Step-by-Step Proof of Vinogradov's Theorem

In the first section, we begin with some lemmas and theorems which will be useful in presenting a step-by-step proof of Vinogradov's theorem, which states that there exists a natural number N, such that every odd positive integer n, with $n \geq N$, can be represented as the sum of three prime numbers. The experienced reader may wish to skip this section.

In the second section, we present the Hardy-Littlewood Circle Method and describe the basic ideas which will be used in the presentation of the proof of Vinogradov's theorem.

The third, and most important section of this chapter, is devoted to Vaughan's proof of Vinogradov's theorem. A number of authors have presented Vaughan's proof in books and expositions. We also present here this proof, but in a step-by-step manner. More specifically, in the beginning of this section we describe in detail how the Circle Method can be applied to attack the Ternary Goldbach Conjecture (TGC). Firstly, we define the appropriate Major and Minor arcs, and afterwards, we investigate their contribution in the integral which describes the number of representations of an integer as the sum of three prime numbers.

The following references have been particularly useful in writing this chapter: [9, 14, 29, 33, 40, 43, 61, 64, 66, 67]. The paper of Vaughan [65] has been of exceptional interest in this monograph.

1 Introductory Lemmas and Theorems

In this section, we state some lemmas and theorems which will be useful during the step-by-step analysis of the proof of Vinogradov's theorem. These lemmas and theorems are essentially independent from each other. The following two lemmas can be easily verified, and thus, we omit the details.

M.T. Rassias, *Goldbach's Problem*, DOI 10.1007/978-3-319-57914-6_2

Lemma 1.1 *The following holds*

$$\sum_{l=1}^{d} e\left(\frac{ln}{d}\right) = \begin{cases} d, \text{ if } d \mid n \\ \\ 0, \text{ if } d \nmid n, \end{cases}$$

where $e(x) = e^{2\pi ix}$, $x \in \mathbb{R}$.

Lemma 1.2 *Let* $a, b \in \mathbb{Z}$. *Then,*

$$\int_0^1 e(ax)e(-bx)dx = \begin{cases} 1, & \text{if } a = b \\ 0, & \text{if } a \neq b. \end{cases}$$

Definition 1.3 Let f be an arithmetic function. The series

$$D(f, s) = \sum_{n=1}^{+\infty} \frac{f(n)}{n^s},$$

where $s \in \mathbb{C}$, is called a Dirichlet series with coefficients $f(n)$.

We shall handle Dirichlet series for s being a real number.
Consider now a Dirichlet series, which is absolutely convergent for $s > s_0$.
If for these values of s it holds

$$\sum_{n=1}^{+\infty} \frac{f(n)}{n^s} = 0,$$

then $f(n) = 0$, for every integer n with $n \geq 1$.
If for these values of s it holds

$$D(f, s) = D(g, s),$$

then by the above argument it holds

$$f(n) = g(n), \text{ for every integer } n \text{ with } n \geq 1.$$

Theorem 1.4 *(1) Let* $D(f_1, s)$ *and* $D(f_2, s)$ *be convergent for* $s \in \mathbb{C}$. *Then the sum of* $D(f_1, s)$ *and* $D(f_2, s)$ *is obtained by*

$$D(f_1, s) + D(f_2, s) = \sum_{n=1}^{+\infty} \frac{f_1(n) + f_2(n)}{n^s}.$$

(2) Let $D(f_1, s)$ and $D(f_2, s)$ be absolutely convergent for $s \in \mathbb{C}$. Then the product of $D(f_1, s)$ and $D(f_2, s)$ is obtained by

$$D(f_1, s) \cdot D(f_2, s) = \sum_{n=1}^{+\infty} \frac{g(n)}{n^s},$$

where

$$g(n) = \sum_{n_1 n_2 = n} f_1(n_1) f_2(n_2).$$

Theorem 1.5 *Let f be a multiplicative function. Then, it holds*

$$D(f, s) = \prod_p \left(\sum_{n=0}^{+\infty} \frac{f(p^n)}{p^{ns}} \right),$$

where the product extends over all prime numbers p.

The basic idea of the proof of the theorem is the following:
It is true that

$$\prod_p \left(\sum_{n=0}^{+\infty} \frac{f(p^n)}{p^{ns}} \right) = \prod_p \left(\frac{f(1)}{1} + \frac{f(p)}{p^s} + \frac{f(p^2)}{p^{2s}} + \cdots \right)$$

$$= \left(\frac{f(1)}{1} + \frac{f(p_1)}{p_1^s} + \frac{f(p_1^2)}{p_1^{2s}} + \cdots \right) \left(\frac{f(1)}{1} + \frac{f(p_2)}{p_2^s} + \frac{f(p_2^2)}{p_2^{2s}} + \cdots \right) \cdots$$

$$= \sum \frac{f(p_1^{a_1}) \cdots f(p_k^{a_k})}{(p_1^{a_1} \cdots p_k^{a_k})^s}, \tag{1}$$

where[1] the sum extends over all possible combinations of multiples of powers of prime numbers. But, since the function $f(n)$ is multiplicative, it is evident that

$$\sum \frac{f(p_1^{a_1}) \cdots f(p_k^{a_k})}{(p_1^{a_1} \cdots p_k^{a_k})^s} = \sum \frac{f(p_1^{a_1} \cdots p_k^{a_k})}{(p_1^{a_1} \cdots p_k^{a_k})^s}$$

$$= \sum_{n=1}^{+\infty} \frac{f(n)}{n^s} = D(f, s).$$

□

[1] Here p_i denotes the ith prime number ($p_1 = 2, p_2 = 3, \ldots$).

Definition 1.6 The zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s} \ ,$$

for all real values of s with $s > 1$.

This function was defined for the first time in 1737 by Leonhard Euler (1707−1783). More than a century after Euler, in 1859, Georg Friedrich Bernhard Riemann (1826−1866) rediscovered the zeta function for complex values of s, while he was trying to prove the prime number Theorem.

Theorem 1.7 (Euler's Identity)

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}}, \ s \in \mathbb{R}, \ \textit{with } s > 1,$$

where the product extends over all prime numbers p.

The proof of Euler's Identity follows directly from Theorem 1.5.

Definition 1.8 Let $n \in \mathbb{N}$. The Möbius function $\mu(n)$ is defined as follows

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1 \\ (-1)^k, & \text{if } n = p_1 p_2 \cdots p_k \text{ where } p_1, p_2, \ldots, p_k \text{ are } k \text{ distinct primes} \\ 0, & \text{in every other case} \end{cases}$$

Theorem 1.9

$$\sum_{d|n} \mu(d) = \begin{cases} 1, & \textit{if } n = 1 \\ 0, & \textit{if } n > 1 \ , \end{cases}$$

where the sum extends over all positive divisors of the positive integer n.

Proof If $n = 1$ then the theorem obviously holds true, since by the definition of the Möbius function we know that $\mu(1) = 1$.
If $n > 1$ we can write

$$n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k},$$

where $p_1, \ p_2, \ldots, \ p_k$ are distinct prime numbers.
Therefore

$$\sum_{d|n} \mu(d) = \mu(1) + \sum_{1 \le i \le k} \mu(p_i) + \sum_{\substack{i \neq j \\ 1 \le i,j \le k}} \mu(p_i p_j) + \cdots + \mu(p_1 p_2 \cdots p_k), \quad (1)$$

where generally the sum

$$\sum_{i_1 \neq i_2 \neq \cdots \neq i_\lambda} \mu(p_{i_1} p_{i_2} \cdots p_{i_\lambda})$$

extends over all possible products of λ distinct prime numbers. Hence, by (1) and the binomial identity, we obtain

$$\begin{aligned}\sum_{d|n} \mu(d) &= 1 + \binom{k}{1}(-1) + \binom{k}{2}(-1)^2 + \cdots + \binom{k}{k}(-1)^k \\ &= (1-1)^k = 0.\end{aligned}$$

Therefore,

$$\sum_{d|n} \mu(d) = 0, \quad \textit{if} \quad n > 1.$$

□

Theorem 1.10 (The Möbius Inversion Formula) *Let* $n \in \mathbb{N}$. *If*

$$g(n) = \sum_{d|n} f(d)$$

then

$$f(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) g(d).$$

The converse also holds.

Proof For every arithmetic function $m(n)$, it holds

$$\sum_{d|n} m(d) = \sum_{d|n} m\left(\frac{n}{d}\right).$$

Therefore, it is evident that

$$\sum_{d|n} \mu\left(\frac{n}{d}\right) g(d) = \sum_{d|n} \mu(d) g\left(\frac{n}{d}\right). \tag{1}$$

But

$$\sum_{d|n} \mu(d) g\left(\frac{n}{d}\right) = \sum_{d|n} \left(\mu(d) \cdot \sum_{\lambda | \frac{n}{d}} f(\lambda)\right).$$

Hence, we get

$$\sum_{d|n} \mu(d) g\left(\frac{n}{d}\right) = \sum_{\lambda d|n} \mu(d) f(\lambda) \, .$$

Similarly,

$$\sum_{\lambda|n} \left(f(\lambda) \cdot \sum_{d|\frac{n}{\lambda}} \mu(d) \right) = \sum_{\lambda d|n} \mu(d) f(\lambda) \, .$$

Thus,

$$\sum_{d|n} \mu(d) g\left(\frac{n}{d}\right) = \sum_{\lambda|n} \left(f(\lambda) \cdot \sum_{d|\frac{n}{\lambda}} \mu(d) \right) . \tag{2}$$

However, by Theorem 1.9 we get

$$\sum_{d|\frac{n}{\lambda}} \mu(d) = 1 \text{ if and only if } \frac{n}{\lambda} = 1 \, ,$$

and in every other case, the sum is equal to zero. Thus, for $n = \lambda$ we obtain

$$\sum_{\lambda|n} \left(f(\lambda) \cdot \sum_{d|\frac{n}{\lambda}} \mu(d) \right) = f(n). \tag{3}$$

Therefore, by (1), (2) and (3) it follows that if

$$g(n) = \sum_{d|n} f(d)$$

then

$$f(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) g(d) \, .$$

Conversely we shall prove that if

$$f(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) g(d) \, ,$$

then

$$g(n) = \sum_{d|n} f(d).$$

We have

$$\begin{aligned}\sum_{d|n} f(d) &= \sum_{d|n} f\left(\frac{n}{d}\right)\\ &= \sum_{d|n}\sum_{\lambda|\frac{n}{d}} \mu\left(\frac{n}{\lambda d}\right) g(\lambda)\\ &= \sum_{d\lambda|n} \mu\left(\frac{n}{\lambda d}\right) g(\lambda)\\ &= \sum_{\lambda|n} g(\lambda) \sum_{d|\frac{n}{\lambda}} \mu\left(\frac{n}{\lambda d}\right)\end{aligned}$$

The sum

$$\sum_{d|\frac{n}{\lambda}} \mu\left(\frac{n}{\lambda d}\right) = 1$$

if and only if $n = \lambda$, and in every other case, it is equal to zero. Hence, for $n = \lambda$ we obtain

$$\sum_{d|n} f(d) = g(n).$$

This completes the proof of the theorem. □

Theorem 1.11 *For $s > 1$,*

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{+\infty} \frac{\mu(n)}{n^s}.$$

Proof By Euler's identity, we have

$$\frac{1}{\zeta(s)} = \prod_p \left(1 - \frac{1}{p^s}\right) \tag{2.1}$$

Additionally, by Theorem 1.5, it is evident that

$$\sum_{n=1}^{+\infty} \frac{\mu(n)}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right) . \tag{2.2}$$

By the above formulas (2.1) and (2.2), the theorem follows. □

Definition 1.12 Let $n \in \mathbb{N}$. The Von Mangoldt function $\Lambda(n)$ is defined as follows

$$\Lambda(n) = \begin{cases} \log p, & \text{if } n = p^k \text{ for some prime number } p \text{ and some } k \in \mathbb{N} \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 1.13 *Let $n \in \mathbb{N}$. Then, it holds*

$$\sum_{d|n} \Lambda(d) = \log n \tag{L1}$$

and

$$\Lambda(n) = -\sum_{d|n} \mu(d) \log d. \tag{L2}$$

Proof It is evident that the theorem holds true in the case when $n = 1$. Hence, let us assume that $n > 1$. If

$$n = p_1^{a_1} p_2^{a_2} \dots p_m^{a_m}$$

is the cannonical representation of n by its prime factors, we get

$$\begin{aligned} \log n &= \sum_{i=1}^{m} a_i \log p_i \\ &= \sum_{i=1}^{m} \sum_{q=1}^{a_i} \log p_i \\ &= \sum_{i=1}^{m} \sum_{q=1}^{a_i} \Lambda(p_i^q) \\ &= \sum_{d|n} \Lambda(d). \end{aligned}$$

This completes the proof of (L1).
By the Möbius inversion formula and (L1) we obtain

$$\begin{aligned} \Lambda(n) &= \sum_{d|n} \mu(d) \log \frac{n}{d} \\ &= \log n \sum_{d|n} \mu(d) - \sum_{d|n} \mu(d) \log d. \end{aligned}$$

Therefore, by Theorem 1.9 and the above formula, (L2) follows. □

Theorem 1.14 *For $s > 1$, it holds*

$$\sum_{n=1}^{+\infty} \frac{\Lambda(n)}{n^s} = -\frac{1}{\zeta(s)} \frac{d\zeta(s)}{ds}.$$

Proof From the definition of the Riemann zeta function and a well-known theorem of termwise differentiation of an infinite series, it follows that

$$\frac{d\zeta(s)}{ds} = -\sum_{n=1}^{+\infty} \frac{\log n}{n^s}. \tag{1}$$

Additionally, by Theorem 1.11, we know that

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{+\infty} \frac{\mu(n)}{n^s}. \tag{2}$$

By Theorem 1.4, if we multiply (1), (2), we obtain

$$\frac{1}{\zeta(s)} \frac{d\zeta(s)}{ds} = -\sum_{n=1}^{+\infty} \frac{g(n)}{n^s} \tag{3}$$

where

$$g(n) = \sum_{n_1 n_2 = n} \mu(n_1) \log n_2.$$

But, by Theorem 1.13, it follows that

$$g(n) = \Lambda(n).$$

Therefore, by (3) and the above relation, the theorem follows. □

Definition 1.15 The exponential sum

$$c_n(m) = \sum_{\substack{1 \le q \le n \\ (q,n)=1}} e\left(\frac{qm}{n}\right),$$

is called the Ramanujan sum $c_n(m)$.

Note By the use of the Chinese Remainder Theorem (cf. [55]) it can be proved that the Ramanujan sum $c_n(m)$ is a multiplicative function of n.

Lemma 1.16 *For the Ramanujan sum $c_n(m)$, we have*

$$c_n(m) = \sum_{d|(n,m)} \mu\left(\frac{n}{d}\right) d\ .$$

Proof By Theorem 1.9, we know that for the Möbius function it holds

$$\sum_{d|n} \mu(d) = \begin{cases} 1, & \text{if } n = 1 \\ 0, & \text{if } n > 1 \end{cases}$$

Therefore, we can write

$$\begin{aligned} c_n(m) &= \sum_{\substack{1 \le q \le n \\ (q,n)=1}} e\left(\frac{qm}{n}\right) \\ &= \sum_{q=1}^{n} \left(e\left(\frac{qm}{n}\right) \sum_{d|(q,n)} \mu(d) \right) \\ &= \sum_{d|n} \sum_{k=1}^{n/d} e\left(\frac{km}{n/d}\right) \mu(d) \\ &= \sum_{d|n} \sum_{k=1}^{d} e\left(\frac{km}{d}\right) \mu\left(\frac{n}{d}\right)\ . \end{aligned}$$

However, by Lemma 1.1, we obtain

$$c_n(m) = \sum_{\substack{d|n \\ d|m}} \mu\left(\frac{n}{d}\right) d\ .$$

But, $d \mid n$ and $d \mid m$ is equivalent to $d \mid (n, m)$. Hence, it follows that

$$c_n(m) = \sum_{d|(n,m)} \mu\left(\frac{n}{d}\right) d\ .$$

□

Lemma 1.17 *Let x be a real number. Then,*

$$\left| \sum_{n=B_1+1}^{B_2} e(xn) \right| \le \min\left\{ \frac{1}{[x]}, B_2 - B_1 \right\}\ ,$$

where B_1*,* B_2 *are integers with* $B_1 < B_2$ *and* $\left[y\right] = \min_{k \in \mathbb{Z}} |y - k|$*, where* $y \in \mathbb{R}$.

Proof Let us suppose that x is not an integer. In this case, we have

$$\left|\sum_{n=B_1+1}^{B_2} e(xn)\right| \leq \frac{\left|e^{2\pi i(B_1+1)x}\right| + \left|e^{2\pi i(B_2+1)x}\right|}{\left|e^{2\pi i(x/2)} - e^{2\pi i(-x/2)}\right|}$$
$$= \frac{2}{\left|e^{\pi ix} - e^{-\pi ix}\right|} .$$

But

$$i \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2}$$

and therefore, we obtain

$$\left|\sum_{n=B_1+1}^{B_2} e(xn)\right| \leq \frac{1}{|2\sin(\pi x)|} .$$

In addition, it is true that

$$\pi x = \pi(k \pm [x]) ,$$

where k is the nearest integer to x. Thus, it follows that

$$\frac{1}{|2\sin(\pi x)|} = \frac{1}{2\sin(\pi [x])} .$$

However, it is a well-known fact that

$$\frac{\sin\theta}{\theta} > \frac{2}{\pi}, \text{ for } -\frac{\pi}{2} < \theta < \frac{\pi}{2}, \ \theta \neq 0 .$$

Hence, for $\theta = \pi [x] > 0$, we get

$$\sin \pi [x] > \frac{2\pi [x]}{\pi} = 2[x] .$$

From all the above, it is evident that

$$\left|\sum_{n=B_1+1}^{B_2} e(xn)\right| \leq \frac{1}{[x]} . \qquad (1)$$

Of course, it is clear that

$$\left|\sum_{n=B_1+1}^{B_2} e(xn)\right| \le \sum_{n=B_1+1}^{B_2} |e(xn)| = B_2 - B_1 \,. \tag{2}$$

Hence, from (1) and (2), we obtain

$$\left|\sum_{n=B_1+1}^{B_2} e(xn)\right| \le \min\left\{\frac{1}{[x]}, B_2 - B_1\right\} \,.$$

□

Theorem 1.18 *Let $\tau(n)$ denote the divisor function, defined by*

$$\tau(n) = \sum_{\substack{d|n \\ d\ge 1}} 1 \,.$$

Then, for any real number x, with $x \ge 2$, it holds

$$\sum_{n\le x} \tau^2(n) \ll x \log^3 x \,.$$

Proof We have

$$\begin{aligned}\sum_{n\le x} \tau^2(n) &= \sum_{n\le x} \left(\sum_{d_1|n} 1\right)\left(\sum_{d_2|n} 1\right) \\ &= \sum_{d_1,d_2\le x} \sum_{\substack{n\le x \\ n\equiv 0 \pmod{\operatorname{lcm}\{d_1,d_2\}}}} 1 \,.\end{aligned}$$

We write $e = (d_1, d_2)$, $d_1 = t_1 e$, $d_2 = t_2 e$.
We have $\operatorname{lcm}\{d_1, d_2\} = t_1 t_2 e$ and thus

$$\begin{aligned}\sum_{n\le x} \tau^2(n) &\le \sum_{e\le x} \sum_{t_1,\, t_2\le x} \left\lfloor \frac{x}{t_1 t_2 e} \right\rfloor \\ &\le x\left(\sum_{e\le x} \frac{1}{e}\right)^3 \\ &\ll x \log^3 x.\end{aligned}$$

This completes the proof of the theorem. $\square$

Theorem 1.19 (LEGENDRE'S THEOREM)
The largest power of p which divides the integer n! is[2]

$$\sum_{k=1}^{+\infty} \left\lfloor \frac{n}{p^k} \right\rfloor$$

Proof The number of factors of $n!$ which are divisible by p, is $\lfloor n/p \rfloor$. More specifically, these factors are the integers:

$$1 \cdot p,\ 2 \cdot p, \ldots, \left\lfloor \frac{n}{p} \right\rfloor \cdot p\,.$$

However, some factors of $n!$ are divisible by at least the second power of p, namely they contain p^2 at least one time. These factors are the integers:

$$1 \cdot p^2,\ 2 \cdot p^2, \ldots, \left\lfloor \frac{n}{p^2} \right\rfloor \cdot p^2\,,$$

which are exactly

$$\left\lfloor \frac{n}{p^2} \right\rfloor$$

in number.
If we continue similarly for higher powers of p, it follows that the integer $n!$ contains the prime number p exactly

$$\left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \cdots + \left\lfloor \frac{n}{p^k} \right\rfloor + \cdots$$

times and therefore that is exactly the largest power of p which divides n !.
The above sum is finite since for $k > r$, where $p^r \geq n$, it holds

$$\left\lfloor \frac{n}{p^k} \right\rfloor = 0.$$

$\square$

Definition 1.20 We define $\pi(x)$ to be the number of primes which do not exceed a given real number x.

[2] By $\lfloor x \rfloor$ we denote the integer part of x and by $\lceil x \rceil$ the least integer, greater than or equal to x.

Theorem 1.21 (CHEBYSHEV'S INEQUALITY)
For every positive integer n, where $n \geq 2$, the following inequality holds

$$\frac{1}{6} \cdot \frac{n}{\log n} < \pi(n) < 6 \cdot \frac{n}{\log n}$$

Proof We claim that

$$2^n \leq \binom{2n}{n} < 4^n . \tag{1}$$

The inequality

$$2^n \leq \binom{2n}{n}$$

follows by mathematical induction.
For $n = 2$ one has

$$4 \leq \binom{4}{2} = 6 ,$$

which holds true. Suppose that (1) is valid for n, i.e.

$$2^n \leq \binom{2n}{n} .$$

It suffices to prove (1) for $n + 1$.
It is clear that

$$\begin{aligned}\binom{2n+2}{n+1} &= \frac{(2n+2)!}{(n+1)!(n+1)!} \\ &= \frac{(2n)!}{n\,!\,n\,!} \frac{(2n+1)(2n+2)}{(n+1)^2} \\ &\geq 2^n \frac{(2n+1)(2n+2)}{(n+1)^2} .\end{aligned}$$

It is enough to prove that

$$\frac{(2n+1)(2n+2)}{(n+1)^2} \geq 2 \, , \text{for } \ n \geq 2 \ .$$

However,

$$\frac{(2n+1)(2n+2)}{(n+1)^2} \geq 2 \Leftrightarrow 2n \geq 0 \, ,$$

which clearly holds true.
Thus,

$$\binom{2n+2}{n+1} \geq 2^{n+1}$$

and therefore, we have proved that

$$2^n \leq \binom{2n}{n}$$

for every positive integer n, where $n \geq 2$.
The proof of the right-hand side of inequality (1) follows from the fact that

$$\binom{2n}{n} < \binom{2n}{0} + \binom{2n}{1} + \cdots + \binom{2n}{2n} = 2^{2n} = 4^n \, .$$

From (1), we get that

$$\log 2^n \leq \log \frac{(2n)!}{n!n!} < \log 4^n$$

and therefore

$$n \log 2 \leq \log(2n)! - 2 \log n! < n \log 4 \, . \tag{2}$$

However, from Legendre's Theorem (see Theorem 1.19), it follows that

$$n! = \prod_{p \leq n} p^{j(n,p)} \, , \tag{3}$$

where

$$j(n,p) = \sum_{k=1}^{+\infty} \left\lfloor \frac{n}{p^k} \right\rfloor \, .$$

From (3), we obtain

$$\begin{aligned}\log n! &= \log \prod_{p\le n} p^{j(n,p)}\\ &= \sum_{p\le n} \log p^{j(n,p)}\\ &= \sum_{p\le n} j(n,p)\log p\,.\end{aligned}$$

By applying this result, we get

$$\begin{aligned}\log(2n)! - 2\log n! &= \sum_{p\le 2n} j(n,p)\log p - 2\sum_{p\le n} j(n,p)\log p\\ &= \sum_{p\le 2n}\left(\sum_{k=1}^{+\infty}\left\lfloor \frac{2n}{p^k}\right\rfloor\right)\log p - 2\sum_{p\le n}\left(\sum_{k=1}^{+\infty}\left\lfloor \frac{n}{p^k}\right\rfloor\right)\log p\,.\end{aligned}$$

However,

$$\sum_{p\le n}\left(\sum_{k=1}^{+\infty}\left\lfloor \frac{n}{p^k}\right\rfloor\right)\log p = \sum_{p\le 2n}\left(\sum_{k=1}^{+\infty}\left\lfloor \frac{n}{p^k}\right\rfloor\right)\log p$$

since for $p > n$ it is true that

$$\left\lfloor \frac{n}{p^k}\right\rfloor = 0\ .$$

Therefore,

$$\begin{aligned}\log(2n)! - 2\log n! &= \sum_{p\le 2n}\left(\sum_{k=1}^{+\infty}\left\lfloor \frac{2n}{p^k}\right\rfloor - 2\sum_{k=1}^{+\infty}\left\lfloor \frac{n}{p^k}\right\rfloor\right)\log p\\ &= \sum_{p\le 2n}\left[\sum_{k=1}^{+\infty}\left(\left\lfloor \frac{2n}{p^k}\right\rfloor - 2\left\lfloor \frac{n}{p^k}\right\rfloor\right)\right]\log p\,.\end{aligned}$$

However, it holds

$$\left\lfloor \frac{2n}{p^k}\right\rfloor - 2\left\lfloor \frac{n}{p^k}\right\rfloor < \frac{2n}{p^k} - 2\left(\frac{n}{p^k} - 1\right) = 2\,.$$

Thus, clearly

$$\left\lfloor \frac{2n}{p^k} \right\rfloor - 2\left\lfloor \frac{n}{p^k} \right\rfloor = 0 \text{ or } 1 .$$

The terms of the infinite summation

$$\sum_{k=1}^{+\infty} \left(\left\lfloor \frac{2n}{p^k} \right\rfloor - 2\left\lfloor \frac{n}{p^k} \right\rfloor \right)$$

assume the value zero for k such that $p^k > 2n$, that means for

$$k > \frac{\log 2n}{\log p} .$$

Thus,

$$\begin{aligned} \sum_{k=1}^{+\infty} \left(\left\lfloor \frac{2n}{p^k} \right\rfloor - 2\left\lfloor \frac{n}{p^k} \right\rfloor \right) &= \sum_{k=1}^{\left\lfloor \frac{\log 2n}{\log p} \right\rfloor} \left(\left\lfloor \frac{2n}{p^k} \right\rfloor - 2\left\lfloor \frac{n}{p^k} \right\rfloor \right) \\ &\le \sum_{k=1}^{\left\lfloor \frac{\log 2n}{\log p} \right\rfloor} 1 . \end{aligned}$$

Hence,

$$\begin{aligned} \log(2n)! - 2\log n! &\le \sum_{p \le 2n} \left(\sum_{k=1}^{\left\lfloor \frac{\log 2n}{\log p} \right\rfloor} 1 \right) \log p \\ &\le \sum_{p \le 2n} \frac{\log 2n}{\log p} \log p \\ &= \sum_{p \le 2n} \log 2n \\ &= \pi(2n) \log 2n . \end{aligned}$$

From this relation and inequality (2), it follows that

$$n \log 2 \le \pi(2n) \log 2n$$

$$\Leftrightarrow \pi(2n) \geq \frac{n\log 2}{\log 2n} > \frac{n/2}{\log 2n} = \frac{2n}{4\log 2n}$$

$$\Leftrightarrow \pi(2n) > \frac{1}{4}\cdot\frac{2n}{\log 2n} > \frac{1}{6}\cdot\frac{2n}{\log 2n}\,. \tag{4}$$

Therefore, the inequality

$$\frac{1}{6}\cdot\frac{n}{\log n} < \pi(n)$$

is satisfied if n is an even integer. It remains to examine the case where n is an odd integer.

It is true that

$$\begin{aligned}\pi(2n+1) \geq \pi(2n) &> \frac{1}{4}\cdot\frac{2n}{\log 2n}\\ &= \frac{1}{4}\cdot\frac{2n}{2n+1}\cdot\frac{2n+1}{\log 2n}\\ &> \frac{1}{4}\cdot\frac{2n}{2n+1}\cdot\frac{2n+1}{\log(2n+1)}\,.\end{aligned}$$

It is evident that

$$\frac{2n}{2n+1} \geq \frac{2}{3}$$

for every positive integer n.

Therefore

$$\begin{aligned}\pi(2n+1) &> \frac{1}{4}\cdot\frac{2}{3}\cdot\frac{2n+1}{\log(2n+1)}\\ &= \frac{1}{6}\cdot\frac{2n+1}{\log(2n+1)}\end{aligned}$$

Hence, the inequality

$$\frac{1}{6}\cdot\frac{n}{\log n} < \pi(n)$$

is also satisfied in the case where n is an odd integer.

Thus

$$\frac{1}{6}\cdot\frac{n}{\log n} < \pi(n),$$

for every positive integer n, with $n \geq 2$.

We will now prove the inequality

$$\pi(n) < 6 \cdot \frac{n}{\log n}$$

for every positive integer n with $n \geq 2$.
We have already proved that

$$\log(2n)! - 2\log n! = \sum_{p \leq 2n} \left[\sum_{k=1}^{+\infty} \left(\left\lfloor \frac{2n}{p^k} \right\rfloor - 2 \left\lfloor \frac{n}{p^k} \right\rfloor \right) \right] \log p \ ,$$

where none of the terms

$$\left\lfloor \frac{2n}{p^k} \right\rfloor - 2 \left\lfloor \frac{n}{p^k} \right\rfloor$$

is negative.
Therefore, it is clear that

$$\left\lfloor \frac{2n}{p} \right\rfloor - 2 \left\lfloor \frac{n}{p} \right\rfloor \leq \sum_{k=1}^{+\infty} \left(\left\lfloor \frac{2n}{p^k} \right\rfloor - 2 \left\lfloor \frac{n}{p^k} \right\rfloor \right) .$$

Thus

$$\begin{aligned} \log(2n)! - 2\log n! &\geq \sum_{p \leq 2n} \left(\left\lfloor \frac{2n}{p} \right\rfloor - 2 \left\lfloor \frac{n}{p} \right\rfloor \right) \log p \\ &\geq \sum_{n < p \leq 2n} \left(\left\lfloor \frac{2n}{p} \right\rfloor - 2 \left\lfloor \frac{n}{p} \right\rfloor \right) \log p \ . \end{aligned}$$

However for the prime numbers p, such that $n < p \leq 2n$ one has

$$\left\lfloor \frac{2n}{p} \right\rfloor - 2 \left\lfloor \frac{n}{p} \right\rfloor = 1 \ ,$$

since

$$\left\lfloor \frac{2n}{p} \right\rfloor = 1 \text{ and } \left\lfloor \frac{n}{p} \right\rfloor = 0 \ .$$

Hence,

$$\log(2n)! - 2\log n! \geq \sum_{n < p \leq 2n} \log p \ . \tag{5}$$

By the definition of Chebyshev's function $\vartheta(x)$, one has

$$\vartheta(x) = \sum_{p \le x} \log p \ .$$

Therefore, (5) can be written as follows:

$$\log(2n)! - 2\log n! \ge \vartheta(2n) - \vartheta(n) \ .$$

Thus by meansof (2), we obtain

$$\vartheta(2n) - \vartheta(n) < n \log 4 \ . \tag{6}$$

Suppose that the positive integer n can be expressed as an exact power of 2. Then from (6), it follows

$$\vartheta(2 \cdot 2^m) - \vartheta(2^m) < 2^m \log 2^2$$

and therefore

$$\vartheta(2^{m+1}) - \vartheta(2^m) < 2^{m+1} \log 2 \ .$$

For $m = 1,\ 2,\ \ldots\ ,\ \lambda - 1,\ \lambda$ the above inequality, respectively, yields

$$\left.\begin{aligned} \vartheta(2^2) - \vartheta(2) &< \quad 2^2 \log 2 \\ \vartheta(2^3) - \vartheta(2^2) &< \quad 2^3 \log 2 \\ &\vdots \\ \vartheta(2^\lambda) - \vartheta(2^{\lambda-1}) &< \quad 2^\lambda \log 2 \\ \vartheta(2^{\lambda+1}) - \vartheta(2^\lambda) &< 2^{\lambda+1} \log 2 \end{aligned}\right\}$$

Adding up the above inequalities, we get

$$\vartheta(2^{\lambda+1}) - \vartheta(2) < (2^2 + 2^3 + \ldots + 2^\lambda + 2^{\lambda+1}) \log 2 \ .$$

But $\vartheta(2) = \log 2$, therefore

$$\begin{aligned} \vartheta(2^{\lambda+1}) &< (1 + 2^2 + 2^3 + \ldots + 2^\lambda + 2^{\lambda+1}) \log 2 \\ &= (2^{\lambda+2} - 1) \log 2 \ . \end{aligned}$$

Hence

$$\vartheta(2^{\lambda+1}) < 2^{\lambda+2} \log 2 \ . \tag{7}$$

For every positive integer n we can choose a suitable integer m such that

$$2^m \leq n \leq 2^{m+1} .$$

Then

$$\vartheta(n) = \sum_{p \leq n} \log p \leq \sum_{p \leq 2^{m+1}} \log p = \vartheta(2^{m+1})$$

and by means of (7), it follows that

$$\vartheta(n) < 2^{m+2} \log 2 = 2^2 \cdot 2^m \log 2 \leq 4n \log 2 . \tag{8}$$

Let N be the number of primes p_i, such that

$$n^r < p_i \leq n$$

where $0 < r < 1$, for $i = 1, 2, \cdots, N$. Then

$$\left.\begin{array}{rl} \log n^r < & \log p_1 \\ \log n^r < & \log p_2 \\ \vdots & \\ \log n^r < & \log p_N \end{array}\right\} \Rightarrow N \log n^r < \sum_{n^r < p \leq n} \log p .$$

and therefore

$$(\pi(n) - \pi(n^r)) \log n^r < \sum_{n^r < p \leq n} \log p . \tag{9}$$

It is obvious that

$$\vartheta(n) \geq \sum_{n^r < p \leq n} \log p . \tag{10}$$

Therefore by means of (8), (9) and (10), one has

$$\begin{aligned} (\pi(n) - \pi(n^r)) \log n^r &< 4n \log 2 \\ \Leftrightarrow \pi(n) \log n^r &< 4n \log 2 + \pi(n^r) \log n^r \\ \Leftrightarrow \pi(n) &< \frac{4n \log 2}{\log n^r} + \pi(n^r) \\ &< \frac{4n \log 2}{r \log n} + n^r . \end{aligned}$$

Thus, equivalently, we obtain

$$\pi(n) < \frac{n}{\log n}\left(\frac{4\log 2}{r} + n^{r-1}\log n\right) . \tag{11}$$

Consider the function defined by the formula

$$f(x) = \frac{\log x}{x^{1-r}} , \ x \in \mathbb{R}^+ .$$

Then

$$f'(x) = \frac{\frac{1}{x}x^{1-r} - (1-r)x^{-r}\log x}{(x^{1-r})^2} .$$

It is clear that

$$f'(x) = 0$$

if

$$x^{-r} = (1-r)x^{-r}\log x \Leftrightarrow \log x = \frac{1}{1-r} ,$$

that means

$$x = e^{1/(1-r)} .$$

For $x = e^{1/(1-r)}$ the function $f(x)$ attains its maximal value.
Thus

$$f(x) \leq \frac{1}{e(1-r)} \Rightarrow f(n) \leq \frac{1}{e(1-r)} ,$$

and therefore

$$n^{r-1}\log n \leq \frac{1}{e(1-r)} . \tag{12}$$

From (11) and (12), it follows

$$\pi(n) < \frac{n}{\log n}\left(\frac{4\log 2}{r} + \frac{1}{e(1-r)}\right) .$$

Set $r = \frac{2}{3}$. Then

$$\pi(n) < \frac{n}{\log n}\left(6\log 2 + \frac{3}{e}\right) .$$

However, it holds

$$6\log 2+\frac{3}{e}<6 \text{ and thus } \pi(n)<6\cdot\frac{n}{\log n}$$

Hence for every positive integer n, where $n \geq 2$, the following inequality holds

$$\frac{1}{6}\cdot\frac{n}{\log n}<\pi(n)<6\cdot\frac{n}{\log n}$$ □

Theorem 1.22 (DIRICHLET'S APPROXIMATION THEOREM) *Let A be a real number and n a natural number. Then, there exists an integer b, such that $0 < b \leq n$, and an integer c, for which the following holds*

$$|Ab-c|<\frac{1}{n}\,.$$

Proof Let $\{Ai\} = Ai - \lfloor Ai \rfloor$, for $i = 0, 1, 2, \ldots, n$. It is clear that $0 \leq \{Ai\} < 1$. We now construct the intervals

$$\left[\frac{x}{n},\frac{x+1}{n}\right)\,,$$

where $0 \leq x < n$.
Since there are $n+1$ real numbers $\{Ai\}$, such that $0 \leq \{Ai\} < 1$, by the Pigeonhole Principle it follows that at least one of the intervals $[x/n, (x+1)/n)$ will contain two of these numbers.
Let us suppose that

$$\{Ak\}\,,\ \{Al\}\in\left[\frac{x}{n},\frac{x+1}{n}\right)\,,$$

for some $0 \leq x < n$.
Therefore

$$|\{Ak\}-\{Al\}|<\frac{1}{n}$$

or

$$|Ak-\lfloor Ak\rfloor-(Al-\lfloor Al\rfloor)|<\frac{1}{n}$$

or

$$|A(k-l)-(\lfloor Ak\rfloor-\lfloor Al\rfloor)|<\frac{1}{n}\,.$$

Thus, we distinguish the following cases:

- If $k - l > 0$, then we set $b = k - l$ and $c = \lfloor Ak \rfloor - \lfloor Al \rfloor$.
- If $k - l < 0$, then we set $b = l - k$ and $c = \lfloor Al \rfloor - \lfloor Ak \rfloor$.

Hence, we obtain

$$|Ab - c| < \frac{1}{n} .$$

□

Corollary 1.23 *Let A be a real number and n a natural number. Then, there exists an integer b, such that* $0 < b \leq n$*, and an integer c relatively prime to b, for which it holds*

$$\left| A - \frac{c}{b} \right| < \frac{1}{b^2} .$$

Proof By Dirichlet's Approximation Theorem, we have

$$|Ab - c| < \frac{1}{n} .$$

Thus, since b is a positive integer, we can write

$$\frac{|Ab - c|}{b} < \frac{1}{nb} \leq \frac{1}{b^2}$$

or

$$\left| A - \frac{c}{b} \right| < \frac{1}{b^2} .$$

□

The following theorem as well as other related theorems can be found in [9, 23].

Theorem 1.24 (Siegel- Walfisz Theorem)
Let D be a positive constant. Then there exists a positive constant $C(D)$ *such that the following holds: Assume that r is a real number and* a, q *are integers such that* $(a, q) = 1$ *with* $q \leq \log^D r$*. Then*

$$\sum_{\substack{n \leq r \\ n \equiv a (\bmod q)}} \Lambda(n) = \frac{r}{\phi(q)} + O\left(r \exp\left(-C(D)\sqrt{\log r} \right) \right) ,$$

where $\Lambda(n)$ *denotes the Von Mangoldt function and* $\phi(n)$ *the Euler totient function.*

2 The Circle Method

The Circle Method was introduced for the first time in a paper by Hardy and Ramanujan [21] concerning partitions. Moreover, Hardy and Littlewood developed that method so that it could be used to connect exponential sums with general problems of additive number theory.[3] For recent developments and generalizations of the Hardy-Littlewood method to additive number theory, the interested reader is referred to the paper of Green [17].
A characteristic problem to which the Circle Method finds an application is the following:

Problem 2.1 *Let S be a subset of $\mathbb{N}$ and $k \in \mathbb{N}$. Determine*

$$\{s_1 + s_2 + \cdots + s_k \mid s_1, s_2, \ldots, s_k \in S\} \cap \mathbb{N}\,.$$

In other words, determine which natural numbers can be represented as the sum of k elements of the set S and in how many ways.

Remark 2.2 If we set $S = \mathbb{P}$, where $\mathbb{P}$ denotes the set of all prime numbers, then

1. For $k = 2$, the statement of Problem 2.1 becomes:
   Determine the set

   $$E = \{p_1 + p_2 \mid p_1, p_2 \in \mathbb{P}\} \cap \mathbb{N}\,.$$

   The Goldbach conjecture states that the set E is the set of all positive even integers.
2. For $k = 3$, the statement of Problem 2.1 becomes:
   Determine the set

   $$O = \{p_1 + p_2 + p_3 \mid p_1, p_2, p_3 \in \mathbb{P}\} \cap \mathbb{N}\,.$$

In 1937, I. M. Vinogradov [66, 67] proved that every large enough odd positive integer is included in the set O.
Generally, the starting point of the Circle Method is to consider a generating function of the form:

$$F_S(x) = \sum_{s \in S} x^s\,.$$

[3]Hardy and Littlewood in a paper published in 1923 have used the Circle Method to prove that on assumption of a modified form of the Riemann Hypothesis there exists a natural number N, such that every odd integer $n \geq N$ can be expressed as the sum of three prime numbers.

Questions of convergence may be avoided if S is a finite set, which we shall assume in the following. We write

$$F_S(x)^k = \sum_{n=1}^{+\infty} R(n, k, S)x^n .$$

It can be proved that the coefficient $R(n, k, S)$ is equal to the number of ways that n can be represented as the sum of k elements of the set S.
Moreover, it follows from Cauchy's formula that

$$R(n, k, S) = \frac{1}{2\pi i} \int_C \frac{F_S(z)^k}{z^{n+1}} dz , \tag{1}$$

where C is the unit circle oriented counterclockwise.
However, if we substitute $x = e^{2\pi i u}$ and

$$f_S(u) = F_S(x),$$

we obtain

$$R(n, k, S) = \int_0^1 f_S(u)^k e^{-2\pi i n u} du .$$

In addition, for every natural number $n \leq N$, it holds

$$R(n, k, S) = R_N(n, k, S) = \int_0^1 f_N(x)^k e^{-2\pi i n x} dx ,$$

where $R_N(n, k, S)$ is equal to the number of ways that n can be represented as the sum of k elements of the set S, where each element is at most N.
The key feature of the Circle Method is to split C into two disjoint pieces, generally referred to as the *Major* and *Minor arcs* $\mathfrak{M}$ and $\mathfrak{m}$, respectively.
Therefore, we obtain

$$R(n, k, S) = R_N(n, k, S) = \int_{\mathfrak{M}} f_N(x)^k e^{-2\pi i n x} dx + \int_{\mathfrak{m}} f_N(x)^k e^{-2\pi i n x} dx$$

or equivalently

$$R(n, k, S) = R_N(n, k, S) = \int_{\mathfrak{M}} f_N(x)^k e(-nx) dx + \int_{\mathfrak{m}} f_N(x)^k e(-nx) dx .$$

The basic idea behind the choice of the Major and Minor arcs is the following: The Major arcs are constructed in such a way, so that the function in the integral

$$\int_{\mathfrak{M}}$$

can be evaluated asymptotically and that the contribution of the Minor arcs is of lower order.

3 Proof of Vinogradov's Theorem

The purpose of this section is to present R. C. Vaughan's proof of Vinogradov's theorem.

Theorem 3.1 (VINOGRADOV'S THEOREM)
There exists a natural number N, such that every odd positive integer n, with $n \geq N$, can be represented as the sum of three prime numbers.

Before we define the appropriate function f and construct the relevant Major and Minor arcs, in order to apply the Circle Method, we observe that

$$\begin{aligned} R(n, 3, \mathbb{P}) &= \sum_{n=p_1+p_2+p_3} 1 \\ &> \sum_{n=p_1+p_2+p_3} \frac{\log p_1 \cdot \log p_2 \cdot \log p_3}{\log^3 (p_1 + p_2 + p_3)} \\ &= \sum_{n=p_1+p_2+p_3} \frac{\log p_1 \cdot \log p_2 \cdot \log p_3}{\log^3 n} \end{aligned}$$

or equivalently

$$R(n, 3, \mathbb{P}) > \frac{1}{\log^3 n} \sum_{n=p_1+p_2+p_3} \log p_1 \cdot \log p_2 \cdot \log p_3 \,, \tag{a}$$

where $\mathbb{P}$ denotes the set of all prime numbers and consequently p_1, p_2, p_3 are prime numbers.
Therefore, instead of working with the sum

$$\sum_{n=p_1+p_2+p_3} 1$$

we shall work with the sum

$$\sum_{n=p_1+p_2+p_3} \log p_1 \cdot \log p_2 \cdot \log p_3$$

More specifically, Vinogradov succeeded in proving that

$$\sum_{n=p_1+p_2+p_3} \log p_1 \cdot \log p_2 \cdot \log p_3 \gg n^2 .$$

Thus, by (a), we obtain

$$R(n, 3, \mathbb{P}) \gg \frac{n^2}{\log^3 n} ,$$

from which it is obvious that there exists a natural number N, such that every $n \geq N$, can be represented as the sum of three prime numbers.
Let us now proceed to the details of the proof of Vinogradov's Theorem by the use of the Circle Method.
Let

$$f(x) = \sum_{p \leq N} \log p \cdot e(xp)$$

and

$$f_r(x) = \sum_{p \leq r} \log p \cdot e(xp) ,$$

where p is a prime number and x, r are real numbers.
In addition, let

$$\overline{R}_N(m, k) = \sum_{\substack{m=p_1+p_2+\cdots+p_k \\ p_i \leq N}} \log p_1 \cdot \log p_2 \cdots \log p_k ,$$

where $p_1, p_2, \ldots, p_k$ are prime numbers.
Then, it follows that

$$\overline{R}_N(m, k) = \int_0^1 f^k(x) e(-mx) dx$$

and in our case, for $k = 3$, one has

$$\overline{R}_N(m, 3) = \sum_{\substack{m=p_1+p_2+p_3 \\ p_i \leq N}} \log p_1 \cdot \log p_2 \cdot \log p_3 .$$

We shall now construct the Major and Minor arcs. As we briefly mentioned in the section concerning the Circle Method, we have to split the unit circle C into two disjoint pieces (equivalently we can split the interval [0, 1] into two disjoint pieces).
Since in this problem we are going to make use of the Siegel-Walfisz Theorem 1.24, it is evident that we must first consider a positive constant D and set

$$L = \log^D N .$$

More specifically, we consider D, such that $D > 10$.
We define the Major arcs as follows:

$$\mathfrak{M} = \bigcup_{\substack{1 \le q \le L \\ (a,q)=1}} \mathfrak{M}_{(a,q)} ,$$

where

$$\mathfrak{M}_{(a,q)} = \left\{ x \in \left[\frac{L}{N}, 1 + \frac{L}{N} \right] : \left| x - \frac{a}{q} \right| \le \frac{L}{N} \right\}$$

and $a \in \{1, 2, \ldots, q\}$.
At this point, we shall prove a useful lemma.

Lemma 3.2 *Let a, q be positive integers such that $1 \le a \le q$, $1 \le q \le L$ and $(a, q) = 1$. Then, for all sufficiently large N, the Major arcs $\mathfrak{M}$ can be expressed as a disjoint union of $\mathfrak{M}_{(a,q)}$.*

Proof Let us suppose that there exists $x \in \mathfrak{M}_{(a_1,q_1)} \cap \mathfrak{M}_{(a_2,q_2)}$, with

$$\left| \frac{a_1}{q_1} - \frac{a_2}{q_2} \right| > 0 .$$

Then, it is evident that

$$|a_1 q_2 - a_2 q_1| > 0$$

or

$$|a_1 q_2 - a_2 q_1| \ge 1 .$$

However,

$$\begin{aligned}\frac{2L}{N} &\geq \left|x - \frac{a_2}{q_2}\right| + \left|\frac{a_1}{q_1} - x\right| \geq \left|\frac{a_1}{q_1} - \frac{a_2}{q_2}\right| \\ &= \left|\frac{a_1q_2 - a_2q_1}{q_1q_2}\right| \geq \frac{1}{q_1q_2} \\ &\geq \frac{1}{L^2} \, .\end{aligned}$$

Therefore, we have

$$2L^3 \geq N \, .$$

But, by the definition of L we obtain

$$2\log^{3D} N \geq N \, ,$$

which is not true for large values of N. Hence, we have arrived to a contradiction. This completes the proof of the lemma. □

We define now the Minor arcs $\mathfrak{m}$ as follows:

$$\mathfrak{m} = \left[\frac{L}{N}, 1 + \frac{L}{N}\right] \setminus \mathfrak{M} \, .$$

3.1 The Contribution of the Major Arcs

In this section, we shall investigate the contribution of the Major arcs by proving two basic theorems. The first one provides an approximation of $f(x)$ for $x \in \mathfrak{M}_{(a,q)}$ and the second one provides an approximation of the integral

$$\int_{\mathfrak{M}} f^3(x)e(-xN)dx \, .$$

Theorem 3.3 *Let $x \in \mathfrak{M}_{(a,q)}$. Then there exists a positive constant C, such that*

$$f(x) - \frac{\mu(q)}{\phi(q)} \sum_{n=1}^{N} e\left(\left(x - \frac{a}{q}\right)n\right) \ll N \exp\left(-C\sqrt{\log N}\right) \, .$$

Proof Let r be a real number, such that $r \in [1, N]$. Then, it holds

$$f_r\left(\frac{a}{q}\right) = \sum_{p \leq r} \log p \cdot e\left(\frac{a}{q}\, p\right) \, .$$

But, it is clear that

$$p \equiv t \pmod{q} ,$$

for some integer t with $1 \le t \le q$.
Therefore, we can write

$$\begin{aligned}
f_r\left(\frac{a}{q}\right) &= \sum_{t=1}^{q} \sum_{\substack{p\equiv t (\bmod q)\\ p\le r}} \log p \cdot e\left(\frac{a}{q}\,p\right)\\
&= \sum_{t=1}^{q} \sum_{\substack{p\equiv t (\bmod q)\\ p\le r}} \log p \cdot e\left(\frac{a}{q}\,t\right)\\
&= \sum_{t=1}^{q} \left(e\left(\frac{a}{q}\,t\right) \sum_{\substack{p\equiv t (\bmod q)\\ p\le r}} \log p \right)\\
&= \sum_{\substack{t=1\\ (t,q)=1}}^{q} e\left(\frac{a}{q}\,t\right) \sum_{\substack{p\equiv t (\bmod q)\\ p\le r}} \log p + \sum_{\substack{t=1\\ (t,q)>1}}^{q} e\left(\frac{a}{q}\,t\right) \sum_{\substack{p\equiv t (\bmod q)\\ p\le r}} \log p \,.
\end{aligned}$$

Hence,

$$\left| f_r\left(\frac{a}{q}\right) - \frac{r}{\phi(q)} \sum_{\substack{t=1\\ (t,q)=1}}^{q} e\left(\frac{at}{q}\right) \right|$$

$$= \left| \sum_{\substack{t=1\\ (t,q)=1}}^{q} e\left(\frac{at}{q}\right) \left(\sum_{\substack{p\le r\\ p\equiv t (\bmod q)}} \log p - \frac{r}{\phi(q)} \right) + \sum_{\substack{t=1\\ (t,q)>1}}^{q} e\left(\frac{at}{q}\right) \sum_{\substack{p\le r\\ p\equiv t (\bmod q)}} \log p \right|$$

$$\le \sum_{\substack{t=1\\ (t,q)=1}}^{q} \left| e\left(\frac{at}{q}\right) \right| \left| \sum_{\substack{p\le r\\ p\equiv t (\bmod q)}} \log p - \frac{r}{\phi(q)} \right| + \sum_{\substack{t=1\\ (t,q)>1}}^{q} \left| e\left(\frac{at}{q}\right) \right| \left(\sum_{\substack{p\le r\\ p\equiv t (\bmod q)}} \log p \right) .$$

Therefore, we get

$$\left| f_r\left(\frac{a}{q}\right) - \frac{r}{\phi(q)} \sum_{\substack{t=1\\(t,q)=1}}^{q} e\left(\frac{at}{q}\right) \right|$$

$$\leq \sum_{\substack{t=1\\(t,q)=1}}^{q} \left| \sum_{\substack{p\leq r\\p\equiv t(\bmod q)}} \log p - \frac{r}{\phi(q)} \right| + \sum_{\substack{t=1\\(t,q)>1}}^{q} \left(\sum_{\substack{p\leq r\\p\equiv t(\bmod q)}} \log p \right). \quad (1)$$

By Theorem 1.24, we have

$$\sum_{\substack{p\leq r\\p\equiv t(\bmod q)}} \log p - \frac{r}{\phi(q)} = O\left(r\exp\left(-C_D\sqrt{\log r}\right)\right).$$

Thus,

$$\left| \sum_{\substack{p\leq r\\p\equiv t(\bmod q)}} \log p - \frac{r}{\phi(q)} \right| \ll r\exp\left(-C_D\sqrt{\log r}\right). \quad (2)$$

Since, $1 \leq r \leq N$, we get

$$\begin{aligned} r\exp\left(-C_D\sqrt{\log r}\right) &= re^{-C_D\sqrt{\log r}} = e^{\log r - C_D\sqrt{\log r}} \\ &= e^{\sqrt{\log r}\left(\sqrt{\log r}-C_D\right)} \leq e^{\sqrt{\log N}\left(\sqrt{\log N}-C_D\right)} \\ &= e^{\log N - C_D\sqrt{\log N}} = N\exp\left(-C_D\sqrt{\log N}\right). \end{aligned}$$

Therefore, by (2), we obtain

$$\left| \sum_{\substack{p\leq r\\p\equiv t(\bmod q)}} \log p - \frac{r}{\phi(q)} \right| \ll N\exp\left(-C_D\sqrt{\log N}\right). \quad (3)$$

However, by the definition of the Euler ϕ function, we know that

$$\phi(q) = \sum_{\substack{t=1 \\ (t,q)=1}}^{q} 1 ,$$

and thus, by (3), it is clear that

$$\sum_{\substack{t=1 \\ (t,q)=1}}^{q} \left| \sum_{\substack{p \le r \\ p \equiv t (\bmod q)}} \log p - \frac{r}{\phi(q)} \right| \ll \sum_{\substack{t=1 \\ (t,q)=1}}^{q} N \exp\left(-C_D\sqrt{\log N}\right)$$
$$= N \exp\left(-C_D\sqrt{\log N}\right) \cdot \sum_{\substack{t=1 \\ (t,q)=1}}^{q} 1 ,$$

or

$$\sum_{\substack{t=1 \\ (t,q)=1}}^{q} \left| \sum_{\substack{p \le r \\ p \equiv t (\bmod q)}} \log p - \frac{r}{\phi(q)} \right| \ll N \exp\left(-C_D\sqrt{\log N}\right) \cdot \phi(q) . \quad (4)$$

We also have

$$\sum_{\substack{t=1 \\ (t,q)>1}}^{q} \left(\sum_{\substack{p \le r \\ p \equiv t (\bmod q)}} \log p \right) \ll \sum_{p|q} \log p . \quad (5)$$

Hence, by (1), (4) and (5), it follows that

$$\left| f_r\left(\frac{a}{q}\right) - \frac{r}{\phi(q)} \sum_{\substack{t=1 \\ (t,q)=1}}^{q} e\left(\frac{at}{q}\right) \right| \ll \phi(q) N \exp\left(-C_D\sqrt{\log N}\right) + \sum_{p|q} \log p .$$

Since $q \le L = \log^D N$, it is evident that

$$\phi(q) = \sum_{\substack{t=1 \\ (t,q)=1}}^{q} 1 \le L .$$

Therefore,

$$\left| f_r\left(\frac{a}{q}\right) - \frac{r}{\phi(q)} \sum_{\substack{t=1 \\ (t,q)=1}}^{q} e\left(\frac{at}{q}\right) \right| \ll LN \exp\left(-C_D\sqrt{\log N}\right) + \sum_{p|q} \log p$$
$$\leq LN \exp\left(-C_D\sqrt{\log N}\right) + \log N\,.$$

By the above relation, it is clear that

$$\left| f_r\left(\frac{a}{q}\right) - \frac{r}{\phi(q)} \sum_{\substack{t=1 \\ (t,q)=1}}^{q} e\left(\frac{at}{q}\right) \right| \ll N \exp\left(-C\sqrt{\log N}\right)\,, \tag{6}$$

for any positive constant $C < C_D$.
However, by the definition of the Ramanujan sum, we have

$$c_q(a) = \sum_{\substack{t=1 \\ (t,q)=1}}^{q} e\left(\frac{at}{q}\right)$$

and thus (6) takes the form

$$\left| f_r\left(\frac{a}{q}\right) - \frac{rc_q(a)}{\phi(q)} \right| \ll N \exp\left(-C\sqrt{\log N}\right)\,, \tag{7}$$

But, by the hypothesis of the theorem and Lemma 1.16, it follows that in this case

$$c_q(a) = \mu(q)\,.$$

Therefore, (7) is equivalent to

$$\left| f_r\left(\frac{a}{q}\right) - r\frac{\mu(q)}{\phi(q)} \right| \ll N \exp\left(-C\sqrt{\log N}\right)\,, \tag{8}$$

Now, let

$$E_d := (\pi(d) - \pi(d-1))\, e\left(\frac{ad}{q}\right) \log d - \frac{\mu(q)}{\phi(q)}\,,$$

where $\pi(x)$ denotes the prime counting function[4].

[4] It is evident that if d is a prime number, then $\pi(d) - \pi(d-1) = 1$, and thus,

We have

$$\left| f(x) - \frac{\mu(q)}{\phi(q)} \sum_{d=1}^{N} e(wd) \right| = \left| \sum_{d=1}^{N} E_d e(wd) \right| ,$$

where $w = x - a/q$.
But, it is clear that

$$e(wd) = e(wN) - \int_d^N \frac{d}{dy}\, e(wy) dy .$$

Hence, we obtain

$$\begin{aligned}
\left| f(x) - \frac{\mu(q)}{\phi(q)} \sum_{d=1}^{N} e(wd) \right| &= \left| e(wN) \sum_{d=1}^{N} E_d - \sum_{d=1}^{N} \left(E_d \int_d^N \frac{d}{dy}\, e(wy) dy \right) \right| \\
&\le \left| e(wN) \sum_{d=1}^{N} E_d \right| + \left| \sum_{d=1}^{N} \left(E_d \int_d^N \frac{d}{dy}\, e(wy) dy \right) \right| \\
&= \left| \sum_{d=1}^{N} E_d \right| + \left| \int_0^N \left(\frac{d}{dy}\, e(wy) \sum_{d=1}^{y} E_d \right) dy \right| \\
&\le \left| \sum_{d=1}^{N} E_d \right| + \int_0^N |2\pi i w e(wy)| \left| \sum_{d=1}^{y} E_d \right| dy .
\end{aligned}$$

However,

$$\sum_{d=1}^{r} E_d = f_r\left(\frac{a}{q}\right) - r\frac{\mu(q)}{\phi(q)} .$$

Thus, by (8), we get that each one of

$$\left| \sum_{d=1}^{N} E_d \right| , \ \left| \sum_{d=1}^{y} E_d \right| \ll N \exp\left(-C\sqrt{\log N}\right)$$

and therefore, since

$$|w| \le \frac{L}{N} ,$$

(Footnote 4 continued)

$$E_d = e(ad/q) \log d - \mu(q)/\phi(q) .$$

On the other hand, if d is a composite number $\pi(d) = \pi(d-1)$, which yields $E_d = -\mu(q)/\phi(q)$.

we obtain

$$\begin{aligned}\left|f(x)-\frac{\mu(q)}{\phi(q)}\sum_{d=1}^{N}e(wd)\right| &\ll N\exp\left(-C\sqrt{\log N}\right)+(2\pi\,|w|\,N)\,N\exp\left(-C\sqrt{\log N}\right)\\ &\le N\exp\left(-C\sqrt{\log N}\right)+\left(2\pi\frac{L}{N}\right)N\exp\left(-C\sqrt{\log N}\right)\\ &= (1+2\pi L)\,N\exp\left(-C\sqrt{\log N}\right)\\ &\ll N\exp\left(-C'\sqrt{\log N}\right)\ ,\end{aligned}$$

for any constant $C' < C$.
This completes the proof of Theorem 3.3. □

Theorem 3.4 *Let*

$$G(N) := \sum_{q=1}^{+\infty}\frac{\mu(q)c_q(N)}{\phi(q)^3}\ ,$$

where $c_q(N)$ stands for the Ramanujan sum.
Then,

$$\int_{\mathfrak{M}} f^3(x)e(-xN)dx-\frac{N^2}{2}G(N)\ll N^2\log^{-D/2}N\ .$$

Proof Let $w = x - a/q$. Then,

$$\left|f^3(x)-\frac{\mu(q)^3}{\phi(q)^3}\left(\sum_{d=1}^{N}e(wd)\right)^3\right|$$

$$=\left|f(x)-\frac{\mu(q)}{\phi(q)}\sum_{d=1}^{N}e(wd)\right|\cdot\left|f^2(x)-f(x)\frac{\mu(q)}{\phi(q)}\sum_{d=1}^{N}e(wd)+\frac{\mu(q)^2}{\phi(q)^2}\left(\sum_{d=1}^{N}e(wd)\right)^2\right|$$

$$\le\left|f(x)-\frac{\mu(q)}{\phi(q)}\sum_{d=1}^{N}e(wd)\right|\left(\left|f^2(x)\right|+|f(x)|\left|\frac{\mu(q)}{\phi(q)}\sum_{d=1}^{N}e(wd)\right|+\left|\frac{\mu(q)^2}{\phi(q)^2}\left(\sum_{d=1}^{N}e(wd)\right)^2\right|\right).$$

However, it is evident that

$$|f(x)|\le\pi(N)\log N$$

and by Chebyshev's inequality (Theorem 1.21), it follows that

$$|f(x)| \ll N .$$

In addition, it is clear that

$$\left| \frac{\mu(q)}{\phi(q)} \sum_{d=1}^{N} e(wd) \right| \ll N .$$

Moreover, since all the possible values of $\mu(q)$ are -1, 0 and 1, it is obvious that $\mu(q)^3 = \mu(q)$. Therefore, by all the above, we obtain

$$\left| f^3(x) - \frac{\mu(q)}{\phi(q)^3} \left(\sum_{d=1}^{N} e(wd) \right)^3 \right| \ll 3N^2 \left| f(x) - \frac{\mu(q)}{\phi(q)} \sum_{d=1}^{N} e(wd) \right| . \tag{1}$$

But, by the previous theorem, we know that for $x \in \mathfrak{M}_{(a,q)}$, it holds

$$f(x) - \frac{\mu(q)}{\phi(q)} \sum_{d=1}^{N} e(wd) \ll N \exp\left(-C\sqrt{\log N}\right) .$$

Thus, by (1) we get

$$\left| f^3(x) - \frac{\mu(q)}{\phi(q)^3} \left(\sum_{d=1}^{N} e(wd) \right)^3 \right| \ll N^3 \exp\left(-C\sqrt{\log N}\right) .$$

Since,

$$\mathfrak{M} = \bigcup_{\substack{1 \le q \le L \\ (a,q)=1}} \mathfrak{M}_{(a,q)} ,$$

in order to obtain the integral over the Major arcs $\mathfrak{M}$, we must integrate over $\mathfrak{M}_{(a,q)}$ and sum over all q, $1 \le q \le L$ and all a, $1 \le a \le q$ with $(a, q) = 1$. However,

$$\sum_{q=1}^{L} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \int_{\mathfrak{M}_{(a,q)}} \left(f^3(x) - \frac{\mu(q)}{\phi(q)^3} \left(\sum_{d=1}^{N} e(wd) \right)^3 \right) e(-xN)\, dx$$

$$
\begin{aligned}
&\le \sum_{q=1}^{L} \sum_{\substack{a=1\\(a,q)=1}}^{q} \int_{\mathfrak{M}_{(a,q)}} \left| f^3(x) - \frac{\mu(q)}{\phi(q)^3} \left(\sum_{d=1}^{N} e(wd) \right)^3 \right| dx \\
&\ll \sum_{q=1}^{L} \sum_{\substack{a=1\\(a,q)=1}}^{q} \int_{\mathfrak{M}_{(a,q)}} N^3 \exp\left(-C\sqrt{\log N}\right) dx \\
&\le L^2 \cdot 2\,\frac{L}{N} \cdot N^3 \exp\left(-C\sqrt{\log N}\right), \text{ since } |\mathfrak{M}_{(a,q)}| = 2\,\frac{L}{N} \\
&= 2\,\frac{L^{3+1/2}}{L^{1/2}} \cdot N^2 \exp\left(-C\sqrt{\log N}\right) .
\end{aligned}
$$

But, by the definition of L, we have

$$
L^g = \log^{gD} N = \exp\left(\log\left(gD \log N\right)\right) \ll \exp\left(C'\sqrt{\log N}\right) ,
$$

for any positive constant C' and $g \ge 1$.
Thus, for $g = 3 + 1/2$ and $C' = C$, we obtain

$$
\begin{aligned}
S &:= \sum_{q=1}^{L} \sum_{\substack{a=1\\(a,q)=1}}^{q} \int_{\mathfrak{M}_{(a,q)}} \left(f^3(x) - \frac{\mu(q)}{\phi(q)^3} \left(\sum_{d=1}^{N} e(wd) \right)^3 \right) e(-xN) dx \\
&\ll \frac{N^2 \exp\left(C\sqrt{\log N}\right)}{L^{1/2}} \cdot \exp\left(-C\sqrt{\log N}\right) \\
&= \frac{N^2}{L^{1/2}} .
\end{aligned}
$$

Moreover, we have

$$
\begin{aligned}
S &= \int_{\mathfrak{M}} f^3(x) e(-xN) dx - \sum_{q=1}^{L} \sum_{\substack{a=1\\(a,q)=1}}^{q} \int_{\mathfrak{M}_{(a,q)}} \frac{\mu(q)}{\phi(q)^3} \left(\sum_{d=1}^{N} e(wd) \right)^3 e(-xN) dx \\
&= \int_{\mathfrak{M}} f^3(x) e(-xN) dx \\
&\quad - \sum_{q=1}^{L} \sum_{\substack{a=1\\(a,q)=1}}^{q} \int_{\mathfrak{M}_{(a,q)}} \frac{\mu(q)}{\phi(q)^3} \left(\sum_{d=1}^{N} e(wd) \right)^3 e\left(-\left(x - \frac{a}{q}\right) N\right) \cdot e\left(-\frac{a}{q} N\right) dx \\
&= \int_{\mathfrak{M}} f^3(x) e(-xN) dx - \sum_{q=1}^{L} \sum_{\substack{a=1\\(a,q)=1}}^{q} e\left(-\frac{aN}{q}\right) \int_{\mathfrak{M}_{(a,q)}} \frac{\mu(q)}{\phi(q)^3} \left(\sum_{d=1}^{N} e(wd) \right)^3 e(-wN) dw
\end{aligned}
$$

(note that $w = x - a/q$) and since we proved that

$$S \ll \frac{N^2}{L^{1/2}} ,$$

it is evident that

$$\int_{\mathfrak{M}} f^3(x)e(-xN)dx - \sum_{q=1}^{L} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} e\left(-\frac{aN}{q}\right) \frac{\mu(q)}{\phi(q)^3} \int_{-\frac{L}{N}}^{\frac{L}{N}} \left(\sum_{d=1}^{N} e(wd)\right)^3 e(-wN)dw \ll \frac{N^2}{L^{1/2}} . \tag{2}$$

Therefore, by (2) we see that we must also determine a bound for the integral

$$I = \int_{-\frac{L}{N}}^{\frac{L}{N}} \left(\sum_{d=1}^{N} e(wd)\right)^3 e(-wN)dw .$$

However, we observe that

$$\begin{aligned} I' &:= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\sum_{d=1}^{N} e(wd)\right)^3 e(-wN)dw \\ &= \sum_{\substack{d_1+d_2+d_3=N \\ d_i \geq 1}} 1 \\ &= \frac{(N-1)(N-2)}{2} \end{aligned} \tag{3}$$

and therefore

$$\left| I' - \frac{N^2}{2} \right| \leq 2N .$$

Since we know the exact value of I', we shall try to correlate the integral I with the integral I'. Let

$$h(w) = \left(\sum_{d=1}^{N} e(wd)\right)^3 e(-wN) .$$

Therefore, we have

$$\begin{aligned}
|I' - I| &= \left| \int_{-\frac{1}{2}}^{\frac{1}{2}} h(w)dw - \left(\int_{-\frac{L}{N}}^{-\frac{1}{2}} h(w)dw + \int_{-\frac{1}{2}}^{\frac{1}{2}} h(w)dw + \int_{\frac{1}{2}}^{\frac{L}{N}} h(w)dw \right) \right| \\
&= \left| \int_{-\frac{L}{N}}^{-\frac{1}{2}} h(w)dw + \int_{\frac{1}{2}}^{\frac{L}{N}} h(w)dw \right| .
\end{aligned}$$

If we substitute w with $-w$, we get

$$\begin{aligned}
|I' - I| &= \left| - \int_{\frac{L}{N}}^{\frac{1}{2}} h(-w)dw + \int_{\frac{1}{2}}^{\frac{L}{N}} h(w)dw \right| \\
&= \left| - \int_{\frac{L}{N}}^{\frac{1}{2}} h(-w)dw - \int_{\frac{L}{N}}^{\frac{1}{2}} h(w)dw \right| \\
&\le \int_{\frac{L}{N}}^{\frac{1}{2}} \left| \sum_{d=1}^{N} e(-wd) \right|^3 dw + \int_{\frac{L}{N}}^{\frac{1}{2}} \left| \sum_{d=1}^{N} e(wd) \right|^3 dw \\
&\le 2 \int_{\frac{L}{N}}^{\frac{1}{2}} \left| \sum_{d=1}^{N} e(-wd) \right|^3 dw .
\end{aligned}$$

But, by Lemma 1.17, it follows that

$$\left| \sum_{d=1}^{N} e(-wd) \right| \le \min \left\{ \frac{1}{[-w]}, N \right\} = \frac{1}{[w]} .$$

Thus,

$$|I' - I| \le 2 \int_{\frac{L}{N}}^{\frac{1}{2}} \frac{1}{[w]^3} \, dw = 2 \int_{\frac{L}{N}}^{\frac{1}{2}} \frac{1}{w^3} \, dw = \frac{N^2}{L^2} - 4 < \left(\frac{N}{L} \right)^2 .$$

It follows that

$$\left| \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \frac{\mu(q)}{\phi(q)^3} e\left(-\frac{aN}{q} \right) \right| = \left| \frac{\mu(q)}{\phi(q)^3} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} e\left(-\frac{aN}{q} \right) \right|$$

$$\leq \frac{|\mu(q)|}{\phi(q)^3} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} 1$$

$$\leq \frac{1}{\phi(q)^3}\phi(q) = \frac{1}{\phi(q)^2} .$$

But, it can be shown that if β is a positive real number, then

$$\lim_{n\to\infty} \frac{n^{1-\beta}}{\phi(n)} = 0 .$$

Thus, it is evident that for $\beta = 1/4$ there exists sufficiently large N, such that

$$\frac{N^{1-1/4}}{\phi(N)} < 1$$

or

$$N^{3/4} < \phi(N) .$$

Therefore, it is clear that

$$\frac{1}{\phi(q)^2} \ll \frac{1}{q^{3/2}} .$$

Hence, we have with

$$G(N) := \sum_{q=1}^{+\infty} \frac{\mu(q)c_q(N)}{\phi(q)^3} ,$$

that

$$\left| G(N) - \sum_{q=1}^{L} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \frac{\mu(q)}{\phi(q)^3} e\left(-\frac{aN}{q}\right) \right| \ll \sum_{q=L+1}^{+\infty} \frac{1}{q^{3/2}}$$

$$\leq \int_{L+1}^{+\infty} \frac{1}{x^{3/2}} dx$$

$$= \frac{2}{(L+1)^{1/2}} .$$

Thus,

$$\left| G(N) - \sum_{q=1}^{L} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \frac{\mu(q)}{\phi(q)^3} e\left(-\frac{aN}{q}\right) \right| \ll \frac{1}{L^{1/2}} . \tag{4}$$

By (2) and (3) we obtain

$$\begin{aligned} \int_{\mathfrak{M}} f^3(x)e(-xN)dx - \left(\sum_{q=1}^{L} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} e\left(-\frac{aN}{q}\right) \frac{\mu(q)}{\phi(q)^3} \right) \cdot \frac{N^2}{2} &\ll \frac{N^2}{L^{1/2}} + \frac{N^2}{L^2} + N \\ &= (1 + L^{-3/2}) \frac{N^2}{L^{1/2}} \\ &\ll \frac{N^2}{L^{1/2}} . \end{aligned}$$

By the above relation and (3), we get

$$\int_{\mathfrak{M}} f^3(x)e(-xN)dx - G(N)\frac{N^2}{2} \ll \frac{N^2}{L^{1/2}} = N^2 \log^{-D/2} N .$$

This completes the proof of Theorem 3.4. □

3.2 *The Contribution of the Minor Arcs*

In this section, we shall investigate the contribution of the Minor arcs $\mathfrak{m}$. Our ultimate goal is to prove that

$$\int_{\mathfrak{m}} f^3(x)e(-xN)dx \ll \frac{N^2}{\log^c N} ,$$

for any positive constant c with $c \leq D/2 - 5$ and D as in Theorem 1.24 where

$$f(x) = \sum_{p \leq N} \log p \cdot e(xp)$$

We need to prove some more theorems and lemmas.
The following lemma is presented without a proof, since it is a classical result in approximation theory and analytic number theory.

Lemma 3.5 *For any real numbers x, r_1, r_2, with $r_1, r_2 \geq 1$, and integers q, a, such that*

$$\left| x - \frac{a}{q} \right| \leq \frac{1}{q^2}\ , \ (a, q) = 1$$

and $q \geq 1$, it holds

$$\sum_{n<r_1} \min\left\{ \frac{1}{[xn]}, \frac{r_1 r_2}{n} \right\} \ll \left(\frac{r_1 r_2}{q} + r_1 + q \right) \log(2 r_1 q)\ . \tag{1}$$

The following result is a special case of an identity due to R. C. Vaughan [64].

Lemma 3.6 *Let r be a real number, such that $1 \leq r \leq \sqrt{N}$.* [5]
Then,

$$\sum_{r<k\leq N} \Lambda(k) e(xk) = \sum_{d\leq r} \sum_{m\leq \frac{N}{d}} \log m \cdot \mu(d) e(xdm)$$

$$- \sum_{r<d\leq N} \sum_{r<m\leq \frac{N}{d}} \sum_{\substack{q|d \\ q\leq r}} \mu(q) \Lambda(m) e(xdm)$$

$$- \sum_{d\leq r^2} \sum_{m\leq \frac{N}{d}} \sum_{\substack{q|d \\ q\leq r \\ d\leq rq}} \mu(q) \Lambda\left(\frac{d}{q}\right) e(xdm)\ , \tag{L1}$$

where $\mu(n)$ and $\Lambda(n)$ denote the Möbius and the von Mangoldt function, respectively.

Proof Throughout the proof, we assume that $Re\{s\} > 1$. All the Dirichlet series considered in the proof will be absolutely convergent, and their terms can be rearranged arbitrarily. By the definition of the Riemann zeta function $\zeta(s)$, it follows that

$$\zeta'(s) = -\sum_{n\geq 1} \frac{\log n}{n^s} \quad \left(\zeta' \text{ denoting } \frac{d\zeta}{ds} \right).$$

We have

$$-\frac{\zeta'}{\zeta}(s) = \sum_{m\geq 1} \frac{\Lambda(m)}{m^s}.$$

[5] Throughout this subsection, r will always stand for a real number, such that $1 \leq r \leq \sqrt{N}$, unless otherwise stated.

We define

$$D_1(s) := \sum_{n\le r} \frac{\Lambda(n)}{n^s}\ ,\ \ D_2(s) := \sum_{n\le r} \frac{\mu(n)}{n^s}\ .$$

It follows that

$$0 = -\zeta'(s)D_2(s) - \zeta(s)D_1(s)D_2(s) - \zeta(s)D_2(s)\left(-\frac{\zeta'}{\zeta}(s) - D_1(s)\right).$$

$$= \sum_{n\ge 1} \frac{1}{n^s}\left(\sum_{\substack{m\le r\\ d\cdot m=n}} (\log d)\,\mu(m)\right) - \sum_{n\ge 1}\frac{1}{n^s} \sum_{\substack{d\cdot m=n\\ m\le r^2}} \sum_{\substack{q|d\\ q\le r}} \mu(q)\Lambda(m)$$

$$-\sum_{n\ge 1}\frac{1}{n^s}\sum_{\substack{m\cdot d=n\\ r<m}}\left(\sum_{\substack{t|d\\ t\le r}}\mu(t)\right)\Lambda(m)\ .$$

Therefore, equating the coefficients of the Dirichlet series above we obtain:

$$\sum_{\substack{d\cdot m=n\\ m\le r}} \mu(m)\log d - \sum_{\substack{d\cdot m=n\\ m\le r^2}}\left(\sum_{\substack{q|d\\ q\le r}}\mu(q)\Lambda(m)\right) - \sum_{r<m}\left(\sum_{\substack{d|m\\ d\le r}}\mu(d)\right)\Lambda(m) = 0 \qquad (*)$$

From (*) by multiplying by $e(xn)$ and adding

$$\sum_{r<n\le N}\Lambda(n)e(xn)$$

we obtain

$$\sum_{r<n\le N}\Lambda(n)e(xn) = \sum_{1\le n\le N} e(xn)\sum_{\substack{m\le r\\ d\cdot m=n}}(\log d)\mu(m)$$

$$-\sum_{1\le n\le N} e(xn)\sum_{\substack{m\le r^2\\ d\cdot m=n}}\left(\sum_{\substack{q|d\\ q\le r}}\mu(q)\right)\Lambda(m)$$

$$-\left(\sum_{1\le n\le N} e(xn)\sum_{\substack{d\cdot m=n\\ r<m}}\left(\sum_{\substack{q|m\\ q\le r}}\mu(q)\right)\Lambda(m) - \sum_{r\le n\le N}\Lambda(n)e(xn)\right)\ .$$

For $d \leq r$, we have

$$\sum_{\substack{q|d\\q\leq r}} \mu(q) = \begin{cases} 1, & \text{if } d = 1 \\ 0, & \text{otherwise} \end{cases}$$

and therefore the proof of Lemma 3.6 is finished. □

Lemma 3.7 *Let*

$$A := \sum_{d\leq r} \sum_{m\leq \frac{N}{d}} \log m \cdot \mu(d) \cdot e(xdm)$$

Then

$$|A| \ll \log N \sum_{d\leq r^2} \min\left\{\frac{1}{[xd]}, \frac{N}{d}\right\} ,$$

where $\left[y\right] = \min_{k\in\mathbb{Z}} |y-k|$.

Proof We have

$$\sum_{1\leq m\leq N/d} (\log m) e(xdm) = \sum_{1\leq m\leq N/d} e(xdm) \int_1^m \frac{du}{u} = \int_1^{N/d} \left(\sum_{u<m\leq N/d} e(xdm)\right) \frac{du}{u} .$$

Therefore,

$$\begin{aligned} |A| &\leq \sum_{d\leq r} \left| \sum_{m\leq N/d} (\log m) e(xdm) \right| \\ &\leq \sum_{d\leq r} \min\left\{ \sum_{m\leq N/d} \log m \, , \left| \int_1^{N/d} \sum_{u<m\leq N/d} e(xdm) \frac{du}{u} \right| \right\} \\ &\ll \sum_{d\leq r} \min\left\{ \frac{N}{d} \log N \, , \frac{\log N}{[xd]} \right\} \end{aligned}$$

Thus the proof of the lemma now follows. □

Lemma 3.8 *Let*

$$B := \sum_{d \le r^2} \sum_{m \le \frac{N}{d}} \sum_{\substack{q \mid d \\ q \le r \\ d \le rq}} \mu(q) \Lambda\left(\frac{d}{q}\right) e(xdm) \,.$$

Then

$$|B| \ll \log N \sum_{d \le r^2} \min\left\{\frac{1}{[xd]}, \frac{N}{d}\right\} \,.$$

Proof We have

$$\begin{aligned} |B| &\le \sum_{d \le r^2} \left| \sum_{\substack{q \mid d \\ q \le r \\ d \le rq}} \Lambda\left(\frac{d}{q}\right) \right| \left| \sum_{m \le \frac{N}{d}} e(xdm) \right| \\ &\le \sum_{d \le r^2} \sum_{q \mid d} \Lambda(q) \left| \sum_{m \le \frac{N}{d}} e(xdm) \right| \,. \end{aligned}$$

By Theorem 1.13, we know that

$$\sum_{q \mid d} \Lambda(q) = \log d \,.$$

In addition, by Lemma 1.17 it follows that

$$\left| \sum_{m \le \frac{N}{d}} e(xdm) \right| \le \min\left\{\frac{1}{[xd]}, \frac{N}{d}\right\} \,.$$

Hence, we obtain

$$|B| \le \sum_{d \le r^2} \left((\log d) \cdot \min\left\{\frac{1}{[xd]}, \frac{N}{d}\right\} \right),$$

and therefore

$$|B| \ll \log N \sum_{d \le r^2} \min\left\{\frac{1}{[xd]}, \frac{N}{d}\right\} ,$$

which proves the lemma. □

Lemma 3.9 *Let*

$$C := \sum_{r<d\le N} \sum_{r<m\le \frac{N}{d}} \sum_{\substack{q|d\\ q\le r}} \mu(q)\Lambda(m)e(xdm)\,.$$

Then

$$|C| \ll \sum_{i=1}^{t} \left(\left(2^i r \log^5 N\right) \sum_{r<d\le \frac{N}{2^i r}} \left(2^i r + \sum_{1\le s\le \frac{N}{2^i r}} \min\left\{ \frac{1}{[xs]}, \frac{N}{s} \right\} \right) \right)^{1/2},$$

where

$$t = \left\lfloor \frac{\log(N/r^2)}{\log 2} \right\rfloor .$$

Proof If we observe the indices under the first two sums of the definition of C, we see that

$$r < d \le N$$

and

$$r < m \le \frac{N}{d}\,. \tag{1}$$

But, for $N/r < d \le N$ it holds $N/d < r$ and thus (1) does not hold true. In that case, the second sum of C does not contain any terms. Therefore, it is evident that

$$C = \sum_{r<d\le \frac{N}{r}} \sum_{r<m\le \frac{N}{d}} \sum_{\substack{q|d\\ q\le r}} \mu(q)\Lambda(m)e(xdm)\,.$$

Let

$$D(r) = \sum_{r<m\le \frac{N}{d}} \sum_{\substack{q|d\\ q\le r}} \mu(q)\Lambda(m)e(xdm)\,.$$

Then, we can write

$$C = \sum_{r<d\le 2r} D(r) + \sum_{2r<d\le 4r} D(r) + \cdots + \sum_{2^t r<d\le 2^{t+1} r} D(r)\,, \tag{2}$$

where t is an integer, such that

$$2^t r < \frac{N}{r} \le 2^{t+1} r \, .$$

From,

$$2^t r < \frac{N}{r} \le 2^{t+1} r \, ,$$

we have

$$2^t < \frac{N}{tr^2} \le 2^{t+1} \, .$$

Thus,

$$t < \frac{\log \left(N/r^2 \right)}{\log 2} \, .$$

However, for $0 \le i \le t$, we have by the definition of $D(r)$ that

$$\left| \sum_{2^i r < d \le 2^{i+1} r} D(r) \right|^2 = \left| \sum_{2^i r < d \le 2^{i+1} r} \left(\sum_{r < m \le \frac{N}{d}} \left(\sum_{\substack{q|d \\ q \le r}} \mu(q) \right) \Lambda(m) e(xdm) \right) \right|^2$$

and by the Cauchy-Schwarz-Buniakowsky inequality we obtain

$$\left| \sum_{2^i r < d \le 2^{i+1} r} D(r) \right|^2$$

$$\le \left(\sum_{2^i r < d \le 2^{i+1} r} \left| \sum_{\substack{q|d \\ q \le r}} \mu(q) \right|^2 \right) \cdot \left(\sum_{2^i r < d \le 2^{i+1} r} \left| \sum_{r < m \le \frac{N}{d}} \Lambda(m) e(xdm) \right|^2 \right) \qquad (3)$$

Because of the fact that $|\mu(q)| \le 1$, we have

$$\sum_{2^i r < d \le 2^{i+1} r} \left| \sum_{\substack{q|d \\ q \le r}} \mu(q) \right|^2 \le \sum_{2^i r < d \le 2^{i+1} r} \left| \sum_{\substack{q|d \\ q \le r}} 1 \right|^2 = \sum_{2^i r < d \le 2^{i+1} r} \tau^2(d) \, .$$

By Theorem 1.18 and due to the fact that $2^i r \le N$, we obtain

$$\begin{aligned}\sum_{2^i r<d\le 2^{i+1}r}\left|\sum_{\substack{q|d\\ q\le r}}\mu(q)\right|^2 &\ll (2^i r)\log^3(2^i r)\\ &\le (2^i r)\log^3 N\,.\end{aligned}$$

Hence, by the above relation and (3), we get

$$\left|\sum_{2^i r<d\le 2^{i+1}r} D(r)\right|^2 \ll (2^i r)\log^3 N \sum_{2^i r<d\le 2^{i+1}r}\left|\sum_{r<m\le \frac{N}{d}}\Lambda(m)e(xdm)\right|^2 \tag{4}$$

In addition, for the remaining sums in (4), we can write

$$\begin{aligned}&\sum_{2^i r<d\le 2^{i+1}r}\left|\sum_{r<m\le \frac{N}{d}}\Lambda(m)e(xdm)\right|^2\\ &=\sum_{2^i r<d\le 2^{i+1}r}\left(\sum_{r<m\le \frac{N}{d}}\Lambda(m)e(xdm)\cdot\sum_{r<s\le \frac{N}{d}}\Lambda(s)e(-xds)\right)\\ &\le \sum_{r<m\le \frac{N}{2^i r}}\left(\sum_{r<s\le \frac{N}{2^i r}}\Lambda(m)\Lambda(s)\left|\sum_{2^i r<d\le \min\left\{2^{i+1}r,\frac{N}{m},\frac{N}{s}\right\}} e(xd(m-s))\right|\right)\,.\end{aligned}$$

However, $\Lambda(m),\,\Lambda(s)\le \log N$, for every $x<n,\ s\le N/2^i r$. Thus,

$$\Lambda(m)\Lambda(s)\le \log^2 N\,.$$

In addition, by Lemma 1.17, we have

$$\left|\sum_{2^i r<d\le 2^{i+1}r} e(x(m-s)d)\right| \le \min\left\{\frac{1}{[x(m-s)]},\,2^i r\right\}\,.$$

Therefore, we obtain

$$\begin{aligned}\sum_{2^i r<d\le 2^{i+1}r}\left|\sum_{r<m\le\frac{N}{d}}\Lambda(m)e(xdm)\right|^2 &\le \log^2 N\sum_{r<m\le\frac{N}{2^i r}}\sum_{r<s\le\frac{N}{2^i r}}\min\left\{\frac{1}{[x(m-s)]},2^i r\right\}\\ &\ll \log^2 N\sum_{r<m\le\frac{N}{2^i r}}\sum_{0<s\le\frac{N}{2^i r}}\min\left\{\frac{1}{[xs]},2^i r\right\}\\ &= \log^2 N\sum_{r<m\le\frac{N}{2^i r}}\left(2^i r+\sum_{1<s\le\frac{N}{2^i r}}\min\left\{\frac{1}{[xs]},2^i r\right\}\right).\end{aligned}$$

Hence,

$$\sum_{2^i r<d\le 2^{i+1}r}\left|\sum_{r<m\le\frac{N}{d}}\Lambda(m)e(xdm)\right|^2 \le \log^2 N\sum_{r<m\le\frac{N}{2^i r}}\left(2^i r+\sum_{1<s\le\frac{N}{2^i r}}\min\left\{\frac{1}{[xs]},\frac{N}{s}\right\}\right).$$

By the above relation and (4), we obtain

$$\left|\sum_{2^i r<d\le 2^{i+1}r}D(r)\right| \ll \left((2^i r)\log^5 N\sum_{r<m\le\frac{N}{2^i r}}\left(2^i r+\sum_{1<s\le\frac{N}{2^i r}}\min\left\{\frac{1}{[xs]},\frac{N}{s}\right\}\right)\right)^{1/2}.$$

By the above relation and (2), it is evident that

$$|C| \ll \sum_{i=1}^{t}\left(\left(2^i r\log^5 N\right)\sum_{r<d\le\frac{N}{2^i r}}\left(2^i r+\sum_{1\le s\le\frac{N}{2^i r}}\min\left\{\frac{1}{[xs]},\frac{N}{s}\right\}\right)\right)^{1/2}.$$

This completes the proof of Lemma 3.9. □

Corollary 3.10 *We have the following estimate*

$$|C| \ll \log^4 N\cdot\left(\frac{N}{\sqrt{r}}+\frac{N}{\sqrt{q}}+\sqrt{Nq}\right).$$

Proof By the previous lemma and Lemma 3.5, it follows that

$$
\begin{aligned}
|C| &\ll \sum_{1\le i\le t}\left(N\log^5 N\sum_{r<d\le \frac{N}{2^i r}}\left(2^i r+\left(\frac{N}{q}+\frac{N}{2^i r}+q\right)\log N\right)\right)^{1/2}\\
&= \sum_{1\le i\le t}\left(N\log^6 N\left(2^i r+\frac{N}{q}+\frac{N}{2^i r}+q\right)\right)^{1/2}\\
&\ll \sum_{1\le i\le t}\sqrt{N}\log^3 N\left(2^i r+\frac{N}{q}+\frac{N}{2^i r}+q\right)^{1/2}\\
&\le \sum_{1\le i\le t}\sqrt{N}\log^3 N\left(\sqrt{2^i r}+\sqrt{N/q}+\sqrt{N/2^i r}+\sqrt{q}\right)\,.
\end{aligned}
$$

However, we have shown that

$$t=\left\lfloor\frac{\log\left(N/r^2\right)}{\log 2}\right\rfloor\,.$$

Thus,

$$t\ll\log N\,,$$

which implies that

$$
\begin{aligned}
|C| &\ll \sqrt{N}\log^4 N\left(\sqrt{2^i r}+\sqrt{N/q}+\sqrt{N/2^i r}+\sqrt{q}\right)\\
&= \log^4 N\left(\sqrt{2^i rN}+\frac{N}{\sqrt{q}}+\frac{N}{\sqrt{2^i r}}+\sqrt{Nq}\right)\,.
\end{aligned}
$$

But, since $r\le 2^i r\le N/r$, we obtain

$$|C|\ll\log^4 N\cdot\left(\frac{N}{\sqrt{r}}+\frac{N}{\sqrt{q}}+\sqrt{Nq}\right)\,,$$

which proves the corollary. □

Lemma 3.11 *Let r be a real number, such that*

$$\frac{N}{\sqrt{r}}=r^2\,,$$

then

$$\sum_{r<k\le N}\Lambda(k)e(xk)\ll\log^4 N\left(N^{4/5}+\frac{N}{\sqrt{q}}+\sqrt{Nq}\right)\,.$$

Proof For $N/\sqrt{r} = r^2$, by the previous corollary, we get

$$|C| \ll \log^4 N \left(r^2 + \frac{N}{\sqrt{q}} + \sqrt{Nq} \right) .$$

From Lemmas 3.5, 3.7 and 3.8, we also get:

$$|A| , \ |B| \ll \log^4 N \left(r^2 + \frac{N}{\sqrt{q}} + \sqrt{Nq} \right) .$$

Therefore, we have

$$|A| , \ |B| , \ |C| \ll \log^4 N \left(r^2 + \frac{N}{\sqrt{q}} + \sqrt{Nq} \right)$$

or

$$|A| , \ |B| , \ |C| \ll \log^4 N \left(N^{4/5} + \frac{N}{\sqrt{q}} + \sqrt{Nq} \right) . \tag{1}$$

But, by Lemma 3.6, we know that

$$\sum_{r<k\leq N} \Lambda(k)e(xk) = A - C - B . \tag{2}$$

Thus, by (1) and (2) we obtain that

$$\sum_{r<k\leq N} \Lambda(k)e(xk) \ll \log^4 N \left(N^{4/5} + \frac{N}{\sqrt{q}} + \sqrt{Nq} \right) .$$

This completes the proof of the lemma. □

Theorem 3.12 *(a) If $x \in \mathfrak{m}$, then*

$$f(x) \ll N \log^{5-D/2} N ,$$

where D is a positive constant, such that $D > 10$.[6]
(b) If c is a positive constant, such that $c \leq D/2 - 5$, then

$$\int_{\mathfrak{m}} f^3(x)e(-xN)dx \ll \frac{N^2}{\log^c N} .$$

[6] We mentioned the constant D in Theorem 1.24 (Siegel-Walfisz Theorem).

Proof (a) We have

$$
\begin{aligned}
\left| f(x) - \sum_{r<k\le N} \Lambda(k)e(xk) \right| &= \left| \sum_{p\le r} \log p \cdot e(xp) + \sum_{r<p\le N} \log p \cdot e(xp) - \sum_{r<k\le N} \Lambda(k)e(xk) \right| \\
&\le \left| \sum_{p\le r} \log p \cdot e(xp) \right| + \left| \sum_{r<p\le N} \log p \cdot e(xp) - \sum_{r<k\le N} \Lambda(k)e(xk) \right| \\
&= \sum_{p\le r} \log p + \left| \sum_{\substack{r<k\le N \\ k=p^m \\ m\ge 2}} \log p \cdot e(xp) \right| .
\end{aligned}
$$

Thus,

$$
\left| f(x) - \sum_{r<k\le N} \Lambda(k)e(xk) \right| \le \sum_{p\le r} \log p + \sum_{\substack{r<k\le N \\ k=p^m \\ m\ge 2}} \log p . \tag{1}
$$

But, as far as the second summand is concerned, we observe that

$$
r < p^m \le N , \quad \text{or} \;\; p \le \sqrt[m]{N} , \quad m \ge 2 .
$$

For

$$
m = \frac{\log N}{\log 2}
$$

we get

$$
p^m \ge 2^m = 2^{\log N/\log 2} = e^{\log 2 \frac{\log N}{\log 2}} = e^{\log N} = N
$$

or

$$
p^m \ge N .
$$

Therefore, from (1) and the above observation, it follows that

$$
\left| f(x) - \sum_{r<k\le N} \Lambda(k)e(xk) \right| \le \sum_{p\le r} \log p + \sum_{2\le m\le \left\lfloor \frac{\log N}{\log 2} \right\rfloor} \sum_{p^m\le N} \log p . \tag{2}
$$

However,

$$\begin{aligned}\sum_{2\le m\le\left\lfloor\frac{\log N}{\log 2}\right\rfloor}\sum_{p^m\le N}\log p &\le \sum_{p^2\le N}\log p+\log N\sum_{3\le m\le\left\lfloor\frac{\log N}{\log 2}\right\rfloor}\sum_{p^m\le N}1\\ &\le \sum_{p^2\le N}\log p+\log N\sum_{3\le m\le\left\lfloor\frac{\log N}{\log 2}\right\rfloor}\sum_{p^3\le N}1\\ &= \sum_{p^2\le N}\log p+\log N\left(\sum_{3\le m\le\left\lfloor\frac{\log N}{\log 2}\right\rfloor}1\right)\cdot\left(\sum_{p^3\le N}1\right).\end{aligned}$$

But, since $\log 2 > 1/2$, we obtain

$$\begin{aligned}\sum_{2\le m\le\left\lfloor\frac{\log N}{\log 2}\right\rfloor}\sum_{p^m\le N}\log p &\le \sum_{p^2\le N}\log p+\log N\left(\sum_{3\le m\le\left\lfloor\frac{\log N}{\log 2}\right\rfloor}1\right)\cdot\left(\sum_{p^3\le N}1\right)\\ &\le \log N\left(\sum_{p^2\le N}1+2\log N\sum_{p^3\le N}1\right).\end{aligned}$$

By the above relation and (2), we obtain

$$\left|f(x)-\sum_{r<k\le N}\Lambda(k)e(xk)\right|\le\log N\left(\sum_{p\le r}1+\sum_{p\le\sqrt{N}}1+2\log N\sum_{p\le\sqrt[3]{N}}1\right) \tag{3}$$

However, by Chebyshev's inequality, we know that for every positive integer n, where $n\ge 2$, it holds

$$\frac{1}{6}\cdot\frac{n}{\log n}<\pi(n)<6\cdot\frac{n}{\log n}.$$

Therefore, it is evident that

$$\sum_{p\le r}1\ll\frac{r}{\log r},\quad\sum_{p\le\sqrt{N}}1\ll\frac{\sqrt{N}}{\log\sqrt{N}},\quad\sum_{p\le\sqrt[3]{N}}1\ll\frac{\sqrt[3]{N}}{\log\sqrt[3]{N}}.$$

In addition, we have

$$\log N = 2 \cdot \frac{1}{2} \log N = 2 \log \sqrt{N} \ll \log \sqrt{N}$$

and similarly

$$\log N \ll \log \sqrt[3]{N} .$$

Hence, by the above arguments and (3), we get

$$\left| f(x) - \sum_{r<k\leq N} \Lambda(k) e(xk) \right| \ll r + \sqrt{N} \log N + 2\sqrt[3]{N} \log N$$
$$\ll \sqrt{N} \log N . \tag{4}$$

But, by Lemma 3.11, we know that

$$\sum_{r<k\leq N} \Lambda(k) e(xk) \ll \log^4 N \left(N^{4/5} + \frac{N}{\sqrt{q}} + \sqrt{Nq} \right) .$$

Thus, by (4), it follows that

$$f(x) \ll \log^5 N \left(N^{4/5} + \frac{N}{\sqrt{q}} + \sqrt{Nq} \right)$$

or

$$f(x) \ll N \log^5 N \left(N^{-1/5} + \frac{1}{\sqrt{q}} + \sqrt{\frac{q}{N}} \right) . \tag{5}$$

By Dirichlet's Approximation Theorem (see Theorem 1.22 for a proof), we know that for any real number x and natural number n, there exists an integer q, such that $0 < q \leq n$, and an integer a relatively prime to b, for which it holds

$$\left| x - \frac{a}{q} \right| < \frac{1}{nq} .$$

Since Dirichlet's Approximation Theorem holds for every $x \in \mathbb{R}$ and $n \in \mathbb{N}$, let us assume that $x \in \mathfrak{m}$ and n is a natural number, such that $n \geq N/L$ and $n - 1 < N/L$, where $L = \log^D N$. Then, we have

$$\left| x - \frac{a}{q} \right| \leq \frac{L}{Nq} \leq \frac{L}{N} .$$

Therefore, by the definition of the Major arcs, it follows that $x \in \mathfrak{M}$. But, it is impossible for x to belong in both the Major and the Minor arcs.

Thus, since

$$\mathfrak{M} = \bigcup_{\substack{1 \le q \le L \\ (a,q)=1}} \mathfrak{M}_{(a,q)} ,$$

it is evident that it must hold $q > L$. Hence, it is clear that

$$L < q < \frac{N}{L} ,$$

for $x \in \mathfrak{m}$.
Consequently, by (5), we obtain

$$\begin{aligned} f(x) &\ll N \log^5 N \left(N^{-1/5} + \frac{2}{\sqrt{L}} \right) \\ &= N \log^5 N \left(N^{-1/5} + 2 \log^{-D/2} N \right) \\ &\ll N \log^{5-D/2} N . \end{aligned}$$

(b) We proved above that

$$f(x) \ll N \log^{5-D/2} N .$$

Therefore, we have

$$\int_{\mathfrak{m}} f^3(x) e(-xN) dx \le \int_0^1 \left| f^2(x) \right| |f(x)| \, dx \ll N \log^{5-D/2} N \int_0^1 \left| f^2(x) \right| dx \tag{1}$$

However,

$$\begin{aligned} \int_0^1 \left| f^2(x) \right| dx &= \int_0^1 f(x) f(-x) dx \\ &= \int_0^1 \sum_{p_1 \le N} \log p_1 \cdot e(x p_1) \sum_{p_2 \le N} \log p_2 \cdot e(-x p_2) dx \\ &= \sum_{p_1 \le N} \log p_1 \sum_{p_2 \le N} \log p_2 \int_0^1 e((p_1 - p_2)x) dx . \end{aligned}$$

But, by Lemma 1.2, we know that

$$\int_0^1 e((p_1 - p_2)x) dx = \begin{cases} 1, & \text{if } p_1 = p_2 \\ 0, & \text{if } p_1 \neq p_2 . \end{cases}$$

Thus, it is evident that

$$\int_0^1 \left|f^2(x)\right| dx \le \sum_{p\le N} \log^2 p \le \sum_{p\le N} \log^2 N$$
$$= \pi(N) \log^2 N \,.$$

Hence, by (1) and Chebyshev's inequality, we obtain

$$\begin{aligned}\int_{\mathfrak{m}} f^3(x)e(-xN)dx &\ll N \log^{5-D/2} N \cdot \log^2 N \cdot \frac{N}{\log N} \\ &= N^2 \log^{6-D/2} N \\ &= \frac{N^2}{\log^c N} \,.\end{aligned}$$

This proves Theorem 3.12. □

3.3 *Putting It All Together*

In this section we use the results obtained in the previous sections in order to prove Vinogradov's theorem.

Theorem 3.13 (VINOGRADOV'S THEOREM) *There exists a natural number N_0, such that every odd positive integer N with $N \ge N_0$, can be represented as the sum of three prime numbers.*

Proof Recall that by the arguments presented in the section related to the Circle Method, in order to prove Vinogradov's theorem it suffices to prove that

$$\overline{R}_N(m,3) = \sum_{\substack{m=p_1+p_2+p_3 \\ p_i\le N}} \log p_1 \cdot \log p_2 \cdot \log p_3 \gg N^2 \,.$$

However, we have

$$\overline{R}_N(m,3) = \int_{\mathfrak{M}} f^3(x)e(-mx)dx + \int_{\mathfrak{m}} f^3(x)e(-mx)dx \,.$$

In addition, by Theorems 3.4 and 3.12, we know that

$$\int_{\mathfrak{M}} f^3(x)e(-xN)dx - \frac{N^2}{2}\, G(N) \ll N^2 \log^{-D/2} N$$

and

$$\int_{\mathfrak{m}} f^3(x)e(-xN)dx \ll N^2 \log^{-c} N \,,$$

where c is a positive constant, such that $c \leq D/2 - 5$ and

$$G(N) = \sum_{q=1}^{+\infty} \frac{\mu(q)c_q(N)}{\phi(q)^3} \,.$$

Therefore,

$$\overline{R}_N(m,3) - \frac{N^2}{2}\, G(N) \ll \frac{N^2}{\log^w N} \,, \tag{1}$$

where w is a positive constant, such that $w \leq D/2 - 5$.
Generally, for any Dirichlet series with coefficients $f(n)$, where $f(n)$ is a multiplicative arithmetic function, by Theorem 1.5, it holds

$$\sum_{n=1}^{+\infty} \frac{f(n)}{n^s} = \prod_p \left(\sum_{n=0}^{+\infty} \frac{f(p^n)}{p^{ns}} \right) ,$$

where the product extends over all prime numbers p.
Therefore, for $s = 0$, we get

$$\sum_{n=1}^{+\infty} f(n) = \prod_p \left(\sum_{n=0}^{+\infty} f(p^n) \right) .$$

In our case, since the arithmetic function

$$\frac{\mu(q)c_q(n)}{\phi(q)^3}$$

is multiplicative, we can write

$$G(N) = \sum_{q=1}^{+\infty} \frac{\mu(q)c_q(n)}{\phi(q)^3} = \prod_p \left(\sum_{n=0}^{+\infty} \frac{\mu(p^n)c_{p^n}(N)}{\phi(p^n)^3} \right) .$$

However, for $n > 1$ we have $\mu(p^n) = 0$. Thus,

$$G(N) = \prod_p \left(\frac{\mu(1)c_1(N)}{\phi(1)^3} + \frac{\mu(p)c_p(N)}{\phi(p)^3} \right) = \prod_p \left(1 + \frac{(-1)c_p(N)}{(p-1)^3} \right) \tag{2}$$

But, by Lemma 1.16, we know that

$$c_p(N) = \sum_{d|(p,N)} \mu\left(\frac{p}{d}\right) d\ .$$

Hence, if $p \mid N$, then the only possible values of d are 1 and p. Thus,

$$c_p(N) = p - 1\ .$$

Similarly, if $p \nmid N$, we get

$$c_p(N) = -1\ .$$

Therefore, by (2), it follows that

$$\begin{aligned} G(N) &= \prod_{p|N}\left(1 - \frac{p-1}{(p-1)^3}\right)\prod_{p\nmid N}\left(1 + \frac{1}{(p-1)^3}\right) \\ &= \prod_{p|N}\left(1 - \frac{1}{(p-1)^2}\right)\prod_{p\nmid N}\left(1 + \frac{1}{(p-1)^3}\right)\ . \end{aligned}$$

For odd integer N, we have

$$1 - \frac{1}{(p-1)^2} > 0$$

for all $p \mid N$.
Furthermore, the infinite series

$$\sum_p \frac{1}{(p-1)^2} \quad \text{and} \quad \sum_p \frac{1}{(p-1)^3}$$

are absolutely convergent, and thus, the infinite products

$$\prod_{p|N}\left(1 - \frac{1}{(p-1)^2}\right) \quad \text{and} \quad \prod_{p\nmid N}\left(1 + \frac{1}{(p-1)^3}\right)$$

are bounded from below and from above by bounds that are independent from N. Hence, Vinogradov's theorem is now proved. $\square$

http://www.springer.com/978-3-319-57912-2

Goldbach's Problem
Selected Topics
Rassias, M.T.
2017, XV, 122 p., Softcover
ISBN: 978-3-319-57912-2