{\infty} |l(x_n)|^p < \infty$ for all $l \in X'$ implies that $\sum_{n=1}^{\infty} \|T(x_n)\|_Y^p < \infty$. In this case there is a number $C > 0$ such that

$$\sum_{n=1}^{\infty} \|Tx_n\|_Y^p \leqslant C \sup_{\|l\| \leqslant 1} \sum_{n=1}^{\infty} |l(x_n)|^p \tag{2.10.1}$$

for every sequence $\{x_n\} \subset X$. Indeed, otherwise for every m we could find a finite set $x_{m,1}, \ldots, x_{m,k}$ for which

$$\sup_{\|l\| \leqslant 1} \sum_{i=1}^{k} |l(x_{m,i})|^p \leqslant 2^{-m} \quad \text{and} \quad \sum_{i=1}^{k} \|Tx_{m,i}\|_Y^p \geqslant 1,$$

which leads to a contradiction. The smallest possible C is denoted by $\pi_p(T)$.

These classes are stable under left and right compositions with bounded operators. By the Dvoretzky–Rogers theorem (Corollary 2.9.15) each infinite-dimensional Banach space X contains an unconditionally, but not absolutely converging series. If X has no subspaces isomorphic to c_0 (and only in this case), then unconditional convergence of the series of x_n is equivalent to the weak 1-summability of $\{x_n\}$ (see Kadec, Kadec [**244**, Chapters 3, 4]). If X and Y are two Hilbert spaces, then the class of absolutely summing operators coincides with the class of absolutely 2-summing operators and with the class of *Hilbert–Schmidt operators*, i.e., operators $T \in \mathcal{L}(X, Y)$ such that $\sum_\alpha \|Te_\alpha\|_Y^2 < \infty$ for some (and then every) orthonormal basis $\{e_\alpha\}$ in X, see Bogachev, Smolyanov [**72**, Proposition 7.10.26] or Pietsch [**388**, § 2.5].

Let us describe one important special absolutely 2-summing embedding (see [**72**, Theorem 7.10.27] or Pietsch [**389**, § 17.3]).

2.10.13. Theorem. *Let μ be a probability measure. Then the identical embedding $L^\infty(\mu) \to L^2(\mu)$ is absolutely 2-summing. In particular, if μ is a Radon measure on a topological space, then the embedding $C_b(\Omega) \to L^2(\mu)$ is absolutely 2-summing.*

It is worth noting (see Pietsch [**388**, Lemma 3.3.4]) that every absolutely summing mapping $T\colon X \to Y$ between Banach spaces can be written as a composition $T = T_1 \circ j \circ T_2$, where $j\colon C(\Omega) \to L^2(\mu)$ is the identical embedding for some Radon measure μ on a compact space Ω, $T_1 \in \mathcal{L}\bigl(X, C(\Omega)\bigr)$, $T_2 \in \mathcal{L}\bigl(L^2(\mu), Y\bigr)$.

As the following theorem due to Pietsch shows, in the case of general Banach spaces absolutely 2-summing operators are also connected with the space L^2 (about measures, see Chapter 5).

2.10.14. Theorem. *Let X and Y be two Banach spaces, let B' be the closed unit ball in the space X' equipped with the topology $\sigma(X', X)$ (making it compact), and let $\sigma\big(C(B')\big)$ be the σ-algebra generated by continuous functions on the compact space B'. An operator $T \in \mathcal{L}(X, Y)$ is absolutely 2-summing precisely when there exists a bounded nonnegative measure μ on $\sigma\big(C(B')\big)$ such that*

$$\|Tx\|_Y^2 \leqslant \int_{B'} |\xi(x)|^2 \, \mu(d\xi).$$

Any nuclear operator between normed spaces is absolutely summing. Indeed, let $\{x_i\}$ be a weakly absolutely summing sequence and let $Tx = \sum_{n=1}^{\infty} \lambda_n l_n(x) y_n$ be a nuclear operator, where $\lambda_n \geqslant 0$, $\sum_{n=1}^{\infty} \lambda_n < \infty$, $\|l_n\| \leqslant 1$, $\|y_n\| \leqslant 1$. As we have shown in the lemma above, there exists $C > 0$ such that

$$\sup_n \sum_{i=1}^{\infty} |l_n(x_i)| \leqslant C.$$

Therefore,

$$\sum_{i=1}^{\infty} \|Tx_i\| \leqslant \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} \lambda_n |l_n(x_i)| \, \|y_n\| \leqslant C \sum_{n=1}^{\infty} \lambda_n.$$

In Pietsch [**388**, Chapter 3] one can find the proof of the following theorem due to Grothendieck.

2.10.15. Theorem. *The composition of two absolutely summing operators between normed spaces is nuclear.*

With the aid of absolutely summing operators one can characterize nuclear spaces. For example, according to Theorem 2.9.14, a Fréchet space is nuclear precisely when all unconditionally converging series in it converge absolutely, i.e., the identity operator is absolutely summing.

2.10(iii). Local completeness

In relation to Grothdieck's construction discussed in § 2.5 the following useful property of completeness emerged. For simplicity of terminology, we shall use here the term a *disc* to denote a bounded absolutely convex set in a topological vector space.

2.10.16. Definition. *A Hausdorff locally convex space is called locally complete if every closed disc in it is a Banach disc.*

It follows from the results in § 2.5 that the sequential completeness yields the local completeness, but the converse is false. For example, the space c_0 of all sequences converging to zero equipped with the weak topology $\sigma(c_0, l^1)$ is not sequentially complete, but it is locally complete, since it is Banach with respect to its standard norm and weakly bounded sets are norm bounded.

2.10.17. Definition. *We shall say that a sequence $\{x_n\}$ in a locally convex space E converges to x in the sense of Mackey, or is Mackey convergent, if there is a disc $B \subset E$ such that $p_B(x - x_n) \to 0$. Similarly, a sequence is called Mackey fundamental if for some disc B it is fundamental in E_B.*

2.10.18. Lemma. *A sequence $\{x_n\}$ in a locally convex space E Mackey converges to zero precisely when there exists an increasing sequence of positive numbers $\lambda_n \to +\infty$ for which $\lambda_n x_n \to 0$ in E.*

PROOF. If $\{x_n\}$ is Mackey converging to zero, then such numbers obviously exist: whenever $p_B(x_n) > 0$ we can take $\lambda_n = p_B(x_n)^{-1/2}$. Conversely, if such a sequence $\{\lambda_n\}$ exists, then the closed absolutely convex hull B of the sequence $\{\lambda_n x_n\}$ is a closed disc and $p_B(x_n) = \lambda_n^{-1} \to 0$. □

It follows from this lemma that in any metrizable locally convex space convergence of a sequence is equivalent to its Mackey convergence.

2.10.19. Proposition. *The following conditions for a Hausdorff locally convex space E are equivalent*:

(i) *the space E is locally complete*;

(ii) *every Mackey fundamental sequence in E is Mackey convergent*;

(iii) *every bounded set in E is contained in some Banach disc.*

PROOF. The implications (i)⇒(ii)⇒(iii) are trivial. In order to prove (iii)⇒(i), given a closed disc D we find a Banach disc B containing it. Then the closure of D in E_B will be a Banach disc. We observe that D coincides with this closure, since every limit point for D in E_B will be a limit point in E, but D is closed. □

2.10.20. Corollary. *Any locally complete space E will remain locally complete in all locally convex topologies on E for which the dual space remains E'.*

The description of such topologies that agree with the given duality will be obtained in § 3.2.

2.10.21. Theorem. *The following conditions for a Hausdorff locally convex space E are equivalent*:

(i) *the space E is locally complete*;

(ii) *every sequence in E that is Mackey converging to zero has a compact closed absolutely convex span*;

(iii) *every sequence converging to zero in $\sigma(E, E')$ has a $\sigma(E, E')$-compact closed absolutely convex span*;

(iv) *every sequence converging to zero in E has a compact closed absolutely convex span.*

PROOF. We verify the implication (i)⇒(iii). Let $x_n \to 0$ in $\sigma(E, E')$ and let B be the closed absolutely convex hull of $\{x_n\}$ in the topology $\sigma(E, E')$. Then B is bounded and closed in the original topology, hence is a Banach disc. Proposition 2.5.14 gives the compactness of B in the topology $\sigma(E, E')$.

The implication (iii)⇒(iv) is clear from the fact that the closed absolutely convex hull of any sequence $\{x_n\}$ converging to zero in E is weakly compact by condition (iii), and since it is precompact, by Corollary 1.8.12 we obtain its compactness in the original topology.

The implication (iv)⇒(ii) is trivial. Let us deduce (i) from (ii). Let B be a closed disc in E and let a sequence $\{x_n\}$ be Cauchy in E_B. We show that it has a limit in E_B. Passing to a subsequence, we can assume that $p_B(x_{n+1}-x_n) \leqslant 2^{-2n}$, i.e., $x_{n+1} - x_n \in 2^{-2n}B$. Set $y_n = 2^n(x_{n+1} - x_n)$. Then $p_B(y_n) \leqslant 2^{-n}$. By condition (ii) the absolutely convex hull K of the sequence $\{y_n\}$ is compact in E. The sequence $x_{k+1}-x_1 = \sum_{n=1}^k 2^{-n}y_n$ is contained in K, hence has a limit point in K. Hence $\{x_n\}$ has a limit point $x \in K$. We observe that then $p_B(x_n-x) \to 0$. Indeed, let $\varepsilon > 0$. For all $n, k \geqslant m$, where $2^{-m} < \varepsilon$, we have $x_n - x_k \in \varepsilon B$. Since B is closed and x is a limit point of $\{x_k\}$, we obtain $x_n - x \in \varepsilon B$ for all indices $n \geqslant m$, i.e., $p_B(x_n - x) \leqslant \varepsilon$, as required. □

2.10.22. Example. For metrizable locally convex spaces the local completeness is equivalent to completeness. Indeed, if a sequence $\{x_n\}$ is Cauchy in a locally complete metrizable space E, then it converges in the completion of E. Hence in the completion it is Mackey convergent, hence is Mackey fundamental, which yields that it is Mackey fundamental in E, then it is Mackey converging in E.

The Mackey convergence and its desription by Lemma 2.10.18 will be used in § 4.3. See Pérez Carreras, Bonet [**385**, Chapter 5] for additional information.

Exercises

2.10.23. Construct an example of an incomplete direct spectre $\varinjlim E_\alpha$ of complete spaces (E_α, τ_α).

2.10.24. (i) There exists an incomplete separated inductive limit of an increasing sequence of separable Banach spaces B_n with continuous embeddings $B_n \subset B_{n+1}$.

(ii) There exists a separated inductive limit of an increasing sequence of separable Banach spaces B_n with continuous embeddings $B_n \subset B_{n+1}$ in which there is a bounded set contained entirely in no B_n.

(iii) There exists a separated inductive limit of an increasing sequence of separable Banach spaces B_n with continuous embeddings $B_n \subset B_{n+1}$ in which there is a bounded set contained in B_1 which is unbounded in each B_n.

(iv) There exists an inductive limit of strictly increasing sequence of separable Banach spaces B_n with continuous embeddings $B_n \subset B_{n+1}$ in which there is no neighborhood of zero distinct from the whole space.

HINT: in (i), (ii) consider the following example (see Makarov [**331**], [**332**]). Let B_n be the space of double sequences $x = (x_{i,j})$ with a finite limit $\lim\limits_{j\to\infty} x_{i,j}$ for all $i > n$ such that one has $\lim\limits_{j\to\infty} x_{i,j}/(1+j) = 0$ for $i \leqslant n$, set $\|x\|_n = \sup_{i,j} |x_{i,j}|/c_{i,j,n}$, $c_{i,j,n} = 1+j$ if $i \leqslant n$, $c_{i,j,n} = 1$ if $i > n$; consider the countable set A of elements $a^{k,n}$, where $a^{k,n} = (a^{k,n}_{i,j})$, $a^{k,n}_{i,j} = -1$ if $j = 2m$, $m \leqslant k$, $i \leqslant n$, $a^{k,n}_{i,j} = 1$ in other cases; verify that A is bounded in B_1, but the closure of A contains $\{y^n\}$, where $y^n_{i,j} = -1$ if $j = 2m$, $i \leqslant n$, $y^n_{i,j} = 1$ in other cases, and $y^n \in B_n \backslash B_{n-1}$; verify that the sequence of partial sums of the series of $n^{-1}y^n$ is fundamental, but does not converge. For examples in (iii), (iv), see [**332**].

2.10.25. Generalize the example (ii) as follows. Suppose that for every $k \in \mathbb{N}$ we have a sequence of numbers $a_n^k > 0$ with $a_n^k \leqslant a_n^{k+1}$. Denote by E_k the Banach space of sequences $x = (x_n)$ for which $x_n/a_n^k \to 0$ as $n \to \infty$, equipped with the norm $\|x\|_k = \sup_n |x_n/a_n^k|$. Let $E = \mathrm{ind}_k E_k$. In place of an index n we now use a pair (m, n) and define $a_{(m,n)}^k$ as follows: $a_{(m,n)}^k = n$ if $m < k$, $a_{(m,n)}^k = 1$ in other cases. Let vectors v^k be such that $v_{(m,n)}^1 = 0$ for all m, n, $v_{(m,n)}^k = k^{-1}$ if $m = k - 1$, $v_{(m,n)}^k = 0$ in other cases. Show that $v^k \in E_k$, $v^{k+1} \notin E_k$ and $v^k \to 0$ in E. Thus, we obtain a sequence converging in $\mathrm{ind}_k E_k$, but contained in no E_k.

2.10.26. On the product $X = \prod_{t \in T} X_t$ of nonempty topological spaces the *box topology* is generated by a base consisting of the products of the form $\prod_{t \in T} U_t$, where U_t is a nonempty open subset of T (not necessarily equal to X_t for all t except for a finite number as in Tychonoff's topology). (i) Investigate whether an infinite power of a compact interval is compact in the box topology. (ii) Let E_t, where $t \in T$, be an infinite family of separated nonzero topological vector spaces over $\mathbb{R}$ or $\mathbb{C}$. Let us equip their product with the box topology. Show that this topology does not agree with the vector structure. Does it agree with the structure of an additive group?

HINT: (i) observe that the box topology is stronger than Tychonoff's topology; (ii) if $x(t_n) > 0$ for infinitely many $t_n \in T$, then take a box neighborhood of zero W such that $\lambda x \notin W$ for all $\lambda \in (0, 1)$.

2.10.27. Let E be a locally convex space in which there is a closed linear subspace E_0 such that E_0 and E/E_0 are metrizable. Prove that E is metrizable as well. Show that if E_0 and E/E_0 are Fréchet spaces, then E is a Fréchet space as well.

HINT: see Pérez Carreras, Bonet [**385**, p. 51].

2.10.28. Prove Corollary 2.7.10.

2.10.29. Let A be a bounded complete subset of a separated locally convex space E. Show that there exists a closed absolutely convex set D such that A is complete in E_D.

HINT: let D be the closed absolutely convex hull of A and let B be the closure of D in the completion of E; observe that A is closed in the Banach space E_B and E_D is a linear subspace in E_B with the same norm.

2.10.30. Let V be a complete bounded convex set in a Hausdorff locally convex space E. Show that its absolutely convex hull is a Banach disc.

HINT: see Pérez Carreras, Bonet [**385**, p. 88].

2.10.31. Let $m_0(\Omega)$ be the space of all functions with finitely many values on a nonempty set Ω equipped with the norm $\|x\| = \sup_\omega |x(\omega)|$. Show that every Banach disc in $m_0(\Omega)$ is finite-dimensional. Prove that in any infinite-dimensional separable Banach space there exists a hypersubspace F not containing infinite-dimensional Banach discs.

HINT: see Pérez Carreras, Bonet [**385**, p. 90].

2.10.32. Let D be a closed disc in a locally convex space E and let $A \subset D$ be an absolutely convex and precompact set in E_D. Prove that the closures of A in E and E_D coincide.

HINT: see Pérez Carreras, Bonet [**385**, p. 169].

2.10.33. A set A in a locally convex space E will be called hyperprecompact if there exists a closed disc D such that A is precompact in E_D. (i) Prove that the closed absolutely convex hull of any hyperprecompact set is hyperprecompact. (ii) Prove that if a sequence $\{x_n\}$ converges to zero in E_D for some disc D, then its closed absolutely convex hull in E is hyperprecompact. (iii) Prove that every hyperprecompact set is contained in the closed absolutely convex hull of some sequence which, for some disc D, tends to zero in E_D. (iv) Prove that for every hyperprecompact set A there is a closed absolutely convex hyperprecompact set C such that A is precompact in E_C.

HINT: see Pérez Carreras, Bonet [**385**, Proposition 6.1.13].

2.10.34. Let a normed space B be continuously embedded into a sequentially complete locally convex space X such that the closed unit ball of B is closed in X. Prove that B is complete.

HINT: apply Theorem 2.5.1.

2.10.35. Construct an example of a Banach space X and an incomplete normed space E continuously embedded into X by means of an injective operator T for which the extension of T by continuity to the completion of E is not injective.

2.10.36. Let E be the inductive limit of an increasing sequence of Hausdorff locally convex spaces E_n such that in each E_n there is an absolutely convex neighborhood of zero with weakly compact closure in E_{n+1}. Show that if a set A is such that $A \cap E_n$ is weakly closed in E_n for all n, then A is closed in E (if A is convex, then it suffices to have the closedness of $A \cap E_n$ in E_n in the original topology of E_n). Hence if E_n are reflexive Banach spaces and Z is a closed subspace in E, then all sequentially continuous linear functions on Z are continuous.

HINT: see Pérez Carreras, Bonet [**385**, Proposition 8.5.28].

2.10.37. Let E be the inductive limit of an increasing sequence of Hausdorff locally convex spaces E_n such that every bounded set in E is contained and bounded in some E_n. Prove that for every continuous linear operator T from a metrizable locally convex space F to E there exists n such that $T(F) \subset E_n$ and the mapping $T\colon\ F \to E_n$ is continuous.

HINT: see Pérez Carreras, Bonet [**385**, Proposition 8.5.38].

2.10.38. Let E be a Hausdorff inductive limit of an increasing sequence of Hausdorff locally convex spaces E_n such that there exist absolutely convex neighborhoods of zero $U_n \subset E_n$ with $U_n \subset U_{n+1}$. Prove that if a set A is bounded in E, then there exists n such that $A \subset n\overline{U_n}$, where the closure is taken in E.

HINT: see Pérez Carreras, Bonet [**385**, Proposition 8.5.20].

2.10.39. (i) Let E be a topological vector space and let Q be a sequentially closed convex set in E which for some $\beta > 0$ contains all sets αQ with $|\alpha| < \beta$. Prove that if Q absorbs every point of some absolutely convex sequentially complete set A, then Q absorbs A.

(ii) Deduce from (i) that if E is sequentially complete and Q is an absolutely convex sequentially closed absorbent set, then it absorbs every bounded absolutely convex set.

HINT: see Edwards [**150**, Proposition 7.4.1, Corollary 7.4.2].

2.10.40. (i) Suppose that E is a nuclear locally convex space and $V \subset E$ is an absolutely convex neighborhood of zero. Show that the space $\widetilde{E}_V$ is separable. (ii) Show that if E' is metrizable in the strong topology, then E is separable.

2.10.41. Let E and F be two vector spaces in duality, let E be equipped with the topology $\sigma(E,F)$, let V be a barrel in E, and let $\widetilde{E}_V$ be the completion of the normed space (E_V, p_V), where p_V is the Minkowski functional of the set V. Prove that the canonical mapping $j_V\colon\ E \to \widetilde{E}_V$ has a closed graph and $V = j_V^{-1}\bigl(j_V(V)\bigr)$.

HINT: if $p_V(x_\alpha) \to 0$ and $x_\alpha \to x$, where $x \notin V/n$, then x and V/n are separated by a hyperplane.

2.10.42. Let F be a Fréchet space. Prove that F is linearly homeomorphic to a closed subspace in a countable product of Banach spaces.

HINT: let the topology of F be defined by seminorms p_n and let X_n be the completion of $X/p_n^{-1}(0)$ with respect to the norm generated by p_n; the embedding of F into $\prod_{n=1}^{\infty} X_n$ is defined by the formula $x \mapsto (x, x, \ldots)$.

2.10.43. Let X be an infinite-dimensional Banach space. Prove that for every separable Banach space E there exists an injective continuous linear operator $A\colon E \to X$. Prove also that such an operator exists from l^∞ to X.

HINT: construct a continuous linear operator $T\colon l^2 \to X$ with an infinite-dimensional range and restrict it to the orthogonal complement of the kernel; construct explicitly an injective operator from l^∞ to l^2; verify that every separable Banach space can be injectively embedded into l^∞.

2.10.44. Show that the vector sum of two Banach discs is again a Banach disc, the intersection of any family of Banach discs is a Banach disc, and the image of a Banach disc under a continuous linear operator to a Hausdorff locally convex space is a Banach disc.

2.10.45.° Prove that the topology of the spaces $\mathcal{S}(\mathbb{R}^n)$, $H(U)$ and $C^\infty(U)$ from Example 2.9.10 (see also § 1.3) cannot be defined by a norm.

2.10.46. Justify the examples from the propositions in § 2.10(i).

2.10.47.° Let E be a sequentially complete locally convex space and let a sequence $\{x_n\}$ converge to zero in the topology $\sigma(E, E')$. Show that its closed absolutely convex hull coincides with the set $\bigl\{\sum_{n=1}^{\infty} \lambda_n x_n\colon\ \sum_{n=1}^{\infty} |\lambda_n| \leqslant 1\bigr\}$.

2.10.48. Let $F_j \in \mathcal{D}'(\mathbb{R}^1)$, $j \in \mathbb{N}$ and $F_j(\varphi) \to 0$ for every $\varphi \in \mathcal{D}(\mathbb{R}^1)$. Prove that there exist a seminorm p of the form indicated in Example 1.3.21 and numbers $\varepsilon_j \to 0$ for which $|F_j(\varphi)| \leqslant \varepsilon_j p(\varphi)$ for all j.

HINT: show first that this is true for each space $\mathcal{D}_k$ in place of $\mathcal{D}(\mathbb{R})$, since by the Banach–Steinhaus theorem there exist $r \in \mathbb{N}$ and $C > 0$ such that $|F_j(\varphi)|$ does not exceed $C \max_{t\in[-k,k]} |\varphi^{(r)}(t)|$ for all j. By the compactness of the natural embedding $C_0^{r+1}[-k,k] \to C_0^r[-k,k]$ one has $\sup_j |F_j(\varphi)| \to 0$ uniformly on the set of functions $\varphi \in \mathcal{D}_k$ with $\max_{t\in[-k,k]} |\varphi^{(r+1)}(t)| \leqslant 1$. Find functions $\zeta_k \in C_0^\infty(k-2/3, k+2/3)$, $k \in \mathbb{Z}$, with $0 \leqslant \zeta_k \leqslant 1$ and $\sum_{k=-\infty}^{\infty} \zeta_k = 1$. Use numbers $\varepsilon_{k,j} \to 0$ and $r_k \in \mathbb{N}$ for which $|F_j(\varphi)| \leqslant \varepsilon_{k,j} \max_{t\in[-k,k]} |\varphi^{(r_k)}(t)|$ for all $\varphi \in \mathcal{D}_k$ and all j to construct the desired numbers ε_j. To this end, estimate $|F(\varphi)|$ by the series of $|F(\zeta_k\varphi)|$.

2.10.49.° (i) Justify the assertions in § 2.10(i) by proving that the topology τ on the space $\mathcal{D}(\mathbb{R}^1)$ introduced there is strictly weaker than the topology $\tau_{\rm top}$ on $\mathcal{D}(\mathbb{R}^1)$ in which open sets are by definition those sets that give open intersections with all $\mathcal{D}_n$. To this end show that the quadratic function $F(\varphi) = \sum_{n=1}^{\infty} \varphi(n)\varphi^{(n)}(0)$ is discontinuous in the topology τ, but is continuous in the topology $\tau_{\rm top}$.

(ii) Prove that the topology τ is strictly stronger than the topology τ_2 on $\mathcal{D}(\mathbb{R}^1)$ generated by the norms $p_\psi(\varphi) = \sup |\psi(x)\varphi^{(m)}(x)|$, where we take arbitrary nonnegative integer numbers m and positive locally bounded functions ψ. To this end, verify that the linear function $F(\varphi) = \sum_{n=1}^{\infty} \varphi^{(n)}(n)$ is continuous in the topology τ, but is discontinuous in the topology τ_2.

Therefore, the topologies on $\mathcal{D}$ used in Kirillov, Gvishiani [**271**] and Kolmogorov, Fomin [**284**] differ from the standard one and are distinct.

2.10.50. Let L be a linear subspace in $\mathcal{D}(\mathbb{R}^1)$ of finite codimension such that all intersections $L \cap \mathcal{D}_m$ are closed. Prove that L is closed.

HINT: assuming for simplicity that L is of codimension 1, find k such that $L \cap \mathcal{D}_k$ is of codimension 1, i.e., has the form $\mathcal{D}_k \cap f_k^{-1}(0)$ for some $f_k \in \mathcal{D}'$. Next, there is $f_{k+1} \in \mathcal{D}'$ with $L \cap \mathcal{D}_{k+1} = DD_{k+1} \cap f_{k+1}^{-1}(0)$. Hence f_{k+1} and f_k are proportional on $\mathcal{D}_k$ and f_{k+1} can be taken such that $f_{k+1} = f_k$ on $\mathcal{D}_k$. Continuing inductively we find functionals $f_n \in \mathcal{D}'$, $n > k$, such that $f_{n+1} = f_n$ on $\mathcal{D}_n$ and $L \cap \mathcal{D}_n = \mathcal{D}_n \cap f_n^{-1}(0)$. This yields a functional $f \in \mathcal{D}'$ such that L is its kernel.

2.10.51. Show that every complete Hausdorff locally convex space is linearly homeomorphic to a closed linear subspace in some product of Banach spaces.

2.10.52. (i) Let E be a Hausdorff locally convex space and let $C \subset E$ be a closed convex balanced set. Show that if the restriction of a linear function f to C is continuous in the original topology, then it is continuous in the topology $\sigma(E, E')$.

(ii) Construct an example showing that even for a Banach space E the restriction of f to the linear space generated by C need not be continuous.

HINT: (i) use Theorem 1.11.17; (ii) consider the space $E = l^1$ with the standard norm and $C = \{(x_n) \in l^1 \colon |x_n| \leqslant 4^{-n}\}$. Let E_C be the linear span of C and define f by $f(x) = \sum_{n=1}^{\infty} 2^n x_n$ if $x \in E_C$. Extend f by linearity to all of E. Then f is not continuous on E_C in the topology of E, since $f(2^{-n} e_n) = 1$, where $\{e_n\}$ is the standard basis in l^1, but $2^{-n} e_n \to 0$ in l^1. However, f is continuous on C. Indeed, let $x \in C$. Then, whenever $y \in C$ and $\|x - y\| < 4^{-k}$, one has $|f(y) - f(x)| < 3 \cdot 2^{-k}$.

2.10.53.$^\circ$ Let E and F be Hausdorff locally convex spaces and let $S \colon E \to F$ be a sequentially continuous linear mapping. Prove that S takes Cauchy sequences to Cauchy sequences.

2.10.54. Give an example of an absolutely convex precompact set V in l^2 with the closure $\overline{V}$ such that V is not dense in $\overline{V}$ with respect to the norm $p_{\overline{V}}$.

HINT: take the set V of vectors $x = (x_n)$ with finitely many nonzero coordinates such that $|x_n| \leqslant n^{-1}$ for all n; observe that $p_{\overline{V}}(x) = \sup_n |n x_n|$ and $v = (n^{-1}) \in \overline{V}$ does not belong to the closure of V in the norm $p_{\overline{V}}$.

2.10.55. Show that in any locally convex space any sequentially closed convex set with a nonempty interior is closed. Show that this is false without the assumption about interior points.

HINT: assuming that 0 is in the interior of the given set V, verify that $V = \{p_V \leqslant 1\}$; consider the topological dual to l^2 in the algebraic dual.

2.10.56.$^\circ$ Show that a linear functional on a topological vector space bounded on precompact sets is bounded. Verify also that if in two vector topologies precompact sets are the same, then also bounded sets are the same.

2.10.57.$^\circ$ A sequence in a locally convex space converges weakly to zero precisely when zero is contained in the closed convex hull of every subsequence in this sequence.

2.10.58. (Gorin, Mityagin [**198**]) Let F be a Fréchet space whose topology is defined by norms p_n with $p_n \leqslant p_{n+1}$ compatible in the following sense: any sequence Cauchy in p_n and converging to zero in p_{n+1} also converges to zero in p_n. Suppose that bounded sets in F are precompact. Then one can find two collections of norm $\{q_n\}$ and $\{r_n\}$ defining the original topology and possessing the following property: for every $f \in F'$ we have $\lim\limits_{n\to\infty} \|f\|_{q_n^*} = 0$, $\lim\limits_{n\to\infty} \|f\|_{r_n^*} > 0$ if $f \neq 0$, where $\|f\|_{q_n^*}$ and $\|f\|_{r_n^*}$ denote the norms of f on (F, q_n) and (F, r_n).

2.10.59. (Banaszczyk [**41**], [**43**]) Let F be a Fréchet space. Suppose that the series of vectors $\varphi_n \in F$ converges to $s_0 \in F$, G is the subspace in F' consisting of all functionals f with $\sum_{n=1}^{\infty} |f(\varphi_n)| < \infty$, and $\mathfrak{S}(\{\varphi_n\})$ is the set of all sums of convergent rearrangements of the series of the vectors φ_n. Then the nuclearity of F is equivalent to the equality $\mathfrak{S}(\{\varphi_n\}) = s_0 + G_F^{\circ}$ for all series of φ_n converging in F.

2.10.60. (i) (Drewnowski [**136**]) Let E be a Hausdorff locally convex space and let A be its subset that is separable and metrizable in the induced topology. Then on the linear span of A there is a metrizable locally convex topology that is majorized by the original topology and the restriction of which to A coincides with the original topology.

(ii) (Larman, Rogers [**310**]) If, in addition, A is locally bounded (for every $a \in A$ there is a neighborhood of zero U such that $(a + U) \cap A$ is bounded), then on the linear span of A there is a norm generating on A the original topology (but the topology generated by this norm is not always majorized by the original topology on the linear span of A).

2.10.61. Let C be a convex closed set in a normed space with a closed unit ball U and let $qU \subset C + U$ for some $q > 1$. Show that C has a nonempty interior.

HINT: let $x_1 \in U$. By condition there exist vectors $y_1 \in C, x_2 \in U$ for which $qx_1 = y_1 + x_2$,i.e., $x_1 = (y_1 + x_2) = q$. By induction construct $x_n \in U$ and $y_n \in C$ with $qx_n = y_n + x_{n+1}$. Then $x_1 = y_1/q + y_2/q^2 + \cdots + y_n/q^n + x_{n+1}/q^{n+1}$. Set $z_n = y_1/q + y_2/q^2 + \cdots + y_n/q^n + y_1/(q^n(q-1))$. Then $z_n \in C/(q-1)$ since C is convex and the sum of the coefficients at the elements $y_i \in C$ is $1/(q-1)$. The sequence $\{z_n\}$ converges to x_1 by the estimate

$$Z - n - x_1\| = \left\| \frac{y_1}{q^n(q-1)} - \frac{x_{n+1}}{q^n} \right\| \leqslant \frac{\|y_1\|}{q^n(q-1)} + \frac{1}{q^n} \to 0.$$

Since C is closed, $x_1 \in C/(q-1)$. It follows that $U \subset C/(q-1)$, hence C has inner points.

2.10.62. A locally convex space E is locally complete precisely when, for every sequence x_n converging to zero in E and every absolutely convergent series of numbers λ_n, the series $\sum_{n=1}^{\infty} \lambda_n x_n$ converges in E. This is also equivalent to convergence of such series for all bounded sequences $\{x_n\}$. One more equivalent description: convergence of the series $\sum_{n=1}^{\infty} \lambda_n x_n$ for all $\{\lambda_n\} \in l^2$ and all $\{x_n\} \in E$ such that $\{p(x_n)\} \in l^2$ for all continuous seminorms p.

HINT: see Qiu [**401**], Saxon, Sánchez Ruiz [**433**].

2.10.63. (Słowikowski [**470**]) The space $\mathcal{D}(\mathbb{R}^1)$ possesses a closed linear subspace Z on which there is a discontinuous sequentially continuous linear function (hence it cannot be extended to a sequentially continuous linear function on all of $\mathcal{D}(\mathbb{R}^1)$). Therefore, Z is not the inductive limit of $Z \cap \mathcal{D}_n$. Cf. Exercise 2.10.36.

2.10.64. Let $F(\varphi) = \min\Bigl(\sum_{n=1}^{\infty} \sqrt{|\varphi(n)\varphi^{(n)}(0)}, \max_t |\varphi(t)|\Bigr)$, $\varphi \in \mathcal{D}(\mathbb{R})$. Then $0 \leqslant F(\varphi) \leqslant \max_t |\varphi(t)|$, $F(t\varphi) = |t|F(\varphi)$ and F is sequentially continuous. Show that F is not continuous.

HINT: take $\varphi_0 \in \mathcal{D}(\mathbb{R})$ with $|\varphi_0| \leqslant 1$, $\varphi_0(0) = 1$, $\varphi_0(n) = \varphi_0^{(n)}(0) = 0$ for all $n \in \mathbb{N}$. Observe that every neighborhood of φ_0 in $\mathcal{D}(\mathbb{R})$ contains a function φ such that $F(\varphi) > 1/2$.

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