

Internal structures

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In category theory we have two major streams of interest:

- (1) enriched category theory,
- (2) theory of internal structures.

We shall not enter the first one which is developed in [25] for instance, see also [28] (chapter VIII) or [2] (vol. 2, chapter 6). In these notes we shall only enter the second one. There is a point of convergence for these two streams with the notion of additive categories, which emphasizes the importance of this concept developed in ► Chap. 6. In a way, we already entered in this second stream with the notion of internal equivalence relations.

2.1 Internal unitary magmas and monoids

Let $\mathbb{E}$ be a category with finite limits.

Definition 2.1.1

An internal unitary magma is a triple (M, m, e) , in which M is an object in $\mathbb{E}$, $m : M \times M \rightarrow M$ a morphism in $\mathbb{E}$ giving rise to an internal binary operation and a morphism $e : 1 \rightarrow M$ giving rise to an internal unit, namely making the following triangles commute:

$$\begin{array}{ccccc} M & \xrightarrow{(1_M, e \cdot \tau_M)} & M \times M & \xleftarrow{(e \cdot \tau_M, 1_M)} & M \\ & \searrow 1_M & \downarrow m & \swarrow 1_M & \\ & & M & & \end{array}$$

It is an internal monoid when moreover the binary operation is associative, namely when the following diagram commute:

$$\begin{array}{ccc} M \times M \times M & \xrightarrow{m \times M} & M \times M \\ \downarrow M \times m & & \downarrow m \\ M \times M & \xrightarrow{m} & M \end{array}$$

When there is no ambiguity we shall denote this structure by the unique symbol M . A unitary magma (resp. monoid) homomorphism $f : (M, m, e) \rightarrow (M', m', e')$ is a map $f : M \rightarrow M'$ which preserves the unit and the binary operation.

We shall denote by $\text{UMg}(\mathbb{E})$ (resp. $\text{Mon}(\mathbb{E})$) the category of internal unitary magmas (resp. monoids) in $\mathbb{E}$. We shall denote by the same symbol $\mathcal{U}_{\mathbb{E}}$ the respective forgetful functors: $\text{UMg}(\mathbb{E}) \rightarrow \mathbb{E}$ and $\text{Mon}(\mathbb{E}) \rightarrow \mathbb{E}$ associating the underlying object M with any structure (M, m, e) .

Example 2.1.2

The category $\text{Mon}(\text{Top})$ is the category of topological monoids, i.e. monoids $(M, \cdot, 1)$ endowed with a topological structure which makes continuous the binary operation.

According to the Eckmann-Hilton argument, the category $\text{Mon}(\text{Mon})$ coincides with the category CoM of commutative monoids.

Accordingly the forgetful functor $\mathcal{U}_{\text{Mon}} : \text{Mon}(\text{Mon}) \rightarrow \text{Mon}$ is just the inclusion $\text{CoM} \hookrightarrow \text{Mon}$ and the result of a violent contraction in comparison with the forgetful functor $\mathcal{U}_{\text{Top}} : \text{Mon}(\text{Top}) \rightarrow \text{Top}$.

Actually the proof of the Eckmann-Hilton argument does not use the associativity axiom and allows a much stronger observation;

i Proposition 2.1.3 Any internal unitary magma in the category UMg is such that the two law coincide and produce a commutative monoid. So the categories $\text{UMg}(\text{UMg})$, $\text{UMg}(\text{Mon})$ and $\text{Mon}(\text{UMg})$ coincide with CoM .

Proof

Given a unitary magma inside the category UMg provides us with a unitary magma homomorphism $*$:

$$(M, \cdot) \times (M, \cdot) \xrightarrow{*} (M, \cdot) \xleftarrow{e} (1, \cdot)$$

We already proved that the two binary operations $\cdot$ and $*$ coincide, and this makes the operation $\cdot$ commutative. Checking that the operation $\cdot$ is associative follows from:

$$(x \cdot x') \cdot (y \cdot y') = (x \cdot y) \cdot (x' \cdot y')$$

with $x' = 1$. □

Definition 2.1.4

A functor $H : \mathbb{E} \rightarrow \mathbb{F}$ is said to be left exact when H preserves finite limits. It is said to be conservative (or reflecting isomorphisms), when $f : A \rightarrow B$ is an isomorphism as soon as so is $H(f)$.

Exercise 2.1.5

Show that the left exact (resp. conservative) functors are stable under composition. Show that the conservative functors are stable under left cancellation.

Exercise 2.1.6

Show that when the categories $\mathbb{E}$ and $\mathbb{F}$ are finitely complete, and the functor $H : \mathbb{E} \rightarrow \mathbb{F}$ is left exact, then H is conservative if and only if H is “conservative on monomorphisms”, namely when any monomorphism $m : A' \hookrightarrow A$ is an isomorphism, as soon as so is $H(m)$.

Hint: Starting with any map $f : X \rightarrow Y$, complete it by its kernel equivalence relation $R[f]$. When $H(f)$ is an isomorphism, so is $H(s_0^f)$ since the functor H is left exact. Accordingly the monomorphism s_0^f is itself an isomorphism, and the map f is a monomorphism, see Exercise 1.6.13.

We shall leave to the reader the proof of the following proposition which generalizes the properties of the forgetful functor $\mathcal{U} : \text{Mon} \rightarrow \text{Set}$:

Proposition 2.1.7 The forgetful functors $\mathcal{U}_{\mathbb{E}} : \text{UMg}(\mathbb{E}) \rightarrow \mathbb{E}$ and $\mathcal{U}_{\mathbb{E}} : \text{Mon}(\mathbb{E}) \rightarrow \mathbb{E}$ are left exact and conservative. Conclude that they are faithful.

Definition 2.1.8

A unitary magma (resp. monoid) (M, m, e) is said to be commutative when the following diagram commute

$$\begin{array}{ccc} M \times M & \xrightarrow{tw_{M,M}} & M \times M \\ & \searrow m \quad \swarrow m & \\ & M & \end{array}$$

where $tw_{X,Y}$ is the twisting isomorphism, see Exercise 1.9.5.

From now on, we shall suppose that any category $\mathbb{E}$ is such that the collections $\text{Hom}_{\mathbb{E}}(X, X')$ are actually sets.

Exercise 2.1.9

- (i) Let (M, m, e) be an internal monoid in $\mathbb{E}$. Show that the functor $\text{Hom}_{\mathbb{E}}(-, M)$ has a factorization:

$$\begin{array}{ccc} \mathbb{E}^{op} & \dashrightarrow & \text{Mon} \\ \text{Hom}_{\mathbb{E}}(-, M) \downarrow & \swarrow \mathcal{U} & \\ \text{Set} & & \end{array}$$

i.e. $\forall X \in \mathbb{E}$, $\text{Hom}_{\mathbb{E}}(X, M)$ is a monoid, and $\forall f : X \rightarrow X'$ in $\mathbb{E}$

$$\text{Hom}_{\mathbb{E}}(f, M) : \text{Hom}_{\mathbb{E}}(X', M) \rightarrow \text{Hom}_{\mathbb{E}}(X, M)$$

is a monoid homomorphism.

- (ii) Show that there are as many internal monoid structures on an object M as there are factorizations of this kind.
- (iii) Show that a monoid (M, m, e) is commutative if and only if, for any object X the previous monoid $\text{Hom}_{\mathbb{E}}(X, M)$ is commutative, or, in other words the functor $\text{Hom}_{\mathbb{E}}(-, M)$ has a factorization:

$$\begin{array}{ccc} \mathbb{E}^{op} & \dashrightarrow & \text{CoM} \\ \text{Hom}_{\mathbb{E}}(-, M) \downarrow & \swarrow \mathcal{U} & \\ \text{Set} & & \end{array}$$

2.2 Internal groups

Everybody knows that a group can be equivalently defined as a monoid $(G, \cdot, 1)$ in which:

- (i) any element x has an inverse,
which needs an extra data, namely the inverse mapping: $()^{-1} : M \rightarrow M$, or
- (ii) any element x is invertible, which is a property.

These two ways, although equivalent, are not of same nature.

Exercise 2.2.1

Show that:

- (i) a monoid (M, m, e) is a group if and only if the following commutative square in $\mathbf{Set}$:

$$\begin{array}{ccc} M \times M & \xrightarrow{m} & M \\ p_0^M \downarrow & & \downarrow \tau_M \\ M & \xrightarrow{\tau_M} & 1 \end{array}$$

is a pullback.

- (ii) produce the inverse mapping $()^{-1}$ from property (i).

Definition 2.2.2

An internal group in $\mathbb{E}$ is an internal monoid (M, m, e) such that the previous square is a pullback in $\mathbb{E}$. A internal abelian group is an internal commutative monoid (M, m, e) with the same property.

We shall denote by $\mathbf{Gp}(\mathbb{E})$ (resp. $\mathbf{Ab}(\mathbb{E})$) the full subcategory of $\mathbf{Mon}(\mathbb{E})$ whose objects are the internal groups (resp. abelian groups).

i Proposition 2.2.3 The forgetful functors $\mathcal{U}_{\mathbb{E}} : \mathbf{Gp}(\mathbb{E}) \rightarrow \mathbb{E}$, $\mathcal{U}_{\mathbb{E}} : \mathbf{Ab}(\mathbb{E}) \rightarrow \mathbb{E}$ are left exact and conservative.

Example 2.2.4

The category $\mathbf{Gp}(\mathbf{Top})$ is the category of topological groups, i.e groups $(G, \cdot, 1)$ endowed with a topological structure which makes continuous the binary operation $\cdot$ and the inverse mapping $()^{-1}$ as well.

According to the Eckmann-Hilton argument, the categories $\mathbf{Gp}(\mathbf{Mon})$ and $\mathbf{Ab}(\mathbf{Mon})$ coincide with the category $\mathbf{Ab}$ of abelian groups.

Similarly, the categories $\mathbf{UMg}(\mathbf{Gp})$ and $\mathbf{Mon}(\mathbf{Gp})$ coincide with the category $\mathbf{Ab}$ of abelian groups. The forgetful functor $\mathcal{U}_{\mathbf{Mon}} : \mathbf{Gp}(\mathbf{Mon}) \rightarrow \mathbf{Mon}$ is just the inclusion $\mathbf{Ab} \hookrightarrow \mathbf{Mon}$ and again appears as the result of a violent contraction in comparison with the forgetful functor $\mathcal{U}_{\mathbf{Top}} : \mathbf{Gp}(\mathbf{Top}) \rightarrow \mathbf{Top}$.

Exercise 2.2.5

Let (M, m, e) be an internal monoid in $\mathbb{E}$. Show that:

- (i) it is an internal group if and only if for all object $X \in \mathbb{E}$, $\text{Hom}_{\mathbb{E}}(X, M)$ is a group and $\forall f : X \rightarrow X'$ in $\mathbb{E}$

$$\text{Hom}_{\mathbb{E}}(f, M) : \text{Hom}_{\mathbb{E}}(X', M) \rightarrow \text{Hom}_{\mathbb{E}}(X, M)$$

is a group homomorphism, or, in other words, if and only if there is a factorization:

$$\begin{array}{ccc} \mathbb{E}^{op} & \dashrightarrow & Gp \\ \text{Hom}_{\mathbb{E}}(-, M) \downarrow & \swarrow \text{U} & \\ Set & & \end{array}$$

- (ii) it is an internal abelian group if and only if for all object $X \in \mathbb{E}$, the previous group $\text{Hom}_{\mathbb{E}}(X, M)$ is an abelian group or, in other words, if and only if there is a factorization:

$$\begin{array}{ccc} \mathbb{E}^{op} & \dashrightarrow & Ab \\ \text{Hom}_{\mathbb{E}}(-, M) \downarrow & \swarrow \text{U} & \\ Set & & \end{array}$$

2.3 Yoneda embedding and internal structures

In this section we shall be interested in the global meaning of the last exercise, and for that we have to go deeper into the notion of functor between two given categories $\mathbb{E}$ and $\mathbb{F}$.

2.3.1 Natural transformations and functor categories

Given a parallel pair $(H, K) : \mathbb{E} \rightrightarrows \mathbb{F}$ of functors, a natural transformation $\theta : H \Rightarrow H'$ between them assigns to any object $X \in \mathbb{E}$ a map $\theta_X : H(X) \rightarrow H'(X)$ such that, for any map $f : X \rightarrow X'$ in $\mathbb{E}$, the following square commutes in $\mathbb{F}$:

$$\begin{array}{ccc} H(X) & \xrightarrow{H(f)} & H(X') \\ \theta_X \downarrow & & \downarrow \theta_{X'} \\ H'(X) & \xrightarrow{H'(f)} & H'(X') \end{array}$$

Clearly the natural transformations give rise to a composition which determines the functor category $\mathcal{F}(\mathbb{E}, \mathbb{F})$ whose object are the functors between the categories $\mathbb{E}$ and $\mathbb{F}$ and whose maps are the natural transformations between such functors.

Exercise 2.3.1

Show that the isomorphisms in $\mathcal{F}(\mathbb{E}, \mathbb{F})$ are the natural isomorphisms, namely the natural transformations θ such that θ_X is an isomorphism for any $X \in \mathbb{E}$.

2.3.2 Yoneda embedding

We already demanded in ► Sect. 2.1 that any object $X \in \mathbb{E}$ produces a functor:

$$Y(X) = \text{Hom}_{\mathbb{E}}(-, X) : \mathbb{E}^{op} \rightarrow \text{Set}$$

This, in turn, produces a functor:

$$Y : \mathbb{E} \rightarrow \mathcal{F}(\mathbb{E}^{op}, \text{Set})$$

which is called the *Yoneda embedding*. The following proposition is fundamental in category theory and its (easy) proof left to the reader.

i Proposition 2.3.2 The Yoneda embedding is fully faithful, and left exact when in addition the category $\mathbb{E}$ is finitely complete.

The left exactness property is a synthetic translation of the universal properties of the finite limits.

2.3.3 Embedding for internal structures

Given (M, m, e) an internal unitary magma in $\mathbb{E}$, Exercise 2.1.9 shows that there is a factorization $\tilde{Y}_{UMg}$ making the following diagram commute:

$$\begin{array}{ccc} \text{UMg}(\mathbb{E}) & \xrightarrow{\tilde{Y}_{UMg}} & \mathcal{F}(\mathbb{E}^{op}, \text{UMg}) \\ \downarrow \mathcal{U}_{UMg} & & \downarrow \mathcal{F}(\mathbb{E}^{op}, \mathcal{U}) \\ \mathbb{E} & \xrightarrow{Y} & \mathcal{F}(\mathbb{E}^{op}, \text{Set}) \end{array}$$

i Proposition 2.3.3 The functor $\tilde{Y}_{UMg}$ is fully faithful and left exact.

Proof

By Proposition 2.1.7 the functor $\mathcal{U}_{UMg}$ is faithful. So, according to the previous proposition and Exercise 1.6.28, the functor $\tilde{Y}_{UMg}$ is faithful.

Now, given a pair (M, M') of unitary magmas in $\mathbb{E}$, any natural transformation $\theta : \text{Hom}_{\mathbb{E}}(-, M) \rightarrow \text{Hom}_{\mathbb{E}}(-, M')$ in $\mathcal{F}(\mathbb{E}^{op}, \text{UMg})$ has an underlying natural transformation $\theta : \text{Hom}_{\mathbb{E}}(-, M) \rightarrow \text{Hom}_{\mathbb{E}}(-, M')$ in $\mathcal{F}(\mathbb{E}^{op}, \text{Set})$. From it, Proposition 2.3.2 produces a map $f : M \rightarrow M'$ in $\mathbb{E}$; it remains to check that it is a homomorphism of unitary magma, i.e. that some diagrams commute in $\mathbb{E}$. This, again, can be checked via the Yoneda embedding.

The left exactness of the embedding $\tilde{Y}_{UMg}$ is a consequence of the left exactness of the Yoneda embedding Y itself. $\square$

Exercise 2.3.4

In the same way, show that we have the following left exact fully faithful *structure embeddings*:

- (i) $\tilde{Y}_{Mon} : \text{Mon}(\mathbb{E}) \rightarrow \mathcal{F}(\mathbb{E}^{op}, \text{Mon})$
- (ii) $\tilde{Y}_{CoM} : \text{CoM}(\mathbb{E}) \rightarrow \mathcal{F}(\mathbb{E}^{op}, \text{CoM})$
- (iii) $\tilde{Y}_{Gp} : \text{Gp}(\mathbb{E}) \rightarrow \mathcal{F}(\mathbb{E}^{op}, \text{Gp})$
- (iv) $\tilde{Y}_{Ab} : \text{Ab}(\mathbb{E}) \rightarrow \mathcal{F}(\mathbb{E}^{op}, \text{Ab})$.



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