Chapter 2
Coordinate Transformations

2.1 Introduction

Partial differential equations in the physical domain X^n can be solved on a numerical grid obtained by mapping a reference grid in the logical region Ξ^n into X^n with a coordinate transformation $\boldsymbol{x}(\boldsymbol{\xi}) : \Xi^n \to X^n$. The mapping approach also gives an alternative way to obtain a numerical solution to a partial differential equation, by solving the transformed equation with respect to the new independent variables ξ^i on the reference grid in the logical domain Ξ^n. Some notions and relations concerning the coordinate transformations yielding grids are discussed in this chapter. These notions and relations are used to represent some conservation-law equations in the new logical coordinates in a convenient form. The relations presented will be used in Chap. 3 to formulate various grid properties.

Conservation-law equations in curvilinear coordinates are typically deduced from the equations in Cartesian coordinates through the classical formulas of tensor calculus, through procedures which include the substitution of tensor derivatives for ordinary derivatives. The formulation and evaluation of the tensor derivatives is rather difficult, and they retain some elements of mystery. However, these derivatives are based on specific transformations of tensors, modeling in the equations some dependent variables, e.g. the components of a fluid velocity vector, which after the transformation have a clear interpretation in terms of the contravariant components of the vector. With this concept, the conservation-law equations are readily written out in this chapter without application to the tensor derivatives, but utilizing instead only some specific transformations of the dependent variables, ordinary derivatives, and one basic identity of coordinate transformations derived from the formula for differentiation of the Jacobian.

For generality, the transformations of the coordinates are mainly considered for arbitrary n-dimensional domains, though in practical applications, the dimension n equals $1, 2, 3$, or 4 for time-dependent transformations of three-dimensional domains. We also apply chiefly a standard vector notation for the coordinates, as variables with indices. Sometimes, however, particularly in figures, the ordinary designation

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for three-dimensional coordinates, namely x, y, z for the physical coordinates and ξ, η, ζ for the logical ones, is used to simplify the presentation.

2.2 General Notions and Relations

This section presents some basic relations between Cartesian and curvilinear coordinates.

2.2.1 Jacobi Matrix

Let

$$\boldsymbol{x}(\boldsymbol{\xi}) : \Xi^n \to X^n , \qquad \boldsymbol{\xi} = (\xi^1, \dots, \xi^n) , \qquad \boldsymbol{x} = (x^1, \dots, x^n) , \tag{2.1}$$

be a smooth invertible coordinate transformation of the physical region $X^n \subset R^n$ from the parametric domain $\Xi^n \subset R^n$. If Ξ^n is a standard logical domain, then, in accordance with Chap. 1, this coordinate transformation can be used to generate a structured grid in X^n. Here and later, R^n presents the Euclidean space with the Cartesian basis $\boldsymbol{e}_1, \dots, \boldsymbol{e}_n$, which represents an orthogonal system of vectors, i.e.

$$\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \begin{cases} 1 & \text{if} \quad i = j \ , \\ 0 & \text{if} \quad i \neq j \ . \end{cases}$$

Thus, we have

$$\begin{aligned} \boldsymbol{x} &= x^1 \boldsymbol{e}_1 + \dots + x^n \boldsymbol{e}_n \ , \\ \boldsymbol{\xi} &= \xi^1 \boldsymbol{e}_1 + \dots + \xi^n \boldsymbol{e}_n \ . \end{aligned}$$

The values x^i, (ξ^i), $i = 1, \dots, n$, are called the Cartesian coordinates of the vector $\boldsymbol{x}$, $(\boldsymbol{\xi})$ in X^n. Analogously, the values ξ^i, $i = 1, \dots, n$, are called the Cartesian coordinates of the vector $\boldsymbol{\xi}$ in Ξ^n. The coordinate transformation (2.1) defines, in the domain X^n, new coordinates $\xi^1, \dots, \xi^n$, which are called the curvilinear coordinates. The matrix

$$\jmath = \left\{ \frac{\partial x^i}{\partial \xi^j} \right\} , \qquad i, j = 1, \dots, n ,$$

is referred to as the Jacobi matrix, and its Jacobian is designated by J:

$$J = \det \left\{ \frac{\partial x^i}{\partial \xi^j} \right\} , \qquad i, j = 1, \dots, n . \tag{2.2}$$

The inverse transformation to the coordinate mapping $\boldsymbol{x}(\boldsymbol{\xi})$ is denoted by

$$\boldsymbol{\xi}(\boldsymbol{x}) : X^n \to \Xi^n.$$

This transformation can be considered analogously as a mapping introducing a curvilinear coordinate system $x^1, \dots, x^n$ in the domain $\Xi^n \subset R^n$. It is obvious that the inverse to the matrix $\jmath$ is

$$\jmath^{-1} = \left\{\frac{\partial \xi^i}{\partial x^j}\right\}, \qquad i, j = 1, \dots, n \ ,$$

and consequently

$$\det\left\{\frac{\partial \xi^i}{\partial x^j}\right\} = \frac{1}{J} , \qquad i, j = 1, \dots, n \ . \tag{2.3}$$

In the case of two-dimensional space, the elements of the matrices $(\partial x^i / \partial \xi^j)$ and $(\partial \xi^i / \partial x^j)$ are connected by

$$\begin{aligned} \frac{\partial \xi^i}{\partial x^j} &= (-1)^{i+j} \frac{\partial x^{3-j}}{\partial \xi^{3-i}} \Big/ J \ , \\ \frac{\partial x^i}{\partial \xi^j} &= (-1)^{i+j} J \frac{\partial \xi^{3-j}}{\partial x^{3-i}} \ , \qquad i, j = 1, 2 \ , \end{aligned} \tag{2.4}$$

with fixed indices i and j. Similar relations between the elements of the corresponding three-dimensional matrices have the form

$$\begin{aligned} \frac{\partial \xi^i}{\partial x^j} &= \frac{1}{J}\left(\frac{\partial x^{j+1}}{\partial \xi^{i+1}}\frac{\partial x^{j+2}}{\partial \xi^{i+2}} - \frac{\partial x^{j+1}}{\partial \xi^{i+2}}\frac{\partial x^{j+2}}{\partial \xi^{i+1}}\right) , \\ \frac{\partial x^i}{\partial \xi^j} &= J\left(\frac{\partial \xi^{j+1}}{\partial x^{i+1}}\frac{\partial \xi^{j+2}}{\partial x^{i+2}} - \frac{\partial \xi^{j+1}}{\partial x^{i+2}}\frac{\partial \xi^{j+2}}{\partial x^{i+1}}\right) , \qquad i, j = 1, 2, 3 \ , \end{aligned} \tag{2.5}$$

where i and j are fixed indices and for each superscript or subscript index, say l, $l + 3$ is equivalent to l. With this condition, the sequence of indices $(l, l + 1, l + 2)$ is the result of a cyclic permutation of $(1, 2, 3)$ and vice versa; the indices of a cyclic sequence (i, j, k) satisfy the relation $j = i + 1, k = i + 2$.

2.2.2 Tangential Vectors

The value of the function $\boldsymbol{x}(\boldsymbol{\xi}) = [x^1(\boldsymbol{\xi}), \dots, x^n(\boldsymbol{\xi})]$ in the Cartesian basis $(\boldsymbol{e}_1, \dots, \boldsymbol{e}_n)$, i.e.

$$\boldsymbol{x}(\boldsymbol{\xi}) = x^1(\boldsymbol{\xi})\boldsymbol{e}_1 + \dots + x^n(\boldsymbol{\xi})\boldsymbol{e}_n \ ,$$

is a position vector for every $\boldsymbol{\xi} \in \Xi^n$. This vector-valued function $\boldsymbol{x}(\boldsymbol{\xi})$ generates the nodes, edges, faces, etc. of the cells of the coordinate grid in the domain X^n. Each edge of the cell corresponds to a coordinate line ξ^i for some i and is defined by the vector

$$\Delta_i \boldsymbol{x} = \boldsymbol{x}(\boldsymbol{\xi} + h\boldsymbol{e}^i) - \boldsymbol{x}(\boldsymbol{\xi}) ,$$

where h is the step size of the uniform grid in the ξ^i direction in the logical domain Ξ^n. We have

$$\Delta_i \boldsymbol{x} = h\boldsymbol{x}_{\xi^i} + \boldsymbol{t} ,$$

where

$$\boldsymbol{x}_{\xi^i} = \left(\frac{\partial x^1}{\partial \xi^i}, \dots, \frac{\partial x^n}{\partial \xi^i}\right)$$

is the vector tangential to the coordinate curve ξ^i, and $\boldsymbol{t}$ is a residual vector whose length does not exceed the following quantity:

$$\frac{1}{2} \max |\boldsymbol{x}_{\xi^i \xi^i}| h^2 .$$

Thus, the cells in the domain X^n whose edges are formed by the vectors $h\boldsymbol{x}_{\xi^i}$, $i = 1, \dots, n$, are approximately the same as those obtained by mapping the uniform coordinate cells in the computational domain Ξ^n with the transformation $\boldsymbol{x}(\boldsymbol{\xi})$. Consequently, the uniformly contracted n-dimensional parallelepiped spanned by the tangential vectors $\boldsymbol{x}_{\xi^i}$, $i = 1, \dots, n$, represents, to a high order of accuracy with respect to h, the cell of the coordinate grid at the corresponding point in X^n (see Fig. 2.1 for $n = 2$). In particular, for the length l_i of the ith grid edge, we have

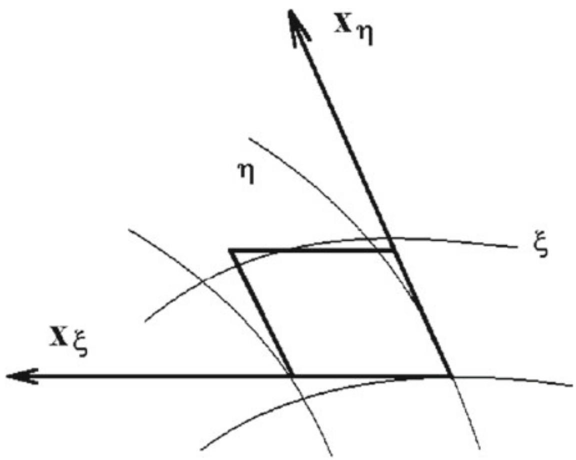

Fig. 2.1 Grid cell and contracted parallelogram

$$l_i = h|\boldsymbol{x}_{\xi^i}| + O(h^2) \, .$$

The volume V_h (area in two dimensions) of the cell is expressed as follows:

$$V_h = h^n V + O(h^{n+1}) \, ,$$

where V is the volume of the n-dimensional parallelepiped determined by the tangential vectors $\boldsymbol{x}_{\xi^i}$, $i = 1, \ldots, n$.

The tangential vectors $\boldsymbol{x}_{\xi^i}$, $i = 1, \ldots, n$, are called the base covariant vectors since they comprise a vector basis. The sequence $\boldsymbol{x}_{\xi^1}, \ldots, \boldsymbol{x}_{\xi^n}$ of the tangential vectors has a right-handed orientation if the Jacobian of the transformation $\boldsymbol{x}(\boldsymbol{\xi})$ is positive. Otherwise, the base vectors $\boldsymbol{x}_{\xi^i}$ have a left-handed orientation.

The operation of the dot product on these vectors produces elements of the covariant metric tensor. These elements generate the coefficients that appear in the transformed governing equations that model the conservation-law equations of mechanics. Besides this, the metric elements play a primary role in studying and formulating various geometric characteristics of the coordinate grid cells.

2.2.3 Normal Vectors

For a fixed i, the vector

$$\left(\frac{\partial \xi^i}{\partial x^1}, \ldots, \frac{\partial \xi^i}{\partial x^n} \right) ,$$

which is the gradient of $\xi^i(\boldsymbol{x})$ with respect to the Cartesian coordinates $x^1, \ldots, x^n$, is denoted by $\nabla \xi^i$. The set of the vectors $\nabla \xi^i$, $i = 1, \ldots, n$, is called the set of base contravariant vectors.

Similarly, as the tangential vectors relate to the coordinate curves, the contravariant vectors $\nabla \xi^i$, $i = 1, \ldots, n$, are connected with their respective $(n-1)$-dimensional coordinate surfaces (curves in two dimensions). A coordinate surface is defined by the equation $\xi^i = \xi^i_0$; i.e. along the surface, all of the coordinates $\xi^1, \ldots, \xi^n$ except ξ^i are allowed to vary. For all of the tangent vectors $\boldsymbol{x}_{\xi^j}$ to the coordinate lines on the surface $\xi^i = \xi^i_0$, we have the obvious identity

$$\boldsymbol{x}_{\xi^j} \cdot \nabla \xi^i = 0 \, , \qquad i \neq j \, ,$$

and thus the vector $\nabla \xi^i$ is normal to the coordinate surface $\xi^i = \xi^i_0$. Therefore, the vectors $\nabla \xi^i$, $i = 1, \ldots, n$, are also called the normal base vectors.

Since

$$\boldsymbol{x}_{\xi^i} \cdot \nabla \xi^i = 1$$

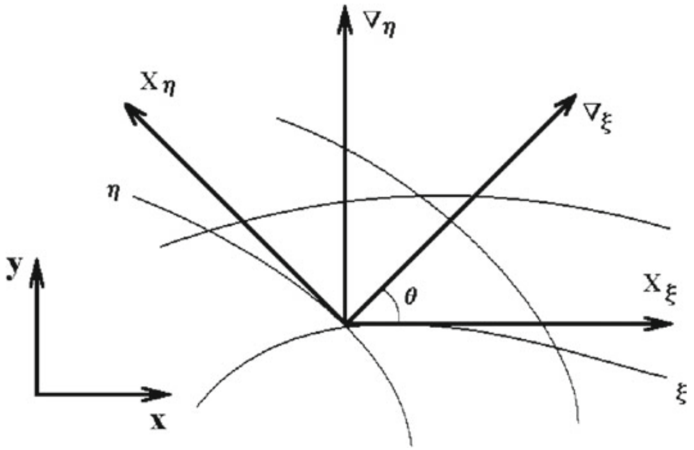

Fig. 2.2 Disposition of the base tangential and normal vectors in two dimensions

for each fixed $i = 1, \dots, n$, the vectors $\boldsymbol{x}_{\xi^i}$ and $\nabla\xi^i$ intersect each other at an angle θ which is less than $\pi/2$. Now, taking into account the orthogonality of the vector $\nabla\xi^i$ to the surface $\xi^i = \xi^i_0$, we find that these two vectors $\boldsymbol{x}_{\xi^i}$ and $\nabla\xi^i$ are directed to the same side of the $(n-1)$-dimensional coordinate surface (curve in two dimensions). An illustration of this fact in two dimensions is given in Fig. 2.2.

The length of any normal base vector $\nabla\xi^i$ is linked to the distance d_i between the corresponding opposite boundary segments (joined by the vector $\boldsymbol{x}_{\xi^i}$) of the n-dimensional parallelepiped formed by the base tangential vectors, namely,

$$d_i = 1/|\nabla\xi^i| \,, \qquad |\nabla\xi^i| = \sqrt{\nabla\xi^i \cdot \nabla\xi^i} \,.$$

To prove this relation, we recall that the vector $\nabla\xi^i$ is normal to all of the vectors $\boldsymbol{x}_{\xi^j}$, $j \neq i$, and therefore to the boundary segments formed by these $n-1$ vectors. Hence, the unit normal vector $\boldsymbol{n}_i$ to these segments is expressed by

$$\boldsymbol{n}_i = \nabla\xi^i/|\nabla\xi^i| \,.$$

Now, taking into account that

$$d_i = \boldsymbol{x}_{\xi^i} \cdot \boldsymbol{n}_i \,,$$

we readily obtain

$$d_i = \boldsymbol{x}_{\xi^i} \cdot \nabla\xi^i/|\nabla\xi^i| = 1/|\nabla\xi^i| \,.$$

Let l_i denote the distance between a grid point on the coordinate surface $\xi^i = c$ and the nearest point on the neighboring coordinate surface $\xi^i = c + h$; then,

$$l_i = hd_i + O(h^2) = h/|\nabla\xi^i| + O(h^2) \,.$$

This equation shows that the inverse length of the normal vector $\nabla\xi^i$ multiplied by h represents with high accuracy the distance between the corresponding faces of the coordinate cells in the domain X^n.

Note that the volume of the parallelepiped spanned by the tangential vectors equals J, so we find from (2.3) that the volume of the n-dimensional parallelepiped defined by the normal vectors $\nabla\xi^i$, $i = 1, \dots, n$, is equal to $1/J$. Thus, both the base normal vectors $\nabla\xi^i$ and the base tangential vectors $\boldsymbol{x}_{\xi^i}$ have the same right-handed or left-handed orientation.

If the coordinate system $\xi^1, \dots, \xi^n$ is orthogonal, i.e.

$$\boldsymbol{x}_{\xi^i} \cdot \boldsymbol{x}_{\xi^j} = P(\boldsymbol{x})\delta^i_j \ , \qquad P(\boldsymbol{x}) > 0 \ , \qquad i, j = 1, \dots, n \ ,$$

then for each fixed $i = 1, \dots, n$ the vector $\nabla\xi^i$ is parallel to $\boldsymbol{x}_{\xi^i}$. Here and later, δ^i_j is the Kronecker symbol, i.e.

$$\delta^i_j = 0 \quad \text{if} \quad i \neq j \ , \qquad \delta^i_j = 1 \quad \text{if} \quad i = j \ .$$

2.2.4 *Representation of Vectors Through the Base Vectors*

If there are n independent base vectors $\boldsymbol{a}_1, \dots, \boldsymbol{a}_n$ of the Euclidean space R^n, then any vector $\boldsymbol{b}$ with components $b^1, \dots, b^n$ in the Cartesian basis $\boldsymbol{e}_1, \dots, \boldsymbol{e}_n$ is represented through the vectors $\boldsymbol{a}^i$, $i = 1, \dots, n$, by

$$\boldsymbol{b} = a^{ij}(\boldsymbol{b} \cdot \boldsymbol{a}_j)\boldsymbol{a}_i, \qquad i, j = 1, \dots, n \ , \tag{2.6}$$

where a^{ij} are the elements of the matrix (a^{ij}) which is the inverse of the tensor (a_{ij}), $a_{ij} = \boldsymbol{a}_i \cdot \boldsymbol{a}_j$, $i, j = 1, \dots, n$. It is assumed in (2.6) and later that a summation is carried out over repeated indices unless otherwise noted.

The components of the vector $\boldsymbol{b}$ in the natural basis of the tangential vectors $\boldsymbol{x}_{\xi^i}$, $i = 1, \dots, n$, are called contravariant. Let them be denoted by $\overline{b}^i$, $i = 1, \dots, n$. Thus,

$$\boldsymbol{b} = \overline{b}^1 \boldsymbol{x}_{\xi^1} + \cdots + \overline{b}^n \boldsymbol{x}_{\xi^n} \ .$$

Assuming in (2.5) $\boldsymbol{a}_i = \boldsymbol{x}_{\xi^i}$, $i = 1, \dots, n$, we obtain

$$\overline{b}^i = a^{mj}\left(b^k \frac{\partial x^k}{\partial \xi^j}\right)\frac{\partial x^i}{\partial \xi^m} \ , \qquad i, j, k, m = 1, \dots, n \ , \tag{2.7}$$

where $b^1, \dots, b^n$ are the components of the vector $\boldsymbol{b}$ in the Cartesian basis $\boldsymbol{e}_1, \dots, \boldsymbol{e}_n$. Since

$$a_{ij} = \frac{\partial x^k}{\partial \xi^i}\frac{\partial x^k}{\partial \xi^j} \ , \qquad i, j, k = 1, \dots, n \ ,$$

we have

$$a^{ij} = \frac{\partial \xi^i}{\partial x^k} \frac{\partial \xi^j}{\partial x^k} , \qquad k = 1, \ldots, n .$$

Therefore, from (2.7),

$$\overline{b}^i = b^j \frac{\partial \xi^i}{\partial x^j} , \qquad i, j = 1, \ldots, n , \tag{2.8}$$

or, using the dot product notation,

$$\overline{b}^i = \boldsymbol{b} \cdot \nabla \xi^i , \qquad i = 1, \ldots, n . \tag{2.9}$$

Thus, in this case, (2.6) has the form

$$\boldsymbol{b} = (\boldsymbol{b} \cdot \nabla \xi^i) \boldsymbol{x}_{\xi^i} , \qquad i = 1, \ldots, n . \tag{2.10}$$

For example, the normal base vector $\nabla \xi^i$ is expanded through the base tangential vectors $\boldsymbol{x}_{\xi^j}$, $j = 1, \ldots, n$, by the following formula:

$$\nabla \xi^i = \frac{\partial \xi^i}{\partial x^j} \frac{\partial \xi^k}{\partial x^j} \boldsymbol{x}_{\xi^k} , \qquad i, j, k, = 1, \ldots, n . \tag{2.11}$$

Analogously, a component $\overline{b}_i$ of the vector $\boldsymbol{b}$ in the basis $\nabla \xi^i, i = 1, \ldots, n$, is expressed by the formula

$$\overline{b}_i = b^j \frac{\partial x^j}{\partial \xi^i} = \boldsymbol{b} \cdot \boldsymbol{x}_{\xi^i} , \qquad i = 1, \ldots, n , \tag{2.12}$$

and consequently

$$\boldsymbol{b} = \overline{b}_i \nabla \xi^i = (\boldsymbol{b} \cdot \boldsymbol{x}_{\xi^i}) \nabla \xi^i , \qquad i = 1, \ldots, n . \tag{2.13}$$

These components $\overline{b}_i, i = 1, \ldots, n$, of the vector $\boldsymbol{b}$ are called covariant. In particular, the base tangential vector $\boldsymbol{x}_{\xi^i}$ is expressed through the base normal vectors $\nabla \xi^j$, $j = 1, \ldots, n$, as follows:

$$\boldsymbol{x}_{\xi^i} = \frac{\partial x^j}{\partial \xi^i} \frac{\partial x^j}{\partial \xi^k} \nabla \xi^k , \qquad i, j, k = 1, \ldots, n . \tag{2.14}$$

2.2.5 Metric Tensors

Many grid generation algorithms, in particular those based on the calculus of variations, are typically formulated in terms of fundamental features of coordinate transformations and the corresponding mesh cells. These features are compactly described with the use of the metric notation, which is discussed in this subsection.

Covariant Metric Tensor

The matrix

$$\{g_{ij}\}\ , \qquad i,j=1,\dots,n\ ,$$

whose elements g_{ij} are the dot products of the pairs of the basic tangential vectors $\boldsymbol{x}_{\xi^i}$,

$$g_{ij}=\boldsymbol{x}_{\xi^i}\cdot\boldsymbol{x}_{\xi^j}=\frac{\partial x^k}{\partial \xi^i}\frac{\partial x^k}{\partial \xi^j}\ , \qquad i,j,k=1,\dots,n\ , \tag{2.15}$$

is called a covariant metric tensor of the domain X^n in the coordinates $\xi^1,\dots,\xi^n$. Geometrically, each diagonal element g_{ii} of the matrix $\{g_{ij}\}$ is the length of the tangent vector $\boldsymbol{x}_{\xi^i}$ squared:

$$g_{ii}=|\boldsymbol{x}_{\xi^i}|^2\ , \qquad i=1,\dots,n\ .$$

Also,

$$g_{ij}=|\boldsymbol{x}_{\xi^i}||\boldsymbol{x}_{\xi^j}|\cos\theta=\sqrt{g_{ii}}\sqrt{g_{jj}}\cos\theta\ , \tag{2.16}$$

where θ is the angle between the tangent vectors $\boldsymbol{x}_{\xi^i}$ and $\boldsymbol{x}_{\xi^j}$. In these expressions, for g_{ii} and g_{ij}, the subscripts ii and jj are fixed, i.e. here, the summation over the repeated indices is not carried out.

The matrix $\{g_{ij}\}$ is called the metric tensor because it defines distance measurements with respect to the coordinates $\xi^1,\dots,\xi^n$:

$$\mathrm{d}s=\sqrt{g_{ij}\mathrm{d}\xi^i\mathrm{d}\xi^j}, \qquad i,j=1,\dots,n\ .$$

Thus, the length s of the curve in X^n prescribed by the parametrization

$$\boldsymbol{x}[\boldsymbol{\xi}(t)]:[a,b]\rightarrow X^n$$

is computed by the formula

$$s=\int_a^b\sqrt{g_{ij}\frac{\mathrm{d}\xi^i}{\mathrm{d}t}\frac{\mathrm{d}\xi^j}{\mathrm{d}t}}\mathrm{d}t\ .$$

We designate by g the Jacobian of the covariant matrix $\{g_{ij}\}$. It is evident that

$$\{g_{ij}\} = \jmath\jmath^T \ ,$$

and hence

$$J^2 = g \ .$$

The covariant metric tensor is a symmetric matrix, i.e. $g_{ij} = g_{ji}$. If a coordinate system at a point $\boldsymbol{\xi}$ is orthogonal, then the tensor $\{g_{ij}\}$ has a simple diagonal form at this point. Note that these advantageous properties are in general not possessed by the Jacobi matrix $\{\partial x^i/\partial \xi^j\}$ from which the covariant metric tensor $\{g_{ij}\}$ is defined.

Contravariant Metric Tensor

The contravariant metric tensor of the domain X^n in the coordinates $\xi^1, \ldots, \xi^n$ is the matrix

$$\{g^{ij}\} \ , \qquad i, j = 1, \ldots, n \ ,$$

inverse to $\{g_{ij}\}$, i.e.

$$g_{ij} g^{jk} = \delta_i^k \ , \qquad i, j, k = 1, \ldots, n \ . \tag{2.17}$$

Therefore,

$$\det\{g^{ij}\} = \frac{1}{g} \ .$$

It is easily shown that (2.16) is satisfied if and only if

$$g^{ij} = \nabla \xi^i \cdot \nabla \xi^j = \frac{\partial \xi^i}{\partial x^k} \frac{\partial \xi^j}{\partial x^k} \ , \qquad i, j, k = 1, \ldots, n \ . \tag{2.18}$$

Thus, each diagonal element g^{ii} (where i is fixed) of the matrix $\{g^{ij}\}$ is the square of the length of the vector $\nabla \xi^i$:

$$g^{ii} = |\nabla \xi^i|^2 \ . \tag{2.19}$$

Geometric Interpretation

Now we discuss the geometric meaning of a fixed diagonal element g^{ii}, say g^{11}, of the matrix $\{g^{ij}\}$. Let us consider a three-dimensional coordinate transformation $\boldsymbol{x}(\boldsymbol{\xi}) : \Xi^3 \to X^3$. Its tangential vectors $\boldsymbol{x}_{\xi^1}, \boldsymbol{x}_{\xi^2}, \boldsymbol{x}_{\xi^3}$ represent geometrically the edges of the parallelepiped formed by these vectors. For the distance d_1 between the opposite faces of the parallelepiped which are defined by the vectors $\boldsymbol{x}_{\xi^2}$ and $\boldsymbol{x}_{\xi^3}$, we have

$$d_1 = \boldsymbol{x}_{\xi^1} \cdot \boldsymbol{n}_1 \ ,$$

where $\boldsymbol{n}_1$ is the unit normal to the plane spanned by the vectors $\boldsymbol{x}_{\xi^2}$ and $\boldsymbol{x}_{\xi^3}$. It is clear that

$$\nabla_{\xi^1} \cdot \boldsymbol{x}_{\xi^j} = 0 \,, \qquad j = 2, 3 \,,$$

and hence the unit normal $\boldsymbol{n}_1$ is parallel to the normal base vector ∇_{ξ^1}. Thus, we obtain

$$\boldsymbol{n}_1 = \nabla_{\xi^1}/|\nabla_{\xi^1}| = \nabla_{\xi^1}/\sqrt{g^{11}} \,.$$

Therefore,

$$d_1 = \nabla_{\xi^1} \cdot \nabla_{\xi^1}/\sqrt{g^{11}} = 1/\sqrt{g^{11}} \,,$$

and consequently

$$g^{11} = 1/(d_1)^2 \,.$$

Analogous relations are valid for g^{22} and g^{33}, i.e. in three dimensions, the diagonal element g^{ii} for a fixed i means the inverse square of the distance d_i between those faces of the parallelepiped which are connected by the vector $\boldsymbol{x}_{\xi^i}$. In two-dimensional space, the element g^{ii} (where i is fixed) is the inverse square of the distance between the edges of the parallelogram defined by the tangential vectors $\boldsymbol{x}_{\xi^1}$ and $\boldsymbol{x}_{\xi^2}$.

The same interpretation of g^{ii} is valid for general multidimensional coordinate transformations:

$$g^{ii} = 1/(d_i)^2, \qquad i = 1, \dots, n \,, \tag{2.20}$$

where the index i is fixed, and d_i is the distance between those faces of the n-dimensional parallelepiped which are linked by the tangential vector $\boldsymbol{x}_{\xi^i}$.

Relations Between Covariant and Contravariant Elements

Now, in analogy with (2.4) and (2.5), we write out very convenient formulas for natural relations between the contravariant elements g^{ij} and the covariant ones g_{ij} in two and three dimensions.

For $n = 2$,

$$\begin{aligned} g^{ij} &= (-1)^{i+j} \frac{g_{3-i\ 3-j}}{g} \,, \\ g_{ij} &= (-1)^{i+j} g g^{3-i\ 3-j} \,, \qquad i, j = 1, 2 \,, \end{aligned} \tag{2.21}$$

where the indices i, j on the right-hand side of the relations (2.21) are fixed, i.e. summation over the repeated indices is not carried out here. For $n = 3$, we have

$$\begin{aligned} g^{ij} &= \frac{1}{g}(g_{i+1\ j+1}\, g_{i+2\ j+2} - g_{i+1\ j+2}\, g_{i+2\ j+1}) \,, \\ g_{ij} &= g(g^{i+1\ j+1}\, g^{i+2\ j+2} - g^{i+1\ j+2}\, g^{i+2\ j+1}) \,, \qquad i, j = 1, 2, 3 \,, \end{aligned} \tag{2.22}$$

with the convention that any index, say l, is identified with $l \pm 3$, so, for instance, $g_{45} = g_{12}$.

We also note that, in accordance with the expressions (2.15) and (2.18) for g_{ij} and g^{ij}, respectively, the relations (2.11) and (2.14) between the basic vectors $\boldsymbol{x}_{\xi^i}$ and $\nabla\xi^j$ can be written in the form

$$\begin{aligned} \boldsymbol{x}_{\xi^i} &= g_{ik}\nabla\xi^k \ , \\ \nabla\xi^i &= g^{ik}\boldsymbol{x}_{\xi^k} \ , \qquad i,k = 1,\dots,n \ . \end{aligned} \tag{2.23}$$

So, the first derivatives $\partial x^i/\partial\xi^j$ and $\partial\xi^k/\partial x^m$ of the transformations $\boldsymbol{x}(\boldsymbol{\xi})$ and $\boldsymbol{\xi}(\boldsymbol{x})$, respectively, are connected through the metric elements:

$$\begin{aligned} \frac{\partial x^i}{\partial\xi^j} &= g_{mj}\frac{\partial\xi^m}{\partial x^i} \ , \\ \frac{\partial\xi^i}{\partial x^j} &= g^{mi}\frac{\partial x^j}{\partial\xi^m} \ , \qquad i,j,m = 1,\dots,n \ . \end{aligned} \tag{2.24}$$

2.2.6 *Cross Product*

In addition to the dot product, there is another important operation on three-dimensional vectors. This is the cross product, $\times$, which for any two vectors $\boldsymbol{a} = (a^1, a^2, a^3)$, $\boldsymbol{b} = (b^1, b^2, b^3)$ is expressed as the determinant of a matrix:

$$\boldsymbol{a}\times\boldsymbol{b} = \det\left\{\begin{matrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ a^1 & a^2 & a^3 \\ b^1 & b^2 & b^3 \end{matrix}\right\} \ , \tag{2.25}$$

where $(\boldsymbol{e}_1,\ \boldsymbol{e}_2,\ \boldsymbol{e}_3)$ is the Cartesian vector basis of the Euclidean space R^3. Thus,

$$\boldsymbol{a}\times\boldsymbol{b} = (a^2b^3 - a^3b^2,\ a^3b^1 - a^1b^3,\ a^1b^2 - a^2b^1) \ ,$$

or, with the previously mentioned convention in three dimensions of the identification of any index j with $j \pm 3$,

$$\boldsymbol{a}\times\boldsymbol{b} = (a^{i+1}b^{i+2} - a^{i+2}b^{i+1})\boldsymbol{e}_i \ , \qquad i = 1,2,3 \ . \tag{2.26}$$

We will now state some facts connected with the cross product operation.

Geometric Meaning

We can readily see that $\boldsymbol{a}\times\boldsymbol{b} = \boldsymbol{0}$ if the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel. Also, from (2.26), we find that $\boldsymbol{a}\cdot(\boldsymbol{a}\times\boldsymbol{b}) = 0$ and $\boldsymbol{b}\cdot(\boldsymbol{a}\times\boldsymbol{b}) = 0$, i.e. the vector $\boldsymbol{a}\times\boldsymbol{b}$ is orthogonal to each of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. Thus, if these vectors are not parallel, then

$$\boldsymbol{a}\times\boldsymbol{b} = \alpha|\boldsymbol{a}\times\boldsymbol{b}|\boldsymbol{n} \ , \tag{2.27}$$

where $\alpha = 1$ or $\alpha = -1$ and $\boldsymbol{n}$ is a unit normal vector to the plane determined by the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

Now we show that the length of the vector $\boldsymbol{a}\times\boldsymbol{b}$ equals the area of the parallelogram formed by the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, i.e.

$$|\boldsymbol{a} \times \boldsymbol{b}| = |\boldsymbol{a}||\boldsymbol{b}| \sin\theta \ , \tag{2.28}$$

where θ is the angle between the two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. To prove (2.28), we first note that

$$|\boldsymbol{a}|^2|\boldsymbol{b}|^2 \sin^2\theta = |\boldsymbol{a}|^2|\boldsymbol{b}|^2(1-\cos^2\theta) = |\boldsymbol{a}|^2|\boldsymbol{b}|^2 - (\boldsymbol{a}\cdot\boldsymbol{b})^2 \ .$$

We have, furthermore,

$$\begin{aligned}
|\boldsymbol{a}|^2|\boldsymbol{b}|^2 - |\boldsymbol{a}\cdot\boldsymbol{b}|^2 &= \left(\sum_{i=1}^{3} a^i a^i\right)\left(\sum_{j=1}^{3} b^j b^j\right) - \left(\sum_{k=1}^{3} a^k b^k\right)^2 \\
&= \sum_{k=1}^{3}[(a^l)^2(b^m)^2 + (a^m)^2(b^l)^2 - 2a^l b^l a^m b^m] \\
&= \sum_{k=1}^{3}(a^l b^m - a^m b^l)^2 \ ,
\end{aligned}$$

where (k, l, m) are cyclic, i.e. $l = k+1, \ m = k+2$ with the convention that $j+3$ is equivalent to j for any index j. According to (2.26), the quantity $a^l b^m - a^m b^l$ for the cyclic sequence (k, l, m) is the kth component of the vector $\boldsymbol{a}\times\boldsymbol{b}$, so we find that

$$|\boldsymbol{a}||\boldsymbol{b}| \sin^2\theta = |\boldsymbol{a}|^2|\boldsymbol{b}|^2 - |\boldsymbol{a}\cdot\boldsymbol{b}|^2 = |\boldsymbol{a}\times\boldsymbol{b}|^2 \ , \tag{2.29}$$

which proves (2.28). Thus, we obtain the result that if the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are not parallel, then the vector $\boldsymbol{a}\times\boldsymbol{b}$ is orthogonal to the parallelogram formed by these vectors and its length equals the area of the parallelogram. Therefore, the three vectors $\boldsymbol{a}, \ \boldsymbol{b}$ and $\boldsymbol{a}\times\boldsymbol{b}$ are independent in this case and represent a base vector system in the three-dimensional space R^3. Moreover, the vectors $\boldsymbol{a}, \ \boldsymbol{b}$ and $\boldsymbol{a}\times\boldsymbol{b}$ form a right-handed triad, since $\boldsymbol{a}\times\boldsymbol{b} \neq \boldsymbol{0}$, and consequently the Jacobian of the matrix determined by $\boldsymbol{a}, \ \boldsymbol{b}$, and $\boldsymbol{a}\times\boldsymbol{b}$, is positive; it equals

$$\boldsymbol{a}\times\boldsymbol{b}\cdot\boldsymbol{a}\times\boldsymbol{b} = (\boldsymbol{a}\times\boldsymbol{b})^2 \ .$$

Relation to Volumes

Let $\boldsymbol{c} = (c^1, c^2, c^3)$ be one more vector. The volume V of the parallelepiped whose edges are the vectors $\boldsymbol{a}, \ \boldsymbol{b}$ and $\boldsymbol{c}$ equals the area of the parallelogram formed by the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ multiplied by the modulas of the dot product of the vector $\boldsymbol{c}$ and the unit normal $\boldsymbol{n}$ to the parallelogram. Thus,

$$V = |\boldsymbol{a} \times \boldsymbol{b}||\boldsymbol{n} \cdot \boldsymbol{c}|$$

and from (2.27), we obtain

$$V = |(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c}| \ . \tag{2.30}$$

Taking into account (2.26), we obtain

$$(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c} = c^1(a^2b^3 - a^3b^2) + c^2(a^3b^1 - a^1b^3) + c^3(a^1b^2 - a^2b^1) \ .$$

The right-hand side of this equation is the Jacobian of the matrix whose rows are formed by the vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$, i.e.

$$(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c} = \det \begin{Bmatrix} a^1 & a^2 & a^3 \\ b^1 & b^2 & b^3 \\ c^1 & c^2 & c^3 \end{Bmatrix} \ . \tag{2.31}$$

From this equation, we readily obtain

$$(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c} = \boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{c} \times \boldsymbol{a}) \cdot \boldsymbol{b} \ .$$

Thus, the volume of the parallelepiped determined by the vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ equals the Jacobian of the matrix formed by the components of these vectors. In particular, we obtain from (2.2) that the Jacobian of a three-dimensional coordinate transformation $\boldsymbol{x}(\boldsymbol{\xi})$ is expressed as follows:

$$J = \boldsymbol{x}_{\xi^1} \cdot (\boldsymbol{x}_{\xi^2} \times \boldsymbol{x}_{\xi^3}) \ . \tag{2.32}$$

Relation to Base Vectors

Applying the operation of the cross product to two base tangential vectors $\boldsymbol{x}_{\xi^l}$ and $\boldsymbol{x}_{\xi^m}$, we find that the vector $\boldsymbol{x}_{\xi^l} \times \boldsymbol{x}_{\xi^m}$ is normal to the coordinate surface $\xi^i = \xi^i_0$ with (i, l, m) cyclic. The base normal vector $\nabla \xi^i$ is also orthogonal to the surface, and therefore it is a scalar multiple of $\boldsymbol{x}_{\xi^l} \times \boldsymbol{x}_{\xi^m}$, i.e.

$$\nabla \xi^i = c(\boldsymbol{x}_{\xi^l} \times \boldsymbol{x}_{\xi^m}) \ .$$

Multiplying this equation for a fixed i by $\boldsymbol{x}_{\xi^i}$, using the operation of the dot product, we obtain, using (2.32),

$$1 = c\,J \ ,$$

and therefore

$$\nabla \xi^i = \frac{1}{J}(\boldsymbol{x}_{\xi^l} \times \boldsymbol{x}_{\xi^m}) \ . \tag{2.33}$$

Thus, the elements of the three-dimensional contravariant metric tensor $\{g^{ij}\}$ are computed through the tangential vectors $\boldsymbol{x}_{\xi^i}$ by the formula

$$g^{ij} = \frac{1}{g}(\boldsymbol{x}_{\xi^{i+1}} \times \boldsymbol{x}_{\xi^{i+2}}) \cdot (\boldsymbol{x}_{\xi^{j+1}} \times \boldsymbol{x}_{\xi^{j+2}}) \,, \qquad i, j = 1, 2, 3 \,.$$

Analogously, every base vector $\boldsymbol{x}_{\xi^i}$, $i = 1, 2, 3$, is expressed by the tensor product of the vectors $\nabla\xi^j$, $j = 1, 2, 3$:

$$\boldsymbol{x}_{\xi^i} = J(\nabla\xi^l \times \nabla\xi^k), \qquad i = 1, 2, 3 \,, \tag{2.34}$$

where $l = i+1$, $k = i+2$, and m is equivalent to $m+3$ for any index m. Accordingly, we have

$$g_{ij} = g(\nabla\xi^{i+1} \times \nabla\xi^{i+2}) \cdot (\nabla\xi^{j+1} \times \nabla\xi^{j+2}) \,, \qquad i, j = 1, 2, 3 \,.$$

Using the relations (2.33) and (2.34) in (2.32), we also obtain

$$\frac{1}{J} = \nabla\xi^1 \cdot \nabla\xi^2 \times \nabla\xi^3 \,. \tag{2.35}$$

Thus, the volume of the parallelepiped formed by the base normal vectors $\nabla\xi^1$, $\nabla\xi^2$, and $\nabla\xi^3$ is the modulus of the inverse of the Jacobian J of the transformation $\boldsymbol{x}(\boldsymbol{\xi})$.

2.3 Relations Concerning Second Derivatives

The elements of the covariant and contravariant metric tensors are defined by the dot products of the base tangential and normal vectors, respectively. These elements are suitable for describing the internal features of the cells such as the lengths of the edges, the areas of the faces, their volumes, and the angles between the edges and the faces. However, as they are derived from the first derivatives of the coordinate transformation $\boldsymbol{x}(\boldsymbol{\xi})$, the direct use of the metric elements is not sufficient for the description of the dynamic features of the grid (e.g. curvature), which reflect changes between adjacent cells. This is because the formulation of these grid features relies not only on the first derivatives but also on the second derivatives of $\boldsymbol{x}(\boldsymbol{\xi})$. Therefore, there is a need to study relations connected with the second derivatives of the coordinate parametrizations.

This section presents some notations and formulas which are concerned with the second derivatives of the components of the coordinate transformations. These notations and relations will be used to describe the curvature and eccentricity of the coordinate lines and to formulate some equations of mechanics in new independent variables.

2.3.1 Christoffel Symbols

The edge of a grid cell in the ξ^i direction can be represented with high accuracy by the base vector $\boldsymbol{x}_{\xi^i}$ contracted by the factor h, which represents the step size of a uniform grid in Ξ^n. Therefore, the local change of the edge in the ξ^j direction is characterized by the derivative of $\boldsymbol{x}_{\xi^i}$ with respect to ξ^j, i.e. by $\boldsymbol{x}_{\xi^i\xi^j}$.

Since the second derivatives may be used to formulate quantitative measures of the grid, we describe these vectors $\boldsymbol{x}_{\xi^i\xi^j}$ through the base tangential and normal vectors using certain three-index quantities known as Christoffel symbols, The Christoffel symbols are commonly used in formulating measures of the mutual interaction of the cells and in formulas for differential equations.

Let us denote by Γ^k_{ij} the kth contravariant component of the vector $\boldsymbol{x}_{\xi^i\xi^j}$ in the base tangential vectors $\boldsymbol{x}_{\xi^k}$, $k = 1, \dots, n$. The superscript k in this designation relates to the base vector $\boldsymbol{x}_{\xi^k}$ and the subscript ij corresponds to the mixed derivative with respect to ξ^i and ξ^j. Thus,

$$\boldsymbol{x}_{\xi^i\xi^j} = \Gamma^k_{ij}\,\boldsymbol{x}_{\xi^k}\ , \qquad i, j, k = 1, \dots, n\ , \tag{2.36}$$

and consequently

$$\frac{\partial^2 x^p}{\partial \xi^j \partial \xi^k} = \Gamma^m_{kj} \frac{\partial x^p}{\partial \xi^m}\ , \qquad j, k, m, p = 1, \dots, n\ . \tag{2.37}$$

In accordance with (2.8), we have

$$\Gamma^i_{kj} = \frac{\partial^2 x^l}{\partial \xi^k \partial \xi^j} \frac{\partial \xi^i}{\partial x^l}\ , \qquad i, j, k, l = 1, \dots, n\ , \tag{2.38}$$

or in vector form,

$$\Gamma^i_{kj} = \boldsymbol{x}_{\xi^k\xi^j} \cdot \nabla \xi^i\ . \tag{2.39}$$

Equation (2.38) is also obtained by multiplying (2.37) by $\partial \xi^i / \partial x^p$ and summing over p.

The quantities Γ^i_{kj} are called the space Christoffel symbols of the second kind and the expression (2.36) is a form of the Gauss relation representing the second derivatives of the position vector $\boldsymbol{x}(\boldsymbol{\xi})$ through the tangential vectors $\boldsymbol{x}_{\xi^i}$.

Analogously, the components of the second derivatives of the position vector $\boldsymbol{x}(\boldsymbol{\xi})$ expanded in the base normal vectors $\nabla \xi^i$, $i = 1, \dots, n$, are referred to as the space Christoffel symbols of the first kind. The mth component of the vector $\boldsymbol{x}_{\xi^k\xi^j}$ in the base vectors $\nabla \xi^i$, $i = 1, \dots, n$, is denoted by $[kj, m]$. Thus, according to (2.12),

$$[kj, m] = \boldsymbol{x}_{\xi^k\xi^j} \cdot \boldsymbol{x}_{\xi^m} = \frac{\partial^2 x^l}{\partial \xi^k \partial \xi^j} \frac{\partial x^l}{\partial \xi^m}\ , \qquad j, k, l, m = 1, \dots, n\ , \tag{2.40}$$

and consequently

$$\boldsymbol{x}_{\xi^k \xi^j} = [kj, m] \nabla \xi^m \ . \tag{2.41}$$

So, in analogy with (2.37), we obtain

$$\frac{\partial^2 x^l}{\partial \xi^j \partial \xi^k} = [kj, m] \frac{\partial \xi^m}{\partial x^i} \ , \qquad i, j, k, m = 1, \dots, n \ . \tag{2.42}$$

Multiplying (2.40) by g^{im} and summing over m, we find that the space Christoffel symbols of the first and second kind are connected by the following relation:

$$\Gamma^i_{kj} = g^{im}[kj, m] \ , \qquad i, j, k, m = 1, \dots, n \ . \tag{2.43}$$

Conversely, from (2.38),

$$[kj, m] = g_{ml} \Gamma^l_{kj}, \qquad j, k, l, m = 1, \dots, n \ . \tag{2.44}$$

The space Christoffel symbols of the first kind $[kj, m]$ can be expressed through the first derivatives of the covariant elements g_{ij} of the metric tensor (g_{ij}) by the following readily verified formula:

$$[kj, m] = \frac{1}{2} \left(\frac{\partial g_{jm}}{\partial \xi^k} + \frac{\partial g_{km}}{\partial \xi^j} - \frac{\partial g_{kj}}{\partial \xi^m} \right) \ , \qquad i, j, k, m = 1, \dots, n \ . \tag{2.45}$$

Thus, taking into account (2.43), we see that the space Christoffel symbols of the second kind Γ^i_{kj} can be written in terms of metric elements and their first derivatives. In particular, in the case of an orthogonal coordinate system ξ^i, we obtain, from (2.43) and (2.45),

$$\Gamma^i_{kj} = \frac{1}{2} g^{ii} \left(\frac{\partial g_{ii}}{\partial \xi^k} + \frac{\partial g_{ii}}{\partial \xi^j} - \frac{\partial g_{kj}}{\partial \xi^i} \right) \ .$$

Here, the index i is fixed, i.e. the summation over i is not carried out.

2.3.2 Differentiation of the Jacobian

Of critical importance in obtaining compact conservation-law equations with coefficients derived from the metric elements in new curvilinear coordinates $\xi^1, \dots, \xi^n$ is the formula for differentiation of the Jacobian

$$\frac{\partial J}{\partial \xi^k} \equiv J \frac{\partial^2 x^i}{\partial \xi^k \partial \xi^m} \frac{\partial \xi^m}{\partial x^i} \equiv J \frac{\partial}{\partial x^i} \left(\frac{\partial x^i}{\partial \xi^k} \right) \equiv J \mathrm{div}_x \frac{\partial \boldsymbol{x}}{\partial \xi^k} \ ,$$
$$i, k, m = 1, \dots, n \ . \tag{2.46}$$

In accordance with (2.38), this identity can also be expressed through the space Christoffel symbols of the second kind Γ^i_{kj} by

$$\frac{\partial J}{\partial \xi^k} = J\Gamma^i_{ik} , \qquad i, k = 1, \ldots, n ,$$

with the summation convention over the repeated index i.

In order to prove the identity (2.46), we note that in the case of an arbitrary matrix $\{a_{ij}\}$, the first derivative of its Jacobian with respect to ξ^k is obtained through the process of differentiating the first row (the others are left unchanged), then performing the same operation on the second row, and so on with all of the rows of the matrix. The summation of the Jacobians of the matrices derived in such a manner gives the first derivative of the Jacobian of the original matrix $\{a_{ij}\}$. Thus,

$$\frac{\partial}{\partial \xi^k} \det\{a_{ij}\} = \frac{\partial a_{im}}{\partial \xi^k} G^{im} , \qquad i, j, k, m = 1, \ldots, n , \tag{2.47}$$

where G^{im} is the cofactor of the element a_{im}. For the Jacobi matrix $\{\partial x^i/\partial \xi^j\}$ of the coordinate transformation $\boldsymbol{x}(\boldsymbol{\xi})$, we have

$$G^{im} = J\frac{\partial \xi^m}{\partial x^i}, \qquad i, j = 1, \ldots, n .$$

Therefore, applying (2.47) to the Jacobi matrix, we obtain (2.46).

2.3.3 Basic Identity

The identity (2.46) implies the extremely important relation

$$\frac{\partial}{\partial \xi^j}\left(J\frac{\partial \xi^j}{\partial x^i}\right) \equiv 0 , \qquad i, j = 1, \ldots, n , \tag{2.48}$$

which leads to specific forms of new dependent variables for conservation-law equations. To prove (2.48), we first note that

$$\frac{\partial^2 \xi^p}{\partial x^k \partial x^j}\frac{\partial x^l}{\partial \xi^p} = -\frac{\partial^2 x^l}{\partial \xi^p \partial \xi^m}\frac{\partial \xi^m}{\partial x^k}\frac{\partial \xi^p}{\partial x^j} , \qquad j, k, l, m, p = 1, \ldots, n .$$

Multiplying this equation by $\partial \xi^i/\partial x^l$ and summing over l, we obtain a formula representing the second derivative $\partial^2 \xi^i/\partial x^k \partial x^m$ of the functions $\xi^i(\boldsymbol{x})$ through the second derivatives $\partial^2 x^m/\partial \xi^l \partial \xi^p$ of the functions $x^m(\boldsymbol{\xi})$, $m = 1, \ldots, n$:

$$\frac{\partial^2 \xi^i}{\partial x^k \partial x^m} = -\frac{\partial^2 x^p}{\partial \xi^l \partial \xi^j} \frac{\partial \xi^j}{\partial x^k} \frac{\partial \xi^l}{\partial x^m} \frac{\partial \xi^i}{\partial x^p} \,, \qquad i, j, k, l, m, p = 1, \dots, n \,. \tag{2.49}$$

Now, using this relation and the formula (2.46) for differentiation of the Jacobian in the identity

$$\frac{\partial}{\partial \xi^j}\left(J \frac{\partial \xi^j}{\partial x^i}\right) = \frac{\partial J}{\partial \xi^j} \frac{\partial \xi^j}{\partial x^i} + J \frac{\partial^2 \xi^j}{\partial x^i \partial x^k} \frac{\partial x^k}{\partial \xi^j} \,, \qquad i, j, k = 1, \dots, n \,,$$

we obtain

$$\begin{aligned} \frac{\partial}{\partial \xi^j}\left(J \frac{\partial \xi^j}{\partial x^i}\right) &= J \frac{\partial^2 x^k}{\partial \xi^p \partial \xi^j} \frac{\partial \xi^p}{\partial x^k} \frac{\partial \xi^j}{\partial x^i} - J \frac{\partial^2 x^p}{\partial \xi^l \partial \xi^m} \frac{\partial \xi^m}{\partial x^i} \frac{\partial \xi^l}{\partial x^k} \frac{\partial \xi^j}{\partial x^p} \frac{\partial x^k}{\partial \xi^j} \\ &= J \frac{\partial^2 x^k}{\partial \xi^p \partial \xi^j} \frac{\partial \xi^p}{\partial x^k} \frac{\partial \xi^j}{\partial x^i} - J \frac{\partial^2 x^p}{\partial \xi^l \partial \xi^m} \frac{\partial \xi^l}{\partial x^p} \frac{\partial \xi^m}{\partial x^i} = 0 \,, \end{aligned}$$

$$i, j, k, l, m, p = 1, \dots, n \,,$$

i.e. (2.48) has been proved.

The identity (2.48) is obvious when $n = 1$ or $n = 2$. For example, for $n = 2$, we have, from (2.4),

$$J \frac{\partial \xi^j}{\partial x^i} = (-1)^{i+j} \frac{\partial x^{3-i}}{\partial \xi^{3-j}} \,, \qquad i, j = 1, 2 \,,$$

with fixed indices i and j, and therefore

$$\frac{\partial}{\partial \xi^j}\Big(J \frac{\partial \xi^j}{\partial x^i}\Big) = (-1)^{i+1}\Big(\frac{\partial}{\partial \xi^1} \frac{\partial x^{3-i}}{\partial \xi^2} - \frac{\partial}{\partial \xi^2} \frac{\partial x^{3-i}}{\partial \xi^1}\Big) = 0, \qquad i, j = 1, 2 \,.$$

An inference from (2.48) for $n = 3$ follows from the differentiation of the cross product of the base tangential vectors $\boldsymbol{r}_{\xi^i}$, $i = 1, 2, 3$. Taking into account (2.26), we readily obtain the following formula for the differentiation of the cross product of two three-dimensional vector-valued functions $\boldsymbol{a}$ and $\boldsymbol{b}$:

$$\frac{\partial}{\partial \xi^i}(\boldsymbol{a} \times \boldsymbol{b}) = \frac{\partial}{\partial \xi^i}\boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{a} \times \frac{\partial}{\partial \xi^i}\boldsymbol{b} \,, \qquad i = 1, 2, 3 \,.$$

With this formula, we obtain

$$\sum_{i=1}^{3} \frac{\partial}{\partial \xi^i}(\boldsymbol{x}_{\xi^j} \times \boldsymbol{x}_{\xi^k}) = \sum_{i=1}^{3} \boldsymbol{x}_{\xi^j \xi^i} \times \boldsymbol{x}_{\xi^k} + \sum_{i=1}^{3} \boldsymbol{x}_{\xi^j} \times \boldsymbol{x}_{\xi^k \xi^i} \,, \tag{2.50}$$

where the indices $(i,\ j,\ k)$ are cyclic, i.e. $j = i + 1,\ k = i + 2$, m is equivalent to $m + 3$. For the last summation of the above formula, we obtain

$$\sum_{i=1}^{3} \boldsymbol{x}_{\xi^j} \times \boldsymbol{x}_{\xi^k \xi^i} = \sum_{i=1}^{3} \boldsymbol{x}_{\xi^k} \times \boldsymbol{x}_{\xi^i \xi^j} \ .$$

Therefore, from (2.50),

$$\sum_{i=1}^{3} \frac{\partial}{\partial \xi^i} (\boldsymbol{x}_{\xi^j} \times \boldsymbol{x}_{\xi^k}) = 0 \ ,$$

since

$$\boldsymbol{x}_{\xi^i} \times \boldsymbol{x}_{\xi^j \xi^k} = -\boldsymbol{x}_{\xi^j \xi^k} \times \boldsymbol{x}_{\xi^i}$$

and (2.33) implies (2.48) for $n = 3$.

The identity (2.48) can help one to obtain conservative or compact forms of some differential expressions and equations in the curvilinear coordinates $\xi^1, \ldots, \xi^n$. For example, for the first derivative of a function $f(\boldsymbol{x})$ with respect to x^i, we obtain, using (2.48),

$$\frac{\partial f}{\partial x^i} = \frac{1}{J} \frac{\partial}{\partial \xi^j} \Big(J \frac{\partial \xi^j}{\partial x^i} f \Big) \ , \qquad j = 1, \ldots, n \ . \tag{2.51}$$

For the Laplacian

$$\nabla^2 f = \frac{\partial}{\partial x^j} \frac{\partial f}{\partial x^j} \ , \qquad j = 1, \ldots, n \tag{2.52}$$

we have, substituting the quantity $\partial f / \partial x^i$ for f in (2.51),

$$\begin{aligned} \nabla^2 f &= \frac{1}{J} \frac{\partial}{\partial \xi^j} \Big(J \frac{\partial \xi^j}{\partial x^i} \frac{\partial f}{\partial x^i} \Big) = \frac{1}{J} \frac{\partial}{\partial \xi^j} \Big(J \frac{\partial \xi^j}{\partial x^i} \frac{\partial \xi^m}{\partial x^i} \frac{\partial f}{\partial \xi^m} \Big) \\ &= \frac{1}{J} \frac{\partial}{\partial \xi^j} \Big(J g^{mj} \frac{\partial f}{\partial \xi^m} \Big) \ , \qquad i,\ j, m = 1, \ldots, n \ . \end{aligned} \tag{2.53}$$

Therefore, the Poisson equation

$$\nabla^2 f = P \tag{2.54}$$

has the form

$$\frac{1}{J} \frac{\partial}{\partial \xi^j} \Big(J g^{mj} \frac{\partial f}{\partial \xi^m} \Big) = P \ , \qquad j, m = 1, \ldots, n \ , \tag{2.55}$$

with respect to the independent variables $\xi^1, \ldots, \xi^n$.

2.4 Conservation Laws

This section utilizes the relations described in Sects. 2.2 and 2.3, in particular the identity (2.48), in order to describe some conservation-law equations of mechanics in divergent or compact form in new independent curvilinear coordinates $\xi^1, \dots, \xi^n$. For this purpose, the dependent physical variables are also transformed to new dependent variables using some specific formulas. The essential advantage of the equations described here is that their coefficients are derived from the elements of the covariant metric tensor $\{g_{ij}\}$.

2.4.1 Scalar Conservation Laws

Let $\boldsymbol{A}$ be an n-dimensional vector with components $A^i,\ i = 1, \dots, n$, in the Cartesian coordinates $x^1, \dots, x^n$. The operator

$$\mathrm{div}_x \boldsymbol{A} = \frac{\partial A^i}{\partial x^i} \ , \qquad i = 1, \dots, n \ , \tag{2.56}$$

is commonly used in mechanics for the representation of scalar conservation laws, commonly in the form

$$\mathrm{div}_x \boldsymbol{A} = F \ .$$

Using (2.48), we easily obtain

$$\mathrm{div}_x \boldsymbol{A} = \frac{1}{J} \frac{\partial}{\partial \xi^j} (J \overline{A}^j) = F \ , \qquad j = 1, \dots, n \ , \tag{2.57}$$

where $\overline{A}^j$ is the jth contravariant component of the vector $\boldsymbol{A}$ in the coordinates $\xi^i, i = 1, \dots, n$, i.e. in accordance with (2.8):

$$\overline{A}^j = A^i \frac{\partial \xi^j}{\partial x^i} \ , \qquad i, j = 1, \dots, n \ . \tag{2.58}$$

Therefore, a divergent form of the conservation-law equation represented by (2.56) is obtained in the new coordinates when the dependent variables A^i are replaced by new dependent variables $\overline{A}^i$ defined by the rule (2.58). Some examples of scalar conservation-law equations are given below.

Mass Conservation Law

As an example of the application of (2.57), we consider the equation of conservation of mass for steady gas flow

$$\frac{\partial \rho u^i}{\partial x^i} = 0\,, \qquad i = 1, \ldots, n\,, \tag{2.59}$$

where ρ is the gas density, and u^i is the ith component of the flow velocity vector $\boldsymbol{u}$ in the Cartesian coordinates $x^1, \ldots, x^n$. With the substitution $A^i = \rho u^i$, (2.58) is transformed into the following divergent form with respect to the new dependent variables ρ and $\overline{u}^i$ in the coordinates $\xi^1, \ldots, \xi^n$:

$$\frac{\partial}{\partial \xi^i}(J\rho\overline{u}^i) = 0\,, \qquad i = 1, \ldots, n\,. \tag{2.60}$$

Here, $\overline{u}^i$ is the ith contravariant component of the flow velocity vector $\boldsymbol{u}$ in the basis $\boldsymbol{x}_{\xi^i},\ i = 1, \ldots, n$, i.e.

$$\overline{u}^i = u^j \frac{\partial \xi^i}{\partial x^j}\,, \qquad i, j = 1, \ldots, n\,. \tag{2.61}$$

Convection–Diffusion Equation

Another example is the conservation equation for the steady convection–diffusion of a transport variable ϕ, which can be expressed as

$$-\frac{\partial}{\partial x^i}\left(\epsilon \frac{\partial \phi}{\partial x^i}\right) + \frac{\partial}{\partial x^i}(\rho\phi u^i) = S\,, \qquad i = 1, \ldots, n\,, \tag{2.62}$$

where ρ and ϵ denote the density and diffusion coefficient of the fluid, respectively. Taking

$$A^i = \rho\phi u^i - \epsilon \frac{\partial \phi}{\partial x^i}, \qquad i = 1, \ldots, n\,.$$

we obtain, in accordance with the relation (2.58),

$$\overline{A}^j = \rho\phi\overline{u}^j - \epsilon \frac{\partial \phi}{\partial \xi^k} g^{kj}\,, \qquad j, k = 1, \ldots, n\,.$$

Therefore, using (2.57), the convection–diffusion equation (2.62) in the curvilinear coordinates $\xi^1, \ldots, \xi^n$ is expressed by the divergent form

$$\frac{\partial}{\partial \xi^j}\left[J\left(\rho\phi\overline{u}^j - \epsilon g^{kj} \frac{\partial \phi}{\partial \xi^k}\right)\right] = JS\,, \qquad j, k = 1, \ldots, n\,. \tag{2.63}$$

Laplace Equation

Analogously, the Laplace equation

$$\nabla^2 f = \frac{\partial}{\partial x^j} \frac{\partial f}{\partial x^j} = 0\,, \qquad j = 1, \ldots, n\,, \tag{2.64}$$

has the form (2.56) if we take

$$A^i = \frac{\partial f}{\partial x^i} , \qquad i = 1, \dots, n .$$

Using (2.58), we obtain

$$\overline{A}^j = g^{ij} \frac{\partial f}{\partial \xi^i} , \qquad i = 1, \dots, n .$$

Therefore, the Laplace equation (2.64) results in

$$\nabla^2 f = \frac{1}{J} \frac{\partial}{\partial \xi^i} \Big(J g^{ij} \frac{\partial f}{\partial \xi^j} \Big) = 0 , \qquad i, j = 1, \dots, n , \tag{2.65}$$

since (2.57) applies.

2.4.2 Vector Conservation Laws

Many physical problems are also modeled as a system of conservation-law equations in the vector form

$$\frac{\partial A^{ij}}{\partial x^j} = F^i , \qquad i, j = 1, \dots, n . \tag{2.66}$$

For the representation of the system (2.66) in new coordinates $\xi^1, \dots, \xi^n$ in a form which includes only coefficients derived from the elements of the metric tensor, it is necessary to make a transition from the original expression for A^{ij} to a new one $\overline{A}^{ij}$. One convenient formula for such a transition from the dependent variables A^{ij} to $\overline{A}^{ij}$, $i, j = 1, \dots, n$, is

$$\overline{A}^{ij} = A^{km} \frac{\partial \xi^i}{\partial x^k} \frac{\partial \xi^j}{\partial x^m} , \qquad i, j, k, m = 1, \dots, n . \tag{2.67}$$

This relation between A^{ij} and $\overline{A}^{km}$ is, in fact, composed of transitions of the kind (2.58) for the rows and columns of the tensor A^{ij}. In tensor analysis, the quantity $\overline{A}^{ij}$ means the (i, j) component of the second-rank contravariant tensor (A^{ij}) in the coordinates $\xi^1, \dots, \xi^n$.

Multiplying (2.67) by $(\partial x^p / \partial \xi^i)(\partial x^l / \partial \xi^j)$ and summing over i and j, we also obtain a formula for the transition from the new dependent variables $\overline{A}^{ij}$ to the original ones A^{ij}:

$$A^{ij} = \overline{A}^{km} \frac{\partial x^i}{\partial \xi^k} \frac{\partial x^j}{\partial \xi^m} , \qquad i, j, k, m = 1, \dots, n . \tag{2.68}$$

Therefore, we can obtain a system of equations for the new dependent variables $\overline{A}^{ij}$ by replacing the dependent quantities A^{ij} in (2.66) with their expressions (2.68). As a result, we obtain

$$\begin{aligned}\frac{\partial A^{ij}}{\partial x^j} &= \frac{\partial}{\partial x^j}\Big(\overline{A}^{km}\frac{\partial x^i}{\partial \xi^k}\frac{\partial x^j}{\partial \xi^m}\Big)\\ &= \frac{\partial \overline{A}^{km}}{\partial \xi^m}\frac{\partial x^i}{\partial \xi^k} + \overline{A}^{km}\frac{\partial^2 x^i}{\partial \xi^k \partial \xi^m} + \overline{A}^{km}\frac{\partial x^i}{\partial \xi^k}\frac{\partial}{\partial x^j}\Big(\frac{\partial x^j}{\partial \xi^m}\Big) = F^i ,\\ &\quad i,j,k,m = 1,\dots,n .\end{aligned}$$

The use of the formula (2.46) for differentiation of the Jacobian in the summation in the equation above yields

$$\frac{\partial A^{ij}}{\partial x^j} = \frac{\partial \overline{A}^{km}}{\partial \xi^m}\frac{\partial x^i}{\partial \xi^k} + \overline{A}^{km}\frac{\partial^2 x^i}{\partial \xi^k \partial \xi^m} + \frac{1}{J}\overline{A}^{km}\frac{\partial x^i}{\partial \xi^k}\frac{\partial J}{\partial \xi^m} = F^i ,$$

$$i,j,k,l,m = 1,\dots,n .$$

Multiplying this system by $\partial \xi^p/\partial x^i$ and summing over i, we obtain, after simple manipulations,

$$\frac{1}{J}\frac{\partial}{\partial \xi^j}(J\overline{A}^{ij}) + \frac{\partial^2 x^l}{\partial \xi^k \partial \xi^j}\frac{\partial \xi^i}{\partial x^l}\overline{A}^{kj} = \overline{F}^i , \qquad i,j,k,l = 1,\dots,n , \tag{2.69}$$

where

$$\overline{F}^i = F^j \frac{\partial \xi^i}{\partial x^j} , \qquad i,j = 1,\dots,n ,$$

is the ith contravariant component of the vector $\boldsymbol{F} = (F^1,\dots,F^n)$ in the basis $\boldsymbol{x}_{\xi^1},\dots,\boldsymbol{x}_{\xi^n}$. The quantities $(\partial^2 x^l/\partial \xi^k \partial \xi^j)(\partial \xi^i/\partial x^l)$ in (2.69) are the space Christoffel symbols of the second kind Γ^i_{kj}. Thus, the system (2.69) has, using the notation Γ^i_{jk}, the form

$$\frac{1}{J}\frac{\partial}{\partial \xi^j}(J\overline{A}^{ij}) + \Gamma^i_{kj}\overline{A}^{kj} = \overline{F}^i , \qquad i,j,k = 1,\dots,n . \tag{2.70}$$

We see that all coefficients of (2.70) are derived from the metric tensor $\{g_{ij}\}$.

Equations of the form (2.70), in contrast to (2.66), do not have a conservative form. The conservative form of (2.66) in new dependent variables is obtained, in analogy with (2.57), from the system

$$\frac{1}{J}\frac{\partial}{\partial \xi^j}(J\overline{A}^j_i) = F^i , \qquad i,j = 1,\dots,n , \tag{2.71}$$

where $\overline{A}_i^j$ is the jth component of the vector $\boldsymbol{A}_i = (A^{i1}, \dots, A^{in})$ in the basis $\boldsymbol{x}_{\xi^j},\ j = 1, \dots, n$, i.e.

$$\overline{A}_i^j = A^{ik} \frac{\partial \xi^j}{\partial x^k} , \qquad i, j, k = 1, \dots, n . \tag{2.72}$$

In fact, (2.71) is the result of the application of (2.57) to the ith line of (2.66). Therefore, in the relations (2.66, 2.71, 2.72), we can assume an arbitrary range for the index i, i.e. the matrix A^{ij} in (2.66) can be a nonsquare matrix with $i = 1, \dots, m,\ j = 1, \dots, n$.

Though the system (2.71) is conservative and more compact than (2.70), it has its drawbacks. In particular, mathematical simulations of fluid flows are generally formulated in the form (2.66) with the tensor A^{ij} represented as

$$A^{ij} = B^{ij} + \rho u^i u^j , \qquad i, j = 1, \dots, n ,$$

where $u^i,\ i = 1, \dots, n$, are the Cartesian components of the flow velocity. The transformation of the tensor $\rho u^i u^j$ by the rule (2.72),

$$\rho u^i u^k \frac{\partial \xi^j}{\partial x^k} = \rho u^i \overline{u}^j , \qquad i, j, k = 1, \dots, n ,$$

results in equations with an increased number of dependent variables, namely u^i and $\overline{u}^j$. The substitution of u^i for $\overline{u}^j$ or vice versa leads to equations whose coefficients are derived from the elements $\partial x^i / \partial \xi^j$ of the Jacobi matrix and not from the elements of the metric tensor $\{g_{ij}\}$.

Example

As an example of (2.66), we consider the stationary equation of a compressible gas flow

$$\frac{\partial}{\partial x^j}(\rho u^i u^j) + \frac{\partial p}{\partial x^i} - \frac{\partial}{\partial x^j} \mu \frac{\partial u^i}{\partial x^j} = \rho F^i , \qquad i, j = 1, \dots, n , \tag{2.73}$$

where u^i is the ith Cartesian component of the vector of the fluid velocity $\boldsymbol{u}$, ρ is the density, p is the pressure and μ is the viscosity. The tensor form of (2.66) is given by

$$A^{ij} = \rho u^i u^j + \delta^i_j p - \mu \frac{\partial u^i}{\partial x^j} , \qquad i, j = 1, \dots, n .$$

From (2.67), we obtain, in this case,

$$\overline{A}^{ij} = \rho \overline{u}^i \overline{u}^j + g^{ij} p - \mu \frac{\partial u^l}{\partial x^k} \frac{\partial \xi^i}{\partial x^l} \frac{\partial \xi^j}{\partial x^k} , \qquad i, j, k, l = 1, \dots, n , \tag{2.74}$$

where $\overline{u}^i$ is the ith component of $\boldsymbol{u}$ in the basis $\boldsymbol{x}_{\xi^i}$, i.e. $\overline{u}^i$ is computed from the formula (2.61). It is obvious that

$$u^l = \overline{u}^j \frac{\partial x^l}{\partial \xi^j}\ , \qquad j, l = 1, \dots, n\ . \tag{2.75}$$

Therefore,

$$\begin{aligned}\frac{\partial u^l}{\partial x^k} &= \frac{\partial}{\partial \xi^m}\left(\overline{u}^p \frac{\partial x^l}{\partial \xi^p}\right)\frac{\partial \xi^m}{\partial x^k}\\ &= \frac{\partial \overline{u}^p}{\partial \xi^m}\frac{\partial x^l}{\partial \xi^p}\frac{\partial \xi^m}{\partial x^k} + \overline{u}^p \frac{\partial^2 x^l}{\partial \xi^p \partial \xi^m}\frac{\partial \xi^m}{\partial x^k}\ , \qquad k, l, m, p = 1, \dots, n\ .\end{aligned}$$

Using this equation, we obtain, for the last term of (2.74),

$$\mu \frac{\partial u^l}{\partial x^k}\frac{\partial \xi^i}{\partial x^l}\frac{\partial \xi^j}{\partial x^k} = \mu g^{mj}\left(\frac{\partial \overline{u}^i}{\partial \xi^m} + \Gamma^i_{pm}\overline{u}^p\right)\ , \qquad i, j, m, p = 1, \dots, n\ ,$$

since (2.38) applies. Thus, (2.74) has the form

$$\overline{A}^{ij} = \rho \overline{u}^i \overline{u}^j + g^{ij} p - \mu g^{mj}\left(\frac{\partial \overline{u}^i}{\partial \xi^m} + \Gamma^i_{pm}\overline{u}^p\right)\ , \quad i, j, m, p = 1, \dots, n\ , \tag{2.76}$$

and, applying (2.70), we obtain the following system of stationary equations (2.73) with respect to the new dependent variables $\rho, \overline{u}^i$, and p and the independent variables ξ^i:

$$\begin{aligned}&\frac{1}{J}\frac{\partial}{\partial \xi^j}\left\{J\left[\rho \overline{u}^i \overline{u}^j + g^{ij} p - \mu g^{mj}\left(\frac{\partial \overline{u}^i}{\partial \xi^m} + \Gamma^i_{pm}\overline{u}^p\right)\right]\right\}\\ &+ \Gamma^i_{kj}\left[\rho \overline{u}^k \overline{u}^j + g^{kj} p - \mu g^{mj}\left(\frac{\partial \overline{u}^k}{\partial \xi^m} + \Gamma^k_{pm}\overline{u}^p\right)\right] = \rho \overline{F}^i\ ,\\ &i, j, k, m, p = 1, \dots, n\ .\end{aligned} \tag{2.77}$$

The application of (2.71)–(2.73) yields the following system of stationary equations:

$$\begin{aligned}&\frac{1}{J}\frac{\partial}{\partial \xi^j}\left[J\left(\rho u^i \overline{u}^j + \frac{\partial \xi^j}{\partial x^i} p - \mu \frac{\partial u^i}{\partial \xi^k} g^{kj}\right)\right] = \rho F^i\ ,\\ &\overline{u}^j = u^k \frac{\partial \xi^j}{\partial x^k}\ , \qquad i, j, k = 1, \dots, n\ .\end{aligned} \tag{2.78}$$

Now, as an example of the utilization of the Christoffel symbols of the second kind Γ^i_{kj}, we write out the expression for the transformed elements of the tensor

$$\sigma^{ij} = \mu\Big(\frac{\partial u^i}{\partial x^j} + \frac{\partial u^j}{\partial x^i}\Big) , \qquad i, j = 1 \dots, n , \tag{2.79}$$

in the coordinates $\xi^1, \dots, \xi^n$, obtained in accordance with the rule (2.67). This tensor is very common and is important in applications simulating deformation in the theory of elasticity and deformation rate in fluid mechanics. Using the notations described above, the tensor $\overline{\sigma}^{ij}$ can be expressed in the coordinates $\xi^1, \dots, \xi^n$ through the metric elements and the Christoffel symbols of the second kind. For the component $\overline{\sigma}^{ij}$, we have

$$\begin{aligned}\overline{\sigma}^{ij} &= \sigma^{mk} \frac{\partial \xi^i}{\partial x^m} \frac{\partial \xi^j}{\partial x^k} \\ &= \mu\Big(g^{jl} \frac{\partial \overline{u}^i}{\partial \xi^l} + g^{il} \frac{\partial \overline{u}^j}{\partial \xi^l} + (g^{jl} \Gamma^i_{pl} + g^{il} \Gamma^j_{pl}) \overline{u}^p\Big) , \\ &\quad i, j, l, p = 1, \dots, n . \end{aligned} \tag{2.80}$$

This formula is obtained rather easily. For this purpose, one can use the relation (2.75) for the inverse transition from the contravariant components $\overline{u}^i$ to the Cartesian components u^j of the vector $\boldsymbol{u} = (u^1, \dots, u^n)$ and the formula (2.38). By substituting (2.75) in (2.79), carrying out differentiation by the chain rule, and using the expression (2.38), we obtain (2.80).

2.5 Time-Dependent Transformations

The numerical solution of time-dependent equations requires the application of moving grids and the corresponding coordinate transformations, which are dependent on time. Commonly, such coordinate transformations are determined in the form of a vector-valued time-dependent function

$$\boldsymbol{x}(t, \boldsymbol{\xi}) : \Xi^n \to X^n_t , \qquad \boldsymbol{\xi} \in \Xi^n , \qquad t \in [0, 1] , \tag{2.81}$$

where the variable t represents the time and X^n_t is an n-dimensional domain whose boundary points change smoothly with respect to t. It is assumed that $\boldsymbol{x}(t, \boldsymbol{\xi})$ is sufficiently smooth with respect to ξ^i and t and, in addition, that it is invertible for all $t \in [0, 1]$. Therefore, there is also the time-dependent inverse transformation

$$\boldsymbol{\xi}(t, \boldsymbol{x}) : X^n_t \to \Xi^n \tag{2.82}$$

for every $t \in [0, 1]$. The introduction of these time-dependent coordinate transformations enables one to compute an unsteady solution on a fixed uniform grid in Ξ^n by the numerical solution of the transformed equations.

2.5.1 Reformulation of Time-Dependent Transformations

Many physical problems are modeled in the form of nonstationary conservation-law equations which include the time derivative. The formulas of Sects. 2.3 and 2.4 can be used directly, by transforming the equations at every value of time t. However, such utilization of the formulas does not influence the temporal derivative, which is transformed simply to the form

$$\frac{\partial}{\partial t} + \frac{\partial \xi^i}{\partial t}\frac{\partial}{\partial \xi^i} , \qquad i = 1, \ldots, n ,$$

so that it does not maintain the property of divergency and its coefficients are not derived from the elements of the metric tensor.

Instead, the formulas of Sects. 2.3 and 2.4 can be more successfully applied to time-dependent conservation-law equations if the set of the functions $\boldsymbol{x}(t, \boldsymbol{\xi})$ is expanded to an $(n + 1)$-dimensional coordinate transformation in which the temporal parameter t is considered in the same manner as the spatial variables.

To carry out this process, we expand the n-dimensional computational and physical domains in (2.81) to $(n + 1)$-dimensional ones, assuming

$$\Xi^{n+1} = I \times \Xi^n , \qquad X^{n+1} = \cup_t (t \times X^n_t) .$$

Let the points of these domains be designated by $\boldsymbol{\xi}_0 = (\xi^0, \xi^1, \ldots, \xi^n)$ and $\boldsymbol{x}_0 = (x^0, x^1, \ldots, x^n)$, respectively. The expanded coordinate transformation is defined as

$$\boldsymbol{x}_0(\boldsymbol{\xi}_0) : \Xi^{n+1} \to X^{n+1} , \tag{2.83}$$

where $x^0(\boldsymbol{\xi}_0) = \xi^0$ and $x^i(\boldsymbol{\xi}_0),\ i = 1, \ldots, n$, which coincides with (2.81) with $\xi^0 = t$.

The variables x^0 and ξ^0 in (2.83) represent, in fact, the temporal variable t. For convenience and in order to avoid ambiguity, we shall also designate the variable ξ^0 in Ξ^{n+1} by τ and the variable x^0 in X^{n+1} by t. Thus, $\boldsymbol{x}_0(\boldsymbol{\xi}_0)$ is the $(n+1)$-dimensional coordinate transformation which is identical to $\boldsymbol{x}(\tau, \boldsymbol{\xi})$ at every section $\xi^0 = \tau$.

The inverted coordinate transformation

$$\boldsymbol{\xi}_0(\boldsymbol{x}_0) : X^{n+1} \to \Xi^{n+1} \tag{2.84}$$

satisfies

$$\xi^0(\boldsymbol{x}_0) = x^0 , \qquad \xi^i(\boldsymbol{x}_0) = \xi^i(t, \boldsymbol{x}) , \qquad i = 1, \ldots, n ,$$

where $t = x^0,\ \boldsymbol{x} = (x^1, \ldots, x^n)$, and $\xi^i(t, \boldsymbol{x})$ is defined by (2.82). Thus, (2.84) is identical to (2.82) at each section X_t.

2.5.2 Basic Relations

This subsection discusses some relations and, in particular, identities of the kind (2.46) and (2.48) for the time-dependent coordinate transformations (2.81), using for this purpose the $(n+1)$-dimensional vector functions (2.83) and (2.84) introduced above.

Velocity of Grid Movement

The first derivative $\boldsymbol{x}_\tau$, $\boldsymbol{x} = (x^1, x^2, \dots, x^n)$, of the transformation $\boldsymbol{x}(\xi, \tau)$ has a clear physical interpretation as the velocity vector of grid point movement. Let the vector $\boldsymbol{x}_\tau$, in analogy with the flow velocity vector $\boldsymbol{u}$, be designated by $\boldsymbol{w} = (w^1, \dots, w^n)$, i.e. $w^i = x^i_\tau$. The ith component $\overline{w}^i$ of the vector $\boldsymbol{w}^i$ in the tangential bases $\boldsymbol{x}_{\xi^i}$, $i = 1, \dots, n$, is expressed by (2.7) as

$$\overline{w}^i = w^j \frac{\partial \xi^i}{\partial x^j} = \frac{\partial x^j}{\partial \tau} \frac{\partial \xi^i}{\partial x^j} \,, \qquad i, j = 1, \dots, n \,.$$

Therefore,

$$\boldsymbol{w} = \overline{w}^i \boldsymbol{x}_{\xi^i} \,, \qquad i = 1, \dots, n \,, \tag{2.85}$$

i.e.

$$w^i = \frac{\partial x^i}{\partial \tau} = \overline{w}^j \frac{\partial x^i}{\partial \xi^j} \,, \qquad i, j = 1, \dots, n \,.$$

Differentiation with respect to ξ^0 of the composition of $\boldsymbol{x}_0(\boldsymbol{\xi}_0)$ and $\boldsymbol{\xi}_0(\boldsymbol{x}_0)$ yields

$$\frac{\partial \xi^i}{\partial x^0} \frac{\partial x^0}{\partial \xi^0} + \frac{\partial \xi^i}{\partial x^j} \frac{\partial x^j}{\partial \xi^0} = 0 \,, \qquad i, j = 1, \dots, n \,.$$

Therefore, we obtain the result

$$\frac{\partial \xi^i}{\partial t} = -\frac{\partial x^j}{\partial \tau} \frac{\partial \xi^i}{\partial x^j} = -\overline{w}^i \,, \qquad i, j = 1, \dots, n \,. \tag{2.86}$$

Derivatives of the Jacobian

It is apparent that the Jacobians of the coordinate transformations $\boldsymbol{x}(\tau, \boldsymbol{\xi})$ and $\boldsymbol{x}_0(\boldsymbol{\xi}_0)$ coincide, i.e.

$$\det\left\{\frac{\partial x^i}{\partial \xi^j}\right\} = \det\left\{\frac{\partial x^k}{\partial \xi^l}\right\} = J \,, \qquad i, j = 0, 1, \dots, n \,, \qquad k, l = 1, \dots, n \,.$$

In the notation introduced above, the formula (2.46) for differentiation of the Jacobian of the transformation

$$\boldsymbol{x}_0(\boldsymbol{\xi}_0) : \Xi^{n+1} \to X^{n+1}$$

is expressed by the relation

$$\frac{1}{J}\frac{\partial}{\partial \xi^i}J = \frac{\partial^2 x^k}{\partial \xi^i \partial \xi^m}\frac{\partial \xi^m}{\partial x^k}\ , \qquad i,k,m = 0,1,\dots,n\ , \tag{2.87}$$

differing from (2.46) only by the range of the indices. As a result, we obtain from (2.87) for $i = 0$,

$$\frac{1}{J}\frac{\partial}{\partial \tau}J = \frac{\partial}{\partial \xi^m}\left(\frac{\partial x^k}{\partial \tau}\right)\frac{\partial \xi^m}{\partial x^k} = \mathrm{div}_x \frac{\partial \boldsymbol{x}}{\partial \tau}\ , \qquad k,m = 0,1,\dots,n\ , \tag{2.88}$$

and, taking into account (2.85),

$$\begin{aligned}\frac{1}{J}\frac{\partial}{\partial \tau}J &= \frac{\partial}{\partial \xi^m}\left(\overline{w}^j\frac{\partial x^k}{\partial \xi^j}\right)\frac{\partial \xi^m}{\partial x^k}\\ &= \frac{\partial \overline{w}^m}{\partial \xi^m} + \overline{w}^i\frac{\partial^2 x^k}{\partial \xi^j \partial \xi^m}\frac{\partial \xi^m}{\partial x^k}\ , \qquad j,k,m = 1,\dots,n\ .\end{aligned}$$

Now, taking advantage of the formula for differentiation of the Jacobian (2.46), in the last sum of this equation, we have

$$\frac{1}{J}\frac{\partial}{\partial \tau}J = \frac{\partial \overline{w}^m}{\partial \xi^m} + \frac{1}{J}\overline{w}^j\frac{\partial J}{\partial \xi^j}\ , \qquad j,m = 1,\dots,n\ ,$$

and consequently

$$\frac{1}{J}\frac{\partial}{\partial \tau}J = \frac{1}{J}\frac{\partial}{\partial \xi^j}(J\overline{w}^j)\ , \qquad j = 1,\dots,n\ . \tag{2.89}$$

Basic Identity

Analogously, the system of identities (2.48) has the following form:

$$\frac{\partial}{\partial \xi^j}\left(J\frac{\partial \xi^j}{\partial x^i}\right) = 0\ , \qquad i,j = 0,1,\dots,n\ . \tag{2.90}$$

Therefore, for $i = 0$, we obtain

$$\frac{\partial}{\partial \tau}(J) + \frac{\partial}{\partial \xi^j}\left(J\frac{\partial \xi^j}{\partial t}\right) = 0\ , \qquad j = 1,\dots,n\ , \tag{2.91}$$

and, taking into account (2.86),

$$\frac{\partial}{\partial \tau}J - \frac{\partial}{\partial \xi^j}(J\overline{w}^j) = 0\ , \qquad j = 1,\dots,n\ , \tag{2.92}$$

which corresponds to (2.89). For $i > 0$, the identity (2.90) coincides with (2.48), i.e.

$$\frac{\partial}{\partial \xi^j}\left(J\frac{\partial \xi^j}{\partial x^i}\right) = 0 \ , \qquad i, j = 1, \ldots, n \ .$$

As a result of (2.91), we obtain, in analogy with (2.51),

$$\frac{\partial f}{\partial t} = \frac{1}{J}\frac{\partial}{\partial \xi^j}\left(J\frac{\partial \xi^j}{\partial t} f\right) = \frac{1}{J}\left(\frac{\partial}{\partial \tau}(Jf) - \frac{\partial}{\partial \xi^k}(J\overline{w}^k f)\right) \ ,$$
$$j = 0, 1, \ldots, n \ , \qquad k = 1, \ldots, n \ . \tag{2.93}$$

2.5.3 Equations in the Form of Scalar Conservation Laws

Many time-dependent equations can be expressed in the form of a scalar conservation law in the Cartesian coordinates $t, x^1, \ldots, x^n$:

$$\frac{\partial A^0}{\partial t} + \frac{\partial A^i}{\partial x^i} = F \ , \qquad i = 1, \ldots, n \ . \tag{2.94}$$

Using (2.90), in analogy with (2.57), this equation is transformed in the coordinates $\xi^0, \xi^1, \ldots, \xi^n, \ \xi^0 = \tau$ to

$$\frac{1}{J}\left(\frac{\partial}{\partial \xi^j}(J\overline{A}_0^j)\right) = F \ , \qquad j = 0, 1, \ldots, n \ , \tag{2.95}$$

where by $\overline{A}_0^j$ we denote the jth contravariant component of the $(n+1)$-dimensional vector $\boldsymbol{A}_0 = (A^0, A^1, \ldots, A^n)$ in the basis $\partial \boldsymbol{x}_0/\partial \xi^i, \ i = 0, 1, \ldots, n$, i.e.

$$\overline{A}_0^j = A^i\frac{\partial \xi^j}{\partial x^i} \ , \qquad i, j = 0, 1, \ldots, n \ . \tag{2.96}$$

We can express each component $\overline{A}_0^j, \ j = 1, \ldots, n$, of the vector $\boldsymbol{A}_0$ through the components $\overline{A}^i$ and $\overline{w}^k, \ i, k = 1, \ldots, n$, of the n-dimensional spatial vectors $\boldsymbol{A} = (A^1, \ldots, A^n)$ and $\boldsymbol{w} = (w^1, \ldots, w^n)$ in the coordinates $\xi^l, \ l = 1, \ldots, n$, where $\boldsymbol{A}$ is a vector obtained by projecting the vector $\boldsymbol{A}_0$ into the space R^n, i.e. $P(A^0, A^1, \ldots, A^n) = (A^1, \ldots, A^n)$. Namely,

$$\overline{A}_0^j = A^0\frac{\partial \xi^j}{\partial t} + A^i\frac{\partial \xi^j}{\partial x^i} = \overline{A}^j - A^0\overline{w}^j \ , \qquad i, j = 1, \ldots, n \ ,$$

using (2.86). Furthermore, we have

$$\overline{A}^0_0 = A^k \frac{\partial \xi^0}{\partial x^k} = A^0 , \qquad k = 0, 1, \dots, n .$$

Therefore, (2.95) implies a conservation law in the variables $\tau, \xi^1, \dots, \xi^n$ in the conservative form

$$\frac{1}{J} \left(\frac{\partial}{\partial \tau}(J A^0) + \frac{\partial}{\partial \xi^j}[J(\overline{A}^j - A^0 \overline{w}^j)] \right) = F , \qquad j = 1, \dots, n . \tag{2.97}$$

Examples of Scalar Conservation-Law Equations

As an illustration of the formula (2.97), we write out some time-dependent scalar conservation-law equations presented first in the form (2.94).

Parabolic Equation

For the parabolic equation

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial x^j} \frac{\partial f}{\partial x^j} , \qquad j = 1, \dots, n , \tag{2.98}$$

we obtain from (2.97), with $A^0 = f$ and $A^i = \partial f / \partial x^i$, $i = 1, \dots n$,

$$\frac{\partial J f}{\partial \tau} = \frac{\partial}{\partial \xi^j} \left[J \left(g^{jk} \frac{\partial f}{\partial \xi^k} + f \overline{w}^j \right) \right] , \qquad j, k = 1, \dots n . \tag{2.99}$$

Mass Conservation Law

The scalar mass conservation law for unsteady compressible gas flow

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u^i}{\partial x^i} = F , \qquad i = 1, 2, 3 , \tag{2.100}$$

is expressed in the new coordinates as

$$\frac{\partial J \rho}{\partial t} + \frac{\partial J \rho (\overline{u}^j - \overline{w}^j)}{\partial \xi^j} = J F , \qquad j = 1, 2, 3 . \tag{2.101}$$

Convection–Diffusion Equation

The unsteady convection–diffusion conservation equation

$$\frac{\partial}{\partial t}(\rho \phi) + \frac{\partial}{\partial x^i}(\rho \phi u^i) - \frac{\partial}{\partial x^i} \left(\epsilon \frac{\partial \phi}{\partial x^i} \right) = S , \qquad i = 1, \dots, n , \tag{2.102}$$

has the form in the coordinates $\tau, \xi^1, \dots, \xi^n$

$$\frac{\partial}{\partial \tau}(J\rho\phi) + \frac{\partial}{\partial \xi^j}\Big(J\rho\phi(\overline{u}^j - \overline{w}^j) - Jg^{kj}\epsilon\frac{\partial \phi}{\partial \xi^k}\Big) = JS\ ,\ \ j,k = 1,\ldots,n\ . \quad (2.103)$$

Energy Conservation Law

Analogously, the energy conservation law

$$\frac{\partial}{\partial t}\rho(e + u^2/2) + \frac{\partial}{\partial x^j}\rho u^j(e + u^2/2 + \overline{p}/\rho) = \rho F_j u_j\ ,\quad j = 1,2,3\ , \quad (2.104)$$

where

$$e = e(\rho, p)\ , \qquad \overline{p} = p - \gamma\frac{\partial u_i}{\partial x^i}\ , \qquad i = 1,2,3\ ,$$

$$u^2 = \sum_{i=1}^{3}(u_i)^2\ ,$$

is transformed in accordance with (2.97) to

$$\frac{\partial}{\partial \tau}\Big[J\rho\Big(e + \frac{1}{2}g_{mk}\overline{u}^m\overline{u}^k\Big)\Big] + \frac{\partial}{\partial \xi^j}\Big[J\rho\Big(e + \frac{1}{2}g_{mk}\overline{u}^m\overline{u}^k\Big)(\overline{u}^j - \overline{w}^j) + J\overline{pu}^j\Big]$$

$$= J\rho g_{mk}\overline{f}^m\overline{u}^k\ , \qquad j,m,k = 1,2,3\ , \quad (2.105)$$

where, taking into account (2.57),

$$\overline{p} = p - \frac{\gamma}{J}\frac{\partial}{\partial \xi^i}(J\overline{u}^i)\ , \qquad i = 1,2,3\ .$$

Linear Wave Equation

The linear wave equation

$$u_{tt} = c^2\nabla^2 u \quad (2.106)$$

arises in many areas such as fluid dynamics, elasticity, acoustics, and magnetohydrodynamics. If the coefficient c is constant, then (2.106) has a divergent form (2.94) with

$$A^0 = u_t\ , \qquad A^i = -c^2\frac{\partial u}{\partial x^i}\ , \qquad i = 1,\ldots,n\ ,$$

or, in the coordinates $\tau, \xi^1, \ldots, \xi^n$,

$$A^0 = u_\tau - \overline{w}^i\frac{\partial u}{\partial \xi^i}\ , \qquad A^i = -c^2\frac{\partial u}{\partial \xi^k}\frac{\partial \xi^k}{\partial x^i}\ , \qquad i,k = 1,\ldots,n\ .$$

Therefore, the divergent form (2.95) of (2.106) in the coordinates $\tau, \xi^1, \dots, \xi^n$ has the form

$$\frac{\partial}{\partial \tau}\left[J\left(u_\tau - \overline{w}^i \frac{\partial u}{\partial \xi^i}\right)\right] + \frac{\partial}{\partial \xi^j}\left[J\left(u_\tau \overline{w}^j + (c^2 g^{mj} - \overline{w}^i \overline{w}^j)\frac{\partial u}{\partial \xi^i}\right)\right] = 0\,. \tag{2.107}$$

Another representation of the linear wave equation (2.106) in the coordinates $\tau, \xi^1, \dots, \xi^n$ comes from the formula (2.65) for the Laplace operator and the description of the temporal derivative (2.93). Taking advantage of (2.93), we obtain

$$\begin{aligned} u_{tt} &= \frac{1}{J}\Big[\frac{\partial}{\partial \tau}\Big(J\frac{\partial u}{\partial t}\Big) - \frac{\partial}{\partial \xi^k}\Big(J\overline{w}^k \frac{\partial u}{\partial t}\Big)\Big] \\ &= \frac{1}{J}\frac{\partial}{\partial \tau}\Big[J\Big(u_\tau - \overline{w}^i \frac{\partial u}{\partial \xi^i}\Big)\Big] - \frac{1}{J}\frac{\partial}{\partial \xi^k}\Big[J\overline{w}^k\Big(u_\tau - \overline{w}^i \frac{\partial u}{\partial \xi^i}\Big)\Big]\,, \quad i, k = 1, \dots, n\,. \end{aligned}$$

This equation and (2.65) allow one to derive the following form of (2.106) in the coordinates $\tau, \xi^1, \dots, \xi^n$:

$$\begin{aligned} \frac{\partial}{\partial \tau}\Big[J\Big(u_\tau - \overline{w}^i \frac{\partial u}{\partial \xi^i}\Big)\Big] &= \frac{\partial}{\partial \xi^k}\Big[J\overline{w}^k\Big(u_\tau - \overline{w}^j \frac{\partial u}{\partial \xi^j}\Big)\Big] \\ &+ c^2 \frac{\partial}{\partial \xi^k}\Big(J g^{kj}\frac{\partial u}{\partial \xi^j}\Big)\,, \quad i, j, k = 1, \dots, n\,, \end{aligned} \tag{2.108}$$

which coincides with (2.107) if c^2 is a constant.

Lagrangian Coordinates

One of the most popular systems of coordinates in fluid dynamics is the Lagrangian system. A coordinate ξ^i is Lagrangian if the both the ith component of the flow velocity vector $\boldsymbol{u}$ and the grid velocity $\boldsymbol{w}$ in the tangent basis $\boldsymbol{x}_{\xi^j},\ j = 1, \dots, m$, coincide, i.e.

$$\overline{u}^i - \overline{w}^i = 0\,. \tag{2.109}$$

The examples of gas-dynamics equations described above, which include the terms $\overline{w}^i$, allow one to obtain the equations in Lagrange coordinates by substituting $\overline{u}^i$ for $\overline{w}^i$ in the written-out equations in accordance with the relation (2.109). In such a manner, we obtain the equation of mass conservation, for example, in the Lagrangian coordinates $\xi^1, \dots, \xi^n$, as

$$\frac{\partial J\rho}{\partial \tau} = JF \tag{2.110}$$

from (2.101). Analogously, the convection–diffusion equation (2.103) and the energy conservation law (2.105) have the forms in Lagrangian coordinates ξ^i

$$\frac{\partial}{\partial \tau}(J\rho\phi) + \frac{\partial}{\partial \xi^j}\Big(J g^{kj}\epsilon \frac{\partial \phi}{\partial \xi^k}\Big) = JS\,, \qquad j, k = 1, \dots, n\,,$$

and

$$\frac{\partial}{\partial \tau}\Big[J\rho\Big(e+\frac{1}{2}g_{mk}\overline{u}^m\overline{u}^k\Big)\Big]+\frac{\partial}{\partial \xi^j}\Big(J\overline{pu}^j\Big)=J\rho g_{mk}\overline{f}^m\overline{u}^k\ , \qquad j,m,k=1,2,3\ ,$$

respectively.

In the same manner, the equations can be written in the Euler–Lagrange form. where some coordinates are Lagrangian while the rest are Cartesian coordinates.

2.5.4 Equations in the Form of Vector Conservation Laws

Now we consider a formula for a vector conservation law with time-dependent physical magnitudes A^{ij}

$$\frac{\partial}{\partial x^j}A^{ij}=F^i\ , \qquad i,j=0,1,\ldots,n\ , \tag{2.111}$$

where the independent variable x^0 represents the time variable t, i.e. $x^0=t$. Let the new independent variables $\xi^0,\xi^1,\ldots,\xi^n$ be obtained by means of (2.83). Along with (2.70), which expresses the vector conservation law (2.66) in the coordinates $\xi^1,\ldots,\xi^n$, we find that the transformation (2.111) has the form of the following system of equations for the new dependent quantities $\overline{A}_0^{ij}$, $i,j=0,1,\ldots,n$, with respect to the independent variables $\xi^0,\xi^1,\ldots,\xi^n$, $\xi^0=\tau$:

$$\frac{1}{J}\frac{\partial}{\partial \xi^j}(J\overline{A}_0^{ij})+\overline{\Gamma}_{kj}^i\overline{A}_0^{kj}=\overline{F}_0^i\ , \qquad i,j=0,1,\ldots,n\ , \tag{2.112}$$

where

$$\begin{aligned}\overline{A}_0^{ij}&=A^{mn}\frac{\partial \xi^i}{\partial x^m}\frac{\partial \xi^j}{\partial x^n}\ , \ i,j,m,n=0,1,\ldots,n\ ,\\ \overline{\Gamma}_{kj}^i&=\frac{\partial^2 x^l}{\partial \xi^k\partial \xi^j}\frac{\partial \xi^i}{\partial x^l}\ , \quad i,j,k,l=0,1,\ldots,n\ ,\\ \overline{F}_0^i&=F^j\frac{\partial \xi^i}{\partial x^j}\ , \qquad i,j=0,1,\ldots,n\ .\end{aligned}$$

As in the case of the scalar conservation law, we represent all of the terms of (2.112) through A^{00} and the spatial components:

$$
\begin{aligned}
\overline{A}^{ij} &= A^{km}\frac{\partial\xi^i}{\partial x^k}\frac{\partial\xi^j}{\partial x^m}\ , && i,j,m,n=1,\dots,n\ ,\\
\Gamma^i_{kj} &= \frac{\partial^2 x^l}{\partial\xi^k\partial\xi^j}\frac{\partial\xi^i}{\partial x^l}\ , && i,j,k,l=1,\dots,n\ ,\\
\overline{F}^i &= F^j\frac{\partial\xi^i}{\partial x^j}\ , && i,j=1,\dots,n\ ,\\
\overline{w}^i &= -\frac{\partial\xi^i}{\partial t} = \frac{\partial x^j}{\partial\tau}\frac{\partial\xi^i}{\partial x^j}\ , && i,j=1,\dots,n\ .
\end{aligned}
$$

For $\overline{A}_0^{ij}$, we obtain

$$
\begin{aligned}
\overline{A}_0^{00} &= A^{00}\ ,\\
\overline{A}_0^{0i} &= A^{00}\frac{\partial\xi^i}{\partial t} + A^{0m}\frac{\partial\xi^i}{\partial x^m} = \overline{A}^{0i} - A^{00}\overline{w}^i\ , && i=1,\dots,n\ ,\\
\overline{A}_0^{i0} &= \overline{A}^{i0} - A^{00}\overline{w}^i\ , && i=1,\dots,n\ ,\\
\overline{A}_0^{ij} &= A^{00}\overline{w}^i\overline{w}^j - \overline{A}^{0j}\overline{w}^i - \overline{A}^{i0}\overline{w}^j + \overline{A}^{ij}\ , && i,j=1,\dots,n\ .
\end{aligned}
$$

Analogously, for $\overline{\Gamma}^i_{kj}$ we obtain

$$
\begin{aligned}
\overline{\Gamma}^0_{kj} &= 0\ , && k,j=0,1,\dots,n\ ,\\
\overline{\Gamma}^i_{00} &= \frac{\partial\overline{w}^i}{\partial t} + \overline{w}^l\overline{w}^m\Gamma^i_{lm}\ , && i,l,m=1,\dots,n\ ,\\
\overline{\Gamma}^i_{j0} &= \Gamma^i_{0j} = \frac{\partial\overline{w}^i}{\partial\xi^j} + \overline{w}^l\Gamma^i_{jl}\ , && i,j,l=1,\dots,n\ ,\\
\overline{\Gamma}^i_{jk} &= \Gamma^i_{jk}\ , && i,j,k=1,\dots,n\ ,
\end{aligned}
$$

and for $\overline{F}^i_0$,

$$
\begin{aligned}
\overline{F}^0_0 &= F^0\ ,\\
\overline{F}^i_0 &= \overline{F}^i - A^0\overline{w}^i\ , && i=1,\dots,n\ .
\end{aligned}
$$

Using these expression in (2.112), we obtain a system of equations for the vector conservation law in the coordinates $\tau,\xi^1,\dots,\xi^n$ with an explicit expression for the components of the speed of the grid movement:

$$
\begin{aligned}
&\frac{\partial}{\partial\tau}(JA^{00}) + \frac{\partial}{\partial\xi^j}[J(\overline{A}^{0i} - A^{00}\overline{w}^j)] = JF^0\ ,\\
&\frac{\partial}{\partial\tau}J(\overline{A}^{i0} - A^{00}\overline{w}^j) + \frac{\partial}{\partial\xi^j}J(\overline{A}^{ij} + A^{00}\overline{w}^i\overline{w}^j - \overline{A}^{0j}\overline{w}^j - \overline{A}^{i0}\overline{w}^i)\\
&+ JA^{00}\Big(\frac{\partial\overline{w}^i}{\partial\tau} + \overline{w}^l\frac{\partial\overline{w}^i}{\partial\xi^l} + \overline{w}^l\overline{w}^j\Gamma^i_{lj}\Big)\\
&+ J(\overline{A}^{j0} + \overline{A}^{0j} - 2A^{00}\overline{w}^j)\Big(\frac{\partial\overline{w}^i}{\partial\xi^j} + \overline{w}^l\Gamma^i_{jl}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+ J(\overline{A}^{lj} + A^{00}\overline{w}^l\overline{w}^j - \overline{A}^{0j}\overline{w}^l - \overline{A}^{0l}\overline{w}^j)\Gamma^i_{lj} \\
&= J(\overline{F}^i - F^0\overline{w}^i) , \qquad i, j, l = 1, \dots, n . \qquad (2.113)
\end{aligned}
$$

Another representation of (2.111) in new coordinates can be derived in the form of (2.97) by applying (2.97) to each line of the system (2.111). As a result, we obtain

$$
\frac{1}{J}\left\{\frac{\partial}{\partial \tau}(JA^{i0}) + \frac{\partial}{\partial \xi^j}\left[J\left(A^{ik}\frac{\partial \xi^j}{\partial x^k} - A^{i0}\overline{w}^j\right)\right]\right\} = F , \quad j, k = 1, \dots, n . \qquad (2.114)
$$

Recall that this approach is not restricted to a square form of the system (2.111), i.e. the ranges for the indices i and j can be different.

As an illustration of these equations for a vector conservation law in the curvilinear coordinates $\tau, \xi^1, \dots, \xi^n$, we write out a joint system for the conservation of mass and momentum, which in the coordinates t, x^1, x^2, x^3 has the following form:

$$
\begin{aligned}
&\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x^i}\rho u^i = 0 , \qquad i = 1, 2, 3 , \\
&\frac{\partial \rho u^i}{\partial t} + \frac{\partial}{\partial x^j}(\rho u^i u^j + \overline{p}\delta^i_j) = \rho f^i , \qquad i, j = 1, 2, 3 , \qquad (2.115)
\end{aligned}
$$

where

$$
\overline{p} = p - \gamma\frac{\partial u^i}{\partial x^i} , \quad \delta^i_j = 0 \quad \text{if} \quad i \neq j \quad \text{and} \quad \delta^i_j = 1 \quad \text{if} \quad i = j .
$$

This system is represented in the form (2.111) with

$$
\begin{aligned}
A^{00} &= \rho , \\
A^{0i} &= A^{i0} = \rho u^i, \qquad i = 1, 2, 3 , \\
A^{ij} &= \rho u^i u^j + \delta^{ij} p, \qquad i, j = 1, 2, 3 ,
\end{aligned}
$$

i.e.

$$
(A^{ij}) = \begin{pmatrix} \rho & \rho u^1 & \rho u^2 & \rho u^3 \\ \rho u^1 & \rho u^1 u^1 + \overline{p} & \rho u^1 u^2 & \rho u^1 u^3 \\ \rho u^2 & \rho u^2 u^1 & \rho u^2 u^2 + \overline{p} & \rho u^2 u^3 \\ \rho u^3 & \rho u^3 u^1 & \rho u^3 u^2 & \rho u^3 u^3 + \overline{p} \end{pmatrix} ,
$$

$$
i, j = 0, 1, 2, 3 \qquad (2.116)
$$

and

$$
\begin{aligned}
F^0 &= 0 , \\
F^i &= \rho f^i, \qquad i = 1, 2, 3 .
\end{aligned}
$$

For the coordinate system $\tau, \xi^1, \ldots, \xi^n$, we obtain

$$\begin{aligned}
\overline{A}^{00} &= \rho \,, \\
\overline{A}^{0i} &= \overline{A}^{i0} = \rho(\overline{u}^i - \overline{w}^i) \,, \qquad i = 1, 2, 3 \,, \\
\overline{A}^{ij} &= \rho(\overline{u}^i - \overline{w}^i)(\overline{u}^j - \overline{w}^j) + \overline{p} g^{ij}, \qquad i, j = 1, 2, 3 \,.
\end{aligned}$$

Substituting these expressions in (2.113), we obtain a system of equations for the mass and momentum conservation laws in the coordinates $\tau, \xi^1, \ldots, \xi^n$:

$$\begin{aligned}
&\frac{\partial}{\partial \tau}(J\rho) + \frac{\partial}{\partial \xi^j}[J\rho(\overline{u}^j - \overline{w}^j)] = 0 \,, \qquad i = 1, 2, 3 \,, \\
&\frac{\partial}{\partial \tau}[J\rho(\overline{u}^i - \overline{w}^i)] + \frac{\partial}{\partial \xi^j}[J\rho(\overline{u}^i - \overline{w}^i)(\overline{u}^j - \overline{w}^j) + J\overline{p} g^{ij}] \\
&+ J\rho \frac{\partial \overline{w}^i}{\partial \tau} + J\rho(2\overline{u}^j - \overline{w}^j)\frac{\partial \overline{w}^i}{\partial \xi^j} + J(\rho \overline{u}^l \overline{u}^j + \overline{p} g^{lj})\Gamma^i_{lj} \\
&= J\rho \overline{f}^i \,, \qquad i, j, l = 1, 2, 3 \,.
\end{aligned} \tag{2.117}$$

If the coordinates ξ^i are the Lagrangian ones, i.e. $\overline{u}^i = \overline{w}^i$, then we obtain, from (2.117),

$$\begin{aligned}
&\frac{\partial}{\partial \tau}(J\rho) = 0 \,, \\
&J\rho \frac{\partial \overline{u}^i}{\partial \tau} + \frac{\partial}{\partial \xi^j}(J\overline{p} g^{ij}) + J\rho \overline{u}^j \frac{\partial \overline{u}^i}{\partial \xi^j} + J(\rho \overline{u}^l \overline{u}^j + \overline{p} g^{lj})\Gamma^i_{lj} = J\rho \overline{f}^i \,, \\
&i, j, l = 1, 2, 3 \,.
\end{aligned} \tag{2.118}$$

Note that the first equation of the system (2.117) coincides with (2.101) if $F = 0$; this was obtained as the scalar mass conservation law.

In the same manner, we can obtain an expression for the general Navier–Stokes equations of mass and momentum conservation by inserting the tensor $\{\sigma^{ij}\}$ described by (2.79) in the system (2.115) and the tensor $\{\overline{\sigma}^{ij}\}$ represented by (2.80) in the system (2.117).

A divergent form of (2.115) in arbitrary coordinates $\tau, \xi^1, \ldots, \xi^n$ is obtained by applying (2.114). With this, we obtain the system

$$\begin{aligned}
&\frac{\partial}{\partial \tau}(J\rho) + \frac{\partial}{\partial \xi^j}[J\rho(\overline{u}^j - \overline{w}^j)] = 0 \,, \\
&\frac{\partial}{\partial \tau}(J\rho \overline{u}^i) + \frac{\partial}{\partial \xi^j}\left[J\left(\rho u^i(\overline{u}^j - \overline{w}^j) + \overline{p}\frac{\partial \xi^j}{\partial x^i}\right)\right] = JF^i \,, \\
&\overline{u}^j = u^k \frac{\partial \xi^j}{\partial x^k} \,, \qquad i, j, k = 1, 2, 3 \,,
\end{aligned} \tag{2.119}$$

and, in the Lagrangian coordinates,

$$\frac{\partial}{\partial \tau}(J\rho) = 0 \,,$$
$$\frac{\partial}{\partial \tau}(J\rho\overline{u}^i) + \frac{\partial}{\partial \xi^j}\left(J\overline{p}\frac{\partial \xi^j}{\partial x^i}\right) = JF^i \,, \qquad i, j = 1, 2, 3 \,. \tag{2.120}$$

2.6 Comments

Many of the basic formulations of vector calculus and tensor analysis may be found in the books by Kochin (1951), Sokolnikoff (1964) and Gurtin (1981).

The formulation of general metric and tensor concepts specifically aimed at grid generation was originally performed by Eiseman (1980) and Warsi (1981).

Very important applications of the most general tensor relations to the formulation of unsteady equations in curvilinear coordinates in a strong conservative form were presented by Vinokur (1974). A strong conservation-law form of unsteady Euler equations was also described by Viviand (1974).

A derivation of various forms of the Navier–Stokes equations in general moving coordinates was described by Ogawa and Ishiguto (1987).

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