

Chapter 2

Foliations and Heat Diffusion

In this chapter we briefly recall the theory of foliations. We present the definition of a foliation and illustrate it by a number of examples (product foliation, a foliation given by a submersion, Reeb foliation of a solid torus, a linear foliation of a torus). We also recall the notion of the holonomy of a leaf. We present the foliated Laplace operator and foliated heat diffusion operators semigroup which plays important role in the metric diffusion. We only demonstrate these facts which are necessary for further results. For the complete theory one can refer to [3], one of the best books about foliations. We also take advantage of some results from [13, 17], and [21].

2.1 Basic Facts

A *foliation* $\mathcal{F}$ on an n -dimensional manifold is an equivalence relation with the equivalence classes being connected, immersed submanifolds, all of dimension $p \geq 1$. Locally, the decomposition into equivalence classes can be modelled on the decomposition of $\mathbb{R}^n$ into the cosets $x + \mathbb{R}^p$ of the imbedded subspace $\mathbb{R}^p$. The equivalence classes of $\mathcal{F}$ are called the *leaves* of $\mathcal{F}$.

More formally, let M be an arbitrary n -dimensional C^r -manifold. A C^r -*foliated chart* of codimension q is a pair (U, φ) , where $\varphi : U \rightarrow V \times W$ is a C^r -diffeomorphism of an open subset $U \subset M$ into a set $V \times W \subset \mathbb{R}^{p+q}$, $p = n - q$, and V and W are rectangular neighborhoods in $\mathbb{R}^p$ and $\mathbb{R}^q$, respectively. The set $P_y = \varphi^{-1}(V \times \{y\})$, $y \in W$, is called a *plaque* of this chart.

Let $\mathcal{F} = \{L_\alpha\}_{\alpha \in \mathcal{A}}$ be a decomposition of M into connected, p -dimensional, $p = n - q$, topologically immersed submanifolds (*leaves*). Suppose that M admits an atlas $\{U_\lambda\}_{\lambda \in \Lambda}$ of foliated charts (Figure 2.1) of codimension q such that for each $\lambda \in \Lambda$ and $\alpha \in \mathcal{A}$ the set $L_\alpha \cap U_\lambda$ is a union of plaques. $\mathcal{F}$ is said to be a *foliation* of M , the number $p = n - q$ is called the *dimension* of $\mathcal{F}$, while q the *codimension*. A leaf passing through a point $x \in M$ will be later denoted by L_x .

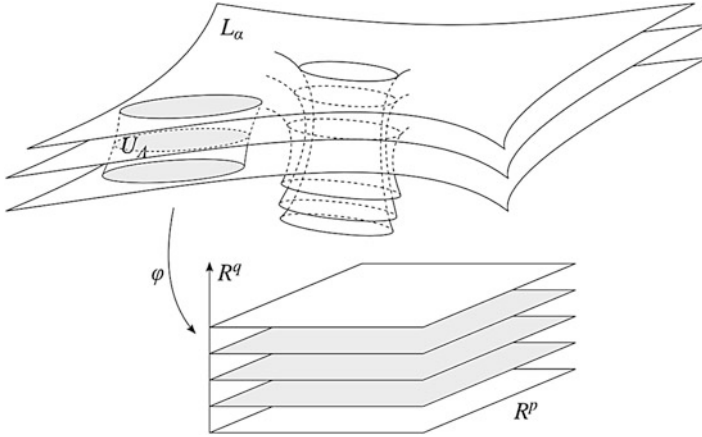


Fig. 2.1 A foliated chart

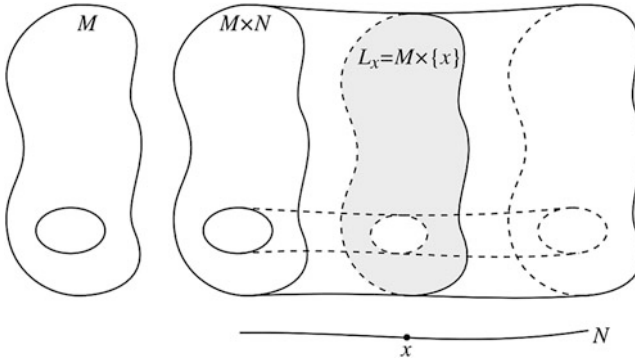


Fig. 2.2 The product foliation

If a foliated atlas is of class C^r , with $0 \leq r \leq \infty$ or $r = \omega$, then $\mathcal{F}$ and the foliated manifold $(M, \mathcal{F})$ are said to be of class C^r .

Example 2.1 The simplest example of a foliation is the *product foliation*. Let M and N be two arbitrary manifolds, $\dim M = m$ and $\dim N = n$. We foliate the product $M \times N$ by submanifolds $M \times \{x\}$, $x \in N$ (Figure 2.2).

Example 2.2 Let M and N be two arbitrary manifolds of dimension m and n , $m > n$, respectively. A smooth submersion $\pi : M \rightarrow N$ provides a foliation on M by the connected components of the non-empty level sets $\pi^{-1}(x)$, $x \in N$ (Figure 2.3).

Example 2.3 As an example of a foliation which is not provided by a single submersion, one can consider the *Reeb foliation* of a solid torus.

Fig. 2.3 A foliation defined by a submersion

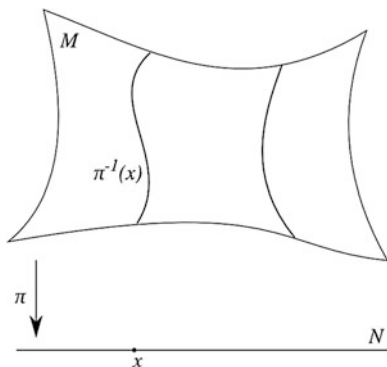
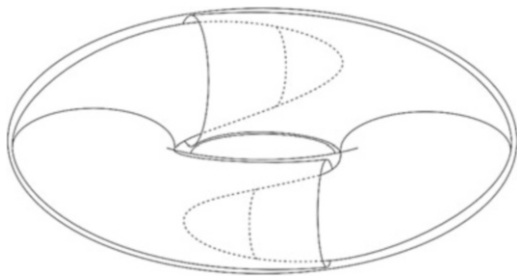


Fig. 2.4 Reeb foliation of a solid torus



Let $M = D^2 \times S^1$, with D^2 denoting a two-dimensional disk, and S^1 being the unit circle. Let $f : (-1, 1) \rightarrow \mathbb{R}$ be a smooth function satisfying

$$f(0) = 0, \quad f(t) \geq 0, \quad f(t) = f(-t),$$

$$\lim_{t \rightarrow \pm 1} \frac{d^k}{dt^k} f(t) = \infty, \quad \lim_{t \rightarrow \pm 1} \frac{d^k}{dt^k} \left(\frac{1}{\frac{df}{dt}(t)} \right) = 0,$$

for all $k = 0, 1, 2, \dots$, for example $f(t) = e^{\frac{t^2}{1-t^2}} - 1$. For $\alpha \in [0, 1)$, define leaves of a foliation by

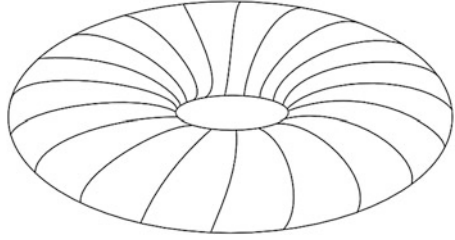
$$L_\alpha = \{(x, y) \in \text{int} D^2 \times S^1 : y = e^{2\pi(\alpha + f(|x|))i}\}.$$

Adding the boundary torus $\partial(D^2 \times S^1)$ as a boundary leaf we obtain a foliation of the solid torus $D^2 \times S^1$ called the *Reeb foliation* (Figure 2.4).

Example 2.4 ([13, Section 2.2]) Consider a foliation $\mathcal{F}$ of the codimension q on a manifold M , and denote by $\mathcal{L}$ the Lie derivative on M . A *transverse metric* on $(M, \mathcal{F})$ is a positive smooth bilinear form g on a module $\mathcal{X}(M)$ of all vector fields on M satisfying

1. $\text{Ker}(g_x) = T_x(\mathcal{F})$,
2. $L_X g = 0$ for any vector field X tangent to $\mathcal{F}$.

Fig. 2.5 A linear foliation of the torus T^2



A foliation $\mathcal{F}$ equipped with a transverse metric g on $(M, \mathcal{F})$ is called a *Riemannian foliation*. One should notice that, for a given foliation, a Riemannian structure on the normal bundle $\mathcal{F}^\perp$ determines the transverse metric if and only if this structure is holonomy invariant.

Example 2.5 The linear foliation of a torus T^2 (Figure 2.5) is also an interesting example. Since a constant vector field

$$\tilde{X} = \begin{bmatrix} a \\ b \end{bmatrix}, \quad a, b \in \mathbb{R}$$

on $\mathbb{R}^2$ is invariant by all translations in $\mathbb{R}^2$, it defines a vector field X on $\mathbb{R}^2/\mathbb{Z}^2$. Assume that $a \neq 0$. The foliation $\tilde{\mathcal{F}}$ on $\mathbb{R}^2$ defined by integral curves (parallel lines of slope $\frac{b}{a}$) of $\tilde{X}$ passes to the foliation $\mathcal{F}$ on T^2 defined by X . Observe that for $\frac{b}{a}$ rational, $\mathcal{F}$ is a foliation of T^2 by circles. Otherwise, each leaf is a one-to-one immersion of $\mathbb{R}$ and is everywhere dense in T^2 .

Example 2.6 *Compact foliations*, i.e., foliations with all leaves compact, is a family of foliations of our special interest. The topological structure of such foliations was deeply studied in [7] and [9]. Compact foliations will be the objects of deeper studies in Chapter 3.

2.2 Holonomy

Some important properties of the leaves of a foliation are described in terms of the *holonomy group* of a leaf, which describes the behavior of leaves in a small neighborhood of this leaf. To understand the notion of holonomy group, we first recall the notion of a germ of a map.

Let M and N be manifolds, and let $x \in M$, $y \in N$. Consider a map $f : U \rightarrow V$, where U and V are open neighborhoods of x and y , respectively. A map $f' : U' \rightarrow V'$, $x \in U'$, $y \in V'$ is said to be equivalent to f iff there exists an open neighborhood $W \subset U \cap U'$ of x such that $f|_W = f'|_W$. The equivalence above defines the equivalence relation in the set of all mappings such that x is mapped to y . An equivalence class $[f]_x$ of this relation is called the *germ* of f at x .

Let us now consider mappings $f, g : M \rightarrow M$ preserving x , i.e., $f(x) = x$. One can compose the germs $[f]_x$ and $[g]_x$ into a germ $[f]_x \circ [g]_x = [f \circ g]_x$. The identity map defines the identity germ $[\text{id}]_x$, and if f is a diffeomorphism on a neighborhood U of x , one can define an inverse germ $[f]_x^{-1} = [f^{-1}]_x$. Hence, the set of all germs of diffeomorphisms preserving x is a group, which will be denoted by Diff_x .

We now return to the definition of the holonomy group of a leaf. Let $L \in \mathcal{F}$ and $x, y \in L$. Consider a curve $\gamma : [0, 1] \rightarrow L$ linking x and y , that is a curve that $\gamma(0) = x$ and $\gamma(1) = y$. Let T_x and T_y be two transversals (smoothly imbedded, compact, connected, q -dimensional manifolds without boundary which are everywhere transverse to $\mathcal{F}$) through x and y , respectively. We associate with γ a germ at x of a diffeomorphism $[h]_{yx}(\gamma)$ defined as follows:

First, let us suppose that $\gamma([0, 1])$ is totally contained in a foliated chart. Since γ is a curve on L , x and y lie on the same plaque (we will denote by P_z the unique plaque containing $z \in M$). There exist an open neighborhood $A_x \subset T_x$ of x and a smooth map $h : A_x \rightarrow T_y$ with $h(x) = y$ and assigning for any $x' \in A$ the unique point $P_{x'} \cap T_y$. Moreover, A can be chosen in such a way that h is a diffeomorphism onto its image (Figure 2.6). We define

$$[h]_{yx}(\gamma) = [h]_x.$$

In the general case (Figure 2.7), let us choose a sequence of foliated charts $(U_1, \dots, U_k)$ covering $\gamma([0, 1])$ for which $U_i \cap U_{i+1} \neq \emptyset$, and $\gamma([\frac{i-1}{k}, \frac{i}{k}]) \subset U_i$. Set $\gamma_i = \gamma|[\frac{i-1}{k}, \frac{i}{k}]$, $i = 1, \dots, k$, and $x_i = \gamma(\frac{i-1}{k})$, $i = 0, \dots, k$. We choose transversals T_i at x_i and define

$$[h]_{yx}(\gamma) = [h]_{x_k x_{k-1}}(\gamma_k) \circ \dots \circ [h]_{x_1 x_0}(\gamma_1).$$

One can check [3] that the holonomy map h does not depend on a choice of a sequence $(U_1, \dots, U_k)$. Moreover, if T_x , T_y , and T_z are transversals at $x, y, z \in L$,

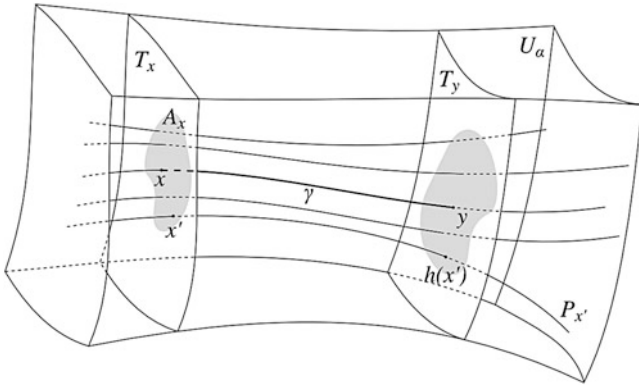


Fig. 2.6 Holonomy map in the single chart U_α

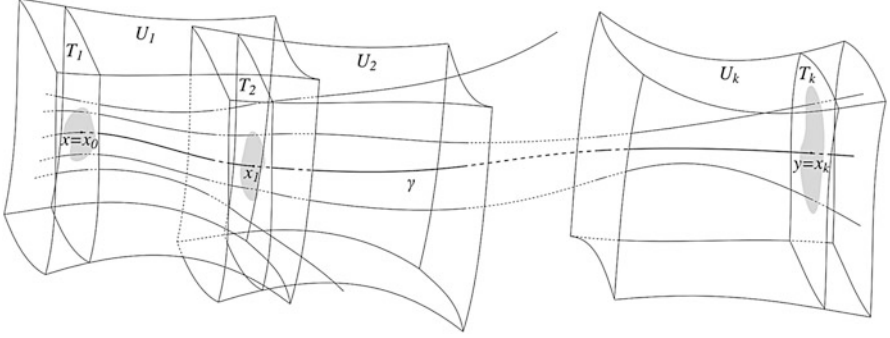


Fig. 2.7 The holonomy of a leaf

respectively, and $\gamma : [0, 1] \rightarrow L$ links y with z , while $\delta : [0, 1] \rightarrow L$ links x with y , then

$$[h]_{zx}(\gamma * \delta) = [h]_{zy}(\gamma) \circ [h]_{yx}(\delta),$$

where

$$(\gamma * \delta)(t) = \begin{cases} \gamma(2t) & \text{for } t \in [0, \frac{1}{2}), \\ \delta(2t - 1) & \text{for } t \in [\frac{1}{2}, 1]. \end{cases}$$

In addition, for any homotopic leaf curves γ and δ linking x with y , the corresponding holonomies are equivalent, i.e.,

$$[h]_{yx}(\gamma) = [h]_{yx}(\delta).$$

Let $\gamma : [0, 1] \rightarrow L$ be a loop at x , i.e., a curve with $\gamma(0) = \gamma(1) = x$. One can now define a holonomy homomorphism from the first fundamental group $\pi_1(L, x)$ at x into the group of germs of diffeomorphisms of a transversal T_x by

$$\Phi_L : \pi(L, x) \ni [\gamma] \mapsto [h]_x(\gamma).$$

The image of Φ_L is called the holonomy group of L . We later denote it by $\mathcal{H}_L$.

2.3 Harmonic Measures and Heat Diffusion

Let $(M, \mathcal{F}, g)$ be a smooth closed oriented foliated manifold of dimension n equipped with a Riemannian tensor g . Let $p = \dim \mathcal{F}$. The *leafwise Laplace operator* Δ defined by

$$\Delta f = \operatorname{div} \nabla f$$

with ∇ denoting the gradient of f , has, in a foliated chart $U = D \times Z$ with coordinates $(x, z) = (x_1, \dots, x_p, z)$ and leafwise metric tensor

$$g = \sum_{i,j=1}^p g_{i,j}(x_1, \dots, x_p, z) dx^i \otimes dx^j,$$

a local expression

$$\Delta f = \frac{1}{\sqrt{|g|}} \sum_{j=1}^p \frac{\partial}{\partial x^j} \left(\sum_{i=1}^p g^{ij} \sqrt{|g|} \frac{\partial}{\partial x^i} f \right),$$

where (g^{ij}) denotes the inverse matrix of the Riemannian tensor matrix (g_{ij}) and $|g| = \det(g_{ij})$. Thus

$$\Delta = \sum_{i,j=1}^p g^{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \text{first order terms}.$$

Since the metric tensor g on $(M, \mathcal{F})$ induces a metric tensor $g|_L$ on each leaf $L \in \mathcal{F}$, a leafwise Laplacian can be defined (following [3] or [21]) as

$$\Delta_L = \Delta|_L.$$

Define the *foliated Laplace operator* $\Delta_{\mathcal{F}}$ by

$$\Delta_{\mathcal{F}} f(x) = \Delta_{L_x} f(x), \quad x \in M,$$

where L_x is a leaf through x , and Δ_{L_x} is the Laplace operator on $(L_x, g|_{L_x})$. The operator $\Delta_{\mathcal{F}}$ acts on bounded measurable functions, which are C^2 -smooth along the leaves.

We say that a probability measure μ on $(M, \mathcal{F}, g)$ is *harmonic* if

$$\int_M \Delta_{\mathcal{F}}(x) f d\mu(x) = 0 \text{ for any } f : M \rightarrow \mathbb{R}.$$

L. Garnet proved [11] that the harmonic measures are related to the differential operator $\Delta_{\mathcal{F}}$. First, we formulate the following.

Theorem 2.1 *On any compact foliated Riemannian manifold, harmonic probability measure exists.*

Lemma 2.1 *Let (X, g) be a Riemannian manifold, and let Δ denote the Laplace operator. If f is a function on M of class C^2 , and $x_0 \in M$ is a local maximum of f , then $\Delta f(x_0) \leq 0$. If $x_0 \in M$ is a local minimum of f , then $\Delta f(x_0) \geq 0$.*

Proof Let x_0 be a local maximum of f . Since

$$\Delta f = \sum_{i,j} g^{ij} \frac{\partial^2 f}{\partial x_i \partial x_j} + \text{first order terms},$$

we can choose the coordinate system that $g^{ij}(x_0)$ is the identity matrix. Moreover, since Δ annihilates constants and all first order derivatives vanish at x_0 , then

$$\Delta f(x_0) = \sum_{i,j} g^{ij} \frac{\partial^2 f}{\partial x_i \partial x_j}(x_0) \leq 0.$$

The second assertion goes in the same way. $\square$

Let $C(M)$ denote the space of continuous functions on M , while $\mathbf{1}$ a constant function $\mathbf{1}(x) = 1$, for all $x \in M$.

Lemma 2.2 *On a compact foliated Riemannian manifold M , the closure of the range of $\Delta_{\mathcal{F}}$ does not contain $\mathbf{1}$.*

Proof Suppose that there is a sequence $\{f_i\}_{i \in \mathbb{N}}$ such that $\Delta_{\mathcal{F}} f_i$ converges in $C(M)$ to $\mathbf{1}$. So, there exists an index $i_0 \in \mathbb{N}$ such that for all $i > i_0$

$$\Delta_{\mathcal{F}}(x)f_i \geq \frac{1}{2} \quad \text{for all } x \in M.$$

Since M is compact, by Lemma 2.1, $\Delta_{\mathcal{F}}(x)f_i \leq 0$ somewhere in M . $\square$

Lemma 2.3 *A continuous linear functional $\Phi : C(M) \rightarrow \mathbb{R}$ is given by the integral with respect to the probability measure μ on M if and only if $\|\Phi\| = 1$ and $\Phi(\mathbf{1}) = 1$.*

Proof To begin, assume that $\|\Phi\| = 1$ and $\Phi(\mathbf{1}) = 1$. By the Riesz Representation Theorem [15], it is only necessary to show that for nonnegative functions $\Phi(f) \geq 0$. We can assume that $0 \leq f \leq 1$, for all $x \in M$. The function $h = 2f - \mathbf{1}$ satisfies $-\mathbf{1} \leq h \leq \mathbf{1}$. Hence $|\Phi(h)| \leq 1$, and $\Phi(h) \geq -1$. This implies $2\Phi(f) \geq 0$.

The converse is immediate. $\square$

Proof (Theorem 2.1) Let $H \subset C(M)$ be the closure of the range of $\Delta_{\mathcal{F}}$ in the uniform norm. Let $a = \inf_{f \in H} \|\mathbf{1} - f\|$. By Lemma 2.2, $a > 0$. Moreover, $a \leq 1$ because H is a subspace, and $a \geq 1$ due to the fact that for any continuous function which is C^2 along the leaves $\mathbf{1} - \Delta g \leq 1$ somewhere on M (due to Lemma 2.1).

Let $\Phi : H + \mathbb{R}\mathbf{1} \rightarrow \mathbb{R}$ be the linear functional defined by $\Phi(h + t\mathbf{1}) = t$. For $v = h + t\mathbf{1} \in H + \mathbb{R}\mathbf{1}$ and $t = 0$ we have $|\Phi(v)| = 0 \leq \|v\|$. If $t \neq 0$, then

$$|\Phi(v)| = |t| \leq |t| \left(\left\| \frac{1}{t}h + \mathbf{1} \right\| \right) = \|v\|.$$

So, by the Hahn–Banach Theorem, there exists a linear extension $\Psi : C(M) \rightarrow \mathbb{R}$ of Φ , such that $|\Psi(g)| \leq \|g\|$ for all $g \in C(M)$. Moreover, $\Psi(\mathbf{1}) = 1$, so $\|\Psi\| = 1$. In addition,

$$\Psi|_H = \Phi_H \equiv 0.$$

By Lemma 2.3, Ψ is the integral associated to the probability measure μ on M , which is the desired harmonic measure. $\square$

Let f be a bounded continuous function on a manifold L . Recall that if the geometry of L is bounded, one can solve on L the heat equation

$$\frac{\partial}{\partial t} u(x, t) = \Delta u(x, t)$$

with the initial condition f , where $u \in C^{2,1}(L \times [0, \infty))$, and $u(x, 0) = f(x)$.

Let L be a leaf of $\mathcal{F}$. The heat equation on $(L, g|_L)$ admits a fundamental solution $p_t(x, y)$, called the *heat kernel*, which satisfies

$$\frac{\partial}{\partial t} p_t(x, y) = \Delta_x p_t(x, y) \text{ for any } y \in L,$$

and for any bounded function f on L

$$D_{L,t} f(x) = \int_L f(y) p_t(x, y) dy$$

is the bounded solution to the heat equation on L with the initial condition f . The operators $D_{L,t}$ form the *semigroup of diffusion operators* on $(L, g|_L)$. The aggregate of $D_{L,t}$ on various leaves defines on M a semigroup D_t of operators satisfying on functions on M

$$D_0 = \text{id}, \quad D_{t+s} = D_t \circ D_s, \quad \frac{d}{dt} D_t|_{t=0} = \Delta_{\mathcal{F}}.$$

Each D_t restricted to a leaf $L \in \mathcal{F}$ coincides with the heat diffusion operators on L . Thus, for suitable functions f on M , $D_t f$ is a function defined at any $x \in M$ by

$$(D_t f)(x) = \int_{L_x} f(y) p_t(x, y) dy$$

with $p_t(x, y)$ being the heat kernel on $(L_x, g|_{L_x})$.

Let μ be a probability measure on M . Following [3] and [21], one can define the *diffused measure* $D_t \mu$ by the formula

$$\int_M f(x) dD_t \mu(x) = \int_M D_t f(x) d\mu(x),$$

where f is bounded measurable function on M . A measure μ is called *diffusion invariant* when $D_t\mu = \mu$.

In addition, we present the important properties of the heat kernel on real line and circle that will be later needed.

Remark 2.1 Let $p_t(x, y)$ be the heat kernel on $\mathbb{R}$, that is

$$p_t(x, y) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{|x-y|^2}{4t}}.$$

Let $R > 0$, $R' > R$ and $t > 0$. One can check that for $s \geq 0$ satisfying $R^2s = (R')^2t$

$$\int_{kR}^{(k+1)R} p_t(x, 0) dx = \int_{kR'}^{(k+1)R'} p_s(x, 0) dx$$

for all $k \in \mathbb{Z}$. Let $\epsilon > 0$. Suppose that $R = l\xi\epsilon$, and $R' = l'\xi\epsilon$, $l, l', \xi \in \mathbb{N}$. Define sets

$$A_i^k = \bigcup_{j=0}^{l-1} [kR + (j\xi + i)\epsilon, kR + (il + j + 1)\epsilon]$$

$$B_i^k = \bigcup_{j=0}^{l'-1} [kR + (jl\xi + i)\epsilon, kR + (il + j + 1)\epsilon]$$

where $k \in \mathbb{Z}$ and $i = 0, 1, \dots, \xi - 1$. Due to uniform equicontinuity of the heat kernel $p_t(x, 0)$ on $\mathbb{R}$, one finds that there exists $T > 0$ such that for all $s, t > T$ satisfying $R^2s = (R')^2t$

$$\sum_{k \in \mathbb{Z}} \sum_{i=0}^{\xi-1} \left| \int_{A_i^k} p_t(x, 0) dx - \int_{B_i^k} p_s(x, 0) dx \right| \leq \epsilon.$$

Remark 2.2 Due to the form of the heat kernel on circle, which is

$$P_t(x, y) = \sum_{n \in \mathbb{Z}} p_t(x + mn, y),$$

with $p_t(x, y)$ being the heat kernel in $\mathbb{R}$, we can extend Remark 2.1 to the sets of the form A_i^k and B_i^k , $k = 0, \dots, m-1$, $i = 1, \dots, \xi-1$, on circles of the length mR and mR' , respectively.

For a detailed description of the theory of harmonic measures and foliated heat diffusion, one should refer to [2], where it is described in full generality in terms of foliated spaces.



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Metric Diffusion Along Foliations

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