Chapter 2
Differential Geometry

2.1 Introduction

A second major development in geometry in the eighteenth century was the study of curves and surfaces in $\mathbf{R}^2$ and $\mathbf{R}^3$ defined by not necessarily algebraic functions. These included two not quite independent developments that took place more or less simultaneously. The first was the development of the now standard elementary transcendental functions: the trigonometric, exponential, and logarithmic functions. In Euler's textbook from 1748 [62] these functions and their algebraic and analytic properties (e.g.,

$$\frac{d}{dx}\sin x = \cos x, \quad \sin(x + y) = \sin x \cos y + \cos x \sin y.$$

etc.) were fully developed and correspond to what one learns in contemporary precalculus and calculus courses in high school today. The second development involved the solution of differential equations (primarily ordinary differential equations) which provided a large variety of functions for analysis and geometrical representation. This led to a large class of special functions that went by the names of the mathematicians who created and developed them: Hermite, Legendre, Bessel, Euler's Gamma function and many others. These functions were tabulated for computational use and their various algebraic and analytical properties were developed, similar to those properties illustrated above for trigonometric functions. Over the course of time these mathematical tools became very important for the applications of mathematics to the worlds of chemistry, physics, biology and other areas of scientific understanding. These methods preceded by one or two centuries contemporary techniques for scientific analysis made available through the use of computers and simulation tools involving modern numerical analysis, which were to diminish the once important role of special functions.

In the latter half of the eighteenth century the differential geometry of curves and surfaces began to develop and flourish. First we consider the development of what

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became known as planar curves and space curves (i.e., smooth curves in $\mathbf{R}^2$ and $\mathbf{R}^3$). *Differential geometry* was named as a concept by Bianchi in 1894 (as noted by Kline [125] on p. 554). This naming of the discipline came long after the most significant developments in the field. It came to mean precisely manifolds equipped with a Riemannian (or more general) metric, or more generally a connection, and where the concepts of curvature played a central role. Indeed, the interaction of differential analysis (i.e. calculus, differential equations, all aspects of analysis involving infinite processes) with geometry is much older and broader than the more precise notion of differential geometry as it is employed today. For instance, the notion of differential topology, which developed in the mid-twentieth century, certainly involves manifolds and analysis, but doesn't formally use the notion of a differential-geometric metric as in differential geometry *per se*. Archimedes knew how to compute areas by the method of exhaustion, and Fermat understood both differentiation of functions (finding maxima and minima and tangents) and how to compute the area under some curves, but he did not know the fundamental theorem of calculus (see [194] for a discussion of these issues). All of these are indeed an interaction of analysis with geometry, and are parts of the foundation of what became differential geometry two centuries later.

2.2 Huygens and Newton

The first important task in differential geometry was to be able to efficiently compute the tangent line to a given curve at a given point and, as any beginning student of calculus knows, this is one of the first applications of the notion of the derivative. A deeper question that we explore in greater detail in this section is: what is curvature? More precisely, what is the curvature of a curve in a plane or in three-dimensional space? What is the curvature of a surface in three-dimensional space? Finally, what is curvature of an abstract two-dimensional or higher-dimensional manifold? This last question is a key part of the geometric developments in the nineteenth century and will be discussed in Part II.

Consider first the simple case of a curve in the plane defined by the graph of a function as in Fig. 2.1. Then one learns in calculus that the curvature of the curve at $P = (x, y)$ is given by

$$K_P = \pm \frac{f''(x)}{[1 + (f'(x)^2]^{\frac{3}{2}}}, \tag{2.1}$$

where the sign is chosen to be positive if the normal vector to the curve at P intersects the approximating circle and is negative otherwise. In the illustration in Fig. 2.1, the normal vector to the curve at P using the usual orientation would be pointing upwards in the figure, away from the approximating circle, whose radius is $1/|K_P|$, and hence in this case the curvature would be negative.

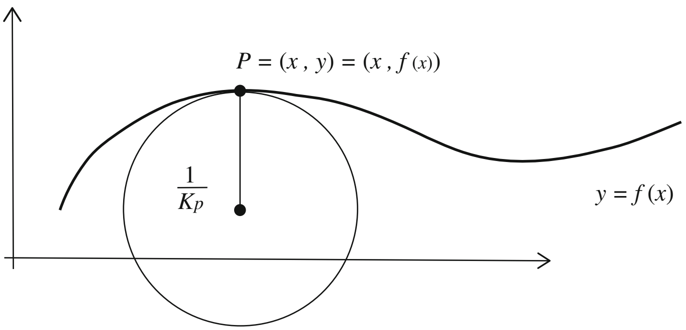


Fig. 2.1 Radius of curvature of a curve at a point

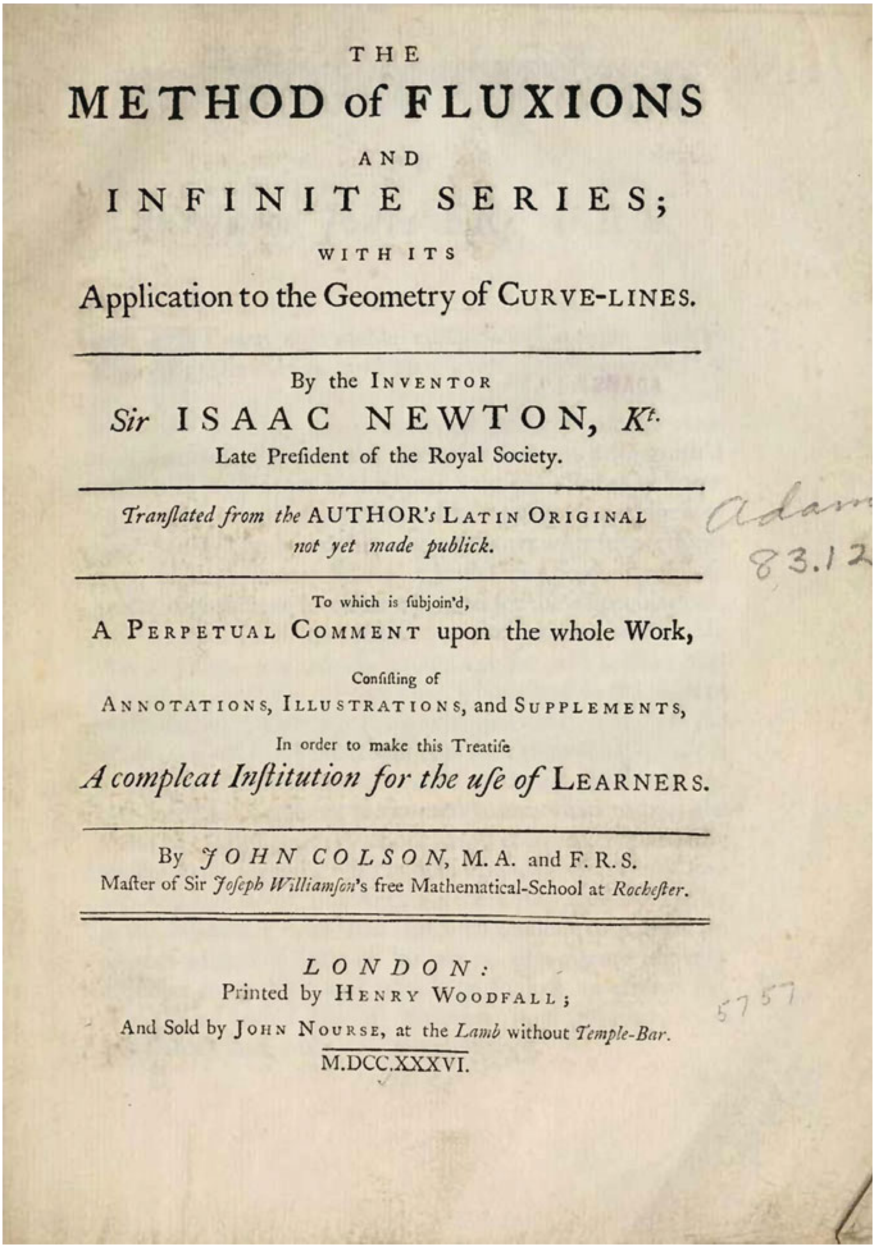
This formula is given for the first time in Newton's monograph of 1736 [169], which was published as an English translation of his original Latin manuscript from 1671, which was never published, but was privately circulated among some of Newton's colleagues. This monograph, published in 1736 after Newton's earlier death, was part of the basis for the controversy between adherents of Newton and Leibniz on who had first invented (or discovered) calculus. Figure 2.2 shows the cover page of this singular monograph, and Fig. 2.3 shows the table of contents, where the curvature of a curve stands out so very distinctly as an object of study. The formula (2.1) appears in the text of Newton's monograph.

The first published account of the curvature of a general curve was due to Christiaan Huygens (1629–1695) in 1673 [114]. In both Newton and Huygens the fundamental definition of the center of curvature (center of the osculating circle at a given point) is the intersection of normal lines to the curve near the given point on the curve (see the figures in Huygens p. 84 [114] and Newton on p. 60 [169], reproduced here in Figs. 2.4 and 2.5).

Huygens didn't have calculus *per se* at his disposal, but he made estimates in terms of normals at an approximating point (like the estimates of slopes of an approximating secant to a tangent line in differential calculus), and using these estimates he was able to compute the curvature for a variety of examples (cycloid, conic sections, etc.).

An interesting historical point is how Huygens came to study this phenomenon. Some 16 years before the appearance of his monograph [114] he had built one of the most important clocks in history: a pendulum whose motion is isochronous. That is, the swing of the pendulum has a constant period of repetition. Huygens showed that a simple pendulum, whose pendant moves in a circular arc, has a period that depends on the size of the oscillations, whereas if the pendant moves in the arc of a cycloid, then the period is fixed independent of the size of the oscillation.

The method Huygens used for making the pendant move in a cycloidal path (which he patented in 1657) was to have the path be the *involute* of a curved plate (which was also a cycloid), i.e., the curve traced out by a fixed string moving from a center attached to a given curve, where initially the fixed string lies along the given curve and moves away from it, with the free straight line portion of the string being



THE

METHOD of FLUXIONS

AND

INFINITE SERIES;

WITH ITS

Application to the Geometry of CURVE-LINES.

By the INVENTOR

Sir ISAAC NEWTON, $K^{t.}$

Late Prefident of the Royal Society.

Tranflated from the AUTHOR's LATIN ORIGINAL

not yet made publick.

To which is fubjoin'd,

A PERPETUAL COMMENT upon the whole Work,

Confifting of

ANNOTATIONS, ILLUSTRATIONS, and SUPPLEMENTS,

In order to make this Treatife

A compleat Inftitution for the ufe of LEARNERS.

By *JOHN COLSON*, M.A. and F.R.S.

Mafter of Sir *Jofeph Williamfon*'s free Mathematical-School at *Rochefter*.

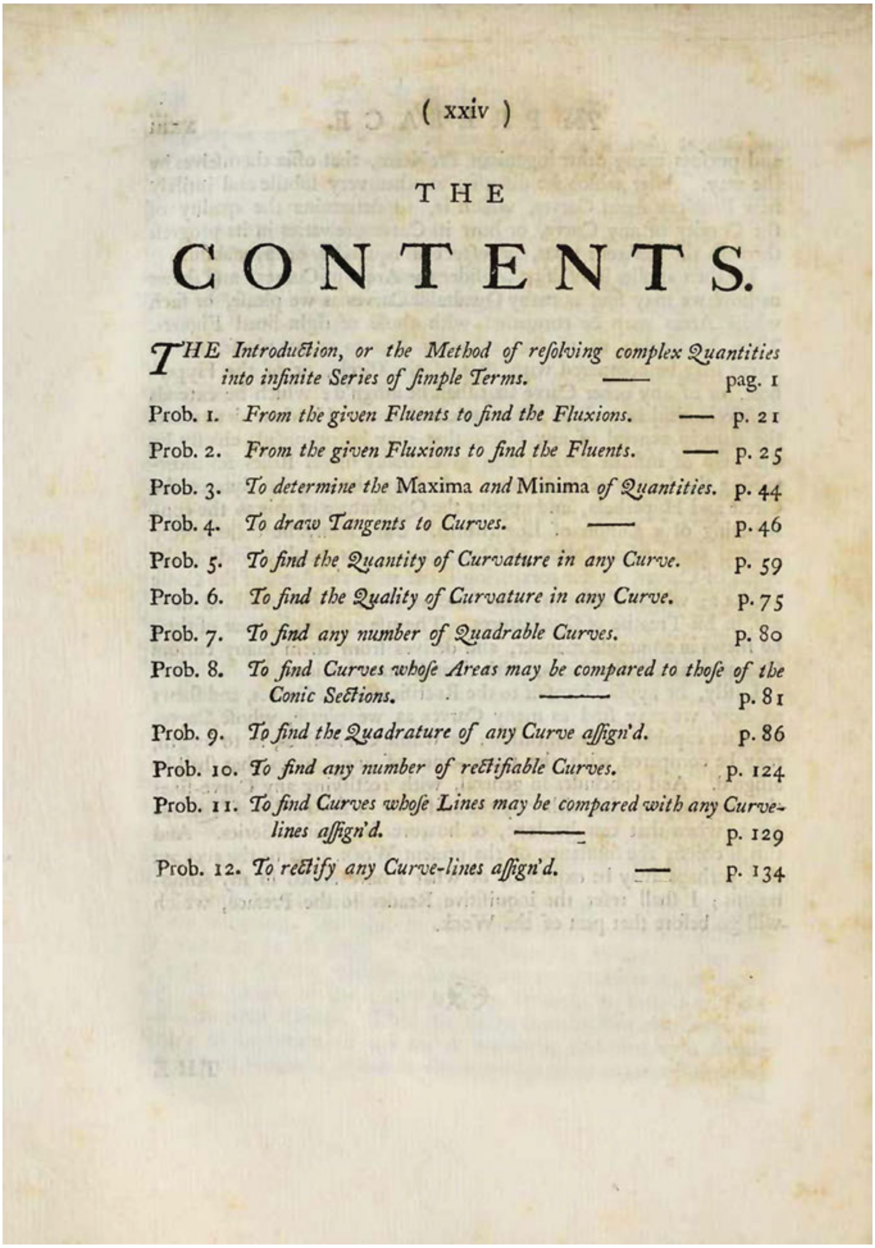
LONDON:

Printed by HENRY WOODFALL;

And Sold by JOHN NOURSE, at the *Lamb* without *Temple-Bar*.

M.DCC.XXXVI.

Fig. 2.2 Title page of Newton's 1736 Monograph on Fluxions



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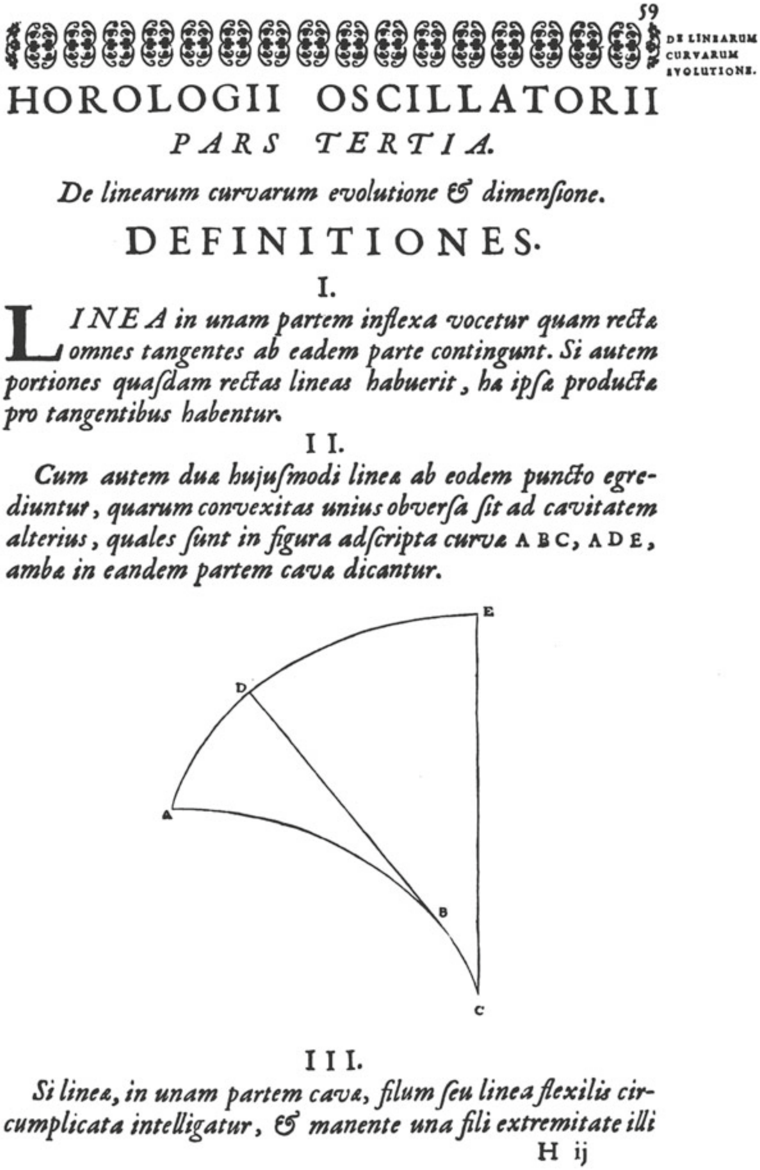
THE

CONTENTS.

Fig. 2.3 Table of contents of Newton's 1736 monograph on fluxions



59

DE LINEARUM CURVARUM EVOLUTIONE.

HOROLOGII OSCILLATORII
PARS TERTIA.

De linearum curvarum evolutione & dimenſione.

DEFINITIONES.

I.

L*INEA in unam partem inflexa vocetur quam rectæ omnes tangentes ab eadem parte contingunt. Si autem portiones quaſdam rectas lineas habuerit, hæ ipſæ productæ pro tangentibus habentur.*

II.

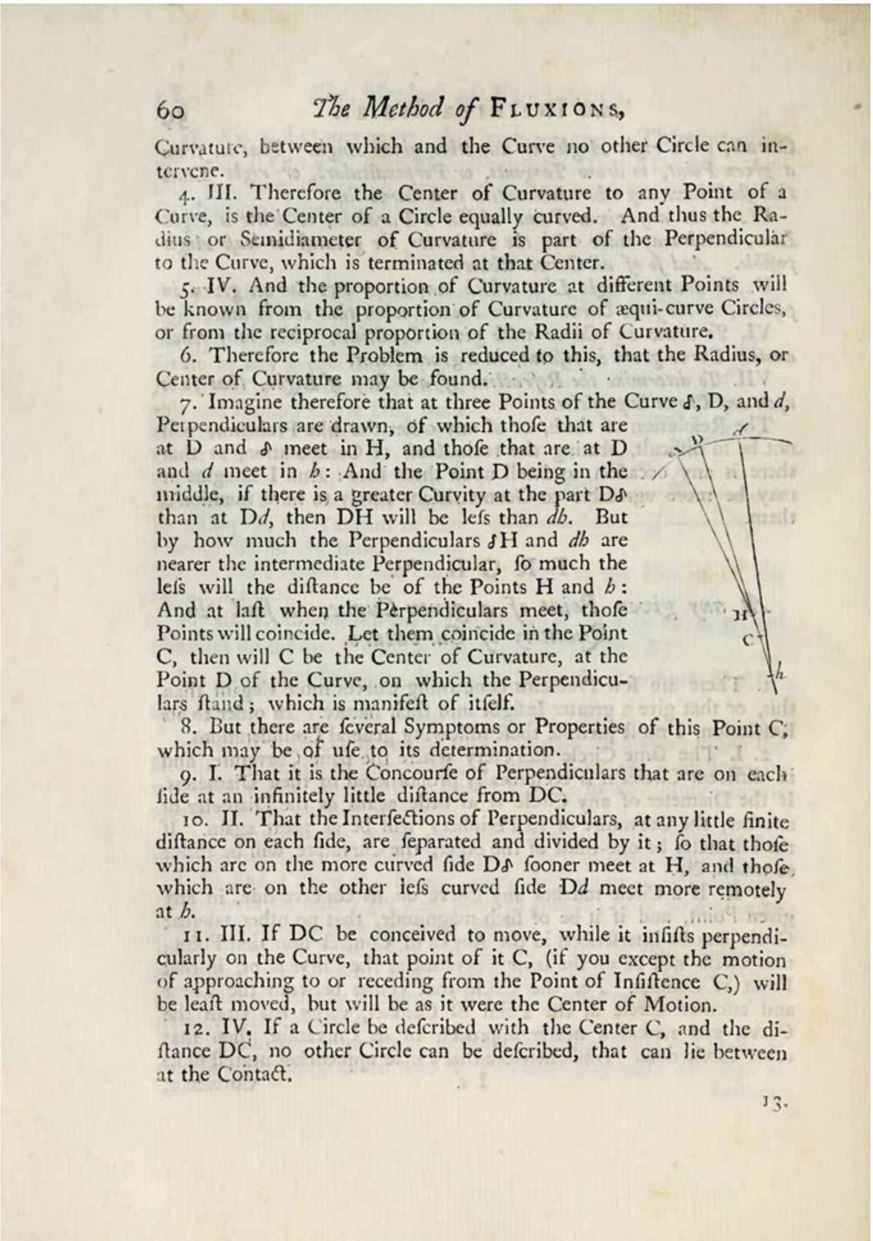
Cum autem duæ hujuſmodi lineæ ab eodem puncto egrediuntur, quarum convexitas unius obverſa ſit ad cavitatem alterius, quales ſunt in figura adſcripta curvæ ABC, ADE, *ambæ in eandem partem cavæ dicantur.*

III.

Si lineæ, in unam partem cavæ, filum ſeu linea flexilis circumplicata intelligatur, & manente una fili extremitate illi

H ij

Fig. 2.4 Huygens's center of curvature from *Horologium Oscillatorium*



60 *The Method of* FLUXIONS,

Curvature, between which and the Curve no other Circle can intervene.

4. III. Therefore the Center of Curvature to any Point of a Curve, is the Center of a Circle equally curved. And thus the Radius or Semidiameter of Curvature is part of the Perpendicular to the Curve, which is terminated at that Center.

5. IV. And the proportion of Curvature at different Points will be known from the proportion of Curvature of æqui-curve Circles, or from the reciprocal proportion of the Radii of Curvature.

6. Therefore the Problem is reduced to this, that the Radius, or Center of Curvature may be found.

7. Imagine therefore that at three Points of the Curve δ, D, and *d*, Perpendiculars are drawn, of which thoſe that are at D and δ meet in H, and thoſe that are at D and *d* meet in *h*: And the Point D being in the middle, if there is a greater Curvity at the part Dδ than at D*d*, then DH will be leſs than *dh*. But by how much the Perpendiculars δH and *dh* are nearer the intermediate Perpendicular, ſo much the leſs will the diſtance be of the Points H and *h*: And at laſt when the Perpendiculars meet, thoſe Points will coincide. Let them coincide in the Point C, then will C be the Center of Curvature, at the Point D of the Curve, on which the Perpendiculars ſtand; which is manifeſt of itſelf.

8. But there are ſeveral Symptoms or Properties of this Point C, which may be of uſe to its determination.

9. I. That it is the Concourſe of Perpendiculars that are on each ſide at an infinitely little diſtance from DC.

10. II. That the Interſections of Perpendiculars, at any little finite diſtance on each ſide, are ſeparated and divided by it; ſo that thoſe which are on the more curved ſide Dδ ſooner meet at H, and thoſe which are on the other leſs curved ſide D*d* meet more remotely at *h*.

11. III. If DC be conceived to move, while it inſiſts perpendicularly on the Curve, that point of it C, (if you except the motion of approaching to or receding from the Point of Inſiſtence C,) will be leaſt moved, but will be as it were the Center of Motion.

12. IV. If a Circle be deſcribed with the Center C, and the diſtance DC, no other Circle can be deſcribed, that can lie between at the Contact.

13.

Fig. 2.5 Newton's center of curvature from *Method of Fluxions*

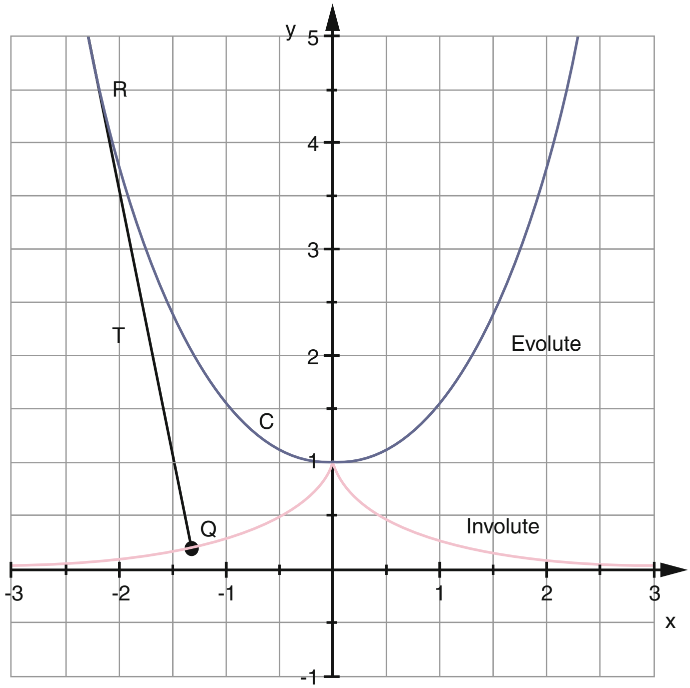


Fig. 2.6 An involute being generated by a string attached to the curve C (called the evolute)

continuously tangent to the given curve (see the illustration in Fig. 2.6). The curve C in Fig. 2.6 is called the *evolute* (which generates the involute traced out by the point Q by the motion of the string). The problem Huygens posed and solved was: given the involute, find the evolute, i.e., find the generating curve. Now the straight line T is normal to the involute at the point Q (as Huygens showed), and, at the point of contact at point R, T is tangent to C. Thus T is normal to the involute at Q, and R can be seen to be the intersections of the normals close to Q (as both Huygens and Newton showed). Hence R is the center of curvature of the involute at the point Q, and the evolute C is the locus of centers of curvature of the involute at points near Q.

In the second illustration of an involute in Fig. 2.7, one sees two "parallel" involutes, the curves C' and C'' being generated from the curve C, and one can see that the involutes are orthogonal to the generating string at the intersection points (as was proved by Huygens). Looking at the illustration from p. 4 (Fig. 2.8) of Huygens's book [114] one sees in Fig. II of the diagrams in Fig. 2.8 the cycloid-shaped curve from which the pendant of the pendulum sweeps out the involute, which is the cycloidal motion of the pendant. Huygens calculated the evolutes for a number of examples, independent of the specific example he used in the design of his clock.

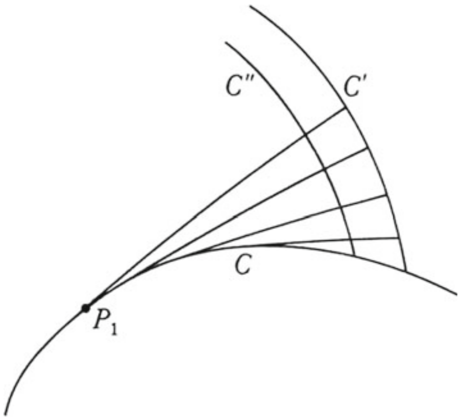


Fig. 2.7 Involutes are orthogonal to the generating string

Some 2000 years earlier, in Book V of his famous work *Conics*, Apollonius was able to compute the curvature of the classical conic sections. Apollonius was in fact trying to solve a different set of problems, and curvature was not explicitly discussed. In Heath's translation [98], he shows what Apollonius did in modern notation. More particularly, on p. 171 one finds that for the parabola of the form

$$\frac{1}{2a}y^2 = x,$$

the evolute (locus of centers of curvature) of this parabola has the form:

$$27ay^2 = 4(x - 2a)^3,$$

which is a semicubical parabola. He finds similar formulas for the ellipse and hyperbola.

Here Apollonius was studying the behavior of normals to conic sections. He showed that each conic section has a unique normal passing through each point. He *defined* a normal as being a straight line which was either a local maximum or a local minimum-length straight line from some point not on the curve. He then showed that such a line was indeed perpendicular to the tangent line at the given point. This leads, by an interesting argument, to the conclusion that Apollonius has calculated the points of the evolute, as Heath points out very explicitly.

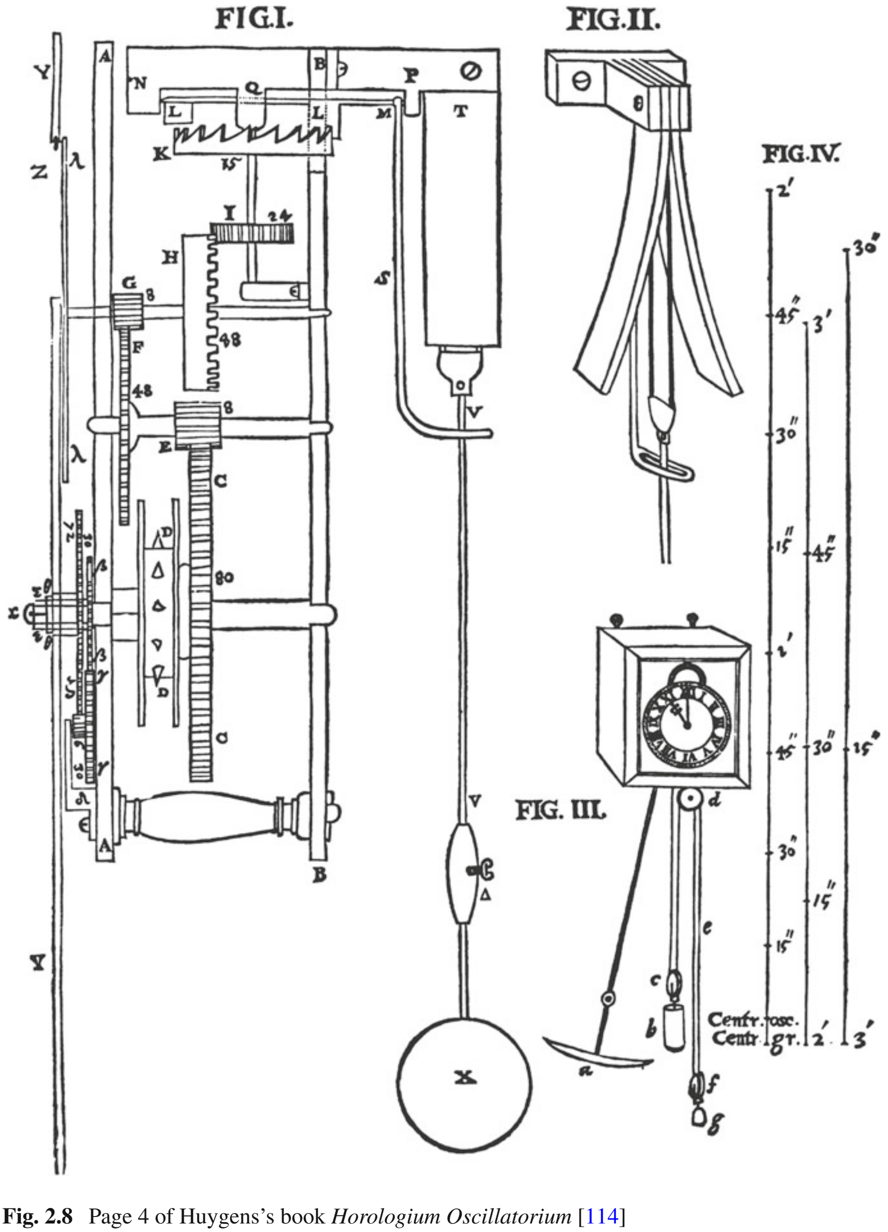


Fig. 2.8 Page 4 of Huygens's book *Horologium Oscillatorium* [114]

2.3 Curves in Space: *Courbes à double courbure*

Since the time of Newton, curvature of a curve in the plane became a standard object of mathematical investigation. The first step in investigating the differential geometry of curves in $\mathbf{R}^3$ was taken by Alexis Claude Clairaut (1713–1765) in his book *Recherches sur les courbes à double courbure* [48], written when he was only 16 years old and published two years later, following up on work he had started when he was 12 years old. We know this from the "Approbation" at the beginning of the book, written by two of the reviewers of the book; and the page where this appears, following the Preface, is the only place Clairaut's name appears in the book, not on the title page! See Fig. 2.9. Clairaut called curves in $\mathbf{R}^3$ "courbes à double courbure",[1] and he says in his book that he was inspired by Descartes, who suggested space curves could be studied in terms of their projections on two orthogonal planes. Clairaut studied the tangent line to a curve, its arc length and the infinite variety of normal lines in the plane perpendicular to the tangent line.

The next steps in the study of space curves were taken by Euler, who primarily looked at space curves which were defined as the intersections of surfaces in $\mathbf{R}^3$ (see Volume 2 of Euler's *Introductio* of 1748 [62]). Michel Ange Lancret (1774–1807) singled out in 1806 the three principal directions of a space curve at any point (tangent, normal, and binormal), and formulated the additional notion of torsion of a curve [132].

The final steps in the study of space curves were taken by Augustin-Louis Cauchy (1789–1857) in 1826 in his *Leçons sur les Applications du Calcul Infinitésimal à la Géométrie* [38], and by Serret [216] and Frenet [75] in their back-to-back papers in 1851 and 1852. Cauchy gave us the formulation of space curves we use today (without the vector notation), and Serret and Frenet gave the final form to the structure equations (which today bear their name, the Frenet–Serret equations), which brought together the formal characterization of space curves in terms of the three principal directions of a curve and its curvature and torsion.

2.4 Curvature of a Surface: Euler in 1767

The concept of the curvature of a curve in $\mathbf{R}^3$ was well understood at the end of the eighteenth century, and the later work of Cauchy, Serret and Frenet completed this set of investigations begun by the young Clairaut a century earlier. The problem arose: how can one define the curvature of a surface defined either locally or globally in $\mathbf{R}^3$? An important contribution is made by Euler in his paper entitled "Recherches sur la courbure des surfaces"[2] [65] from 1767 (note this article is written in French,

[1] "curves with double curvature". The expression "courbes à double courbure" was used to describe space curves for a long time by many mathematicians after the initial impetus of Clairaut.

[2] "Research on the curvature of surfaces".

APPROBATION.

J'AI lû par ordre de M. le Garde des Sceaux un Manuſcrit intitulé *Recherches ſur les Courbes à double courbure*, compoſé par M. CLAIRAUT. Ce Traité que les plus habiles Geometres de notre temps & des ſiecles paſſés ſe ſeroient fait honneur d'avoir compoſé, & qui eſt certainement l'Ouvrage d'un jeune homme de ſeize ans, qui dès l'âge de douze avoit déja donné des marques publiques de ſon habileté dans les Mathematiques, ne merite pas ſeulement d'être imprimé, mais d'être admiré comme un prodige d'imagination, de conception & de capacité. A Paris ce 3 Juin 1730.

J. DE MOLIERES.

EXTRAIT des Regiſtres de l'Academie Royale des Sciences, du 20 Août 1729.

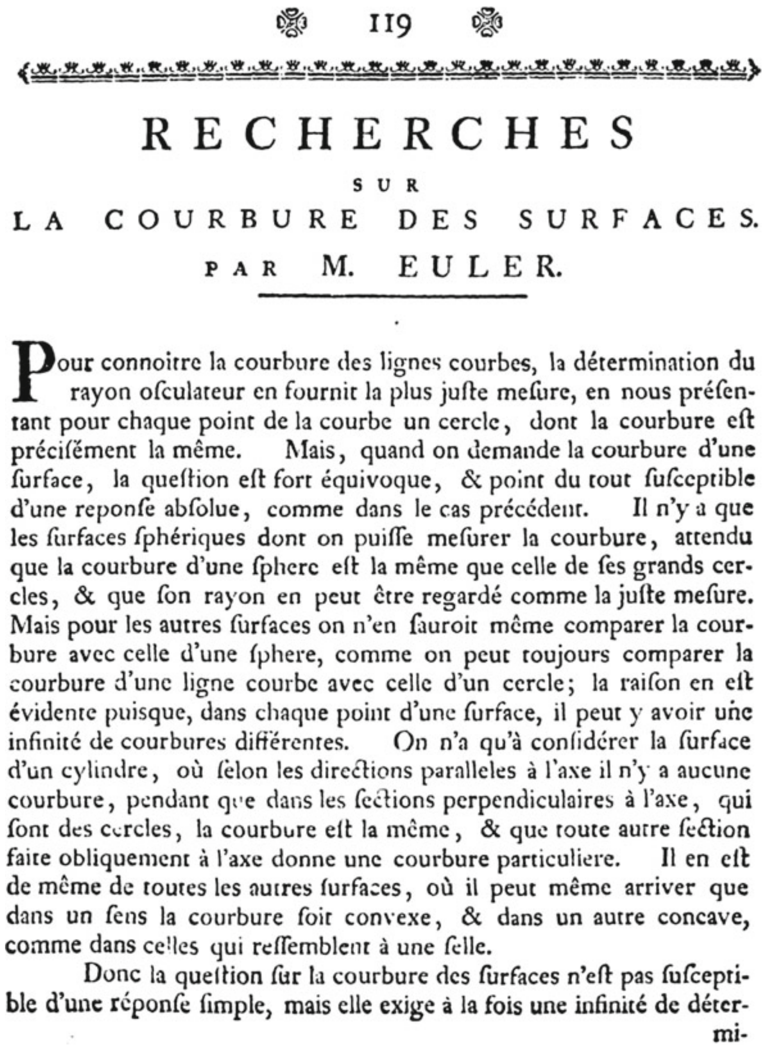
MEſſieurs de Mairan & Nicole qui avoient été nommés pour examiner un Ouvrage de M. CLAIRAUT le Fils, intitulé, *Recherches ſur les Courbes à double courbure*, en ayant fait leur rapport, la Compagnie a jugé que cet Ouvrage contenoit beaucoup de choſes curieuſes & nouvelles ſur ces ſortes de courbes, & montroit non-ſeulement de l'invention dans l'Auteur, qui n'eſt âgé que de ſeize ans, mais encore beaucoup de connoiſſance du Calcul differentiel, & de l'Integral. Fait à Paris ce 23 Août 1729.

FONTENELLE, Sec. perp. de l'Ac. Roy. des Sc.

Fig. 2.9 Excerpt from the beginning of Clairaut's book *Recherches sur les courbes à double courbure* [48]

not like his earlier works, most of which were written in Latin). Figure 2.10 shows the first page of the article and we quote the translation here:

> In order to know the curvature of a curve, the determination of the radius of the osculating circle furnishes us the best measure, where for each point of the curve we find a circle whose curvature is precisely the same. However, when one looks for the curvature of a surface, the question is very equivocal and not at all susceptible to an absolute response, as in the case above. There are only spherical surfaces where one would be able to measure the curvature, assuming the curvature of the sphere is the curvature of its great circles, and whose radius could be considered the appropriate measure. But for other surfaces one doesn't know even how to compare a surface with a sphere, as when one can always compare the curvature of a curve with that of a circle. The reason is evident, since at each point of a surface there are an infinite number of different curvatures. One has to only consider a cylinder, where along the directions parallel to the axis, there is no curvature, whereas in the directions perpendicular to the axis, which are circles, the curvatures are all the same, and all other oblique sections to the axis give a particular curvature. It's the same for all other surfaces, where it can happen that in one direction the curvature is convex, and in another it is concave, as in those resembling a saddle.



119

RECHERCHES

SUR

LA COURBURE DES SURFACES.

PAR M. EULER.

Pour connoitre la courbure des lignes courbes, la détermination du rayon oſculateur en fournit la plus juſte meſure, en nous préſentant pour chaque point de la courbe un cercle, dont la courbure eſt préciſément la même. Mais, quand on demande la courbure d'une ſurface, la queſtion eſt fort équivoque, & point du tout ſuſceptible d'une reponſe abſolue, comme dans le cas précédent. Il n'y a que les ſurfaces ſphériques dont on puiſſe meſurer la courbure, attendu que la courbure d'une ſphere eſt la même que celle de ſes grands cercles, & que ſon rayon en peut être regardé comme la juſte meſure. Mais pour les autres ſurfaces on n'en ſauroit même comparer la courbure avec celle d'une ſphere, comme on peut toujours comparer la courbure d'une ligne courbe avec celle d'un cercle; la raiſon en eſt évidente puisque, dans chaque point d'une ſurface, il peut y avoir une infinité de courbures différentes. On n'a qu'à conſidérer la ſurface d'un cylindre, où ſelon les directions paralleles à l'axe il n'y a aucune courbure, pendant que dans les ſections perpendiculaires à l'axe, qui ſont des cercles, la courbure eſt la même, & que toute autre ſection faite obliquement à l'axe donne une courbure particuliere. Il en eſt de même de toutes les autres ſurfaces, où il peut même arriver que dans un ſens la courbure ſoit convexe, & dans un autre concave, comme dans celles qui reſſemblent à une ſelle.

Donc la queſtion ſur la courbure des ſurfaces n'eſt pas ſuſceptible d'une réponſe ſimple, mais elle exige à la fois une infinité de détermi-

mi-

Fig. 2.10 The opening page of Euler's work on curvature [65]

In this paper Euler describes quite clearly the problem of formulating a concept of curvature of a surface in $\mathbf{R}^3$. In particular, in the quote above one sees that Euler recognized the difficulties in defining curvature for a surface at any given point. He does not resolve this issue in this paper, but he makes extensive calculations and several major contributions to the subject. He considers a surface S in $\mathbf{R}^3$ defined as a graph

$$z = f(x, y)$$

near a given point $P = (x_0, y_0, z_0)$. At the point P he considers planes in $\mathbf{R}^3$ passing through the point P which intersect the surface in a curve in that given plane. For each such plane and corresponding curve he computes explicitly the curvature of the curve at the point P in terms of the given data.

He then restricts his attention to planes which are normal to the surface at P (planes containing the normal vector to the surface at P). There is a one-dimensional family of such planes E_θ, parametrized by an angle θ. He computes explicitly the curvature of the intersections of E_θ with S as a function of θ, and observes that there is a maximum and minimum κ_1 and κ_2 of these curvatures at P, corresponding to two planes E_1 and E_2. These curvatures are called the *principal curvatures* of the surface at the point P. In the generic case, Euler shows that the two planes E_1 and E_2 are orthogonal to each other. Moreover, he shows that the curvature κ_θ for the plane E_θ can be computed in terms of the principal curvatures, namely

$$\kappa_\theta = \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta.$$

This is as far as he goes, but it is a great step forward in understanding the curvature of a surface. He does *not* use this data to define what we now call the *curvature* of the surface S at the point P. This step was taken by Gauss in a visionary and extremely important paper some 60 years later [81].

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