Chapter 2
Vector Spaces

In courses on analytic geometry, basic operations with vectors in three-dimensional Euclidean space are studied. If some basis in the space is fixed, then a one-to-one correspondence between geometric vectors and ordered triples of real numbers (the coordinates of vectors in the basis) is determined, and algebraic operations with vector coordinates can be substituted for geometric operations with the vectors themselves.

Similar situations arise in many other areas of mathematics and its applications when objects under investigation are described by *tuples* (finite ordered lists) of real (or complex) numbers. Then the concept of a multidimensional coordinate space as the set of all tuples with algebraic operations on those tuples naturally arises.

In this chapter we will systematically construct and investigate such spaces. First of all, we will introduce the space $\mathbb{R}^n$ of all n-tuples of real numbers and the space $\mathbb{C}^n$ of all n-tuples of complex numbers. We will begin with definitions and basic properties of these spaces, since later we will introduce and study more general vector spaces. All results that we will obtain for general spaces hold for the vector spaces $\mathbb{R}^n$ and $\mathbb{C}^n$. We will provide also a variety of useful examples of specific bases in finite-dimensional spaces.

2.1 The Vector Spaces $\mathbb{R}^n$ and $\mathbb{C}^n$

2.1.1 The Vector Space $\mathbb{R}^n$

The vector space $\mathbb{R}^n$ is the set of all n-tuples $x = (x_1, x_2, \dots, x_n)$ of real numbers, where $n \geq 1$ is a given integer. Elements of the space $\mathbb{R}^n$ are called *vectors* or *points*; the numbers x_k, $k = 1, 2, \dots, n$, are called the *components* of the vector x.

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Two vectors $x, y \in \mathbb{R}^n$ are *equal* if and only if $x_k = y_k$ for all $k = 1, 2, \dots, n$. The vector all of whose components are zero is called the *zero* vector and is denoted by 0. A vector of the form

$$i_k = (\underbrace{0, \dots, 0}_{k-1}, 1, \underbrace{0, \dots, 0}_{n-k}),$$

where the kth component is equal to one and all other components are zero, is called a *standard unit vector*. In the space $\mathbb{R}^n$ there are exactly n standard unit vectors: $i_1, i_2, \dots, i_n$.

The following linear operations are introduced on the space $\mathbb{R}^n$: *scalar multiplication* (the multiplication of a vector by a scalar) and *vector addition*. Namely, for every real number α and $x, y \in \mathbb{R}^n$, by definition, we put

$$\alpha x = (\alpha x_1, \alpha x_2, \dots, \alpha x_n),$$

$$x + y = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n).$$

These linear operations satisfy the following properties. In the list below, x, y, z are arbitrary vectors in $\mathbb{R}^n$, and α, β are arbitrary real numbers.

1. *Commutativity* of vector addition: $x + y = y + x$.
2. *Associativity* of vector addition: $(x + y) + z = x + (y + z)$.
3. The zero vector is the *identity element* of vector addition: $x + 0 = x$.
4. For every vector x there exists a unique *inverse element* such that $x + (-x) = 0$, where by definition, $-x = (-1)x$.
5. *Distributivity* of scalar multiplication with respect to vector addition:

$$\alpha(x + y) = \alpha x + \alpha y.$$

6. *Distributivity* of scalar multiplication with respect to scalar addition:

$$(\alpha + \beta)x = \alpha x + \beta x.$$

7. Associativity of scalar multiplication: $(\alpha\beta)x = \alpha(\beta x)$.
8. The number 1 is the *identity element* of scalar multiplication: $1x = x$.

Properties 1–8 are called the *vector space axioms*. They follow immediately from the definition of linear operations with elements of the space $\mathbb{R}^n$. It is easy to see that Axioms 1–8 correspond exactly to the properties of linear operations with vectors in three-dimensional Euclidean space.

It is important to note that $\mathbb{R}^1$ is a vector space, but at the same time it is the set of all real numbers. As usual, we denote $\mathbb{R}^1$ by $\mathbb{R}$.

2.1.2 The Vector Space $\mathbb{C}^n$

The vector space $\mathbb{C}^n$ is the set of all n-tuples $x = (x_1, x_2, \dots, x_n)$ of complex numbers, where $n \geq 1$ is a given integer. The elements of the space $\mathbb{C}^n$ are called *vectors* or *points*; the numbers x_k, $k = 1, 2, \dots, n$, are called the *components* of the vector x.

Two vectors $x, y \in \mathbb{C}^n$ are *equal* if and only if $x_k = y_k$ for all $k = 1, 2, \dots, n$. The vector all of whose components are zero is called the *zero* vector and is denoted by 0. The vector i_k whose kth component is equal to one and all other components are equal to zero is called a *standard unit vector*. In the space $\mathbb{C}^n$ there are exactly n standard unit vectors: $i_1, i_2, \dots, i_n$.

The linear operations of *scalar multiplication* and *vector addition* are introduced on the space $\mathbb{C}^n$ in the usual way: for every complex number α and all $x,\ y \in \mathbb{C}^n$, by definition, we put

$$\alpha x = (\alpha x_1, \alpha x_2, \dots, \alpha x_n),$$

$$x + y = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n).$$

Note that we have actually already met this linear space. We can interpret the set of all $m \times n$ matrices with operations of matrix addition and multiplication of matrices by scalars (see p. 32) as the space $\mathbb{C}^{mn}$ of all vectors of length mn. The vectors were written in the form of rectangular arrays, but from the point of view of linear operations with vectors, this fact does not matter.

Properties 1–8 above hold also for the linear operations on the space $\mathbb{C}^n$.

Note that $\mathbb{C}^1$ is a vector space, but at the same time it is the set of all complex numbers. As usual, we denote $\mathbb{C}^1$ by $\mathbb{C}$.

2.2 Abstract Vector Spaces

2.2.1 Definitions and Examples

Two concepts more general than the spaces $\mathbb{R}^n$ and $\mathbb{C}^n$ are widely used in many areas of mathematics. These generalizations are abstract vector spaces: real and complex.

A *real vector space* $\mathbf{X}$ is a set that is closed under the operations of vector addition and scalar multiplication that satisfy the axioms listed below. Elements of $\mathbf{X}$ are called *vectors*. The operation of *vector addition* takes any two elements $x, y \in \mathbf{X}$ and assigns to them a third element $z = x + y \in \mathbf{X}$, which is called the sum of the vectors x and y. The operation of *scalar multiplication* takes any real number α and any element $x \in \mathbf{X}$ and gives an element $\alpha x \in \mathbf{X}$, which is called the product of α and x.

In order for $\mathbf{X}$ to be a real vector space, the following *vector space axioms*, which are analogous to Properties 1–8 of the space $\mathbb{R}^n$ (see p. 56), must hold for all elements $x, y, z \in \mathbf{X}$ and real numbers α, β.

1. *Commutativity* of vector addition: $x + y = y + x$.
2. *Associativity* of vector addition: $(x + y) + z = x + (y + z)$.
3. There exists a unique element $0 \in \mathbf{X}$, called the *zero element* of the space $\mathbf{X}$, such that $x + 0 = x$ for all $x \in \mathbf{X}$.
4. For every element $x \in \mathbf{X}$ there exists a unique element $x' \in \mathbf{X}$, called the *additive inverse* of x, such that $x + x' = 0$.[1]
5. *Distributivity* of scalar multiplication with respect to vector addition:
$$\alpha(x + y) = \alpha x + \alpha y.$$
6. *Distributivity* of scalar multiplication with respect to scalar addition:
$$(\alpha + \beta)x = \alpha x + \beta x.$$
7. Associativity of scalar multiplication: $(\alpha\beta)x = \alpha(\beta x)$.
8. The number 1 is the *identity element* of scalar multiplication: $1x = x$.

If in the definition of the space $\mathbf{X}$ multiplication by complex numbers is allowed, then $\mathbf{X}$ is called a *complex vector space*. It is assumed that Axioms 1–8, where $\alpha, \beta \in \mathbb{C}$, hold.

The proof of the following statements is left to the reader (here $\mathbf{X}$ is an arbitrary vector space):

1. $-0 = 0$ (here 0 is the zero element of $\mathbf{X}$);
2. $\alpha 0 = 0$ for every scalar α;
3. $0x = 0$ for every vector $x \in \mathbf{X}$;
4. if $\alpha x = 0$, $x \in \mathbf{X}$, then at least one of the factors is zero;
5. $-x = (-1)x$ for every $x \in \mathbf{X}$;
6. $y + (x - y) = x$ for every $x, y \in \mathbf{X}$, where, by definition, $x - y = x + (-y)$.

In the remainder of the book we denote vector spaces by the capital letters $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$. Unless otherwise stated, vector spaces are complex. The definitions and the results are mostly true for real spaces, too. Cases in which distinctions must be made in the interpretation of results for real spaces are specially considered.

The proof that the following sets are vector spaces is left to the reader.

1. The set $\mathbf{V}_3$ of all geometric vectors of three-dimensional Euclidean space with the usual definitions of operations of multiplication of a vector by a real scalar and vector addition is a real vector space.
2. The set of all real-valued functions of a real variable is a real vector space if the sum of two functions and the product of a function and a real number are defined as usual.
3. The set of all real-valued functions that are defined and continuous on the closed segment $[a, b]$ of the real axis is a real vector space. This space is denoted by

[1]The vector x' is usually denoted by $-x$.

$C[a, b]$. Hint: recall that the sum of two continuous functions is a continuous function, and the product of a continuous function and a real number is a continuous function.

4. The set of all functions in the space $C[a, b]$ that are equal to zero at a fixed point $c \in [a, b]$ is a real vector space.
5. The set of all polynomials with complex coefficients with the usual definitions of the sum of two polynomials and the product of a polynomial and a complex number is a complex vector space.
6. The set $\mathbf{Q}_n$ of all polynomials of order no more than n, where $n \geq 0$ is a given integer, joined with the zero polynomial, is a complex vector space. Hint: as we have seen in Section 1.1.2, p. 8, the sum of two polynomials is either a polynomial of degree no more than the maximum degree of the summands or the zero polynomial.

The reader can answer the next two questions.

1. Consider the set of all positive functions defined on the real axis and introduce on this set the operation of vector addition as the multiplication of two functions $f \cdot g$ and the operation of scalar multiplication as the calculation of the power function f^α. Is this set a vector space?
2. Consider the set of all even functions defined on the segment $[-1, 1]$ and introduce on this set the operation of vector addition as the multiplication of two functions and the operation of scalar multiplication as usual multiplication of a function by a scalar. Is this set a vector space?

2.2.2 Linearly Dependent Vectors

Two vectors a and b in a vector space $\mathbf{X}$ are said to be *linearly dependent* (*proportional*) if there exist numbers α and β, not both zero, such that

$$\alpha a + \beta b = 0.$$

Clearly, in this case we have either $a = \gamma b$ or $b = \delta a$, where γ, δ are some numbers.

For example, if $k \neq l$, then the standard unit vectors $i_k, i_l \in \mathbb{C}^n$, are nonproportional (prove it!).

Vectors $x_1 = (1 + \mathrm{i}, 3, 2 - \mathrm{i}, 5)$, $x_2 = (2, 3 - 3\mathrm{i}, 1 - 3\mathrm{i}, 5 - 5\mathrm{i}) \in \mathbb{C}^4$ are proportional, since $2/(1 + \mathrm{i}) = (3 - 3\mathrm{i})/3 = (1 - 3\mathrm{i})/(2 - \mathrm{i}) = (5 - 5\mathrm{i})/5 = 1 - \mathrm{i}$.

Let us generalize the concept of linear dependence of two vectors. A set of vectors $\{a_i\}_{i=1}^m = \{a_1, a_2, \ldots, a_m\}$, $m \geq 1$, in a vector space $\mathbf{X}$ is said to be *linearly dependent* if there exist numbers $x_1, x_2, \ldots, x_m$, not all zero, such that

$$x_1 a_1 + x_2 a_2 + \cdots + x_m a_m = 0. \tag{2.1}$$

For instance, the set of vectors

$$a_1 = \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix}, \quad a_2 = \begin{pmatrix} -1 \\ 3 \\ 3 \end{pmatrix}, \quad a_3 = \begin{pmatrix} 9 \\ 7 \\ 5 \end{pmatrix}, \quad a_4 = \begin{pmatrix} 3 \\ 8 \\ 7 \end{pmatrix}$$

in the space $\mathbb{R}^3$ is linearly dependent, since for $x_1 = 4$, $x_2 = -1$, $x_3 = -3$, $x_4 = 2$ we have

$$x_1a_1 + x_2a_2 + x_3a_3 + x_4a_4 = 4\begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ 3 \\ 3 \end{pmatrix} - 3\begin{pmatrix} 9 \\ 7 \\ 5 \end{pmatrix} + 2\begin{pmatrix} 3 \\ 8 \\ 7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0.$$

It is useful to note that there are many other sets of coefficients $x_1,\ x_2,\ x_3,\ x_4$ such that the linear combination $x_1a_1 + x_2a_2 + x_3a_3 + x_4a_4$ is equal to zero. For example,

$$2a_1 + a_2 - a_3 = 2\begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 3 \\ 3 \end{pmatrix} - \begin{pmatrix} 9 \\ 7 \\ 5 \end{pmatrix} = 0,$$

$$3a_2 + a_3 - 2a_4 = 3\begin{pmatrix} -1 \\ 3 \\ 3 \end{pmatrix} + \begin{pmatrix} 9 \\ 7 \\ 5 \end{pmatrix} - 2\begin{pmatrix} 3 \\ 8 \\ 7 \end{pmatrix} = 0.$$

It is useful to write the definition of linear dependence of vectors in matrix form. We use the following notation. Let $\mathcal{A}_m = \{a_1, a_2, \dots, a_m\}$ be a finite ordered list of vectors in the space $\mathbf{X}$. For $x \in \mathbb{C}^m$, by definition, we put

$$\mathcal{A}_m x = x_1a_1 + x_2a_2 + \cdots + x_ma_m.$$

Then we can say that the vectors a_1, a_2,..., a_m are *linearly dependent* if there exists a nonzero vector $x \in \mathbb{C}^m$ such that

$$\mathcal{A}_m x = 0.$$

A vector $a \in \mathbf{X}$ is a *linear combination* of vectors b_1, b_2, ..., b_p, $p \geq 1$, if there exists a vector $x \in \mathbb{C}^p$ such that

$$a = x_1b_1 + x_2b_2 + \cdots + x_pb_p. \tag{2.2}$$

We can write this in matrix form:

$$a = \mathcal{B}_p x.$$

A linear combination of vectors is called *nontrivial* if at least one of the numbers x_1, x_2, …, x_p in (2.2) is not equal to zero.

The proof of the following two theorems is left to the reader.

Theorem 2.1 *A set of vectors is linearly dependent if it contains a linearly dependent subset, in particular if it contains the zero vector.*

Theorem 2.2 *A set of vectors $\{a_i\}_{i=1}^m$ is linearly dependent if and only if it contains a vector a_k that can be represented as a linear combination of other vectors of the set $\{a_i\}_{i=1}^m$.*

Suppose that each vector of the set $\{a_i\}_{i=1}^m$ is a linear combination of the vectors $\{b_i\}_{i=1}^p$, i.e.,

$$a_k = \sum_{j=1}^{p} x_{jk} b_j, \quad k = 1, 2, \ldots, m. \tag{2.3}$$

We can write (2.3) in matrix form:

$$\mathcal{A}_m = \mathcal{B}_p X(p, m), \tag{2.4}$$

where the kth column of the matrix X consists of the coefficients x_{jk} of the kth linear combination in (2.3).

The following property of transitivity holds. If each vector of the set $\{a_i\}_{i=1}^m$ is a linear combination of the vectors $\{b_i\}_{i=1}^p$, and each vector of $\{b_i\}_{i=1}^p$ is a linear combination of the vectors $\{c_i\}_{i=1}^q$, then each vector of $\{a_i\}_{i=1}^m$ is a linear combination of $\{c_i\}_{i=1}^q$. Indeed, using matrix notation, we can write

$$\mathcal{A}_m = \mathcal{B}_p X(p, m), \quad \mathcal{B}_p = \mathcal{C}_q Y(q, p).$$

Substituting $\mathcal{C}_q Y(q, p)$ for $\mathcal{B}_p$ in the first equality, we get

$$\mathcal{A}_m = \mathcal{C}_q Z(q, m),$$

where

$$Z(q, m) = Y(q, p) X(p, m).$$

We say that the two sets of vectors $\{a_i\}_{i=1}^m$ and $\{b_i\}_{i=1}^p$ are *equivalent* if there exist matrices $X(p, m)$ and $Y(m, p)$ such that

$$\mathcal{A}_m = \mathcal{B}_p X(p, m), \quad \mathcal{B}_p = \mathcal{A}_m Y(m, p), \tag{2.5}$$

i.e., each vector of the set $\mathcal{A}_m$ is a linear combination of the vectors of the set $\mathcal{B}_p$ and conversely.

Using the property of transitivity, the reader can easily prove the next statement. Suppose that the sets $\{a_i\}_{i=1}^m$ and $\{b_i\}_{i=1}^p$ are equivalent and the vector $x \in \mathbf{X}$ is a linear combination of the vectors $\{a_i\}_{i=1}^m$. Then x can be represented as a linear combination of the vectors $\{b_i\}_{i=1}^p$.

2.2.3 Linearly Independent Sets of Vectors

A set of vectors $\mathcal{A}_m = \{a_i\}_{i=1}^m$ in a vector space $\mathbf{X}$ is said to be *linearly independent* if $\mathcal{A}_m x = 0$ implies $x = 0$.

Linearly independent sets exist. Let us give some simple examples.

1. Each vector $a \neq 0$ forms a linearly independent set, which consists of one vector.
2. If $m \leq n$, then the standard unit vectors $i_1, i_2, \dots, i_m \in \mathbb{C}^n$ are linearly independent. Indeed, for every $x \in \mathbb{C}^m$, the vector

$$x_1 i_1 + x_2 i_2 + \cdots + x_m i_m \in \mathbb{C}^n$$

   has the form $(x_1, x_2, \dots, x_m, 0, \dots, 0)$ and is equal to zero if and only if $x = 0$.
3. The set of vectors $\varphi_0(z) \equiv 1$, $\varphi_1(z) = z$, ..., $\varphi_k(z) = z^k$, where z is a complex number and $k \geq 0$ is a given integer, is linearly independent in the vector space of polynomials (see p. 59). This statement immediately follows from the fact that if a polynomial is equal to zero, then all its coefficients are equal to zero (see p. 31).

The next theorem is an evident consequence of Theorem 2.1.

Theorem 2.3 *Every subset of a linearly independent set $\{a_i\}_{i=1}^m$ is linearly independent.*

Theorem 2.4 *Every set $\{a_1, a_2, ..., a_n, b\}$ of $n + 1$ vectors in the space $\mathbb{C}^n$ is linearly dependent.*

Proof Suppose that the set of vectors $\{a_i\}_{i=1}^n$ is linearly independent. Denote by A the matrix whose columns are the vectors a_i, $i = 1, 2, \dots, n$. Clearly, $\det A \neq 0$, and the system of linear equations $Ax = b$ has a solution x. Therefore,

$$x_1 a_1 + \cdots + x_n a_n = b,$$

i.e., the set of vectors $\{a_1, a_2, ..., a_n, b\}$ is linearly dependent. □

It follows immediately from Theorem 2.4 that every set $\{a_i\}_{i=1}^m \in \mathbb{C}^n$, $m > n$, is linearly dependent.

Theorem 2.5 *Suppose that the set of vectors $\mathcal{A}_m = \{a_i\}_{i=1}^m$ in the space $\mathbf{X}$ is linearly independent and each vector of the set $\mathcal{A}_m$ is a linear combination of the vectors $\mathcal{B}_p = \{b_i\}_{i=1}^p$. Then $m \leq p$.*

Proof Assume the contrary, i.e., let $m > p$. By definition, there exists a $p \times m$ matrix X such that $\mathcal{A}_m = \mathcal{B}_p X$. Therefore, for every $y \in \mathbb{C}^m$ we have $\mathcal{A}_m y = \mathcal{B}_p X y$. The columns of the matrix X form a set of vectors in the space $\mathbb{C}^p$. The number of vectors in this set is $m > p$, and hence it is linearly dependent. Thus there exists a vector $y \in \mathbb{C}^m$ that is not equal to zero such that $Xy = 0$, but then $\mathcal{A}_m y = 0$, which means, contrary to the assumption, that the set of vectors $a_1, a_2, \ldots, a_m$ is linearly dependent. □

Corollary 2.1 *Every two linearly independent equivalent sets of vectors have the same number of vectors.*

The reader is invited to prove the next theorem (hint: use the reasoning of the proof of Theorem 2.5).

Theorem 2.6 *Suppose that the set $\{a_k\}_{k=1}^m$ is linearly independent and each vector of the set $\{b_k\}_{k=1}^m$ is a linear combination of the vectors $\{a_k\}_{k=1}^m$, i.e., there exists a square matrix X of order m such that $\mathcal{B}_m = \mathcal{A}_m X$. The set $\{b_k\}_{k=1}^m$ is linearly independent if and only if the matrix X is nonsingular.*

It is important to note that in Theorem 2.6 the matrix X is uniquely determined by the sets $\mathcal{A}_m$ and $\mathcal{B}_m$. Indeed, if we assume that there exists another matrix $\widetilde{X}$ such that $\mathcal{B}_m = \mathcal{A}_m \widetilde{X}$, then $\mathcal{A}_m(\widetilde{X} - X) = 0$, and $\widetilde{X} = X$, since the set $\mathcal{A}_m$ is linearly independent.

2.2.4 The Rank of a Set of Vectors

Let $\{a_i\}_{i=1}^m$ be a given set of vectors in the space $\mathbf{X}$. Suppose that not all vectors $\{a_i\}_{i=1}^m$ are equal to zero. Then this set necessarily contains a linearly independent subset of vectors. In particular, the set $\{a_i\}_{i=1}^m$ itself can be linearly independent.

A linearly independent subset $\{a_{i_k}\}_{k=1}^r \subset \{a_i\}_{i=1}^m$ is called *maximal* if including any other vector of the set $\{a_i\}_{i=1}^m$ would make it linearly dependent.

For example, let us consider the following set of vectors:

$$a_1 = \begin{pmatrix} 2 \\ -2 \\ -4 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 1 \\ 9 \\ 3 \end{pmatrix}, \quad a_3 = \begin{pmatrix} -2 \\ -4 \\ 1 \end{pmatrix}, \quad a_4 = \begin{pmatrix} 3 \\ 7 \\ -1 \end{pmatrix} \tag{2.6}$$

in the space $\mathbb{R}^3$. Evidently, the vectors a_1, a_2 are linearly independent and form a maximal linearly independent subset, since the determinants

$$\begin{vmatrix} 2 & 1 & -2 \\ -2 & 9 & -4 \\ -4 & 3 & 1 \end{vmatrix}, \quad \begin{vmatrix} 2 & 1 & 3 \\ -2 & 9 & 7 \\ -4 & 3 & -1 \end{vmatrix},$$

which consist of the components of the vectors a_1, a_2, a_3 and a_1, a_2, a_4, respectively, are equal to zero. Therefore, the sets of vectors a_1, a_2, a_3 and a_1, a_2, a_4 are linearly dependent.

Generally speaking, the set $\{a_i\}_{i=1}^m$ can contain several maximal linearly independent subsets, but the following result is true.

Theorem 2.7 *Every two maximal linearly independent subsets of the set $\{a_i\}_{i=1}^m$ contain the same number of vectors.*

Proof It follows from the definition of a maximal linearly independent subset that each vector of the set $\{a_i\}_{i=1}^m$ is a linear combination of vectors of a maximal linearly independent subset $\{a_{i_k}\}_{k=1}^r$. Obviously,

$$a_{i_k} = a_{i_k} + \sum_{i=1, i\neq i_k}^{m} 0 a_i;$$

hence the converse is also true. Therefore, the set $\{a_i\}_{i=1}^m$ and each of its maximal linearly independent subsets are equivalent. Thus, using Corollary 2.1, we claim that every two maximal linearly independent subsets of the set $\{a_i\}_{i=1}^m$ contain the same number of vectors. $\square$

This result allows us to introduce the following concept. The *rank* of a set of vectors in the space $\mathbf{X}$ is the number of vectors in each of its maximal linearly independent subsets.

For example, the rank of the set of vectors (2.6) is equal to two.

The number of linearly independent vectors in the space $\mathbb{C}^n$ is no more than n. Therefore, the rank of every set of vectors in $\mathbb{C}^n$ is less than or equal to n.

Clearly, a set of vectors $\{a_i\}_{i=1}^m$ in a vector space $\mathbf{X}$ is linearly independent if and only if its rank is equal to m.

2.3 Finite-Dimensional Vector Spaces. Bases

2.3.1 Bases in the Space $\mathbb{C}^n$

A linearly independent set $\{e_k\}_{k=1}^n$ (which consists of n vectors) is called a *basis* in the space $\mathbb{C}^n$. The standard unit vectors $\{i_k\}_{k=1}^n$ form the *standard* (or the *natural*) basis in the space $\mathbb{C}^n$.

It follows from Property 8, p. 23, of determinants that a set $\{e_k\}_{k=1}^n \subset \mathbb{C}^n$ is a basis if and only if the matrix $\mathcal{E}_n$, the columns of which are formed by the vectors $e_1, e_2, \dots, e_n$, is nonsingular.

In the proof of Theorem 2.4, p. 62, we established that if $\{e_k\}_{k=1}^n$ is a basis in the space $\mathbb{C}^n$, then each vector $x \in \mathbb{C}^n$ can be represented as a linear combination

$$x = \xi_1 e_1 + \xi_2 e_2 + \cdots + \xi_n e_n. \tag{2.7}$$

The coefficients in the linear combination (2.7) are uniquely determined by the vector x and satisfy the following system of linear algebraic equations with the nonsingular matrix $\mathcal{E}_n$:

$$\mathcal{E}_n \xi = x. \tag{2.8}$$

Here $\xi = (\xi_1, \xi_2, \dots, \xi_n)$ is the column of coefficients of the expansion of x with respect to the basis $\{e_k\}_{k=1}^n$.

2.3.2 Finite-Dimensional Spaces. Examples

A vector space $\mathbf{X}$ is called *finite-dimensional* if there exist vectors

$$\mathcal{E}_n = \{e_1, e_2, \dots, e_n\}$$

that form a linearly independent set in the space $\mathbf{X}$ and such that each vector $x \in \mathbf{X}$ can be represented as a linear combination

$$x = \sum_{k=1}^{n} \xi_k e_k = \mathcal{E}_n \xi, \quad \xi \in \mathbb{C}^n. \tag{2.9}$$

The set of vectors $\{e_k\}_{k=1}^n$ is called a *basis* of the space $\mathbf{X}$. The number n is called the *dimension* of $\mathbf{X}$, and we denote by $\mathbf{X}_n$ this n-dimensional vector space. The coefficients ξ_1, ξ_2, …, ξ_n in the expansion (2.9) are called the *coordinates* of x with respect to the basis $\{e_k\}_{k=1}^n$.

The coordinates of each vector $x \in \mathbf{X}_n$ are uniquely determined by the basis $\{e_k\}_{k=1}^n$. Indeed, suppose that in addition to (2.9) there exists an expansion $x = \mathcal{E}_n \tilde{\xi}$. Then $\mathcal{E}_n(\xi - \tilde{\xi}) = 0$. Therefore, $\xi = \tilde{\xi}$, since the set of vectors $\{e_k\}_{k=1}^n$ is linearly independent.

Theorem 2.8 *In an n-dimensional vector space* $\mathbf{X}_n$*, every system* $\tilde{\mathcal{E}}_n = \{\tilde{e}_k\}_{k=1}^n$ *consisting of n linearly independent vectors is a basis.*

Proof It is enough to show that each vector $x \in \mathbf{X}_n$ can be represented as a linear combination

$$x = \tilde{\mathcal{E}}_n \tilde{\xi}. \tag{2.10}$$

By the definition of an n-dimensional vector space, a basis $\mathcal{E}_n$ exists in $\mathbf{X}_n$. Therefore, each vector of the set $\tilde{\mathcal{E}}_n$ can be represented as a linear combination of the vectors of $\mathcal{E}_n$. In other words, there exists a square matrix T of order n such that $\tilde{\mathcal{E}}_n = \mathcal{E}_n T$. The matrix T is nonsingular (see p. 63). Since $\mathcal{E}_n$ is a basis, there exists a vector $\xi \in \mathbb{C}^n$ such that $x = \mathcal{E}_n \xi$. Since the matrix T is nonsingular, there exists a vector $\tilde{\xi} \in \mathbb{C}^n$ such that $\xi = T\tilde{\xi}$. Thus we get the relationship $x = \mathcal{E}_n T\tilde{\xi} = \tilde{\mathcal{E}}_n\tilde{\xi}$ of the form (2.10). □

If a vector space is not finite-dimensional, then the space is called *infinite-dimensional*.

Let us give some examples of finite-dimensional and infinite-dimensional vector spaces.

1. Three arbitrary non-coplanar vectors form a basis in the space $\mathbf{V}_3$. The space $\mathbf{V}_3$ is three-dimensional.
2. Evidently, the spaces $\mathbb{C}^n$, $\mathbb{R}^n$ are n-dimensional.
3. The set $\mathbf{Q}_n$ of all polynomials of order no more than n is finite-dimensional. Its dimension is equal to $n + 1$. For example, the set of vectors $\{1, z, \dots, z^n\}$, where z is a complex variable, is a basis in $\mathbf{Q}_n$.
4. The vector space of all polynomials is infinite-dimensional. Indeed, for an arbitrarily large integer k, the set of vectors $\{1, z, \dots, z^k\}$ is linearly independent in this space.
5. The space $C[a, b]$ is infinite-dimensional, since it contains polynomials with real coefficients of arbitrary order.

2.3.3 Change of Basis

Let $\mathcal{E}_n = \{e_k\}_{k=1}^n$, $\tilde{\mathcal{E}}_n = \{\tilde{e}_k\}_{k=1}^n$ be bases in a vector space $\mathbf{X}_n$. As we have shown, the sets $\mathcal{E}_n$ and $\tilde{\mathcal{E}}_n$ are equivalent, and there exist square matrices T and $\widetilde{T}$ of order n such that

$$\mathcal{E}_n = \widetilde{\mathcal{E}}_n\widetilde{T}, \quad \widetilde{\mathcal{E}}_n = \mathcal{E}_n T. \tag{2.11}$$

The matrix T is called the *change of basis matrix* from $\mathcal{E}_n$ to $\widetilde{\mathcal{E}}_n$. The matrices T and $\widetilde{T}$ are mutually inverse. Indeed, substituting $\mathcal{E}_n T$ for $\widetilde{\mathcal{E}}_n$ in the first equality in (2.11), we obtain $\mathcal{E}_n = \mathcal{E}_n T\widetilde{T}$. Thus we get

$$T\widetilde{T} = I, \tag{2.12}$$

since the vectors of the basis $\mathcal{E}_n$ are linearly independent (see the remark after Theorem 2.6, p. 63).

Suppose that we know the vector ξ of the coordinates of an element $x \in \mathbf{X}_n$ with respect to the basis $\mathcal{E}_n$, and we also know the change of basis matrix T from $\mathcal{E}_n$ to the basis $\widetilde{\mathcal{E}}_n$. Let us construct a formula for calculating the vector $\tilde{\xi}$ of coordinates of the same element x with respect to the basis $\widetilde{\mathcal{E}}_n$. Using (2.9), we see that $x = \mathcal{E}_n\xi$, but $\mathcal{E}_n = \widetilde{\mathcal{E}}_n\widetilde{T} = \widetilde{\mathcal{E}}_n T^{-1}$ (see (2.11), (2.12)). Therefore, $x = \widetilde{\mathcal{E}}_n T^{-1}\xi$, which means that

$$\tilde{\xi} = T^{-1}\xi. \tag{2.13}$$

For example, suppose that vectors e_1, e_2, e_3 form a basis in a three-dimensional space $\mathbf{X}_3$. Let us consider the vectors

$$\begin{aligned}\tilde{e}_1 &= 5e_1 - e_2 - 2e_3,\\ \tilde{e}_2 &= 2e_1 + 3e_2,\\ \tilde{e}_3 &= -2e_1 + e_2 + e_3.\end{aligned}$$

Writing these equalities in matrix form, we get $\tilde{\mathcal{E}} = \mathcal{E}T$, where

$$\tilde{\mathcal{E}} = \{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}, \quad \mathcal{E} = \{e_1, e_2, e_3\}, \quad T = \begin{pmatrix} 5 & 2 & -2 \\ -1 & 3 & 1 \\ -2 & 0 & 1 \end{pmatrix}.$$

It is easy to see that $\det T = 1$; hence the matrix T is nonsingular. Therefore, the vectors $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$ also form a basis in the space $\mathbf{X}_3$. Let us consider the vector $a = e_1 + 4e_2 - e_3$. The coordinates of the vector a with respect to the basis $\mathcal{E}$ are the numbers $\xi_1 = 1, \xi_2 = 4, \xi_3 = -1$, i.e., $a = \mathcal{E}\xi$, where $\xi = (\xi_1, \xi_2, \xi_3)$. Now we calculate the coordinates of the same vector, but with respect to the basis $\tilde{\mathcal{E}}$. Calculating the matrix T^{-1}, we get

$$T^{-1} = \begin{pmatrix} 3 & -2 & 8 \\ -1 & 1 & -3 \\ 6 & -4 & 17 \end{pmatrix},$$

and therefore,

$$\tilde{\xi} = T^{-1}\xi = \begin{pmatrix} 3 & -2 & 8 \\ -1 & 1 & -3 \\ 6 & -4 & 17 \end{pmatrix} \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix} = \begin{pmatrix} -13 \\ 6 \\ -27 \end{pmatrix},$$

i.e., $a = -13\tilde{e}_1 + 6\tilde{e}_2 - 27\tilde{e}_3$. Thus we have calculated the coordinate representation of the vector a with respect to the basis $\tilde{\mathcal{E}}$.

Note that infinitely many bases exist in the space $\mathbf{X}_n$. Indeed, if $\mathcal{E}_n$ is a basis, then the set of vectors $\tilde{\mathcal{E}}_n = \mathcal{E}_n T$, where T is an arbitrary nonsingular matrix, also is a basis (see Theorem 2.6, p. 63).

Below are some examples of bases in the space of polynomials of order no more than n with complex coefficients, which are often used in applications.

1. The *natural basis* for this space is the set of vectors $\{1, z, \dots, z^n\}$, where z is a complex variable.
2. The polynomials

$$\Phi_j(z) = \frac{(z - z_0)(z - z_1)\cdots(z - z_{j-1})(z - z_{j+1})\cdots(z - z_n)}{(z_j - z_0)(z_j - z_1)\cdots(z_j - z_{j-1})(z_j - z_{j+1})\cdots(z_j - z_n)},$$

$j = 0, 1, 2, \dots, n$, where z_0, z_1, ..., z_n are arbitrary distinct complex numbers, also form a basis in the space of polynomials (see p. 29). Such a basis is called a *Lagrange basis*.

3. Let us prove that the polynomials

$$\varphi_0(z) \equiv 1,\ \varphi_1(z) = (z - z_0),\ \varphi_2(z) = (z - z_0)(z - z_1), \dots,$$
$$\varphi_n(z) = (z - z_0)(z - z_1) \cdots (z - z_{n-1}), \tag{2.14}$$

where z_0, z_1, ..., z_{n-1} are arbitrary distinct complex numbers, form a basis. As in the case of a Lagrange basis, it is enough to check that for the numbers z_0, z_1, ..., z_{n-1}, and a number z_n that does not coincide with any of the numbers z_0, z_1, ..., z_{n-1}, the system of equations

$$c_0\varphi_0(z_j) + c_1\varphi_1(z_j) + \dots + c_n\varphi_n(z_j) = h_j, \quad j = 0, 1, 2, \dots, n, \tag{2.15}$$

has a unique solution for every choice of h_0, h_1, ...h_n. This fact is evident, since system (2.15) is triangular,

$$\begin{aligned} &c_0 = h_0, \\ &c_0 + c_1(z_1 - z_0) = h_1, \\ &c_0 + c_1(z_2 - z_0) + c_2(z_2 - z_0)(z_2 - z_1) = h_2, \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ &c_0 + c_1(z_n - z_0) + \dots + c_n(z_n - z_0)(z_n - z_1)\cdots(z_n - z_{n-1}) = h_n, \end{aligned} \tag{2.16}$$

and all diagonal coefficients are different from zero. The basis defined in (2.14) is called a *Newton basis*.[2]

[2]Sir Isaac Newton (1642–1727) was an English physicist and mathematician.

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