Chapter 2
Spherical Surface

As it follows from the preceding segments in this volume, the lion share of our work will be devoted to a specific class of applied problems that have never been touched upon before in standard texts on differential equations. Boundary value problems will be analyzed as setup for second-order elliptic partial differential equations that simulate potential fields induced in thin-wall structures. The construction and implementation of Green's functions for such problems will be in our focus. While investigating this topic, one of the primary objectives will be to meet the needs of practitioners who might be potentially interested in the employment of Green's functions in their numerical work.

Chapter 1 was devoted to Green's functions for ordinary differential equations. The intention was to maintain a basis for a productive work with partial differential equations later on. The objective in the current chapter is to describe a workable algorithm developed in [31, 32] for the construction of computer friendly representations of Green's functions for problems formulated in regions that represent fragments of a spherical surface.

The presentation is organized, herein, in such a way that each section deals with a problem set up for a specific fragment of a spherical surface. The idea behind of this kind of organization is that different regions on a sphere reveal different peculiar features of our algorithm. Section 2.1 overviews the algorithm, while each of the following sections focuses on region's shape specificities which require individual considerations. Namely, it is important to know, for example, whether or not the poles of the sphere represent parts of the region's boundary. It is also an issue if the solution of the encountered problem is periodic with respect to one or both the independent variables.

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2.1 Basics of the Resolving Algorithm

In order to make our presentation as explicit and self-explained as possible, this section introduces the reader to the basics of the above-mentioned algorithm. The latter is intended for the efficient construction of Green's functions for boundary value problems that simulate potential fields induced in thin shells.

Regions of different shape will be encountered later in the upcoming sections. In this section however, we do not plan to be maximally specific as to the region's shape, but some specificity is nevertheless required. Let

$$\Omega = \{\vartheta, \varphi | \ \vartheta_0 \leq \vartheta \leq \vartheta_1, \ 0 \leq \varphi \leq \varphi_1\}$$

be the region on a spherical surface of radius a as depicted in Fig. 2.1.

The shape of Ω will be referred to as the *spherical quadrilateral*. It represents a simply connected region bounded, in spherical coordinates, with two parallels $\vartheta = \vartheta_0$ and $\vartheta = \vartheta_1$, as well as with two meridians $\varphi = 0$ and $\varphi = \varphi_1$. This obviously makes Ω of a "rectangular" shape justifying the use of the introduced term *quadrilateral*.

Let a point belonging to Ω have coordinates $x, \ y$ and z in the rectangular Cartesian coordinate system whose origin is located at the center of the encountered spherical surface. This makes the coordinates $x, \ y$ and z expressed in a parametric form in terms of the variables ϑ and φ as

$$x = a \sin\vartheta \cos\varphi, \quad y = a \sin\vartheta \sin\varphi \ \text{ and } \ z = a \cos\vartheta$$

If Ω represents the middle surface of a thin-shell element made of a homogeneous conductive material, then a potential field $u = u(\vartheta, \varphi)$ induced in Ω can be simulated by a boundary value problem where the nonhomogeneous Poisson type equation

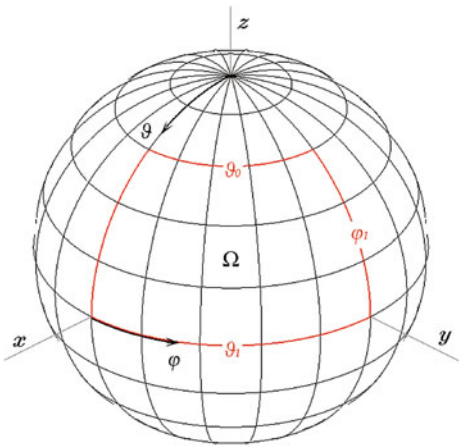


Fig. 2.1 The spherical quadrilateral region

$$\frac{1}{a^2 \sin\vartheta} \frac{\partial}{\partial\vartheta} \left(\sin\vartheta \frac{\partial u}{\partial\vartheta} \right) + \frac{1}{a^2 \sin^2\vartheta} \frac{\partial^2 u}{\partial\varphi^2} = -f(\vartheta, \varphi) \quad \text{in } \Omega \tag{2.1}$$

is subject to the set of boundary conditions

$$B_1 [u(\vartheta, 0)] = 0 \quad \text{and} \quad B_2 [u(\vartheta, \varphi_1)] = 0, \tag{2.2}$$

and

$$B_3 [u(\vartheta_0, \varphi)] = 0 \quad \text{and} \quad B_4 [u(\vartheta_1, \varphi)] = 0, \tag{2.3}$$

where $B_i, i = \overline{1,4}$, represent boundary condition operators of either one of the three standard types (Dirichlet, Neumann, and Robin).

It is evident that, in physical terms, the above formulation pretends that the face surfaces of the considered shell element are insulated, restricting a loss of energy through.

Before we proceed any further, let us assume that the problem in (2.1)–(2.3) is well posed allowing a unique solution. This implies [5, 10, 11, 16, 17, 27, 38–40] that there exists a unique Green's function for the corresponding to (2.1)–(2.3) homogeneous (with $f(\vartheta, \varphi) \equiv 0$) problem.

It is well known (see, for example, [2, 10, 12, 16, 37]) that if $G(\vartheta, \varphi; \tau, \psi)$ represents the Green's function that we have just referred to, then the solution to the problem in (2.1)-(2.3) itself can be expressed in a form of the domain integral

$$u(\vartheta, \varphi) = \iint\limits_{\Omega} G(\vartheta, \varphi; \tau, \psi) f(\tau, \psi) d_{\tau,\psi}\Omega, \tag{2.4}$$

where the element of area $d_{\tau,\psi}\Omega$ reads in spherical coordinates as $a^2 \sin\tau d\tau d\psi$.

This prompts a strategy of obtaining the Green's function $G(\vartheta, \varphi; \tau, \psi)$ for the homogeneous problem corresponding to (2.1)–(2.3). Namely, whenever the Green's function is required, one needs to simply solve the problem in (2.1)–(2.3) itself. But proceeding with this, the resolving routine should be developed in such a way that the solution is eventually expressed in the form of (2.4) delivering the desired Green's function.

Return now to the problem setting in (2.1)–(2.3) and sketch an algorithm of its solution which follows the foregoing type of strategy. Assume that the boundary conditions in (2.2) allow analytic separation of variables, implying that B_1 and B_2 represent either Dirichlet or Neumann operators. If so, then the solution $u(\vartheta, \varphi)$ of the original problem and the right-hand side function $f(\vartheta, \varphi)$ of the governing equation can be expressed in the Fourier sine series form

$$u(\vartheta, \varphi) = \sum_{n=1}^{\infty} u_n(\vartheta) \sin\nu\varphi \tag{2.5}$$

and

$$f(\vartheta, \varphi) = \sum_{n=1}^{\infty} f_n(\vartheta) \sin \nu\varphi, \tag{2.6}$$

where the factor ν is supposed to be directly proportional to the summation index n in the above series. The coefficient of proportionality depends upon specific combination of B_1 and B_2. As an example, if both B_1 and B_2 are Dirichlet condition operators, then $\nu = n\pi/\varphi_1$.

Upon substituting the trigonometric series representations of (2.5) and (2.6) into the governing equation of the boundary value problem in (2.1)–(2.3), we arrive at the set of linear nonhomogeneous ordinary differential equations

$$\frac{d}{d\vartheta}\left(\sin\vartheta \frac{du_n(\vartheta)}{d\vartheta}\right) - \frac{\nu^2 u_n(\vartheta)}{\sin\vartheta} = -\widetilde{f}_n(\vartheta), \quad n = 1, 2, 3, \ldots \tag{2.7}$$

in the coefficients $u_n = u_n(\vartheta)$ of the series in (2.5), where $\widetilde{f}_n(\vartheta) = a^2 \sin\vartheta\, f_n(\vartheta)$. These equations are subject to the boundary conditions

$$B_3[u_n(\vartheta_0)] = 0 \quad \text{and} \quad B_4[u_n(\vartheta_1)] = 0 \tag{2.8}$$

With the above problem formulation, the reader arrives at the familiar territory of Chap. 1, which has provided us with a necessary experience, and equips with a workable instrument for dealing with ordinary differential equations. Our next move will be to properly implement the gained experience while analyzing the problem in (2.7)–(2.8).

In order to solve the problem in (2.7)–(2.8), we recall the method of variation of parameters which just represents one of the potential options for that. As the reader has learned from Chap. 1, this method requires a fundamental set of solutions to the homogeneous equation corresponding to (2.7). But it is not quite trivial to find two linearly independent particular solutions representing that set. The variable coefficients of (2.7) are evidently an issue. However, upon introducing a new independent variable

$$\omega = \ln\left(\tan\frac{\vartheta}{2}\right),$$

the governing differential equation in (2.7) reduces to a trivial form, converting the whole problem in (2.7)–(2.8) to

$$\frac{d^2 u_n(\omega)}{d\omega^2} - \nu^2 u_n(\omega) = -\widetilde{f}_n(\omega) \tag{2.9}$$

$$\widetilde{B}_3[u_n(\omega_1)] = 0 \quad \text{and} \quad \widetilde{B}_4[u_n(\omega_2)] = 0. \tag{2.10}$$

Components of a fundamental set of solutions to the homogeneous equation corresponding to (2.9) can evidently be chosen as

$$e^{\nu\omega} \text{ and } e^{-\nu\omega}.$$

In view of the recent change of the independent variable, the backward substitution reveals the set of functions

$$\tan^{\nu}\frac{\vartheta}{2} \text{ and } \tan^{-\nu}\frac{\vartheta}{2}$$

that represent a fundamental set of solutions to the homogeneous equation corresponding to (2.7). Using the above set of functions and following the standard variation of parameters procedure, we look for the general solution to the problem in (2.7) and (2.8) in the form

$$u_n(\vartheta) = C_1(\vartheta)\tan^{\nu}\frac{\vartheta}{2} + C_2(\vartheta)\tan^{-\nu}\frac{\vartheta}{2}. \tag{2.11}$$

In compliance with the method routine, the form (2.11) yields the well-posed system of linear algebraic equations

$$\begin{pmatrix} \tan^{\nu}\frac{\vartheta}{2} & \tan^{-\nu}\frac{\vartheta}{2} \\ \nu\tan^{\nu}\frac{\vartheta}{2} & -\nu\tan^{-\nu}\frac{\vartheta}{2} \end{pmatrix} \begin{pmatrix} C_1'(\vartheta) \\ C_2'(\vartheta) \end{pmatrix} = \begin{pmatrix} 0 \\ -\widetilde{f}_n(\vartheta) \end{pmatrix}$$

in the derivatives of the functions $C_1(\vartheta)$ and $C_2(\vartheta)$ from (2.11). The solution for this system appears as

$$C_1'(\vartheta) = -\frac{1}{2\nu}\tan^{-\nu}\frac{\vartheta}{2}\,\widetilde{f}_n(\vartheta) \text{ and } C_2'(\vartheta) = \frac{1}{2\nu}\tan^{\nu}\frac{\vartheta}{2}\,\widetilde{f}_n(\vartheta)$$

Upon integrating the above, the functions $C_1(\vartheta)$ and $C_2(\vartheta)$ themselves are found as

$$C_1(\vartheta) = -\frac{1}{2\nu}\int_{\vartheta_0}^{\vartheta}\tan^{-\nu}\frac{\tau}{2}\,\widetilde{f}_n(\tau)d\tau + D_1$$

and

$$C_2(\vartheta) = \frac{1}{2\nu}\int_{\vartheta_0}^{\vartheta}\tan^{\nu}\frac{\tau}{2}\,\widetilde{f}_n(\tau)d\tau + D_2$$

Substituting the found expressions for $C_1(\vartheta)$ and $C_2(\vartheta)$ into (2.11), we come up with the integral-containing form

$$u_n(\vartheta) = \frac{1}{2\nu}\int_{\vartheta_0}^{\vartheta}\frac{\tan^{2\nu}(\tau/2) - \tan^{2\nu}(\vartheta/2)}{\tan^{\nu}(\vartheta/2)\tan^{\nu}(\tau/2)}\widetilde{f}_n(\tau)\,d\tau$$

$$+\ D_1 \tan^{\nu} \frac{\vartheta}{2} + D_2 \tan^{-\nu} \frac{\vartheta}{2} \tag{2.12}$$

for the general solution to the equation in (2.7). The constants of integration D_1 and D_2 can be found upon satisfying the boundary conditions of (2.8). As it follows from the standard variation of parameters procedure, these constants are expressed as definite integrals from ϑ_0 to ϑ_1 of a product of two functions one of which is $\widetilde{f_n}(\tau) = a^2 \sin\tau\, f_n(\tau)$. This allows us to transform the expression for $u_n(\vartheta)$ from (2.12) onto the integral-only representation

$$u_n(\vartheta) = \int_{\vartheta_0}^{\vartheta_1} g_n(\vartheta, \tau)\, a^2 \sin\tau\ f_n(\tau)\, d\tau, \tag{2.13}$$

where the kernel function $g_n(\vartheta, \tau)$ represents the Green's function to the homogeneous ODE boundary value problem corresponding to (2.7) and (2.8). It is expressed in two pieces whose explicit expressions depend on specifics flowing out from the boundary conditions operators of (2.8). While considering particular problems later on, we will discuss this issue in more detail, which at this moment look unnecessary.

Note that the functions $f_n(\tau)$ in (2.13) represent Fourier coefficients of the series in (2.6). At this point in our development, they ought to be expressed in terms of the right-hand side function $f(\vartheta, \varphi)$ of the governing equation (2.1). Applying hence, the Fourier–Euler formula to $f_n(\tau)$

$$f_n(\tau) = \frac{2}{\varphi_1} \int_0^{\varphi_1} f(\tau, \psi) \sin\nu\psi d\psi, \quad n = 1, 2, 3, \ldots$$

we substitute the above expression for $f_n(\tau)$ into (2.13). This yields

$$u_n(\vartheta) = \frac{2}{\varphi_1} \int_0^{\varphi_1} \int_{\vartheta_0}^{\vartheta_1} g_n(\vartheta, \tau) \sin\nu\psi f(\vartheta, \psi)\, a^2 \sin\tau d\tau d\psi, \quad n = 1, 2, 3, \ldots$$

Substituting now the above form of $u_n(\vartheta)$ into (2.5), the solution $u(\vartheta, \varphi)$ to the problem of (2.1)–(2.3) is ultimately found as

$$u(\vartheta, \varphi) = \int_0^{\varphi_1} \int_{\vartheta_0}^{\vartheta_1} \frac{2}{\varphi_1} \sum_{n=1}^{\infty} g_n(\vartheta, \tau) \sin\nu\varphi \sin\nu\psi\ f(\tau, \psi)\ a^2 \sin\tau d\tau d\psi$$

At this point, it is not hard to realize that the integral form we just came up with in an extended version of (2.4), and its kernel function

$$G(\vartheta, \varphi; \tau, \psi) = \frac{2}{\varphi_1} \sum_{n=1}^{\infty} g_n(\vartheta, \tau) \sin\nu\varphi \sin\nu\psi. \tag{2.14}$$

represents, subsequently, the Green's function to the homogeneous boundary value problem corresponding to (2.1)–(2.3).

Thus, we have completed a sketch of the algorithm that appears efficient for the construction of series representations of Green's functions for boundary value problems of the type in (2.1)–(2.3) set up on a spherical surface.

It is important to remind the reader that Green's functions of elliptic boundary value problems in two dimensions, which are targeted in the present study, possess the logarithmic singularity [12, 16, 27, 42]. That is why their series representations (like the one in (2.14), for example) cannot converge uniformly, significantly cutting down their practicality. To fix this unfortunate circumstance, the convergence rate of series representing Green's functions ought to be controlled enhancing their practicality. This is what we are going to especially pay attention to in the coming sections, where particular problems are encountered.

2.2 Triangular Shaped Region

We proceed now with the implementation of our algorithm (which was sketched in the preceding section) for the construction of Green's functions to boundary value problems that simulate potential fields induced in various fragments of a thin spherical shell. In doing so, we consider the simply connected region

$$\Omega = \{\vartheta, \varphi \mid 0 \le \vartheta \le \vartheta_1;\ 0 \le \varphi \le \varphi_1\}$$

representing a fragment of a spherical surface of radius a (see Fig. 2.2).

The region Ω is bounded with two meridians $\varphi = 0$ and $\varphi = \varphi_1$, and a single parallel $\vartheta = \vartheta_1$, where $0 < \vartheta_1 < \pi$ and $0 < \varphi_1 < 2\pi$. In what follows, we will refer to Ω as the *spherical triangle*. The shape of the latter makes the north pole $\vartheta = 0$

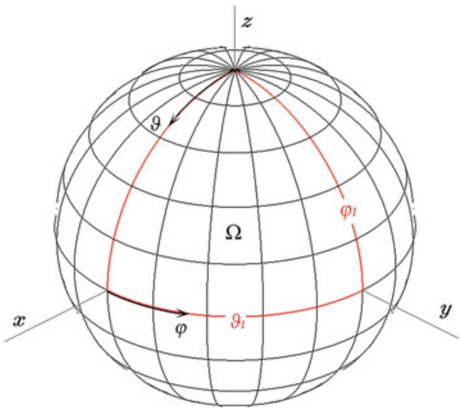


Fig. 2.2 The triangular region on a sphere

a part of the "boundary" for Ω. It will be seen soon that this specific circumstance notably affects the algorithm.

In the described spherical triangle, we consider a boundary value problem, where the governing equation

$$\frac{1}{a^2 \sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\,\frac{\partial u}{\partial\vartheta}\right)+\frac{1}{a^2 \sin^2\vartheta}\frac{\partial^2 u}{\partial\varphi^2}=-\,f\,(\vartheta,\varphi)\quad \text{in } \Omega \tag{2.15}$$

is subject to the boundary conditions

$$u\,(\vartheta,0)=0 \quad \text{and} \quad u\,(\vartheta,\varphi_1)=0, \tag{2.16}$$

and

$$\lim_{\vartheta\to 0}\ |u\,(\vartheta,\varphi)|<\infty \quad \text{and} \quad \frac{\partial u\,(\vartheta_1,\varphi)}{\partial\vartheta}=0. \tag{2.17}$$

Note that there is a reason behind expressing the first condition of (2.17) in its current nontrivial form. The thing is that any standard boundary condition is meaningless at $\vartheta=0$. This is so because $\vartheta=0$ represents a point of singularity to the governing equation in (2.15).

Taking into account the boundary conditions of (2.16), we express the solution function $u\,(\vartheta,\varphi)$ and the right-hand side function $f\,(\vartheta,\varphi)$ of the governing equation in the Fourier sine series form

$$u\,(\vartheta,\varphi)=\sum_{n=1}^{\infty}u_n\,(\vartheta)\sin\frac{n\pi\varphi}{\varphi_1} \tag{2.18}$$

and

$$f\,(\vartheta,\varphi)=\sum_{n=1}^{\infty}f_n\,(\vartheta)\sin\frac{n\pi\varphi}{\varphi_1} \tag{2.19}$$

Implementing the above series representations, one arrives at a boundary value problem, in the Fourier coefficients $u_n(\vartheta)$ of (2.18), for the linear ordinary differential equations

$$\frac{d}{d\vartheta}\left(\sin\vartheta\frac{du_n(\vartheta)}{d\vartheta}\right)-\frac{\nu^2 u_n(\vartheta)}{\sin\vartheta}=-\,\widetilde{f_n}\,(\vartheta)\quad \text{in } (0,\vartheta_1), \tag{2.20}$$

where $\widetilde{f_n}\,(\vartheta)=a^2\sin\vartheta\,f_n(\vartheta)$, and $n=1,2,3,\ldots$, subject to the boundary conditions

$$\lim_{\vartheta\to 0}\ |u_n\,(\vartheta)|<\infty \quad \text{and} \quad \frac{du_n\,(\vartheta_1)}{d\vartheta}=0, \tag{2.21}$$

where the parameter ν is defined as $n\pi/\varphi_1$.

Clearly, the first condition in (2.21) symbolizes the boundness of $u_n(\vartheta)$ at the point of singularity $\vartheta = 0$ of the governing equation.

Following in the footsteps of the procedure described in Sect. 2.1, the general solution to the equation in (2.20) appears as

$$u_n(\vartheta) = \frac{1}{2\nu}\int_0^{\vartheta} \frac{\tan^{2\nu}(\tau/2) - \tan^{2\nu}(\vartheta/2)}{\tan^{\nu}(\vartheta/2)\tan^{\nu}(\tau/2)} \widetilde{f}_n(\tau)\,d\tau$$

$$+ D_1 \tan^{\nu}\frac{\vartheta}{2} + D_2 \tan^{-\nu}\frac{\vartheta}{2}, \tag{2.22}$$

The uniqueness conditions from (2.21) allow us to specify the constants of integration D_1 and D_2. Observing the above form, one realizes that the function $\tan^{-\nu}(\vartheta/2)$ is undefined at $\vartheta = 0$. Thus, the only way to satisfy the first boundary condition in (2.21), which requires for the solution $u_n(\vartheta)$ to be bounded at $\vartheta = 0$, is to let $D_2 = 0$.

Upon applying the second condition of (2.21), the constant D_1 can also be found. Omitting a quite trivial algebra, we present just its ultimate expression

$$D_1 = \frac{1}{2\nu}\int_0^{\vartheta_1} \frac{\Phi^{2n}(\vartheta_1) + \Phi^{2n}(\tau)}{\Phi^{n}(\tau)\,\Phi^{2n}(\vartheta_1)} \widetilde{f}_n(\tau)\,d\tau$$

in which the function $\Phi(x)$ is introduced as

$$\Phi(x) = \tan^{\pi/\varphi_1}\frac{x}{2}\,.$$

After substituting into (2.22) the expressions for D_1 and D_2 just found, the solution to the boundary value problem in (2.20) and (2.21) appears in the form

$$u_n(\vartheta) = \frac{1}{2\nu}\left[\int_0^{\vartheta} \frac{\Phi^{n}(\tau)\left(\Phi^{2n}(\vartheta_1) + \Phi^{2n}(\vartheta)\right)}{\Phi^{n}(\vartheta)\,\Phi^{2n}(\vartheta_1)} \widetilde{f}_n(\tau)\,d\tau \right.$$
$$\left. + \int_{\vartheta}^{\vartheta_1} \frac{\Phi^{n}(\vartheta)\left(\Phi^{2n}(\vartheta_1) + \Phi^{2n}(\tau)\right)}{\Phi^{n}(\tau)\,\Phi^{2n}(\vartheta_1)} \widetilde{f}_n(\tau)\,d\tau\right],$$

which can be written in a single-integral form as

$$u_n(\vartheta) = \int_0^{\vartheta_1} g_n(\vartheta, \tau)\,\widetilde{f}_n(\tau)\,d\tau, \tag{2.23}$$

whose kernel function $g_n(\vartheta, \tau)$, representing, by the way, the Green's function to the homogeneous ODE boundary value problem corresponding to (2.20) and (2.21), is found in two pieces

$$g_n(\vartheta,\tau)=\begin{cases} g_n^-(\vartheta,\tau)\,, & 0\le\tau\le\vartheta \\ g_n^+(\vartheta,\tau)\,, & \vartheta\le\tau\le\vartheta_1 \end{cases} \tag{2.24}$$

The piece $g_n^-(\vartheta,\tau)$ of $g_n(\vartheta,\tau)$ is valid for $0\le\tau\le\vartheta$, and reads as

$$g_n^-(\vartheta,\tau)=\frac{1}{2\nu}\frac{\Phi^n(\tau)\left(\Phi^{2n}(\vartheta_1)+\ \Phi^{2n}(\vartheta)\right)}{\Phi^n(\vartheta)\,\Phi^{2n}(\vartheta_1)},$$

The piece $g_n^+(\vartheta,\tau)$, valid for $\vartheta\le\tau\le\vartheta_1$, appears in the form

$$g_n^+(\vartheta,\tau)=\frac{1}{2\nu}\frac{\Phi^n(\vartheta)\left(\Phi^{2n}(\vartheta_1)+\ \Phi^{2n}(\tau)\right)}{\Phi^n(\tau)\,\Phi^{2n}(\vartheta_1)}\,.$$

Turning back to the expression for $u_n(\vartheta)$ in (2.23), notice that the functions $f_n(\tau)$, as factors of $\widetilde{f}_n(\tau)$, represent Fourier coefficients of the series in (2.19). Expressing them in terms of the right-hand side function $f(\vartheta,\varphi)$ of the governing equation in (2.15), we have

$$f_n(\tau)=\frac{2}{\varphi_1}\int_0^{\varphi_1} f(\tau,\psi)\sin\nu\psi d\psi$$

This subsequently yields

$$u_n(\vartheta)=\frac{2}{\varphi_1}\int_0^{\varphi_1}\int_0^{\vartheta_1} g_n(\vartheta,\tau)\sin\nu\psi f(\tau,\psi)\,a^2\sin\tau d\tau d\psi,\quad n=1,2,3,\ldots$$

Substituting the above into (2.18), one arrives at an ultimate representation for the solution to the problem of (2.15)–(2.17), which reads as

$$u(\vartheta,\varphi)=\iint\limits_{\Omega} G(\vartheta,\varphi;\tau,\psi)\,f(\tau,\psi)\,d_{\tau,\psi}\Omega,$$

where the kernel function $G(\vartheta,\varphi;\tau,\psi)$ written as

$$G(\vartheta,\varphi;\tau,\psi)=\frac{2}{\varphi_1}\sum_{n=1}^{\infty} g_n(\vartheta,\tau)\sin\nu\varphi\sin\nu\psi \tag{2.25}$$

is recognized, in light of (2.4), as the sought-after Green's function to the homogeneous boundary value problem of (2.15)–(2.17).

This leads to that very point in the development at which the focus should be placed on the convergence of the series representation in (2.25). As we emphasized in the last paragraph of Sect. 2.1, its convergence ought to be improved (if possible at all) to make the representation in (2.25) practical and, as a result, more attractive to potential users.

In what follows, we will show that the series representation of the Green's function to the problem in (2.15)–(2.17) which is presented in (2.25) appears completely summable. It reduces to a computer friendly series-free form expressed in terms of elementary functions. This summability ensures high potential of our algorithm and plays an important role in attracting possible Green's functions users.

To begin with the summation procedure, we refer the reader to the standard [1, 19] summation formula

$$\sum_{n=1}^{\infty} \frac{p^n}{n} \cos nx = -\frac{1}{2} \ln\left(1 - 2p\cos x + p^2\right), \tag{2.26}$$

which is valid for $p^2 < 1$ and $0 \leq x < 2\pi$. It is worth noting that this series expansion will play a substantial role in the development that follows.

Proceeding with the summation itself, let us convert the expansion for the Green's function from (2.25) to the equivalent form

$$G(\vartheta, \varphi; \tau, \psi) = \frac{1}{\varphi_1} \sum_{n=1}^{\infty} g_n(\vartheta, \tau)\left[\cos \nu(\varphi - \psi) - \cos \nu(\varphi + \psi)\right] \tag{2.27}$$

Note that the expression for $g_n(\vartheta, \tau)$ presented in (2.24) is given in two pieces. Note also that each of the pieces (either $g_n^-(\vartheta, \tau)$ or $g_n^+(\vartheta, \tau)$) is equivalently feasible for employment in the coming summation procedure. We take $g_n^-(\vartheta, \tau)$, but before going any further, transform it to the equivalent form

$$g_n^-(\vartheta, \tau) = \frac{1}{2\nu}\left[\left(\frac{\Phi(\tau)}{\Phi(\vartheta)}\right)^n + \left(\frac{\Phi(\tau)\,\Phi(\vartheta)}{\Phi^2(\vartheta_1)}\right)^n\right]$$

Upon substituting the above into (2.27), the latter reads as

$$G(\vartheta, \varphi; \tau, \psi) = \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{1}{n}\left[\left(\frac{\Phi(\tau)}{\Phi(\vartheta)}\right)^n + \left(\frac{\Phi(\tau)\,\Phi(\vartheta)}{\Phi^2(\vartheta_1)}\right)^n\right] \cos n\alpha$$

$$- \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{1}{n}\left[\left(\frac{\Phi(\tau)}{\Phi(\vartheta)}\right)^n + \left(\frac{\Phi(\tau)\,\Phi(\vartheta)}{\Phi^2(\vartheta_1)}\right)^n\right] \cos n\beta, \tag{2.28}$$

where

$$\alpha = \frac{\pi(\varphi - \psi)}{\varphi_1} \quad \text{and} \quad \beta = \frac{\pi(\varphi + \psi)}{\varphi_1}$$

The single-variable function $\Phi(x) = \tan^{\pi/\varphi_1}(x/2)$, just recently introduced in this section, is increasing in its domain, and the two-variable function $g_n^-(\vartheta, \tau)$ is defined for $\tau \leq \vartheta$. With this in mind, one realizes that the expansion in (2.28) represents the sum of four similar summable series of the type in (2.26). The summation, hence, yields the following closed form

$$G(\vartheta, \varphi; \tau, \psi) = \frac{1}{2\pi}\left\{\ln\sqrt{1 - 2\frac{\Phi(\tau)}{\Phi(\vartheta)}\cos\beta + \left(\frac{\Phi(\tau)}{\Phi(\vartheta)}\right)^2}\right.$$

$$+\ln\sqrt{1 - 2\frac{\Phi(\tau)\,\Phi(\vartheta)}{\Phi^2(\vartheta_1)}\cos\beta + \left(\frac{\Phi(\tau)\,\Phi(\vartheta)}{\Phi^2(\vartheta_1)}\right)^2}$$

$$-\ln\sqrt{1 - 2\frac{\Phi(\tau)}{\Phi(\vartheta)}\cos\alpha + \left(\frac{\Phi(\tau)}{\Phi(\vartheta)}\right)^2}$$

$$\left.-\ln\sqrt{1 - 2\frac{\Phi(\tau)\,\Phi(\vartheta)}{\Phi^2(\vartheta_1)}\cos\alpha + \left(\frac{\Phi(\tau)\,\Phi(\vartheta)}{\Phi^2(\vartheta_1)}\right)^2}\right\}$$

for the expansion that appeared in (2.28). After an elementary transformation, it can be rewritten in a more compact form as

$$G(\vartheta, \varphi; \tau, \psi) = \frac{1}{2\pi}\left\{\ln\sqrt{\frac{\Phi^2(\vartheta) - 2\Phi(\vartheta)\,\Phi(\tau)\cos\beta + \Phi^2(\tau)}{\Phi^2(\vartheta) - 2\Phi(\vartheta)\,\Phi(\tau)\cos\alpha + \Phi^2(\tau)}}\right.$$

$$\left.+\ln\sqrt{\frac{\Phi^4(\vartheta_1) - 2\Phi^2(\vartheta_1)\,\Phi(\vartheta)\,\Phi(\tau)\cos\beta + \Phi^2(\tau)\,\Phi^2(\vartheta)}{\Phi^4(\vartheta_1) - 2\Phi^2(\vartheta_1)\,\Phi(\vartheta)\,\Phi(\tau)\cos\alpha + \Phi^2(\tau)\,\Phi^2(\vartheta)}}\right\} \tag{2.29}$$

representing the Green's function to the homogeneous boundary value problem corresponding to (2.15)–(2.17). The appearance of the form in (2.29) speaks for itself. It is compact and closed in the conventional sense, implying that it is expressed in terms of elementary functions. That is why we can call this form computer friendly.

While observing the above expression, the reader is advised to turn back to that point in our recent development at which we had chosen the piece $g_n^-(\vartheta, \tau)$ of the two-piece-defined function $g_n(\vartheta, \tau)$. We claimed over there that each piece of $g_n(\vartheta, \tau)$ is equivalently eligible for use. Indeed, in view of the *symmetry* of (2.29), the interchange of ϑ with τ does not affect it at all. If so then it becomes absolutely clear that giving preference to the piece $g_n^+(\vartheta, \tau)$ instead, would never affect the ultimate form of (2.29).

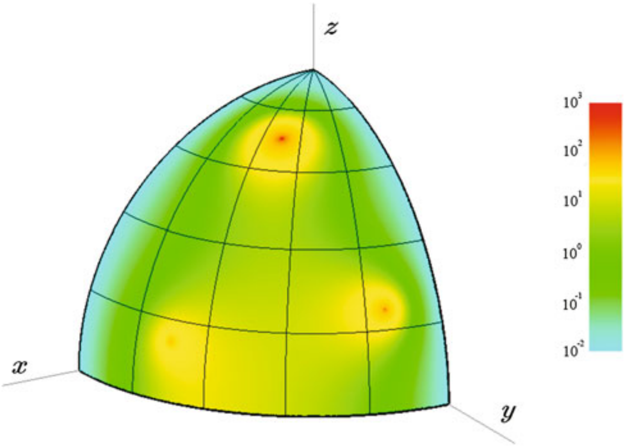


Fig. 2.3 The field induced by three point sources

The workability of the expression for $G(\vartheta, \varphi; \tau, \psi)$ in (2.29) could be justified by Fig. 2.3. The triangular shape of Ω is defined by $\vartheta_1 = \pi/2$ and $\varphi_1 = \pi/2$, which makes a spherical octant. A potential field is depicted as generated in a thin-shell element for which Ω is the middle surface. The field is induced by three point sources of intensities $K_1 = 1$, $K_2 = 10$, and $K_3 = 100$ released at the points $(\tau_1, \psi_1) = (0.425\pi,\ 0.15\pi)$, $(\tau_2, \psi_2) = (0.375\pi,\ 0.425\pi)$, and $(\tau_3, \psi_3) = (0.15\pi,\ 0.25\pi)$, respectively. The superposition

$$\sum_{j=1}^{3} K_j G\left(\vartheta, \varphi; \tau_j, \psi_j\right)$$

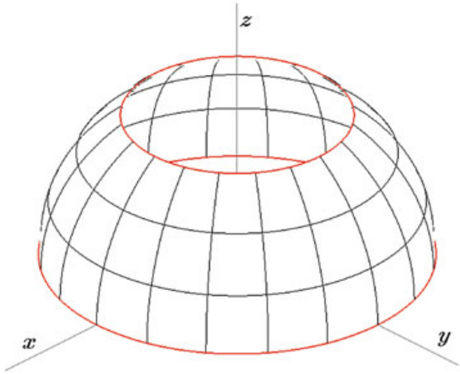
of three profiles of the Green's function shown in (2.29) represents the field.

Appendix segment of this volume provides the reader with an extensive list of computer friendly representations of Green's functions constructed for a number of boundary value problems that simulate potential fields induced in elements of thin shells. Among of many others, Green's functions for some different of (2.15)–(2.17) problems posed in the spherical triangle are also available over there.

2.3 Belt-Shaped Region

The work is continued in this section on the construction of Green's functions for problems stated in regions belonging to a spherical surface. Another practically important shape of a spherical region is brought to the reader's attention. That is the double-connected region

Fig. 2.4 Spherical belt



$$\Omega = \{\vartheta, \varphi |\ \vartheta_0 \leq \vartheta \leq \vartheta_1,\ 0 \leq \varphi < 2\pi\}$$

belonging to a spherical surface of radius a and depicted in Fig. 2.4. The region is bounded with two parallels $\vartheta = \vartheta_0$ and $\vartheta = \vartheta_1$, and will be referred to as the *spherical belt*.

An important feature of any boundary value problem posed in Ω is that its solution has to be 2π-periodic with respect to the longitudinal coordinate φ. This brings new nuances to our approach, which must, of course, be somewhat different of that developed in Sect. 2.2.

For an illustrative example, let the equation

$$\frac{1}{a^2 \sin\vartheta} \frac{\partial}{\partial\vartheta}\left(\sin\vartheta \frac{\partial u}{\partial\vartheta}\right) + \frac{1}{a^2 \sin^2\vartheta} \frac{\partial^2 u}{\partial\varphi^2} = -\ f\,(\vartheta, \varphi) \quad \text{in } \Omega, \tag{2.30}$$

with an integrable in Ω right-hand side function $f\,(\vartheta, \varphi)$, be subject to the Neumann and the Dirichlet conditions

$$\frac{\partial u\,(\vartheta_0, \varphi)}{\partial\vartheta} = 0 \quad \text{and} \quad u\,(\vartheta_1, \varphi) = 0 \tag{2.31}$$

imposed on the boundary fragments $\vartheta = \vartheta_0$ and $\vartheta = \vartheta_1$ of Ω, respectively. The conditions

$$u\,(\vartheta, 0) - u\,(\vartheta, 2\pi) = 0 \tag{2.32}$$

and

$$\frac{\partial u\,(\vartheta, 0)}{\partial\varphi} - \frac{\partial u\,(\vartheta, 2\pi)}{\partial\varphi} = 0 \tag{2.33}$$

have also to be imposed to reflect the 2π-periodicity of the solution we are looking for.

Our target is the Green's function to the homogeneous problem corresponding to the well-posed setting in (2.30)–(2.33).

It is evident that the 2π-periodicity of the above problem requires complete Fourier series expansions for the solution function $u(\vartheta, \varphi)$ and the right-hand side $f(\vartheta, \varphi)$ of (2.30). That is

$$u(\vartheta, \varphi) = \frac{1}{2}u_0(\vartheta) + \sum_{n=1}^{\infty} u_n^{(c)}(\vartheta)\cos n\varphi + \sum_{n=1}^{\infty} u_n^{(s)}(\vartheta)\sin n\varphi \tag{2.34}$$

and

$$f(\vartheta, \varphi) = \frac{1}{2}f_0(\vartheta) + \sum_{n=1}^{\infty} f_n^{(c)}(\vartheta)\cos n\varphi + \sum_{n=1}^{\infty} f_n^{(s)}(\vartheta)\sin n\varphi. \tag{2.35}$$

Evidently, the expansion in (2.34) complies with the conditions in (2.32) and (2.33). Substituting now the above expansions into (2.30) and (2.31), one arrives at the ODE boundary value problem

$$\frac{d}{d\vartheta}\left(\sin\vartheta\,\frac{du_n(\vartheta)}{d\vartheta}\right) - \frac{n^2 u_n(\vartheta)}{\sin\vartheta} = -\widetilde{f}_n(\vartheta), \quad n = 0, 1, 2, \ldots \tag{2.36}$$

$$\frac{du_n(\vartheta_0)}{d\vartheta} = 0 \quad \text{and} \quad u_n(\vartheta_1) = 0 \tag{2.37}$$

in the coefficients $u_n(\vartheta)$ of the series in (2.34). The reader, who delves into every detail, notices perhaps that the functions $u_n(\vartheta)$ and $f_n(\vartheta)$ in (2.36) and (2.37) are not decorated with the superscripts $^{(c)}$ and $^{(s)}$. This is so because for both the cosine and sine coefficients of the Fourier series of (2.34) and (2.35), we arrive at the same problem in $u_n(\vartheta)$, making actually needless the use of these subscripts.

Before we proceed to the solution of the problem in (2.36) and (2.37), it is worth noting that its treatment for the cases of $n = 0$ and of $n \leq 1$ ought to be different. Why so? To answer this question, the reader is advised to recall that the standard method of variation of parameters is in use at this very stage of our procedure (see Sects. 2.1 and 2.2). And a substantial element of this method is a fundamental set of solutions to the homogeneous equation corresponding to (2.36). But fundamental sets of solutions for the cases of $n = 0$ and $n \leq 1$ are obviously different.

Let us focus on the case of $n = 0$ first. The setting in (2.36) and (2.37) reduces, in this case, to

$$\frac{d}{d\vartheta}\left(\sin\vartheta\,\frac{du_0(\vartheta)}{d\vartheta}\right) = -\widetilde{f}_0(\vartheta) \tag{2.38}$$

$$\frac{du_0(\vartheta_0)}{d\vartheta} = 0 \quad \text{and} \quad u_0(\vartheta_1) = 0 \tag{2.39}$$

Components of a fundamental set of solutions to the homogeneous equation corresponding to (2.38) can be chosen [24] as

$$1 \quad \text{and} \quad \ln\left(\tan\frac{\vartheta}{2}\right)$$

and in compliance with the method of variation of parameters, we thus arrive at the general solution to (2.38) in the form

$$u_0(\vartheta) = \int_{\vartheta_0}^{\vartheta} \ln\frac{\Phi_0(\tau)}{\Phi_0(\vartheta)}\widetilde{f_0}(\tau)\,d\tau + D_1\ln\Phi_0(\vartheta) + D_2,$$

where $\Phi_0(x) = \tan(x/2)$ and $\widetilde{f_0}(\tau) = a^2\sin\tau f_0(\tau)$.

The constants of integration D_1 and D_2 are found via the boundary conditions in (2.39) and appear as

$$D_1 = 0 \;\text{ and }\; D_2 = -\int_{\vartheta_0}^{\vartheta_1} \ln\left(\frac{\Phi_0(\tau)}{\Phi_0(\vartheta_1)}\right)\widetilde{f_0}(\tau)\,d\tau.$$

With these at hand, the solution to (2.38) reads as

$$u_0(\vartheta) = \int_{\vartheta_0}^{\vartheta} \ln\frac{\Phi_0(\vartheta_1)}{\Phi_0(\vartheta)}\;\widetilde{f_0}(\tau)\,d\tau + \int_{\vartheta}^{\vartheta_1} \ln\frac{\Phi_0(\vartheta_1)}{\Phi_0(\tau)}\;\widetilde{f_0}(\tau)\,d\tau.$$

Expressing the above in a single-integral form, we have

$$u_0(\vartheta) = \int_{\vartheta_0}^{\vartheta_1} g_0(\vartheta,\tau)\,\widetilde{f_0}(\tau)\,d\tau, \tag{2.40}$$

where

$$g_0(\vartheta,\tau) = \begin{cases} \ln\frac{\Phi_0(\vartheta_1)}{\Phi_0(\tau)}, & \text{if } \vartheta \le \tau \\ \ln\frac{\Phi_0(\vartheta_1)}{\Phi_0(\vartheta)}, & \text{if } \tau \le \vartheta \end{cases} \tag{2.41}$$

represents the Green's function to the homogeneous ODE boundary value problem corresponding to (2.36) and (2.37) in the case of $n = 0$.

The case of $n \ge 1$ turns us back to the problem in (2.36) and (2.37) in its current form, with n representing an integer parameter. Analogously to the case of $n = 0$ just completed, the solution $u_n(\vartheta)$ of the setting in (2.36) and (2.37) can also be obtained with the aid of the method of variation of parameters whose procedure was already explained for this type of problems in our preceding sections. That is why we omit a lengthy but very straightforward routine and present just the ultimate form

$$u_n(\vartheta) = \int_{\vartheta_0}^{\vartheta_1} g_n(\vartheta, \tau)\, \widetilde{f}_n(\tau)\, d\tau \tag{2.42}$$

for $u_n(\vartheta)$ expressed in terms of the Green's function $g_n(\vartheta, \tau)$ of the homogeneous ODE problem corresponding to (2.36) and (2.37). It is customarily found in two pieces. Its expression valid for $\vartheta_0 \leq \tau \leq \vartheta \leq \vartheta_1$ reads as

$$g_n(\vartheta, \tau) = \frac{\Phi_0^n(\vartheta_0)\,\Phi_0^n(\vartheta_1)}{2n\left(\Phi_0^{2n}(\vartheta_0) + \Phi_0^{2n}(\vartheta_1)\right)}$$

$$\times \left(\frac{\Phi_0^n(\tau)}{\Phi_0^n(\vartheta_0)} + \frac{\Phi_0^n(\vartheta_0)}{\Phi_0^n(\tau)}\right)\left(\frac{\Phi_0^n(\vartheta_1)}{\Phi_0^n(\vartheta)} - \frac{\Phi_0^n(\vartheta)}{\Phi_0^n(\vartheta_1)}\right), \tag{2.43}$$

while the expression for $\vartheta_0 \leq \vartheta \leq \tau \leq \vartheta_1$ can be obtained from (2.43) by interchanging ϑ with τ.

In view of the expansions from (2.34) and (2.35), one arrives ultimately at the integral form

$$u(\vartheta, \varphi) = \iint_{\Omega} G(\vartheta, \varphi; \tau, \psi)\, f(\tau, \psi)\, d_{\tau,\psi}\Omega,$$

of the solution to the boundary value problem in (2.30)–(2.33), providing us with an explicit expression for the sought-after Green's function $G(\vartheta, \varphi; \tau, \psi)$ to the corresponding homogeneous problem. It is found in terms of $g_0(\vartheta, \tau)$ and $g_n(\vartheta, \tau)$ just presented in (2.41) and (2.43), and reads as

$$\begin{aligned} G(\vartheta, \varphi; \tau, \psi) &= \frac{1}{2} g_0(\vartheta, \tau) + \sum_{n=1}^{\infty} g_n(\vartheta, \tau) \cos n\varphi \cos n\psi \\ &\quad + \sum_{n=1}^{\infty} g_n(\vartheta, \tau) \sin n\varphi \sin n\psi \\ &= \frac{1}{2} g_0(\vartheta, \tau) + \sum_{n=1}^{\infty} g_n(\vartheta, \tau) \cos n(\varphi - \psi) \end{aligned} \tag{2.44}$$

So, at this stage in our procedure, similarly to the situation in Sect. 2.2, the Green's function that we are looking for is obtained in a series form. Note that both the series in (2.28) and (2.44) are nonuniformly convergent in Ω and both have the same convergence rate. But in contrast to the form in (2.28), which was completely summed up in Sect. 2.2, the form in (2.44) does not allow a complete summation. A significant increase of its convergence rate is nevertheless possible notably enhancing its computational potential.

Convergence of series representing Green's functions for partial differential equations was always an issue for researchers (see, for example, [21, 29]). In [32, 34], we proposed a special technique aimed at the convergence improvement for series of the type in (2.44). We are not going to describe this technique in full detail in relation to the current situation, but its brief sketch would, in our opinion, help the reader.

To figure out a feature which makes the series in (2.44) hard to sum up, let us take a close look at both (2.28) and (2.44), and compare. The form in (2.28) contains four series of the type

$$\sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{P_1}{P_2} \right)^n \cos nx ,$$

where $||P_1|| < ||P_2||$ and $0 \leq x < 2\pi$. This makes the series in (2.28) completely summable with the aid of the standard summation formula of (2.26) which was already implemented in Sect. 2.2. The series in (2.44) is, however, different. Its type can be expressed as

$$\sum_{n=1}^{\infty} \frac{1}{n} \frac{P_1^n P_2^n}{P_2^{2n} + P_1^{2n}} \cos nx , \tag{2.45}$$

where $||P_1|| < ||P_2||$ and $0 \leq x < 2\pi$. It is evident that the formula in (2.26) appears useless in the case of (2.45). But upon transforming it in the way shown below

$$\sum_{n=1}^{\infty} \frac{1}{n} \frac{P_1^n P_2^n}{P_2^{2n} + P_1^{2n}} \cos nx$$

$$= \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{P_1^n P_2^n}{P_2^{2n} + P_1^{2n}} - \frac{P_1^n}{P_2^n} + \frac{P_1^n}{P_2^n} \right) \cos nx$$

$$= \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{P_1^n P_2^n}{P_2^{2n} + P_1^{2n}} - \frac{P_1^n}{P_2^n} \right) \cos nx + \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{P_1}{P_2} \right)^n \cos nx$$

$$= - \sum_{n=1}^{\infty} \frac{1}{n} \frac{P_1^{3n}}{P_2^n \left(P_2^{2n} + P_1^{2n} \right)} \cos nx + \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{P_1}{P_2} \right)^n \cos nx , \tag{2.46}$$

we managed to decompose the series in (2.45) onto two other series, one of which is of the familiar type allowing the complete summation, whereas the convergence rate of the other series in (2.46) is evidently higher then that of (2.45) making it uniformly convergent in Ω.

Applying the technique, a brief sketch of which was just presented, to the series in (2.44), after a tedious but quite straightforward algebra, we obtain the ultimate form

$$G(\vartheta, \varphi; \tau, \psi) = \frac{1}{2\pi} \left\{ \sum_{n=1}^{\infty} R_n(\vartheta, \tau) \cos n(\varphi - \psi) \right.$$

$$+\ln\sqrt{\frac{\Phi_0^2(\vartheta)\Phi_0^4(\vartheta_0)-2\Phi_0^2(\vartheta_0)\Phi_0^2(\vartheta_1)\Phi_0(\vartheta)\Phi_0(\tau)\cos(\varphi-\psi)+\Phi_0^2(\tau)\Phi_0^4(\vartheta_1)}{\Phi_0^4(\vartheta_1)\left[\Phi_0^2(\vartheta)-2\Phi_0(\vartheta)\Phi_0(\tau)\cos(\varphi-\psi)+\Phi_0^2(\tau)\right]}}$$

$$\left.+\ln\sqrt{\frac{\Phi_0^2(\vartheta)\Phi_0^2(\tau)-2\Phi_0^2(\vartheta_1)\Phi_0(\vartheta)\Phi_0(\tau)\cos(\varphi-\psi)+\Phi_0^4(\vartheta_1)}{\Phi_0^2(\vartheta)\Phi_0^2(\tau)-2\Phi_0^2(\vartheta_0)\Phi_0(\vartheta)\Phi_0(\tau)\cos(\varphi-\psi)+\Phi_0^4(\vartheta_0)}}\right\} \tag{2.47}$$

of the Green's function to the boundary value problem in (2.30)–(2.33). The coefficient $R_n(\vartheta,\tau)$ of the series component in (2.47) is defined in two pieces. The piece valid for $\vartheta_0 \le \tau \le \vartheta \le \vartheta_1$ reads as

$$R_n(\vartheta,\tau)=\frac{\Phi_0^{2n}(\vartheta_0)\left[\Phi_0^{2n}(\vartheta)-\Phi_0^{2n}(\vartheta_1)\right]\left[\Phi_0^{2n}(\tau)+\Phi_0^{2n}(\vartheta_0)\right]}{n\,\Phi_0^{2n}(\vartheta_1)\,\Phi_0^{n}(\vartheta)\,\Phi_0^{n}(\tau)\left[\Phi_0^{2n}(\vartheta_0)+\Phi_0^{2n}(\vartheta_1)\right]}$$

while the expression for $R_n(\vartheta,\tau)$ valid for $\vartheta_0 \le \vartheta \le \tau \le \vartheta_1$ can be obtained from the above by interchanging ϑ and τ in its numerator. The denominator is evidently indifferent to this interchange.

Observing the form in (2.47), one might get an impression that it is too heavy loaded for computer implementations, but we can refute such an impression. One of the arguments for this is that the logarithmic components, looking a sort of heavy loaded, are expressed in fact in terms of elementary functions, making them easy to compute. As to the series component, we have already highlighted its uniform convergence in Ω.

Hence, the form in (2.47) is indeed computer friendly, allowing direct implementations in a numerical work.

Illustrating the computability of (2.47), we present, in Fig. 2.5, a potential field induced by a single unit point source in a thin-shell element whose middle surface

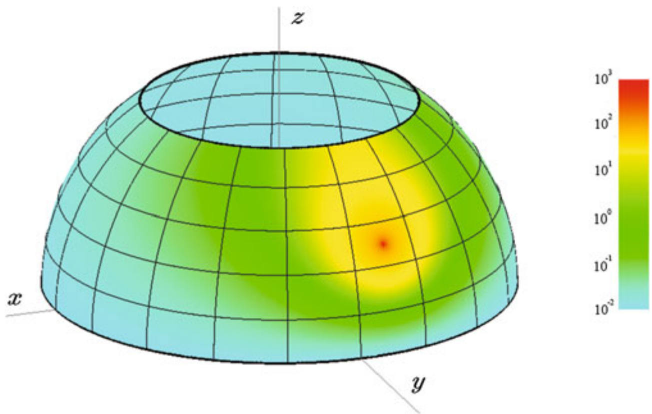


Fig. 2.5 The field induced by a point source in the spherical belt

is a spherical belt. Its shape is defined by $\vartheta_0 = 0.2\pi$ and $\vartheta_1 = 0.5\pi$, and the source is positioned at $(0.35\pi, 0.55\pi)$.

2.4 Quadrilateral-Shaped Region

Another significant peculiarity of our algorithm will be highlighted and clarified in this section. It is associated with the shape of a region hosting a boundary value problem, and boundary conditions imposed on its boundary. To be specific, we take a look at the quadrilateral

$$\Omega = \{\vartheta, \varphi \mid \vartheta_0 \leq \vartheta \leq \vartheta_1;\ 0 \leq \varphi \leq \varphi_1\}$$

on a sphere of radius a, where ranges of the shape defining parameters ϑ_0, ϑ_1, and φ_1 are given as: $0 < \vartheta_0 < \vartheta_1 < \pi$ and $0 < \varphi_1 < 2\pi$.

Consider in Ω the boundary value problem

$$\frac{1}{a^2 \sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial u}{\partial\vartheta}\right) + \frac{1}{a^2\sin^2\vartheta}\frac{\partial^2 u}{\partial\varphi^2} = -f(\vartheta,\varphi), \quad (\vartheta,\varphi) \in \Omega, \tag{2.48}$$

$$u(\vartheta, 0) = 0, \quad \frac{\partial u(\vartheta,\varphi_1)}{\partial\varphi} = 0, \tag{2.49}$$

and

$$u(\vartheta_0, \varphi) = 0, \quad \frac{\partial u(\vartheta_1,\varphi)}{\partial\vartheta} = 0. \tag{2.50}$$

Aiming at a separation of variables for the above problem and taking into account the Dirichlet–Neumann combination of boundary conditions in (2.49), we expand the solution function $u(\vartheta,\varphi)$ and the right-hand side $f(\vartheta,\varphi)$ of the governing equation in the Fourier sine series form

$$u(\vartheta,\varphi) = \sum_{n=1}^{\infty} u_n(\vartheta)\sin\nu\varphi \tag{2.51}$$

and

$$f(\vartheta,\varphi) = \sum_{n=1}^{\infty} f_n(\vartheta)\sin\nu\varphi, \tag{2.52}$$

where the factor ν of the argument of the sine function is specifically defined in terms of the summation index n of the above series as

$$\nu = (2n-1)\pi/(2\varphi_1)$$

This makes $u(\vartheta, \varphi)$ in (2.51) complying with the conditions in (2.49) and yields the boundary value problem

$$\frac{d}{d\vartheta}\left(\sin\vartheta\,\frac{du_n(\vartheta)}{d\vartheta}\right) - \frac{\nu^2 u_n(\vartheta)}{\sin\vartheta} = -\widetilde{f}_n(\vartheta) \quad \text{in } (\vartheta_0, \vartheta_1)$$

$$u_n(\vartheta_0) = 0 \quad \text{and} \quad \frac{du_n(\vartheta_1)}{d\vartheta} = 0$$

in the coefficients $u_n(\vartheta)$ of (2.51).

The above problem has already been encountered in Sect. 2.3, where its solution was found in the form

$$u_n(\vartheta) = \int_{\vartheta_0}^{\vartheta_1} g_n(\vartheta, \tau)\,\widetilde{f}_n(\tau)\,d\tau, \tag{2.53}$$

where $\widetilde{f}_n(\tau) = a^2 \sin\tau\, f(\tau)$ and the expression for $g_n(\vartheta, \tau)$, valid for $\vartheta_0 \le \tau \le \vartheta \le \vartheta_1$, reads as

$$g_n(\vartheta, \tau) = \frac{\Phi^n(\vartheta_0)\,\Phi^n(\vartheta_1)}{2n\left(\Phi^{2n}(\vartheta_0) + \Phi^{2n}(\vartheta_1)\right)}$$

$$\times\left(\frac{\Phi^n(\tau)}{\Phi^n(\vartheta_0)} - \frac{\Phi^n(\vartheta_0)}{\Phi^n(\tau)}\right)\left(\frac{\Phi^n(\vartheta_1)}{\Phi^n(\vartheta)} + \frac{\Phi^n(\vartheta)}{\Phi^n(\vartheta_1)}\right), \tag{2.54}$$

where $\Phi(x) = \tan^{\pi/\varphi_1}(x/2)$.

Note once again that the expression for $g_n(\vartheta, \tau)$ valid for $\vartheta_0 \le \vartheta \le \tau \le \vartheta_1$ can be obtained from (2.54) by interchanging ϑ with τ.

This yields the solution to the boundary value problem in (2.48)–(2.50) as expressed in the integral form

$$u(\vartheta, \varphi) = \iint_{\Omega} G(\vartheta, \varphi; \tau, \psi)\, f(\tau, \psi)\, d_{\tau,\psi}\Omega$$

which reveals the nonuniformly converging in Ω series representation

$$G(\vartheta, \varphi; \tau, \psi) = \frac{1}{\varphi_1}\sum_{n=1}^{\infty} g_n(\vartheta, \tau)\left[\cos\nu(\varphi - \psi) - \cos\nu(\varphi + \psi)\right] \tag{2.55}$$

for the sought-after Green's function of the homogeneous boundary value problem corresponding to (2.48)–(2.50).

Thus, analogously to the situation that took place in Sect. 2.3 with the series of (2.44), we are once again at the familiar point in the development where the Green's function of our interest is expressed in the form of a nonuniformly convergent series. A certain effort is required therefore towards an improvement of its convergence.

In doing so, we will focus on a partial summation of the series in (2.55). It looks like it might potentially be accomplished in the way proposed earlier in Sect. 2.3. Indeed, the transformation sketched over there (see (2.46)) appears also workable for the series in (2.55) splitting it onto two other series. One of them converges uniformly and can therefore be considered as computer friendly. But, as to the second of those series, the summation formula from (2.26) is not unfortunately immediately applicable.

With all the foregoing comments in mind, we recall another standard summation formula [1, 19]

$$\sum_{n=1}^{\infty} \frac{p^{2n-1}}{2n-1} \cos(2n-1)x = \frac{1}{4} \ln \frac{1+2p\cos x + p^2}{1-2p\cos x + p^2}, \tag{2.56}$$

which presumes that its parameters p and x satisfy the conditions: $p^2 < 1$ and $0 \le x < 2\pi$. It perfectly fits the situation with the second of the two series resulting from the just mentioned splitting of the expansion in (2.55).

Omitting a tedious algebra which resembles, in most details, the work done on the series from (2.44) in Sect. 2.3, we reveal just an ultimate representation for the Green's function to the problem in (2.48)–(2.50). It was found in the form

$$G(\vartheta, \varphi; \tau, \psi) = \frac{1}{2\pi} \left\{ \sum_{n=1}^{\infty} R_n(\vartheta, \tau) \sin \nu\varphi \sin \nu\psi \right.$$

$$+ \ln\left(H\left(\frac{\sqrt{\Phi(\vartheta)\Phi(\tau)}}{\Phi(\vartheta_1)}, \frac{\alpha}{2}, \frac{\beta}{2} \right) H\left(\sqrt{\frac{\Phi(\vartheta)}{\Phi(\tau)}}, \frac{\alpha}{2}, \frac{\beta}{2} \right) \right)$$

$$\left. - \ln\left(H\left(\frac{\Phi(\vartheta_0)}{\sqrt{\Phi(\vartheta)\Phi(\tau)}}, \frac{\alpha}{2}, \frac{\beta}{2} \right) H\left(\frac{\Phi(\vartheta_0)}{\Phi(\vartheta_1)} \sqrt{\frac{\Phi(\vartheta)}{\Phi(\tau)}}, \frac{\alpha}{2}, \frac{\beta}{2} \right) \right) \right\}, \tag{2.57}$$

where the three-variable function $H(x, \xi, \eta)$, in terms of which the arguments of the logarithmic functions are expressed, is defined as

$$H(x, \xi, \eta) = \frac{(1+2x\cos\xi + x^2)(1-2x\cos\eta + x^2)}{(1-2x\cos\xi + x^2)(1+2x\cos\eta + x^2)}.$$

The parameters α and β in the function $H(x, \xi, \eta)$ of (2.57) are expressed, in terms of the variables φ and ψ, as

$$\alpha = \frac{\pi}{\varphi_1}(\varphi - \psi) \quad \text{and} \quad \beta = \frac{\pi}{\varphi_1}(\varphi + \psi).$$

The coefficient $R_n(\vartheta, \tau)$ of the series component in (2.57) is found, for $\vartheta \le \tau$, as

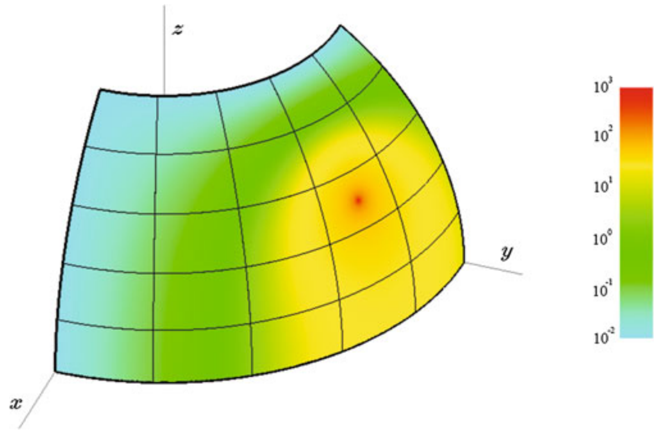


Fig. 2.6 A profile of the Green's function shown in (2.57)

$$R_n(\vartheta,\tau)=\frac{\Phi^n(\vartheta_0)\left[\Phi^n(\vartheta)+\Phi^n(\vartheta_1)\right]\left[\Phi^n(\tau)-\Phi^n(\vartheta_0)\right]}{\nu\Phi^n(\vartheta_1)\sqrt{\Phi^n(\vartheta)\,\Phi^n(\tau)}\left[\Phi^n(\vartheta_0)+\Phi^n(\vartheta_1)\right]},$$

while its expression valid for $\tau \le \vartheta$ can customarily be obtained by the interchange of the variables ϑ and τ.

In Fig. 2.6, the reader finds a profile of the just obtained Green's function, where the parameters defining the region Ω are given as: $\vartheta_0 = 0.2\pi$, $\vartheta_1 = 0.5\pi$, and $\varphi_1 = 0.5\pi$, with a unit source released at the point $(0.35\pi, 0.35\pi)$.

2.5 Robin Problem for Spherical Cap

Note that the boundary condition operators are of either Dirichlet or Neumann kind in all the problems encountered so far in this chapter. But, as we have asserted earlier in Sect. 2.1, the technique, we propose for the construction of Green's functions, is potentially applicable to problems with Robin condition imposed as well. And it is about right time to confirm this assertion. For illustration, we consider a problem stated in the region

$$\Omega=\{\vartheta,\varphi\,|\,0<\vartheta\le\vartheta_1;\;0\le\varphi<2\pi\}$$

on a sphere of radius a, where $0 < \vartheta_1 < \pi$. The region is simply connected and bounded with the parallel $\vartheta = \vartheta_1$ in spherical coordinates. This shape will be referred to as the *spherical cap*.

The shape of Ω combines two specific features each of which has been touched upon earlier in this chapter. Namely, analogously to the spherical belt considered in Sect. 2.3, the spherical cap is closed in the longitudinal direction, and as well as in the case of a spherical triangle (see Sect. 2.2), the north pole $\vartheta = 0$ represents a part of the "boundary" for Ω.

Our objective is the Green's function to the homogeneous boundary value problem corresponding to the well-posed setting

$$\frac{1}{a^2 \sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial u}{\partial\vartheta}\right) + \frac{1}{a^2\sin^2\vartheta}\frac{\partial^2 u}{\partial\varphi^2} = -f(\vartheta,\varphi), \quad (\vartheta,\varphi)\in\Omega \tag{2.58}$$

$$\lim_{\vartheta\to 0}|u(\vartheta,\varphi)| < \infty \quad \text{and} \quad u(\vartheta_1,\varphi) + \lambda\frac{\partial u(\vartheta_1,\varphi)}{\partial\vartheta} = 0 \tag{2.59}$$

and

$$u(\vartheta,0) = u(\vartheta,2\pi) \quad \text{and} \quad \frac{\partial u(\vartheta,0)}{\partial\varphi} = \frac{\partial u(\vartheta,2\pi)}{\partial\varphi}, \tag{2.60}$$

where λ in the Robin condition represents a positive parameter.

In view of the 2π-periodicity of the above problem, the complete Fourier series expansions

$$u(\vartheta,\varphi) = \frac{1}{2}u_0(\vartheta) + \sum_{n=1}^{\infty} u_n^{(c)}(\vartheta)\cos n\varphi + \sum_{n=1}^{\infty} u_n^{(s)}(\vartheta)\sin n\varphi \tag{2.61}$$

and

$$f(\vartheta,\varphi) = \frac{1}{2}f_0(\vartheta) + \sum_{n=1}^{\infty} f_n^{(c)}(\vartheta)\cos n\varphi + \sum_{n=1}^{\infty} f_n^{(s)}(\vartheta)\sin n\varphi \tag{2.62}$$

are required, in this case, for the solution function $u(\vartheta,\varphi)$ and the right-hand side $f(\vartheta,\varphi)$ of (2.58).

This yields the ODE boundary value problem

$$\frac{d}{d\vartheta}\left(\sin\vartheta\,\frac{du_n(\vartheta)}{d\vartheta}\right) - \frac{n^2 u_n(\vartheta)}{\sin\vartheta} = -\,\widetilde{f}_n(\vartheta), \quad n = 0, 1, 2, \ldots \tag{2.63}$$

$$\lim_{\vartheta\to 0}|u_n(\vartheta)| < \infty \quad \text{and} \quad u_n(\vartheta_1) + \lambda\frac{du_n(\vartheta_1)}{d\vartheta} = 0 \tag{2.64}$$

in the coefficients $u_n(\vartheta)$ of the series in (2.61).

In compliance with our approach, whose details relevant to the current situation can be found in Sect. 2.3, the Green's function $G(\vartheta,\varphi;\tau,\psi)$ of the homogeneous problem corresponding to (2.58)–(2.60) appears in the series form

$$G(\vartheta,\varphi;\tau,\psi) = \frac{1}{2}g_0(\vartheta,\tau) + \sum_{n=1}^{\infty} g_n(\vartheta,\tau)\cos n(\varphi-\psi), \tag{2.65}$$

where $g_0(\vartheta, \tau)$ and $g_n(\vartheta, \tau)$ represent Green's functions to the homogeneous problem settings corresponding to (2.63) and (2.64), for the cases $n = 0$ and $n \geq 1$, respectively.

Constructing the Green's function $g_n(\vartheta, \tau)$ to the actual setting in (2.63) and (2.64) by the method variation of parameters, we customarily obtain the general solution to (2.63) as

$$u_n(\vartheta) = \frac{1}{2n} \int_0^{\vartheta} \frac{\Phi_0^{2n}(\tau) - \Phi_0^{2n}(\vartheta)}{\Phi_0^n(\tau)\Phi_0^n(\vartheta)} \widetilde{f}_n(\tau)\, d\tau + D_1 \Phi_0^n(\vartheta) + D_2 \Phi_0^{-n}(\vartheta), \quad (2.66)$$

where $\Phi_0(x) = \tan(x/2)$ and $\widetilde{f}_n(\tau) = a^2 \sin\tau\, f_n(\tau)$.

To determine the constants of integration, we use the boundary conditions of (2.64). It is evident that the boundness condition at $\vartheta = 0$ requires $D_2 = 0$, while the Robin condition in (2.64) allows to obtain

$$D_1 = \frac{1}{2n\Phi_0^n(\vartheta_1)} \int_0^{\vartheta_1} \left[\left(\frac{\sin\vartheta_1 - n\lambda}{\sin\vartheta_1 + n\lambda} \right) \frac{\Phi_0^n(\tau)}{\Phi_0^n(\vartheta_1)} - \frac{\Phi_0^n(\vartheta_1)}{\Phi_0^n(\tau)} \right] \widetilde{f}_n(\tau)\, d\tau$$

Upon substituting the just presented values of the constants of integration D_1 and D_2 into (2.66), the latter reduces to

$$u_n(\vartheta) = \frac{1}{2n} \int_0^{\vartheta} \left[\left(\frac{\sin\vartheta_1 - n\lambda}{\sin\vartheta_1 + n\lambda} \right) \frac{\Phi_0^n(\vartheta)\Phi_0^n(\tau)}{\Phi_0^{2n}(\vartheta_1)} - \frac{\Phi_0^n(\tau)}{\Phi_0^n(\vartheta)} \right] \widetilde{f}_n(\tau)\, d\tau$$

$$+ \frac{1}{2n} \int_{\vartheta}^{\vartheta_1} \left[\left(\frac{\sin\vartheta_1 - n\lambda}{\sin\vartheta_1 + n\lambda} \right) \frac{\Phi_0^n(\vartheta)\Phi_0^n(\tau)}{\Phi_0^{2n}(\vartheta_1)} - \frac{\Phi_0^n(\vartheta)}{\Phi_0^n(\tau)} \right] \widetilde{f}_n(\tau)\, d\tau \quad (2.67)$$

This reveals the Green's function $g_n(\vartheta, \tau)$ to the homogeneous ODE problem corresponding to (2.63) and (2.64). Its representation valid for $\vartheta \leq \tau$ appears as

$$g_n(\vartheta, \tau) = \frac{1}{2n} \left[\left(\frac{\sin\vartheta_1 - n\lambda}{\sin\vartheta_1 + n\lambda} \right) \frac{\Phi_0^n(\vartheta)\Phi_0^n(\tau)}{\Phi_0^{2n}(\vartheta_1)} - \frac{\Phi_0^n(\vartheta)}{\Phi_0^n(\tau)} \right], \quad (2.68)$$

whereas the expression of $g_n(\vartheta, \tau)$ valid for $\tau \leq \vartheta$ can be obtained from the above by interchanging ϑ with τ.

We turn now to the component $g_0(\vartheta, \tau)$ of the series expansion in (2.65). It represents the Green's function to the homogeneous boundary value problem corresponding to

$$\frac{d}{d\vartheta}\left(\sin\vartheta \frac{du_0(\vartheta)}{d\vartheta} \right) = -\widetilde{f}_0(\vartheta) \quad (2.69)$$

$$\lim_{\vartheta \to 0} |u_0(\vartheta)| < \infty \quad \text{and} \quad u_0(\vartheta_1) + \lambda \frac{du_0(\vartheta_1)}{d\vartheta} = 0 \quad (2.70)$$

The above problem is what the setting in (2.63) and (2.64) reduces to in the case of $n = 0$. Via the method of variation of parameters, the general solution of the equation in (2.69) appears as

$$u_0(\vartheta) = \int_0^{\vartheta} \ln \frac{\Phi_0(\tau)}{\Phi_0(\vartheta)}\, \widetilde{f_0}(\tau)\, d\tau + D_{1,0} \ln \Phi_0(\vartheta) + D_{2,0}$$

The boundness condition in (2.70) implies $D_{1,0} = 0$, while the Robin condition at $\vartheta = \vartheta_1$ leads to

$$D_{2,0} = \int_0^{\vartheta_1} \left(\ln \frac{\Phi_0(\vartheta_1)}{\Phi_0(\tau)} + \frac{\lambda}{\sin \vartheta_1} \right) \widetilde{f_0}(\tau)\, d\tau$$

resulting in

$$u_0(\vartheta) = \int_0^{\vartheta} \left(\ln \frac{\Phi_0(\vartheta_1)}{\Phi_0(\vartheta)} + \frac{\lambda}{\sin \vartheta_1} \right) \widetilde{f_0}(\tau)\, d\tau + \int_0^{\vartheta_1} \left(\ln \frac{\Phi_0(\vartheta_1)}{\Phi_0(\tau)} + \frac{\lambda}{\sin \vartheta_1} \right) \widetilde{f_0}(\tau)\, d\tau$$

This reveals the following two-piece-defined expression

$$g_0(\vartheta, \tau) = \begin{cases} \ln \frac{\Phi_0(\vartheta_1)}{\Phi_0(\vartheta)} + \frac{\lambda}{\sin \vartheta_1}, \text{ if } \tau \le \vartheta \\ \ln \frac{\Phi_0(\vartheta_1)}{\Phi_0(\tau)} + \frac{\lambda}{\sin \vartheta_1}, \text{ if } \vartheta \le \tau \end{cases} \tag{2.71}$$

for the Green's function to the homogeneous ODE problem corresponding to (2.69) and (2.70) in the case of $n = 0$.

So, with both $g_0(\vartheta, \tau)$ and $g_n(\vartheta, \tau)$ at hand, the form in (2.65) gives us an explicit nonuniformly convergent series representation for the Green's function $G(\vartheta, \varphi; \tau, \psi)$ to the homogeneous problem corresponding to (2.58)–(2.60). To enhance the computational potential of the series in (2.65), we transform the expression from (2.68) for $g_n(\vartheta, \tau)$, valid for $\vartheta \le \tau$, to the equivalent one

$$g_n(\vartheta, \tau) = \frac{1}{2n} \left[\left(1 - \frac{2n\lambda}{\sin \vartheta_1 + n\lambda} \right) \frac{\Phi_0^n(\vartheta)\Phi_0^n(\tau)}{\Phi_0^{2n}(\vartheta_1)} - \frac{\Phi_0^n(\vartheta)}{\Phi_0^n(\tau)} \right]$$

This yields

$$G(\vartheta, \varphi; \tau, \psi) = \frac{1}{2} \left(\ln \frac{\Phi_0(\vartheta_1)}{\Phi_0(\tau)} + \frac{\lambda}{\sin \vartheta_1} \right)$$

$$+ \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{\Phi_0^n(\vartheta)\Phi_0^n(\tau)}{\Phi_0^{2n}(\vartheta_1)} - \frac{\Phi_0^n(\vartheta)}{\Phi_0^n(\tau)} \right) \cos n(\varphi - \psi)$$

$$-\frac{\lambda}{2\pi}\sum_{n=1}^{\infty}\frac{\Phi_0^n(\vartheta)\Phi_0^n(\tau)}{(\sin\vartheta_1+n\lambda)\,\Phi_0^{2n}(\vartheta_1)}\cos n\,(\varphi-\psi)\,,$$

where the first of the two series is completely summable, while the second is uniformly convergent for any mutual location of the observation and the source points inside of Ω. The partial summation ultimately yields

$$G\,(\vartheta,\varphi;\tau,\psi)=\frac{\lambda}{\sin\vartheta_1}+\frac{\lambda}{2\pi}\sum_{n=1}^{\infty}\frac{\Phi_0^n(\vartheta)\Phi_0^n(\tau)}{(\sin\vartheta_1+n\lambda)\,\Phi_0^{2n}(\vartheta_1)}\cos n\,(\varphi-\psi)$$

$$+\frac{1}{2\pi}\ln\sqrt{\frac{\Phi_0^4(\vartheta_1)-2\Phi_0^2(\vartheta_1)\Phi_0(\vartheta)\Phi_0(\tau)\cos(\varphi-\psi)+\Phi_0^2(\vartheta)\Phi_0^2(\tau)}{\Phi_0^2(\vartheta_1)\left[\Phi_0^2(\tau)-2\Phi_0(\vartheta)\Phi_0(\tau)\cos(\varphi-\psi)+\Phi_0^2(\vartheta)\right]}}\qquad(2.72)$$

which is indeed ready for an immediate computer implementation.

Interestingly enough, the above representation reduces to the closed form

$$\frac{1}{2\pi}\ln\sqrt{\frac{\Phi_0^4(\vartheta_1)-2\Phi_0^2(\vartheta_1)\Phi_0(\vartheta)\Phi_0(\tau)\cos(\varphi-\psi)+\Phi_0^2(\vartheta)\Phi_0^2(\tau)}{\Phi_0^2(\vartheta_1)\left[\Phi_0^2(\tau)-2\Phi_0(\vartheta)\Phi_0(\tau)\cos(\varphi-\psi)+\Phi_0^2(\vartheta)\right]}}$$

of the Green's function for the Dirichlet problem

$$\lim_{\vartheta\to 0}|u\,(\vartheta,\varphi)\,|<\infty\quad\text{and}\quad u\,(\vartheta_1,\varphi)=0$$

$$u\,(\vartheta,0)=u\,(\vartheta,2\pi)\quad\text{and}\quad\frac{\partial u\,(\vartheta,0)}{\partial\varphi}=\frac{\partial u\,(\vartheta,2\pi)}{\partial\varphi}$$

which the statement in (2.58)–(2.60) reduces to, if $\lambda=0$.

On the other hand, the form in (2.72) is undefined, if λ is taken to infinity. This reflects the evident fact that the Neumann problem

$$\lim_{\vartheta\to 0}|u\,(\vartheta,\varphi)\,|<\infty\quad\text{and}\quad\frac{\partial u\,(\vartheta_1,\varphi)}{\partial\vartheta}=0$$

$$u\,(\vartheta,0)=u\,(\vartheta,2\pi)\quad\text{and}\quad\frac{\partial u\,(\vartheta,0)}{\partial\varphi}=\frac{\partial u\,(\vartheta,2\pi)}{\partial\varphi}$$

which the statement in (2.58)–(2.60) reduces to in this case, is ill-posed, and its classical Green's function does not therefore exist.

2.6 Spherical Sector

The term *spherical sector* will be used in this section in reference to the simply connected region

$$\Omega = \{\vartheta, \varphi \mid 0 < \vartheta \leq \pi;\ 0 \leq \varphi < \varphi_1\}$$

on a spherical surface of radius a. It is bounded with two meridians $\varphi = 0$ and $\varphi = \varphi_1$. The term spherical sector (which the reader might be skeptical about) is chosen on account of the way the shape of Ω is formed, analogously to the way the circular sector is obtained from a circle on a plane.

Note that both poles of the spherical surface are parts of the region's "boundary". This notably affects our algorithm for the construction of the Green's function of a problem stated in Ω.

Since the poles represent points of singularity for the governing equation

$$\frac{1}{a^2 \sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial u}{\partial\vartheta}\right) + \frac{1}{a^2\sin^2\vartheta}\frac{\partial^2 u}{\partial\varphi^2} = -f\,(\vartheta,\varphi)\,, \quad (\vartheta,\varphi)\in\Omega \tag{2.73}$$

the solution function $u = u(\vartheta, \varphi)$ ought to be subject to the boundness conditions

$$\lim_{\vartheta\to 0}|u\,(\vartheta,\varphi)\,| < \infty \quad \text{and} \quad \lim_{\vartheta\to\pi}|u\,(\vartheta,\varphi)\,| < \infty \tag{2.74}$$

To complete statement of a well-posed problem in Ω, the Dirichlet conditions

$$u(\vartheta, 0) = 0 \ \text{ and } \ u(\vartheta, \varphi_1) = 0 \tag{2.75}$$

are imposed on the meridians $\varphi = 0$ and $\varphi = \varphi_1$.

The Fourier sine series expansions

$$u\,(\vartheta,\varphi) = \sum_{n=1}^{\infty} u_n\,(\vartheta)\sin\frac{n\pi\varphi}{\varphi_1} \tag{2.76}$$

and

$$f\,(\vartheta,\varphi) = \sum_{n=1}^{\infty} f_n\,(\vartheta)\sin\frac{n\pi\varphi}{\varphi_1}$$

yield the following ODE boundary value problem

$$\frac{d}{d\vartheta}\left(\sin\vartheta\frac{du_n(\vartheta)}{d\vartheta}\right) - \frac{\nu^2 u_n(\vartheta)}{\sin\vartheta} = -\,\widetilde{f}_n\,(\vartheta) \quad \text{in } (0,\pi) \tag{2.77}$$

$$\lim_{\vartheta\to 0}|u_n\,(\vartheta)\,| < \infty \quad \text{and} \quad \lim_{\vartheta\to\pi}|u_n\,(\vartheta)\,| < \infty \tag{2.78}$$

in the coefficients $u_n(\vartheta)$ of the series in (2.76), where $\nu = n\pi/\varphi_1$.

Within the scope of our approach, the Green's function $G(\vartheta, \varphi; \tau, \psi)$ to the homogeneous problem corresponding to (2.73)–(2.75) appears as

$$G(\vartheta, \varphi; \tau, \psi) = \sum_{n=1}^{\infty} g_n(\vartheta, \tau) \sin \nu\varphi \sin \nu\psi, \tag{2.79}$$

where $g_n(\vartheta, \tau)$ represents the Green's functions to the homogeneous problem settings corresponding to (2.77) and (2.78).

Our customary routine, based on the method of variation of parameters, provides us with the general solution to (2.77) in the form

$$u_n(\vartheta) = \frac{1}{2\nu} \int_0^{\vartheta} \frac{\Phi^{2n}(\tau) - \Phi^{2n}(\vartheta)}{\Phi^n(\tau)\Phi^n(\vartheta)} \widetilde{f}_n(\tau)\, d\tau + D_1\Phi^n(\vartheta) + D_2\Phi^{-n}(\vartheta),$$

where $\Phi(x) = \tan^{\pi/\varphi_1}(x/2)$ and $\widetilde{f}_n(\tau) = a^2 \sin\tau f_n(\tau)$.

As to the boundness conditions in (2.78), the current case is slightly different of all considered so far in this volume. The first relation in (2.78), of course, requires that $D_2 = 0$, while the condition at $\vartheta = \pi$ reads

$$\lim_{\vartheta\to\pi} \left| \frac{1}{2\nu} \int_0^{\vartheta} \left(\frac{\Phi^n(\tau)}{\Phi^n(\vartheta)} - \frac{\Phi^n(\vartheta)}{\Phi^n(\tau)} \right) \widetilde{f}_n(\tau)\, d\tau + D_1\Phi^n(\vartheta) \right| < \infty$$

which implies

$$\lim_{\vartheta\to\pi} \left| -\frac{1}{2\nu} \int_0^{\vartheta} \frac{\Phi^n(\vartheta)}{\Phi^n(\tau)} \widetilde{f}_n(\tau)\, d\tau + D_1\Phi^n(\vartheta) \right| < \infty$$

And rewriting the above as

$$\lim_{\vartheta\to\pi} \left| \left(D_1 - \frac{1}{2\nu} \int_0^{\vartheta} \frac{1}{\Phi^n(\tau)} \widetilde{f}_n(\tau)\, d\tau \right) \Phi^n(\vartheta) \right| < \infty$$

we have

$$D_1 = \frac{1}{2\nu} \int_0^{\pi} \frac{1}{\Phi^n(\tau)} \widetilde{f}_n(\tau)\, d\tau$$

This ultimately transforms the solution of the problem in (2.77) and (2.78) to the integral form

$$u_n(\vartheta) = \frac{1}{2\nu} \int_0^{\vartheta} \frac{\Phi^n(\tau)}{\Phi^n(\vartheta)} \widetilde{f}_n(\tau)\, d\tau + \frac{1}{2\nu} \int_{\vartheta}^{\pi} \frac{\Phi^n(\vartheta)}{\Phi^n(\tau)} \widetilde{f}_n(\tau)\, d\tau$$

revealing the Green's function $g_n(\vartheta, \tau)$, to the corresponding homogeneous ODE problem, as defined in two pieces

$$g_n(\vartheta, \tau) = \frac{1}{2\nu} \begin{cases} \frac{\Phi^n(\vartheta)}{\Phi^n(\tau)}, \text{ if } 0 \le \vartheta \le \tau \\ \\ \frac{\Phi^n(\tau)}{\Phi^n(\vartheta)}, \text{ if } \tau \le \vartheta \le \pi \end{cases}$$

So, substituting the above into (2.79), one arrives at the series representation of the Green's function $G(\vartheta, \varphi; \tau, \psi)$ to the homogeneous boundary value problem corresponding to (2.73)–(2.75), which appears summable. The summation can readily be accomplished with the aid of the standard summation formula (2.26) which was presented earlier in Sect. 2.2 and has repeatedly been used in this chapter. The summation yields

$$G(\vartheta, \varphi; \tau, \psi) = \frac{1}{2\pi} \ln \sqrt{\frac{\Phi^2(\vartheta) - 2\Phi(\vartheta)\Phi(\tau)\cos\beta + \Phi^2(\tau)}{\Phi^2(\vartheta) - 2\Phi(\vartheta)\Phi(\tau)\cos\alpha + \Phi^2(\tau)}}, \tag{2.80}$$

where

$$\alpha = \frac{\pi}{\varphi_1}(\varphi - \psi) \quad \text{and} \quad \beta = \frac{\pi}{\varphi_1}(\varphi + \psi).$$

It is interesting to note that the development of this section appears workable for another boundary value problem stated in Ω. Namely, it works smoothly if in (2.75) the condition on the meridian $\varphi = 0$ stays unchanged, but instead of the Dirichlet condition on the meridian $\varphi = \varphi_1$, the Neumann condition

$$u(\vartheta, 0) = 0 \quad \text{and} \quad \frac{\partial u(\vartheta, \varphi_1)}{\partial \varphi} = 0 \tag{2.81}$$

is imposed.

Omitting details, we present just the ultimate closed form

$$G(\vartheta, \varphi; \tau, \psi) = \frac{1}{2\pi} \ln \sqrt{\frac{\Phi(\vartheta) + 2\sqrt{\Phi(\vartheta)\Phi(\tau)}\cos(\alpha/2) + \Phi(\tau)}{\Phi(\vartheta) - 2\sqrt{\Phi(\vartheta)\Phi(\tau)}\cos(\alpha/2) + \Phi(\tau)}}$$
$$- \frac{1}{2\pi} \ln \sqrt{\frac{\Phi(\vartheta) + 2\sqrt{\Phi(\vartheta)\Phi(\tau)}\cos(\beta/2) + \Phi(\tau)}{\Phi(\vartheta) - 2\sqrt{\Phi(\vartheta)\Phi(\tau)}\cos(\beta/2) + \Phi(\tau)}} \tag{2.82}$$

of the Green's function to the homogeneous boundary value problem corresponding to (2.73), (2.74) and (2.81).

The standard summation formula of (2.56), used in Sect. 2.4, is required in this case to transform a series representation of $G(\vartheta, \varphi; \tau, \psi)$ to the closed form of (2.82).

To better comprehend peculiarities of our technique developed in this chapter, the reader is recommended to solve a few of the proposed below problems whose answers are available in the Appendix.

2.7 Chapter Exercises

For the homogeneous equation corresponding to (2.1), *construct the Green's functions to the boundary value problem stated in the indicated region*:

Exercise 2.1

$$u(\vartheta, 0) = 0 \quad \text{and} \quad u(\vartheta, \varphi_1) = 0$$

$$\lim_{\vartheta \to 0} |u(\vartheta, \varphi)| < \infty \quad \text{and} \quad u(\vartheta_1, \varphi) = 0$$

in the spherical triangle

$$\Omega = \{0 \le \vartheta < \vartheta_1,\ 0 \le \varphi \le \varphi_1\}.$$

Exercise 2.2

$$u(\vartheta, 0) = 0 \quad \text{and} \quad \frac{\partial u(\vartheta, \varphi_1)}{\partial \varphi} = 0$$

$$\lim_{\vartheta \to 0} |u(\vartheta, \varphi)| < \infty \quad \text{and} \quad \frac{\partial u(\vartheta_1, \varphi)}{\partial \vartheta} = 0$$

in the spherical triangle

$$\Omega = \{0 \le \vartheta < \vartheta_1,\ 0 \le \varphi \le \varphi_1\}.$$

Exercise 2.3

$$u(\vartheta, 0) = 0 \quad \text{and} \quad u(\vartheta, \varphi_1) = 0$$

$$\lim_{\vartheta \to 0} |u(\vartheta, \varphi)| < \infty \quad \text{and} \quad \frac{\partial u(\vartheta_1, \varphi)}{\partial \vartheta} + \beta u(\vartheta_1, \varphi) = 0$$

in the spherical triangle

$$\Omega = \{0 \le \vartheta < \vartheta_1,\ 0 \le \varphi \le \varphi_1\}.$$

Exercise 2.4

$$u(\vartheta, 0) = u(\vartheta, 2\pi) \quad \text{and} \quad \frac{\partial u(\vartheta, 0)}{\partial \varphi} = \frac{\partial u(\vartheta, 2\pi)}{\partial \varphi}$$

$$u\,(\vartheta_0, \varphi) = 0 \quad \text{and} \quad u\,(\vartheta_1, \varphi) = 0$$

in the spherical belt

$$\Omega = \{\vartheta_0 \le \vartheta < \vartheta_1,\ 0 \le \varphi < 2\pi\}.$$

Exercise 2.5

$$u\,(\vartheta, 0) = u\,(\vartheta, 2\pi) \quad \text{and} \quad \frac{\partial u\,(\vartheta, 0)}{\partial \varphi} = \frac{\partial u\,(\vartheta, 2\pi)}{\partial \varphi}$$

$$u\,(\vartheta_0, 0) = 0 \quad \text{and} \quad u\,(\vartheta_1, \varphi) + \beta \frac{\partial u\,(\vartheta_1, \varphi)}{\partial \vartheta} = 0$$

in the spherical belt

$$\Omega = \{\vartheta_0 \le \vartheta < \vartheta_1,\ 0 \le \varphi < 2\pi\}\,.$$

Exercise 2.6

$$u\,(\vartheta, 0) = 0 \quad \text{and} \quad u\,(\vartheta, \varphi_1) = 0$$

$$u\,(\vartheta_0, \varphi) = 0 \quad \text{and} \quad \frac{\partial u\,(\vartheta_1, \varphi)}{\partial \vartheta} = 0$$

in the spherical quadrilateral

$$\Omega = \{\vartheta_0 \le \vartheta < \vartheta_1,\ 0 \le \varphi \le \varphi_1\}\,.$$

Exercise 2.7

$$u\,(\vartheta, 0) = 0 \quad \text{and} \quad u\,(\vartheta, \varphi_1) = 0$$

$$u\,(\vartheta_0, \varphi) + \beta \frac{\partial u\,(\vartheta_0, \varphi)}{\partial \vartheta} = 0 \quad \text{and} \quad \frac{\partial u\,(\vartheta_1, \varphi)}{\partial \vartheta} = 0$$

in the spherical quadrilateral

$$\Omega = \{\vartheta_0 \le \vartheta < \vartheta_1,\ 0 \le \varphi \le \varphi_1\}\,.$$

Exercise 2.8

$$u\,(\vartheta, 0) = 0 \quad \text{and} \quad u\,(\vartheta, \varphi_1) = 0$$

$$u\,(\vartheta_0, \varphi) + \beta \frac{\partial u\,(\vartheta_0, \varphi)}{\partial \vartheta} = 0 \quad \text{and} \quad u\,(\vartheta_1, \varphi) = 0$$

in the spherical quadrilateral

$$\Omega = \{\vartheta_0 \le \vartheta < \vartheta_1,\ 0 \le \varphi \le \varphi_1\}\,.$$

Exercise 2.9

$$u(\vartheta, 0) = 0 \quad \text{and} \quad \frac{\partial u(\vartheta, \varphi_1)}{\partial \varphi} = 0$$

$$u(\vartheta_0, \varphi) + \beta \frac{\partial u(\vartheta_0, \varphi)}{\partial \vartheta} = 0 \quad \text{and} \quad u(\vartheta_1, \varphi) = 0$$

in the spherical quadrilateral

$$\Omega = \{\vartheta_0 \le \vartheta < \vartheta_1,\ 0 \le \varphi \le \varphi_1\}.$$

Exercise 2.10

$$u(\vartheta, 0) = 0 \quad \text{and} \quad \frac{u(\vartheta, \varphi_1)}{\partial \varphi} = 0$$

$$u(\vartheta, 0) = u(\vartheta, 2\pi) \quad \text{and} \quad \frac{\partial u(\vartheta, 0)}{\partial \varphi} = \frac{\partial u(\vartheta, 2\pi)}{\partial \varphi}$$

in the spherical sector

$$\Omega = \{0 < \vartheta < \pi,\ 0 \le \varphi \le \varphi_1\}.$$

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