

Preface

Discrete curvature is one of the key concepts in modern discrete geometry. Discrete geometry itself is nothing new and boasts on the contrary an ancient and rich history, reaching back to Antiquity and the Platonic solids. The geometry of points, lines, polygons, and polyhedra predates differential geometry. The latter's progress in the last centuries though has made it a crucial field in mathematics and a tool of choice in many applications. And yet, discrete geometry has not ceased to stimulate researchers, as the works of Alexandrov and many others show. It has nevertheless taken a new twist at the end of the twentieth century, first with the onset of computer science and engineering, requiring a science based on discrete objects, bits and integers, molecules or bricks, graphs, and networks. This has greatly stimulated mathematic research and keeps doing so. But another impulse came from mathematics itself, from fields as varied as probabilities, integrable systems theory, or differential calculus; the aim is to construct a modern theory of discrete geometry akin to its smooth cousin while solving applied problems. This double motivation, a need for efficient tools and problems to solve, as well as a renewed approach on geometry in the discrete realm, has been a powerful incentive for research and has generated many advances in discrete geometry. And at the heart of this geometry lies the notion of discrete curvature, as it does in smooth geometry. Indeed, as we know now, curvature is often the key to understanding the metric and topological properties of geometric objects: genus (Gauss-Bonnet), diameter (Bonnet-Myers), functions and spectrum (Poincaré, Sobolev, Lichnerowicz), classification, flows and smoothing (Hamilton and Perelman), local description, etc. However, the challenge is to understand and use curvature in this non-smooth setting. Unlike the differential theory, where definitions have been cast more than a century ago by Riemann, discrete differential geometry still searches for the appropriate analog of the many curvatures (see [17]). It is not clear yet which notion is the most adapted to a given problem nor whether a single notion will (or should) arise. The reader will discover in this book many possible takes on curvature, with complex dependencies. This panoramic approach shows the wealth of contexts and ideas where curvature plays a role. And the material gathered here is but a sample—albeit a wide and representative one—of the dynamic publications currently happening in mathematics and computer science.

A word of warning might be necessary. Given the many possible approaches on discrete curvature, it is not possible to present the ideas with unified formalism and notations. Indeed, there is no unique definition of the various discrete settings, and thus the different notions in this book are different and apply to different objects. In this introduction, we try to provide some organization to this profusion and to present the different chapters in some logical way (see Fig. 1). But achieving such a task is probably wishful thinking. Nevertheless, we are deeply convinced that the interested reader will find her or his pain and efforts rewarded by the various gems that can be found in this monograph.

Discrete curvature has started its modern life with the approximation theory and finite differences, mimicking the smooth definition at a very basic level, derivatives being replaced with differences. This has met the needs of applied science for a

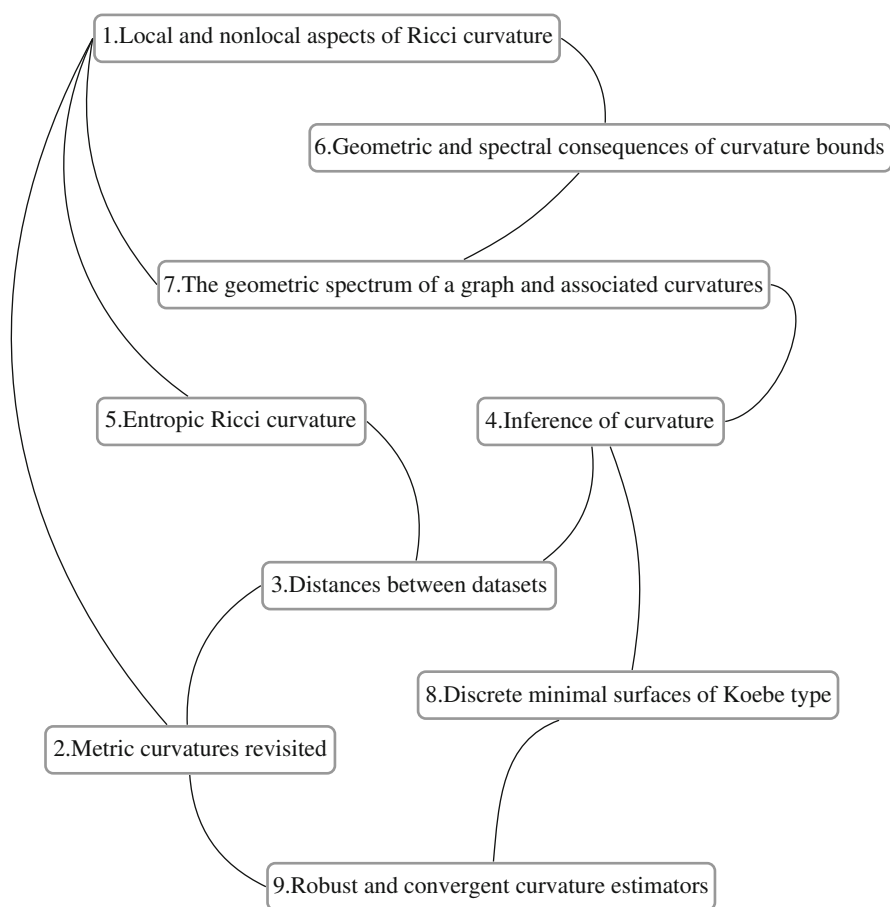


Fig. 1 A reading graph for this book, showing the strongest links between the chapters. The reader is thus invited to choose their own reading path

practical notion of curvature but has soon showed its limits. These were both on the theoretical side, since these approximations hardly ever obeyed the topological properties of their smooth ideal counterparts. But also on the practical side, since, (half-)surprisingly, such notions lack consistency, in general, the limit of such approximated curvatures either does not exist or does not converge to the smooth value. Such indeed was the (negative) answer of Xu et al. [26] to the question posed by Meek and Walton [18]: “Is there a closed-form expression based on local geometric data (lengths and angles) for the curvature of a discretized surface, that converges to the smooth one as the discretization is refined.” This shortcoming challenges the mere idea of discretization. In reality, the lack of convergence of order two quantities is not so surprising to geometers, and counterexamples such as the Schwarz lantern have been known to mathematicians for a long time, especially in geometric measure theory. Yet, this does not preclude convergence results (see, for instance, Chaps. 2 and 9) under proper hypotheses, which are the scope of today’s research. Typically, these rely either on stricter discretization (e.g., thick or fat triangulations (see Chap. 2), and also discretization along curvature lines as in [4]) or on adapting the type of convergence that is expected (e.g., Chaps. 2 and 3 on Gromov-Hausdorff convergence and Chap. 9) or more generally, as in the spirit of this book, on redefining the curvature itself, as a curvature measure (Chap. 4 and many others). A general overview of the lack of pointwise convergence and possible solutions can be found in [14].

Among the new approaches to discrete curvature, one finds a new take on local data (known to fail at pointwise convergence, as mentioned above), averaged as a measure rather than a vertex/edge-based quantity. In a way, the curvature ceases to be a Dirac measure to become more diffuse, by applying a (geometric) smoothing. A better and deeper understanding of this phenomenon can be found using normal cycles (in the Grassmannian) and geometric measure theory (see Chaps. 3 and 4 and the book of Morvan on generalized curvatures [7, 19], especially for extrinsic curvatures). As an advantage, this formalism gives a framework for managing the continuous, the discrete, and the semi-continuous cases, even point clouds. Very little regularity is required of the geometric object, a property which will also be encountered in the metric geometry approach. This is an asset when dealing with noisy real-world data; however this requires the choice of a characteristic length or convolution function to recover actual numeric values. Note that other geometric data such as sharp features can be obtained with these methods.

The starting point of these methods is the Steiner formula, which links the (extrinsic) curvatures to the area of the offsets of the surface under study, a method much used by Alexandrov. Interestingly enough, this approach turns out to be the starting point also of quadrangle/offset formalism for surfaces, developed in Chap. 8. This approach, derived from the integrable systems theory of geometric partial differential equations, is based on the following idea: A specific choice of parametrization (such as conformal, asymptotic, isothermic, etc.) transforms the study of a given (sub-)manifold into the study of a specific (simpler) PDE. In the discrete setup though, there is no such thing as a change of parametrization. On the contrary, it seems that one handles the surface itself instead of a parametrization,

as is the rule in differential geometry (although not in geometric measure theory, as the reader should notice). But even though changes of variables are hardly possible, special parametrizations do exist, e.g., conformal, isothermic, etc., inspired by Thurston's work in complex analysis (circle patterns). This leads to a beautiful theory of quadrangle-based discrete surfaces with special offsets, which allows to define Gaussian and mean curvature through offsets (*à la* Steiner) and minimal as well as constant mean or Gaussian curvature surfaces. The analogy with the smooth cases is justified by the convergence and the existence of similar structural properties (transformations of the space of solutions). In this approach, the geometric properties of the moduli space characterizes the discrete differential geometry. These special surfaces are—as expected—solution to an integrable system given by a Lax pair, analog to the smooth one, or described by conserved quantities [6]. In spite of their abstraction, these definitions often correspond to intuitive three-dimensional geometry; for instance, for K-nets, Steiner-defined curvature coincides with the product of orthogonal osculating circles. And logically but remarkably enough, this theoretical development has direct applications in architecture, in the construction of free form surfaces with constrained faces of fixed width and parallelism (see [24]). Surfaces made of panels of fixed width or with beams of fixed breadth are specific types of geometric quad-surfaces: Koebe and conical meshes. This leads to a new paradigm for discrete surfaces: instead of considering the vertices/edges/faces of one surface, one considers the surface together with its offset or equivalently the surface with its “Gauss map.” By doing so, one avoids the tricky problem of defining the normals (thus guaranteeing convergence), but more than that, we have a consistent theory, even at a low discretization level [5]. Such indeed is the goal of discrete differential geometry.

Another and more classical view of curvature dates back to the early twentieth century. It is based on comparison between distances in a metric (actually a length) space and distances in a model space, usually of constant curvature, the Euclidean space, the round sphere, or the hyperbolic space with varying curvatures. By comparing triangles or quadrangles, Alexandrov, Toponogov, and Wald give bounds on the curvature, when it exists, and define a working curvature in most generalities. This is explored in Chaps. 2 and 3. Since it requires no smoothness, it applies well to discrete spaces (but not only to them). Convergence results are mentioned and we link to point cloud reconstruction as in Chap. 4. Moreover, this theory is deeply intertwined with image processing and other applications. There is also a tentative definition of what the Ricci flow might be for piecewise linear surfaces, a thriving topic in mathematics and computer science, with applications in smoothing (although Ricci curvature is intrinsic, while smoothing often relies on extrinsic curvature).

Adding a measure to a metric space, new ideas came from probability theory and optimal transportation, in the pioneering works of Bakry and Ledoux [3] and Ollivier [22]. The first authors, followed by many others, proposed a notion of curvature via Bochner-type functional inequalities on processes and functions (as explained in the discrete setting in Chap. 1). While coherent with Riemannian geometry, this extends easily to the discrete case (though not exclusively); see, for

instance [16], some recent estimates on graphs. This mixed two simple ideas: (1) work on the curvature via functional inequalities, often using the Laplace operator, and (2) separate the measure from the metric (which are linked in Riemannian geometry). The inspiration of Ollivier is quite different. Ricci curvature, as the rate of divergence of geodesics, is measured as a (logarithmic) difference between optimal transportation distance (Wasserstein distance; see also Chap. 3) and standard distance. This point of view applies very well to discrete spaces, in particular to graphs, and is the focus of Chap. 1 (which as the introductory chapter also recalls many basic definitions in classical geometry). It has been developed since by many authors, including algorithmically. These two approaches to intrinsic curvature are close but different: the former naturally yields lower bounds, while the latter gives an explicit—if difficult to compute—expression. Both cases though rely on a diffusion process (the Laplacian) which is known to be elusive in discrete geometry [25]. They also embody the same vision: By adding measures to a space, one reintroduces continuous objects that are lacking in a discrete space. Such is also the philosophy of Chap. 5, where curvature is defined through entropy. Continuity is realized by the geodesics in the probability space, over which functional inequalities are derived. This relies of course on metric ideas presented in Chaps. 2 and 3. Finally, let us note that optimal transportation appears also in shape reconstruction, to minimize the tension (or the Laplace energy); see [12].

Among discrete spaces, the graphs hold a particular position. It is certainly an old mathematical subject with strong research themes, stimulated by ever-growing applications and also, from our point of view, a geometrization taking place (after earlier works of Gromov). Graphs are interesting per se but may also be seen as a crude version of cell complexes or even discrete manifolds. Whatever works on the former can also be conceived for the latter, and we shall give finer results for those. In this book, Chaps. 1, 6, and 7 are devoted to graphs, with various geometric conditions attached. In these appears with the greatest clarity how much the curvature is linked with the Laplacian. Indeed bounds on the Ollivier-Ricci curvature yield Poincaré estimates, and so do other curvatures. Finer discrete geometric aspects, such as the frequency of triangles, the degree, etc., play a role in the curvature, or equivalently, in the Laplacian, and influence the global geometry, with Myers' and Cartan-Hadamard theorem. The reader will note with interest how different choices of the weights yield different results (combinatorial Laplacian vs. probabilistic, a.k.a. harmonic, Laplacian).

Chapter 7 tackles the “extrinsic” part (which is also present in Chap. 4); curvature appears again via function spaces, i.e., via the geometric spectrum of the Laplacian on graphs. Similar and yet different realizations of the eigenfunctions, as local or global embeddings in space, are presented, the goal being a reconstruction of the geometry. The definitions in Chap. 7 show nevertheless the strong dependence on intrinsic data.

Finally, this work also tackles the topic of digital spaces (i.e., the geometry of \mathbb{Z}^n), another clear application of discrete geometry, with obvious applications to imaging. This is typically a setup where a naive approximation of smooth geometry may lead to vast errors, such as metrication artifacts. Chapter 9 presents

smarter and geometrically meaningful integral invariants as curvature estimators. The issue of approximation is handled through *multigrid convergence* to their continuous counterpart when the digital shape is a digitization of a sufficiently smooth Euclidean shape. Furthermore a good noise robustness is shown.

The crucial problem of connectedness in digital shapes is a key issue in Chap. 9, inducing definitions of curvature, determined by their consistency (the Gauss-Bonnet theorem). The different local configurations that can arise on a digital shape are studied, and curvature indices and normals are introduced, leading to an important transform called digital curvature flow, which can be seen as a digitized version of the classical curvature flow. Several applications of this flow are illustrated, such as thinning and snakes.

There are of course many aspects of discrete curvature that could not fit in this book. A wider overview of the topic may already be found in the 2014 CIRM Proceedings of the Meeting on Discrete Curvature [21]. But, of all the missing aspects, discrete exterior calculus (DEC) is in our opinion one of the most important. Luckily, many fine articles have been written on this topic recently, and we will happily direct the reader to them (see references below).

Let us just make a brief presentation of the topic and recall that discrete curvature, while not the main subject of DEC, is present through the ubiquitous Laplace operator, which is central in discrete calculus. DEC starts by defining the basic objects: discrete differential forms and vector fields. Simple operators are then constructed: the discrete exterior derivative d , the Hodge star \star and the codifferential d^* operating on forms, the discrete wedge product for combining forms, the discrete flat and sharp operators to interchange vector fields and one forms, the discrete interior product operator for combining forms and vector fields, and the Lie derivative. Using these, more sophisticated operators can be constructed, such as the gradient, divergence, and curl operators. Because the basic operators were defined by their algebraic properties, we obtain easily *exact* Green, Gauss, and Stokes theorems (for instance, the differential must commute with the boundary operator: $\int_c d\omega = \int_{\partial c} \omega$).

There is no unique or unequivocally better discretization of differential calculus, and the construction depends on the choice of the building bricks: discrete differential forms and vector fields. Often, these must be thought of as integrals of continuous forms over discrete elements (simplices or cells). For instance, Polthier and Preuß [23] presents a calculus based on finite elements on triangles exclusively. Another approach, followed by Desbrun et al., rests upon the (discrete) dual, which must be endowed with metric as well. For instance, dual vertices, which are primal faces, can be identified with a point in the face, e.g., the center of mass, or the circumcenter when the faces are circular, thus defining dual edge lengths. However, each choice of the dual geometry will generate a different DEC (see [2] for Laplacians on general polygonal meshes). Once again, the problem determines the approach. DEC applications are manifold, including finding geodesics, solving PDEs, remeshing, and optimal transportation again. For further reading, in addition to the works already quoted, we recommend the following papers and monographs: [8, 10, 11, 13, 15] and, for further reading also, [1, 9, 20].

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