Preface

Functional equations express many profound properties in mathematics, classical and quantum physics, and the other sciences, and their unknowns are functions. Thus, the ordinary and partial differential equations (both linear and nonlinear), the pseudo-differential equations, and also the functional equations not involving any derivative at all [e.g., $f(x + y) = f(x) + f(y)$] are all branches of the huge tree of functional equations. A key question is, therefore: How are we going to study the above topics in our monograph?

It has always been my dream to write a book that enables the reader to elevate himself/herself and reach a stage where he/she clearly understands how to achieve a proof or how to set up a conceptual or algorithmic framework. The historical path remains, in my opinion, of high pedagogic value because it shows the sequence of efforts which led to the modern formulations and illustrates what has been gained by the use of modern notations and techniques. If we ignore this path, we run the risk of forming students who think that, since a few brilliant minds have already taken the trouble to understand how the physical and mathematical world can be described, they only have to learn a number of recipes to enable them to apply the well-established general principles. Such an attitude would be disastrous for science because it would fail to provide the fuel necessary to give rise to new ideas and new scientific revolutions.

Now, we can recall that several areas of modern mathematics, with all their physical implications, can indeed be seen as branches of this theory: algebraic functions, Abelian integrals, analytic functions of several complex variables, contact of curves and surfaces, envelopes of curves and surfaces, skew curves, linear and nonlinear ordinary differential equations, calculus of variations, mathematical theory of surfaces and absolute differential calculus, minimal surfaces, and linear and nonlinear partial differential equations, just to mention some of the themes frequently faced by mathematicians and physicists.

It is clear from the above that, if we were to discuss all conceivable topics, we would end up by writing a multivolume work which would hardly be readable, even if feasible. Thus, I instead conceived the following plan. The book consists of eight parts, of unequal length. Part I is devoted to linear and nonlinear ordinary differential equations. Part II studies linear elliptic equations, leading the reader from harmonic to polyharmonic functions and showing the role of the Laplace-Beltrami operator in the theory of surfaces, developing the theory of Sobolev spaces, and presenting the original and the modern derivations of the Caccioppoli–Leray inequality. Spectral theory, mixed boundary-value problems, Morrey and Campanato spaces, and pseudo-holomorphic functions are further topics studied in some detail. Part III introduces the reader to a rigorous formulation of the Euler integral and differential conditions and to variational problems with constraints; thereafter, isoperimetric problems, minimal surfaces, and the Dirichlet boundary-value problem are studied. Part IV is devoted to linear and nonlinear hyperbolic equations. Several basic concepts are defined and discussed: characteristic manifolds for first- and second-order systems, wavelike propagation, hyperbolic equations, Riemann kernel for a hyperbolic equation in two variables, Cauchy's method for integrating a first-order equation, bicharacteristics, the relation between fundamental solution and Riemann's kernel, the characteristic conoid, the Hamiltonian form of geodesic equations, fundamental solution with odd and even number of variables, examples of fundamental solution, and parametrix of the scalar wave equation in curved space-time. Part IV ends with a detailed presentation of the Fourès-Bruhat use of integral equations for solution of linear and nonlinear systems of equations in normal hyperbolic form, with application to the Cauchy problem for general relativity with nonanalytic Cauchy data. Causal structure of space-time and global hyperbolicity are eventually introduced. Part V begins with the elementary theory of parabolic equations, but the detailed evaluation of the fundamental solution of the heat equation prepares the ground for the advanced use of the fundamental solution that Nash made in his proof of continuity of solutions of parabolic and elliptic equations. The moment and G bounds, the overlap estimate, and time continuity are studied following Nash's seminal paper. Part VI studies the celebrated Poincaré work on Fuchsian functions and its application to the solution of a famous nonlinear elliptic equation, which consists in studying the kernel of Laplacian plus exponential. Thus, the reader can see in action the profound link between the theory of functions and the investigation of nonlinear elliptic equations. Part VII studies the Riemann ζ-function, with emphasis on the functional equation for the ξ-function built out of the Γ- and ζ-functions, and on the integral representation for the ξ-function. Eventually, the Riemann hypothesis on nontrivial zeros of the ζ-function is discussed. Last, but not least, Part VIII opens a window on modern theory, defining the geometric setting for pseudo-differential operators on manifolds and discussing the issue of local solvability of partial differential and pseudo-differential equations.

Although I have relied heavily upon my sources in the various chapters, I believe that my navigation path is original, and I hope it will be helpful for young researchers and teachers. I have tried to make it clear to the general reader that an interdisciplinary view of mathematics is essential to make further progress: we need the abstract concepts of algebra, the majorizations and rigor of analysis, the global and local geometric descriptions, and the stunning properties of number theory. This holds true in both mathematical and physical sciences. I have covered a selection of topics and references, frequently writing the chapters as single lectures for a strongly motivated class of graduate students in order to enhance their pedagogic value. The book is aimed at developing the sensibility and skills of readers so that they can begin advanced research after reading it. What will be the basic structures for studying functional equations in the centuries to come? When Forsyth completed his monumental treatise on the theory of differential equations [58–63] at the beginning of the nineteenth century, he was lacking many fundamental tools, e.g., the theory of distributions, weak solutions, and the space-time view provided by special and general relativity. Which tools are we still lacking? Which ones will remain useful?

I leave the reader with these thoughts, and hope that the answers to the above questions will help us, mankind, in our efforts to understand the physical world and its beautiful mathematical language.

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