Chapter 2
Non-linear Equations

2.1 First Examples of Non-linear Ordinary Differential Equations

Before describing some general properties of non-linear equations, we find it useful to present some well-known examples that remain of high pedagogical value, following [72].

(i) **Circles** In analytic geometry, the circles in a plane

$$x^2 + y^2 + 2Ax + 2By + C = 0 \tag{2.1.1}$$

for a three-parameter family of plane curves; the corresponding differential equation is therefore of the third order. By differentiating Eq. (2.1.1) three times, one finds

$$\begin{aligned} &x + y\frac{\mathrm{d}y}{\mathrm{d}x} + A + B\frac{\mathrm{d}y}{\mathrm{d}x} = 0, \\ &1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + (y + B)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0, \\ &3\frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + (y + B)\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = 0. \end{aligned} \tag{2.1.2}$$

The parameter B can be eliminated between the last two equations, and this leads to the non-linear functional equation

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3}\left[1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right] - 3\frac{\mathrm{d}y}{\mathrm{d}x}\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)^2 = 0. \tag{2.1.3}$$

The only plane curves that satisfy this equation are circles and straight lines. Of course, any straight line is an integral curve, because Eq. (2.1.3) is satisfied if we have vanishing second derivative of y, and hence also its third derivative vanishes.

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If instead the second derivative of y does not vanish, one can write Eq. (2.1.3) in the form

$$\frac{\frac{d^3 y}{dx^3}}{\frac{d^2 y}{dx^2}} = \frac{3\frac{dy}{dx}\frac{d^2 y}{dx^2}}{\left[1+\left(\frac{dy}{dx}\right)^2\right]}, \tag{2.1.4}$$

from which one obtains

$$\log\left(\frac{d^2 y}{dx^2}\right) = \frac{3}{2}\log\left[1+\left(\frac{dy}{dx}\right)^2\right] + \log(C_1), \tag{2.1.5}$$

where C_1 is a non-vanishing constant. This formula may be written in the equivalent form

$$\frac{\frac{d^2 y}{dx^2}}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}} = C_1. \tag{2.1.6}$$

Now one can integrate again to find

$$\frac{\frac{dy}{dx}}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}} = C_1 x + C_2, \tag{2.1.7}$$

i.e.

$$\frac{dy}{dx} = \frac{C_1 x + C_2}{\sqrt{1-(C_1 x + C_2)^2}}. \tag{2.1.8}$$

A third integration yields eventually

$$C_1 y + C_3 = -\sqrt{1-(C_1 x + C_2)^2}, \tag{2.1.9}$$

and the equation of a circle is recovered.
(ii) **Conics** If a conic has no asymptote parallel to the y-axis, its equation can be written in the form

$$y = mx + n + \sqrt{Ax^2 + 2Bx + C}. \tag{2.1.10}$$

Upon differentiating twice, one finds

$$\frac{d^2 y}{dx^2} = \frac{(AC - B^2)}{(Ax^2 + 2Bx + C)^{\frac{3}{2}}}, \tag{2.1.11}$$

or the equivalent form

$$\left(\frac{d^2y}{dx^2}\right)^{-\frac{2}{3}} = (AC - B^2)^{-\frac{2}{3}}(Ax^2 + 2Bx + C), \tag{2.1.12}$$

which implies that the left-hand side is a trinomial of second degree in x. Thus, in order to eliminate the three constants A, B, C three differentiations are sufficient, and the desired differential equation can be written as

$$\frac{d^3}{dx^3}\left[\left(\frac{d^2y}{dx^2}\right)^{-\frac{2}{3}}\right] = 0. \tag{2.1.13}$$

Upon performing the three derivatives, one obtains the Halphen non-linear equation

$$40\left(\frac{d^3y}{dx^3}\right)^3 - 45\frac{d^2y}{dx^2}\frac{d^3y}{dx^3}\frac{d^4y}{dx^4} + 9\left(\frac{d^2y}{dx^2}\right)^2\frac{d^5y}{dx^5} = 0. \tag{2.1.14}$$

(iii) **Parabolas** For a parabola, the coefficient A vanishes in the previous formulae, and the fractional power $-\frac{2}{3}$ of the second derivative of y is therefore a binomial of first degree. The differential equation (2.1.13) is then replaced by

$$\frac{d^2}{dx^2}\left[\left(\frac{d^2y}{dx^2}\right)^{-\frac{2}{3}}\right] = 0, \tag{2.1.15}$$

which leads to

$$5\left(\frac{d^3y}{dx^3}\right)^2 - 3\frac{d^2y}{dx^2}\frac{d^4y}{dx^4} = 0. \tag{2.1.16}$$

(iv) **Bernoulli's equation** This is a non-linear equation reading as

$$\frac{dy}{dx} + u_0 y + u_1 y^n = 0, \tag{2.1.17}$$

where the exponent n is any real number different from 0 and 1. Interestingly, this non-linear equation can be reduced to a linear equation by introducing the new dependent variable $z \equiv y^{1-n}$, because

$$z \equiv y^{1-n} \Longrightarrow \frac{1}{(1-n)}\frac{dz}{dx} + u_0 z + u_1 = 0. \tag{2.1.18}$$

In general, one can reduce to the previous type any equation of the form

$$\phi\left(\frac{y}{x}\right)dx + \psi\left(\frac{y}{x}\right)dy + kx^m(x\,dy - y\,dx) = 0, \tag{2.1.19}$$

where k and m are arbitrary numbers. Indeed, upon defining $y \equiv wx$, such an equation can be written in the form

$$[\phi(w) + w\psi(w)]\frac{\mathrm{d}x}{\mathrm{d}w} + x\psi(w) + kx^{m+2} = 0. \tag{2.1.20}$$

At this stage, by considering $z \equiv x^{-(m+1)}$, one obtains a linear equation.
(v) **Riccati equation** This equation occurs frequently in the classical differential geometry of curves and surfaces [11, 30], and provides another enlightening example of a non-linear equation that can be re-expressed exactly in linear form under certain assumptions. Nowadays it is customary to write it as

$$\frac{\mathrm{d}y}{\mathrm{d}x} + u_2 y^2 + u_1 y + u_0 = 0, \tag{2.1.21}$$

although Riccati limited himself to studying the equation [30]

$$\frac{\mathrm{d}y}{\mathrm{d}x} = ay^2 + bx^m, \tag{2.1.22}$$

which is a particular case of Eq. (2.1.21), with

$$u_2 = -a,\ u_1 = 0,\ u_0 = -b,\ m = 0. \tag{2.1.23}$$

In general, Eq. (2.1.21) cannot be integrated by quadratures, but, *if a particular integral is known, the general integral can be found by two quadratures*. Indeed, let y_1 be a particular integral. Upon considering the split $y = y_1 + z$, the equation for z has the same form but does not contain any term independent of z, since $z = 0$ must be an integral. The non-linear equation for z reads as

$$\frac{\mathrm{d}z}{\mathrm{d}x} + (u_1 + 2u_2 y_1)z + u_2 z^2 = 0, \tag{2.1.24}$$

and this equation suggests defining

$$u \equiv \frac{1}{z} \tag{2.1.25}$$

in order to achieve a linear equation. This proves what we anticipated.

This simple result has several important consequences. The general integral of the linear equation for u reads as

$$u = Cf(x) + \phi(x). \tag{2.1.26}$$

By virtue of (2.1.25) and (2.1.26), the general integral of the Riccati equation is

$$y = y_1 + \frac{1}{Cf(x) + \phi(x)} = \frac{Cf_1(x) + \phi_1(x)}{Cf(x) + \phi(x)}. \tag{2.1.27}$$

In other words, *once a particular solution is known, the general integral of the Riccati equation is a rational function of first degree in the integration constant*.

Conversely, every differential equation of first order having this property is a Riccati equation. Indeed, if f, ϕ, f_1, ϕ_1 are any four functions of x, all functions y represented by Eq. (2.1.27), where C is an arbitrary constant, are integrals of an equation of first order, which is obtained by solving Eq. (2.1.27) for C and then deriving with respect to x. This method leads to

$$C = \frac{\phi_1 - y\phi}{yf - f_1}, \tag{2.1.28}$$

and the corresponding differential equation is

$$\begin{aligned} &(yf - f_1)\left(\frac{\mathrm{d}\phi_1}{\mathrm{d}x} - \phi\frac{\mathrm{d}y}{\mathrm{d}x} - y\frac{\mathrm{d}\phi}{\mathrm{d}x}\right) \\ &- (\phi_1 - y\phi)\left(\frac{\mathrm{d}y}{\mathrm{d}x}f + y\frac{\mathrm{d}f}{\mathrm{d}x} - \frac{\mathrm{d}f_1}{\mathrm{d}x}\right) = 0, \end{aligned} \tag{2.1.29}$$

which is of the form (2.1.21).

Let us now consider four particular integrals y_1, y_2, y_3, y_4 that correspond to the values C_1, C_2, C_3 and C_4, respectively, of the constant C. By virtue of the theory of the anharmonic ratio, one can write the relation

$$\frac{(y_4 - y_1)}{(y_4 - y_2)} + \frac{(y_3 - y_1)}{(y_3 - y_2)} = \frac{(C_4 - C_1)}{(C_4 - C_2)} + \frac{(C_3 - C_1)}{(C_3 - C_2)}. \tag{2.1.30}$$

This can be verified also by a patient calculation, and proves that *the anharmonic ratio of any four particular integrals of the Riccati equation is constant*.

This theorem makes it possible to find without quadratures the general integral of a Riccati equation, *provided that one knows three of its particular integrals* y_1, y_2, y_3. Every other integral must be such that the anharmonic ratio

$$\frac{(y - y_1)}{(y - y_2)} + \frac{(y_3 - y_1)}{(y_3 - y_2)}$$

is constant. The general integral is then obtained by equating this ratio to an arbitrary constant. By construction, y is a rational function of first degree in this constant, which proves that the previous property holds only for the Riccati equation.

If instead we knew only two particular integrals, y_1 and y_2, we might complete the integration by performing one quadrature; as a matter of fact, after the split $y = y_1 + z$, the equation obtained for the unknown function z has the integral $y_2 - y_1$.

The linear equation in u has therefore the particular integral $\frac{1}{(y_2-y_1)}$. The general integral of the differential equation in u is then found by a single quadrature.

This paragraph is only an elementary introduction to the Riccati equation. For a thorough investigation from the point of view of differential geometry, we refer the reader to Sect. 7.2.2 of Carinena et al. [22].

2.2 Non-linear Differential Equations in the Complex Domain

The study of solutions of non-linear equations exhibits difficulties resulting from the existence of singularities which are not fixed, but movable. In this section we introduce the reader to the results of Briot and Bouquet, relying upon the monograph by Valiron [134].

If a first-order differential equation is given, here written in the form

$$\frac{dy}{dx} = f(x, y), \tag{2.2.1}$$

we know that it admits a unique solution taking for $x = x_0$ the value y_0, when the function $f(x, y)$ is analytic at the point (x_0, y_0). This solution is holomorphic within a circle centred at x_0, and the coefficients of its expansion can be evaluated one after the other.

Suppose now that, for $x = x_0$, $y = y_0$, the function $f(x, y)$ is not analytic, whereas $\frac{1}{f(x,y)}$ is analytic. Let us set

$$g(x, y) \equiv \frac{1}{f(x, y)}. \tag{2.2.2}$$

One has $g(x_0, y_0) = 0$, otherwise $f(x, y)$ would be analytic. One can thus say that $f(x, y)$ is infinite at the point (x_0, y_0). In order to study the solutions of Eq. (2.2.1) in the neighbourhood of the point (x_0, y_0), one can reduce this point to the origin, by considering the equation

$$\frac{dx}{dy} = g(x, y) = a_{1,0}x + a_{0,1}y + \cdots . \tag{2.2.3}$$

The existence theorem holds for this equation. It admits a unique solution that vanishes at $y = 0$, and which is holomorphic for $|y|$ sufficiently small. Since its derivative vanishes at $y = 0$, this solution is of the form

$$x = c_p y^p + \cdots , \tag{2.2.4}$$

unless it is identically vanishing. In the general case where the solution has the form (2.2.4), one obtains, by performing the inversion,

$$y = \gamma_p x^{\frac{1}{p}} + \cdots \,. \tag{2.2.5}$$

Thus, *there exists a unique solution taking at* $x = 0$ the value 0, *it is an analytic function admitting the origin as an algebraic critical point*.

For the Eq. (2.2.3) to admit for solution 0, it is necessary that the right-hand side vanishes identically at $x = 0$, hence it contains as a factor a power of x, and this is sufficient. In this case, there exists no solution of Eq. (2.2.1) taking at $x = x_0$ the value y_0. As examples, the equations

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2y}, \ \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2xy} \tag{2.2.6}$$

are solved, respectively, by

$$y = \sqrt{x + C}, \ y = \sqrt{\log(Cx)}. \tag{2.2.7}$$

If $C = 0$, the first curve in Eq. (2.2.7) passes through the origin, whereas the second curve in (2.2.7) never meets the origin.

Briot and Bouquet have studied the case in which the function on the right-hand side of Eq. (2.2.1) is indeterminate at the point (x_0, y_0) and analytic about this point. We shall assume that this point is the origin. Let us consider the equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y), \tag{2.2.8}$$

where $f(x, y)$ is analytic about the origin and vanishes at this point. One has

$$f(x, y) = a_{1,0}x + a_{0,1}y + \cdots \,, \tag{2.2.9}$$

and one can write Eq. (2.2.8), by changing the notation:

$$xy' - \lambda y = ax + \varphi(x, y), \tag{2.2.10}$$

where the series $\varphi(x, y)$ is an entire function that begins with terms of second degree. Let us look for a holomorphic solution, necessarily vanishing at the origin, given by

$$y = c_1 x + c_2 x^2 + \cdots \,. \tag{2.2.11}$$

The formal evaluation of coefficients of this series is possible if λ is not a positive integer. One has

$$(-\lambda + 1)c_1 = a, \ldots, (-\lambda + n)c_n = P_n(c_1, c_2, \ldots, c_{n-1}). \tag{2.2.12}$$

With this notation, P_n is a polynomial in $c_1, c_2, \ldots, c_{n-1}$, and with respect to the coefficients of the function $\varphi(x, y)$; the coefficients of this polynomial are positive. One can prove convergence of the series (2.2.11) so obtained by using the method of majorant functions. Let us denote by H the minimum of the numbers $|1 - \lambda|$, $|2 - \lambda|, \ldots$. This minimum exists because these numbers are not vanishing and have as the only accumulation point the value $+\infty$, and is a positive number by construction. Let

$$F(X, Y) = \frac{M}{\left(1 - \frac{X}{r}\right)\left(1 - \frac{Y}{R}\right)} - M - M\frac{Y}{R} \equiv AX + \Phi(X, Y) \tag{2.2.13}$$

be a majorant of the right-hand side of Eq. (2.2.10). Let us consider the implicit equation

$$HY - F(X, Y) = 0. \tag{2.2.14}$$

If one looks for a solution in the form of an entire series $Y = C_1 X + C_2 X^2 + \cdots$, the coefficients will be given by the equalities

$$HC_1 = A, \ldots, \ HC_n = P_n(C_1, C_2, \ldots, C_{n-1}), \tag{2.2.15}$$

where the polynomial P_n, in $C_1, \ldots, C_{n-1}$ and with respect to the coefficients of $\Phi(X, Y)$, is the same as in Eq. (2.2.12). The series Y majorizes therefore the series (2.2.11) because $H \leq |n - \lambda|$. Or, by virtue of the existence theorem of implicit functions in the analytic case, Eq. (2.2.14) admits a holomorphic solution vanishing at $X = 0$ because the derivative of the left-hand side with respect to Y, evaluated at $X = 0$, is $H \neq 0$. The series $Y = C_1 X + C_2 X^2 + \cdots$ converges in a certain circle, hence also the series (2.2.11). We have therefore proved the following result of Briot and Bouquet:

Theorem 2.1 *If λ is not a positive integer, the differential equation (2.1.10), where $\varphi(x, y)$ is analytic about the origin and has neither terms independent of x and y nor terms of first degree in x and y, admits a unique holomorphic solution that vanishes at $x = 0$.*

The assessment of the simplest cases shows that this holomorphic solution is not necessarily the only solution vanishing at the origin. As a first example, let us consider the equation

$$xy' - \lambda y = ax, \tag{2.2.16}$$

where a and λ are some constants. This is homogeneous and can be integrated straight away. One has, if $\lambda - 1 \neq 0$, the general integral

$$y = \frac{a}{(1 - \lambda)} x + Cx^{\lambda}, \tag{2.2.17}$$

where C is an arbitrary constant. If we assume that λ is a non-integer positive number, the solution holomorphic at the origin corresponds to $C = 0$. If λ has a positive real part, all solutions approach 0 if x approaches 0 by following a suitable path. Indeed, if r is the modulus and ψ is the argument of x, and if $\lambda = \alpha + \mathrm{i}\beta$, one has

$$\left|x^{\lambda}\right| = r^{\alpha} e^{-\psi\beta}. \tag{2.2.18}$$

Since α is positive, it suffices that r approaches 0 and ψ remains fixed for y to approach 0.

If $\alpha \leq 0$ and $\beta \neq 0$, it is sufficient to assume that ψ is a function $\psi(r)$ such that

$$\lim_{r \to 0} \Big(\alpha \log(r) - \beta\psi(r)\Big) = -\infty, \tag{2.2.19}$$

for y to approach 0. One can assume that x approaches 0 on a logarithmic spiral $\psi(r) = k \log(r)$, k being a constant such that $\alpha - k\beta > 0$. Thus, the only case in which the holomorphic integral is the only that approaches 0 when x approaches 0 is that one where $\lambda \leq 0$.

2.3 Integrals Not Holomorphic at the Origin

In the general case of Eq. (2.2.10), λ not being a positive integer, Briot and Bouquet obtained integrals that can approach 0 when $x \to 0$ following a suitable path. It is necessary to assume that λ is not a real number ≤ 0.

Let us denote by z the solution holomorphic at $x = 0$ and let us set $y = z + u$. Equation (2.2.10) is then transformed into the equation

$$x \frac{\mathrm{d}u}{\mathrm{d}x} = u \left(\lambda + \alpha_{1,0} x + \alpha_{0,1} u + \cdots\right). \tag{2.3.1}$$

On bearing in mind the investigation of the previous example, let us set

$$u = x^{\lambda} v, \tag{2.3.2}$$

v being the new unknown function, which will solve the equation

$$x \frac{\mathrm{d}v}{\mathrm{d}x} = v \Big[\alpha_{1,0} x + \alpha_{0,1} x^{\lambda} v + \cdots\Big]. \tag{2.3.3}$$

One can look for a solution having the form of double series in the variables x and x^{λ}, i.e.

$$v = \sum_{m,n=0}^{\infty} c_{m,n} x^{m+\lambda n}. \tag{2.3.4}$$

The subsequent coefficients will be evaluated one after the other. One has indeed

$$(m + \lambda n)c_{m,n} = Q_{m,n}(\alpha_{p,q}, c_{\mu,\nu}),\ \ m + n > 0, \tag{2.3.5}$$

$Q_{m,n}$ being a polynomial in $(\alpha_{p,q}, c_{\mu,\nu})$, with $\mu + \nu < m + n$, and having positive coefficients. The coefficient $c_{0,0}$ is arbitrary. One will replace $c_{m,n}$ by a positive number $C_{m,n} > |c_{m,n}|$ if one replaces in $Q_{m,n}$ the $\alpha_{p,q}$ by some positive numbers $A_{p,q} > |\alpha_{p,q}|$, similarly $c_{\mu,\nu}$ by $C_{\mu,\nu} \geq |c_{\mu,\nu}|$, and if $m + \lambda n$ is replaced by a positive number $H < |m + \lambda n|$. If λ is not negative, there exists a positive number H less than all numbers $|m + \lambda n|$, $m \geq 0$, $n \geq 0$. One can take for the $A_{p,q}$ the coefficients of a majorant of the crossed terms of the right-hand of Eq. (2.3.3), expressed in the form

$$\sum_{p,q} A_{p,q} X^{p+\Lambda q V q} = \frac{M}{\left(1 - \frac{X}{r}\right)\left(1 - \frac{V X^{\Lambda}}{R}\right)} - M,\ \ \Lambda \geq |\lambda|. \tag{2.3.6}$$

The coefficients of the expansion of the solution of the equation

$$H(V - C_{0,0}) = \frac{MV}{\left(1 - \frac{X}{r}\right)\left(1 - \frac{V X^{\Lambda}}{R}\right)} - MV,\ \ \Lambda \geq |\lambda| \tag{2.3.7}$$

which takes the value $C_{0,0}$ at $X = 0$, are given by the equalities

$$HC_{m,n} = Q_{m,n}(A_{p,q}, C_{\mu,\nu}),\ \ m + n > 0. \tag{2.3.8}$$

This solution is a majorant of the series (2.3.4). It is legitimate to assume that Λ is an integer; one then says that Eq. (2.3.7) admits a holomorphic solution equal to $C_{0,0}$ at $X = 0$, provided that $C_{0,0}$ is sufficiently small. The series (2.3.4) converges when $|x^{\lambda}|$ and $|x|$ are small enough for the series

$$\sum_{m,n} |c_{m,n}| X^{m+\Lambda n}$$

to converge for $X < X_0$. One obtains in such a way the second theorem of Briot and Bouquet relative to Eq. (2.2.10).

Theorem 2.2 *When λ is neither a positive integer nor a real number ≤ 0, every solution of Eq. (2.2.10), which takes a value sufficiently close to the one of the solution holomorphic at the origin for a value of x sufficiently small, approaches* 0 *when x approaches* 0 *along some paths suitably chosen.*

Case where λ is a Positive Integer. If one reverts to Eq. (2.2.10) and one assumes at the beginning $\lambda = 1$, the equation that provides the coefficient c_1 of the series (2.2.11) reduces to $a = 0$; there is no solution holomorphic at the origin if $\lambda = 1$, $a \neq 0$.

Suppose that $\lambda = 1$, $a = 0$. If one defines $y \equiv xz$, one obtains the equation

$$\frac{dz}{dx} = \frac{\varphi(x, zx)}{x^2}, \tag{2.3.9}$$

whose right-hand side is analytic at the origin. There exists a solution holomorphic at the origin, $z = \psi(x)$, taking at $x = 0$ an arbitrary value z_0 provided that $|z_0|$ is small enough. *Equation* (2.2.10) *admits an infinity of solutions holomorphic at the origin and vanishing at this point*.

If $\lambda = 1, a \neq 0$, the circumstances are different. Let us consider the particular case of the equation

$$xy' - y = ax, \ a = \text{const}, \tag{2.3.10}$$

which is the Euler homogeneous equation whose general integral is

$$y = x[C + a \log x], \tag{2.3.11}$$

C being the arbitrary constant. There is no solution holomorphic at the origin, but all solutions approach 0 when x approaches 0 in a suitable way (if r is the modulus of x and θ its argument, it suffices that θr approaches 0).

This feature is general. One shows that, when $\lambda = 1$, $a \neq 0$, one obtains solutions depending on an arbitrary constant and which are given by an entire series in the variables x and $x \log x$; they approach 0 with x under the conditions which have been specified.

Case $\lambda = n$, n *being a positive integer*. One performs the transformation

$$y = \frac{ax}{(1 - \lambda)} + xz, \tag{2.3.12}$$

which leads to the equation

$$xz' - (\lambda - 1)z = a_1 x + \varphi_1(x, z), \tag{2.3.13}$$

which has indeed the same form as (2.2.10), but with λ replaced by $\lambda - 1$. A finite number of such transformations brings back to the case $\lambda = 1$ that we have already considered. We can therefore state what follows:

In all cases where λ is a positive integer, two circumstances may occur: either there is no solution holomorphic at the origin, or there are infinitely many such solutions. There always exist infinitely many solutions, depending on a parameter, that approach 0 when x approaches 0 along some suitably chosen paths.

General Case If one assumes of dealing with an equation of the form

$$\frac{dy}{dx} = \frac{Y}{X}, \tag{2.3.14}$$

where X and Y are entire series in x and y, convergent in the neighbourhood of the origin and vanishing at the origin:

$$Y = ax + by + \cdots, \ X = a'x + b'y + \cdots, \tag{2.3.15}$$

one can set (as if X and Y were being reduced to their terms of first degree)

$$y = xz. \tag{2.3.16}$$

One obtains therefore the equation

$$x\frac{dz}{dx} = \frac{a + (b - a')z - b'z^2 + x\psi(x, z)}{a' + b'z + x\varphi(x, z)}, \tag{2.3.17}$$

φ and ψ being entire series. If $|a| + |a'| \neq 0$, one is brought back to the case studied previously: if $a \neq 0$, the differential coefficient is infinite at the origin; if $a = 0$, it is not determined. When one passes to the variable $y = xz$, a solution z admitting a simple pole at the origin will provide a solution holomorphic at the origin; a solution z holomorphic and not vanishing at the origin will provide a solution y of Eq. (2.3.14), holomorphic and vanishing at the origin.

In order to obtain the solutions of Eq. (2.3.17) which are holomorphic at the origin, one can set $z = z_0 + Z$, z_0 being the value of this solution at the origin. Hence Eq. (2.3.17) becomes

$$x\frac{dZ}{dx} = \frac{a + (b - a')z_0 - b(z_0)^2 + \cdots}{a' + b'z_0 + \cdots}. \tag{2.3.18}$$

Since the left-hand side vanishes at $x = 0$ if Z is holomorphic, one has the condition

$$a + (b - a')z_0 - b(z_0)^2 = 0, \tag{2.3.19}$$

and if $a' + b'z_0 \neq 0$, one is led to an equation of the type considered when we studied the case when $f(x, y)$ is not determined at the point (x_0, y_0), which admits, in general, a holomorphic solution. Equation (2.3.18) will therefore have, in general, two solutions holomorphic at the origin, corresponding to the two roots of Eq. (2.3.19).

If a and a' vanish, but b and b' are not both 0, one exchanges the roles of x and y, which can provide solutions $y(x)$ approaching 0 when x approaches 0.

If a, a', b, b' are all vanishing, the method used above shows that there exist, in general, three holomorphic integrals of Eq. (2.3.17), and hence three integrals of Eq. (2.3.14) holomorphic and vanishing at the origin.

Note that, when a root of Eq. (2.3.19) is also a root of $a' + b'z_0$, Eq. (2.3.18) for Z can take the new form

$$x^2\frac{dZ}{dx} = \alpha x + \beta Z + \cdots. \tag{2.3.20}$$

There may exist no solution holomorphic at the origin. This will indeed be the case for the equation

$$x^2\frac{dZ}{dx} = \beta Z + \alpha x, \tag{2.3.21}$$

whose general integral

$$Z = e^{-\frac{\beta}{x}} \left[C + \alpha \int e^{\frac{\beta}{x}} \frac{\mathrm{d}x}{x} \right], \; C = \text{const}, \tag{2.3.22}$$

admits, no matter what value is taken by C, the origin as an isolated singularity.

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