

Chapter 2

$\mathcal{Q}_K$ Spaces

As a family of Möbius invariant function spaces, $\mathcal{Q}_K$ spaces were first introduced at the beginning of this century. The theory of $\mathcal{Q}_K$ spaces has since attracted considerable attention and experienced rapid development over the past two decades. In this chapter, we define $\mathcal{Q}_K$ spaces, prove several inclusion relations, and construct some important examples of functions in $\mathcal{Q}_K$. Many of the results and techniques here will be needed later on.

2.1 Preliminaries

In this section, we gather some preliminary results that will be used repeatedly in later chapters. The first one is the classical Schur's test, which is a tool for proving the boundedness of certain integral operators on L^p spaces.

Let (X, μ) be a measure of space. Consider integral operators of the form

$$Tf(x) = \int_X H(x, y)f(y) d\mu(y), \quad (2.1)$$

where H is a nonnegative and measurable function on $X \times X$.

Proposition 2.1 *Suppose $1 < p < \infty$ and $1/p + 1/q = 1$. If there exists a positive and measurable function h on X such that*

$$\int_X H(x, y)h(y)^q d\mu(y) \leq Ch(x)^q$$

for almost all $x \in X$ and

$$\int_X H(x, y) h(x)^p d\mu(x) \leq Ch(y)^p$$

for almost all $y \in X$, where C is a positive constant, then the integral operator T defined in (2.1) is bounded on $L^p(X, d\mu)$. Furthermore, the norm of T on $L^p(X, d\mu)$ does not exceed the constant C .

Proof This is a direct consequence of Hölder's inequality and Fubini's theorem. In fact, if f is in $L^p(X, d\mu)$, then for any $x \in X$ we have

$$|Tf(x)| \leq \int_X H(x, y) h(y) h(y)^{-1} |f(y)| d\mu(y),$$

so Hölder's inequality gives

$$|Tf(x)| \leq \left(\int_X H(x, y) h(y)^q d\mu(y) \right)^{\frac{1}{q}} \left(\int_X H(x, y) h(y)^{-p} |f(y)|^p d\mu(y) \right)^{\frac{1}{p}}.$$

By the first inequality of the assumption, we have

$$|Tf(x)| \leq C^{\frac{1}{q}} h(x) \left(\int_X H(x, y) h(y)^{-p} |f(y)|^p d\mu(y) \right)^{\frac{1}{p}}$$

for all $x \in X$. An application of Fubini's theorem and the second inequality of the assumption then gives

$$\begin{aligned} \int_X |Tf(x)|^p d\mu(x) &\leq C^{\frac{p}{q}} \int_X h(y)^{-p} |f(y)|^p d\mu(y) \int_X H(x, y) h(x)^p d\mu(x) \\ &\leq C^p \int_X |f(y)|^p d\mu(y). \end{aligned}$$

Thus T is a bounded integral operator on $L^p(X, d\mu)$ with norm less than or equal to the constant C .

Specializing to the case $p = 2$, we obtain the following simple criterion for the boundedness of certain integral operators on the Hilbert space L^2 .

Corollary 2.2 *If there exists a positive and measurable function h on X such that*

$$\int_X H(x, y) h(y) d\mu(y) \leq Ch(x)$$

for almost all x and

$$\int_X H(x, y) h(x) d\mu(x) \leq Ch(y)$$

for almost all y , then the operator T defined in (2.1) is bounded on $L^2(X, d\mu)$, and the norm of T on $L^2(X, d\mu)$ is less than or equal to the constant C .

Proof This clearly follows from Proposition 2.1.

Note that Schur's test not only tells when an integral operator is bounded, it also gives an estimate on the norm of the operator. This norm estimate is actually the best possible and will be essential for our analysis later on.

The following estimates have proven very useful in complex analysis on the unit disk. They will be used numerous times in subsequent chapters.

Lemma 2.3 Suppose $s > -1$, t is real, and

$$I(z) = \int_{\mathbb{D}} \frac{(1 - |w|^2)^s dA(w)}{|1 - \bar{z}w|^{2+t+s}}, \quad z \in \mathbb{D}.$$

(a) If $t < 0$, then there exists a constant $C > 0$ such that $I(z) \leq C$ for all $z \in \mathbb{D}$.

(b) If $t = 0$, then there exists a constant $C > 0$ such that

$$I(z) \leq C \log \frac{2}{1 - |z|^2}$$

for all $z \in \mathbb{D}$.

(c) If $t > 0$, then there exists a constant $C > 0$ such that

$$I(z) \leq \frac{C}{(1 - |z|^2)^t}$$

for all $z \in \mathbb{D}$.

Proof This follows easily from integrating the Taylor expansion of the function

$$\frac{1}{(1 - \bar{z}w)^{(2+s+t)/2}}.$$

Details are left to the reader. Part (c) is also a special case of the next result.

Lemma 2.4 Suppose $t \geq 0$ and $t - 1 < s < r - 2$. Then there exists a constant $C > 0$ such that

$$\int_{\mathbb{D}} \frac{(1 - |w|^2)^s dA(w)}{|1 - \bar{w}z|^r |1 - \bar{w}\zeta|^t} \leq \frac{C}{(1 - |z|^2)^{r-s-2} |1 - \bar{\zeta}z|^t}$$

for all z and ζ in $\mathbb{D}$.

Proof Let $I(z, \zeta)$ denote the integral we wish to estimate. The change of variables $w = \varphi_z(u)$ shows that

$$I(z, \zeta) = \frac{1}{(1 - |z|^2)^{r-s-2}} \int_{\mathbb{D}} \frac{(1 - |u|^2)^s dA(u)}{|1 - \bar{z}u|^{4+2s-r} |1 - \bar{\zeta}\varphi_z(u)|^t}.$$

It is easy to check that

$$1 - \bar{\xi}\varphi_z(u) = (1 - \bar{\xi}z) \frac{1 - \overline{u\varphi_z(\xi)}}{1 - \bar{z}u}.$$

It follows that

$$I(z, \xi) = \frac{1}{(1 - |z|^2)^{r-s-2} |1 - \bar{\xi}z|^t} \int_{\mathbb{D}} \frac{(1 - |u|^2)^s dA(u)}{|1 - \bar{z}u|^{4+2s-r-t} |1 - \overline{u\varphi_z(\xi)}|^t}.$$

Since $t \geq 0$ and

$$|1 - \overline{u\varphi_z(\xi)}| \geq 1 - |u|$$

for all u, z , and ξ , we can find a positive constant C (that only depends on t) such that

$$I(z, \xi) \leq \frac{C}{(1 - |z|^2)^{r-s-2} |1 - \bar{\xi}z|^t} \int_{\mathbb{D}} \frac{(1 - |u|^2)^{s-t} dA(u)}{|1 - \bar{z}u|^{4+2s-r-t}}.$$

The desired result then follows from part (a) of Lemma 2.3.

Note that if we take $t = 0$ in Lemma 2.4, the result is part (c) of Lemma 2.3. We will also need the following analogue of Lemma 2.3.

Lemma 2.5 *Let s be real and*

$$J(z) = \int_0^{2\pi} \frac{d\theta}{|1 - ze^{-i\theta}|^s}, \quad z \in \mathbb{D}.$$

(a) *If $s > 1$, then there exists a constant $C > 0$ such that*

$$J(z) \leq \frac{C}{(1 - |z|^2)^{s-1}}$$

for all $z \in \mathbb{D}$.

(b) *If $s = 1$, then there exists a constant $C > 0$ such that*

$$J(z) \leq C \log \frac{2}{1 - |z|^2}$$

for all $z \in \mathbb{D}$.

(c) *If $s < 1$, then there exists a constant $C > 0$ such that $J(z) \leq C$ for all $z \in \mathbb{D}$.*

Proof This also follows from integrating the Taylor series of the function

$$\frac{1}{(1 - ze^{-i\theta})^{s/2}}.$$

Details are left to the reader again.

Let $\beta(z, w)$ denote the Bergman metric (which is also called the hyperbolic metric) between two points z and w in $\mathbb{D}$. Recall that

$$\beta(z, w) = \frac{1}{2} \log \frac{1 + |\varphi_z(w)|}{1 - |\varphi_z(w)|}.$$

For $z \in \mathbb{D}$ and $R > 0$, we use

$$D(z, R) = \{w \in \mathbb{D} : \beta(z, w) < R\}$$

to denote the Bergman metric ball at z with radius R . If R is fixed, then it can be checked that the area of $D(z, R)$, denoted by $|D(z, R)|$, is comparable to $(1 - |z|^2)^2$ as z approaches the unit circle.

Lemma 2.6 *For any $R > 0$, there exists a positive constant C (depending on R) such that*

$$|f(z)|^2 \leq \frac{C}{|D(z, R)|} \int_{D(z, R)} |f(w)|^2 dA(w)$$

for all $z \in \mathbb{D}$ and all analytic functions f in $\mathbb{D}$.

Proof When $z = 0$, $D(0, R)$ is actually a Euclidean disk centered at the origin, so the desired result follows from the sub-mean value property for the subharmonic function $|f|^2$. The general case then follows from the special case above and a change of variables. We leave the details to the interested reader.

Recall that the pseudo-hyperbolic metric in the disk is given by

$$\rho(z, w) = |\varphi_z(w)| = \left| \frac{z - w}{1 - \bar{z}w} \right|.$$

Since

$$\beta(z, w) = \frac{1}{2} \log \frac{1 + \rho(z, w)}{1 - \rho(z, w)},$$

the pseudo-hyperbolic metric disk centered at a with radius $r \in (0, 1)$, namely, the set $\{z \in \mathbb{D} : \rho(z, a) < r\}$, is equal to the hyperbolic metric disk centered at the same point a with the different radius

$$R = \frac{1}{2} \log \frac{1 + r}{1 - r} = \tanh^{-1}(r) \in (0, \infty).$$

On some occasions, it may be more convenient to use the pseudo-hyperbolic metric instead of the hyperbolic metric. Since these two metrics are equivalent, we will use $D(a, r)$ to denote either a hyperbolic disk or a pseudo-hyperbolic disk. Just keep in mind that when we use the hyperbolic metric, the radius can be any positive number, while the radius in the pseudo-hyperbolic metric has to be less than 1. Either way, when the radius is fixed, we always have $|D(a, r)| \approx (1 - |a|^2)^2$ for $a \in \mathbb{D}$.

2.2 The Definition of $\mathcal{Q}_K$

For a nondecreasing function $K : [0, \infty) \rightarrow [0, \infty)$, not identically zero, we define $\mathcal{Q}_K$ as the space of all functions $f \in H(\mathbb{D})$ for which

$$\|f\|_{\mathcal{Q}_K}^2 = \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 K(g(z, a)) dA(z) < \infty, \quad (2.2)$$

where $dA(z)$ is the Euclidean area measure on $\mathbb{D}$ normalized so that $A(\mathbb{D}) = 1$, and

$$g(z, a) = -\log |\varphi_a(z)| = \log \left| \frac{1 - \bar{a}z}{a - z} \right|$$

is the Green function at a . It is elementary to check that $\|\cdot\|_{\mathcal{Q}_K}$ is a semi-norm and

$$\|f\|_{\mathcal{Q}_K}^2 = \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |(f \circ \varphi_a)'(z)|^2 K \left(\log \frac{1}{|z|} \right) dA(z). \quad (2.3)$$

Since every $\varphi \in \mathcal{M}$ can be represented as $\varphi(z) = e^{i\theta} \varphi_a(z)$, it follows from the particular form of the Green function and the rotational invariance of the area measure that

$$\|f\|_{\mathcal{Q}_K}^2 = \sup_{\varphi \in \mathcal{M}} \int_{\mathbb{D}} |f'(z)|^2 K \left(\log \frac{1}{|\varphi(z)|} \right) dA(z) \quad (2.4)$$

$$= \sup_{\varphi \in \mathcal{M}} \int_{\mathbb{D}} |(f \circ \varphi)'(z)|^2 K \left(\log \frac{1}{|z|} \right) dA(z). \quad (2.5)$$

As an immediate consequence of this, we see that $\mathcal{Q}_K$ is Möbius invariant, that is, for any $f \in \mathcal{Q}_K$ and $\varphi \in \mathcal{M}$, we have $f \circ \varphi \in \mathcal{Q}_K$ and $\|f \circ \varphi\|_{\mathcal{Q}_K} = \|f\|_{\mathcal{Q}_K}$.

Our first result determines exactly when the space $\mathcal{Q}_K$ is nontrivial.

Theorem 2.7 *For any nondecreasing function $K : [0, \infty) \rightarrow [0, \infty)$, the following conditions are equivalent:*

- (a) *The space $\mathcal{Q}_K$ contains a nonconstant function.*
- (b) *The space $\mathcal{Q}_K$ contains all polynomials.*
- (c) *The condition*

$$\int_{\mathbb{D}} K \left(\log \frac{1}{|z|} \right) dA(z) = 2 \int_0^1 K \left(\log \frac{1}{r} \right) r dr < \infty \quad (2.6)$$

holds.

- (d) *The function*

$$F(a) = \int_{\mathbb{D}} K \left(\log \frac{1}{|\varphi_a(z)|} \right) dA(z)$$

is bounded on $\mathbb{D}$.

Proof It is obvious that condition (d) implies (c), because

$$F(0) = \int_{\mathbb{D}} K\left(\log \frac{1}{|z|}\right) dA(z) = 2 \int_0^1 K\left(\log \frac{1}{r}\right) r dr$$

by polar coordinates.

The function F is simply the so-called Berezin transform of the function

$$h(z) = K(-\log |z|), \quad z \in \mathbb{D}.$$

Since $K(t)$ is nondecreasing on $[0, \infty)$, we see that $h(z)$ has a finite limit as z approaches the unit circle. It follows from general properties of the Berezin transform (see, e.g., [Zhu1]) that F is bounded and continuously extendable to the closed unit disk.

To avoid the notion of the Berezin transform and to directly prove that condition (c) implies (d), we make a change of variables to get

$$F(a) = \int_{\mathbb{D}} K\left(\log \frac{1}{|z|}\right) \frac{(1 - |a|^2)^2}{|1 - \bar{a}z|^4} dA(z).$$

Write $F(a) = F_1(a) + F_2(a)$, where $F_1(a)$ is the integral over $|z| < 1/2$ and $F_2(a)$ is the integral over $1/2 < |z| < 1$. Condition (c) implies that $F_1(a)$ belongs to $\mathbb{C}_0(\mathbb{D})$, the space of continuous functions on $\mathbb{D}$ that vanish on the unit circle. In particular, there exists a positive constant C such that $0 \leq F_1(a) \leq C$ for all $a \in \mathbb{D}$. For $1/2 < |z| < 1$, the nonnegative function $K(-\log |z|)$ is bounded by $K(\log 2)$. Therefore,

$$0 \leq F_2(a) \leq K(\log 2) \int_{\mathbb{D}} \frac{(1 - |a|^2)^2}{|1 - \bar{a}z|^4} dA(z) = K(\log 2)$$

for all $a \in \mathbb{D}$. This shows that condition (c) implies (d), so (c) and (d) are equivalent.

It is trivial that condition (b) implies (a).

If condition (d) holds, then every function whose derivative is bounded on $\mathbb{D}$ belongs to $\mathcal{Q}_K$. In particular, every polynomial belongs to $\mathcal{Q}_K$. Thus, condition (d) implies (b).

Finally, if condition (a) holds, then there exists a nonconstant function $f \in \mathcal{Q}_K$. For any $a \in \mathbb{D}$, we consider the integral

$$\begin{aligned} I(a) &= \int_{\mathbb{D}} |f'(z)|^2 K(g(a, z)) dA(z) \\ &= \int_{\mathbb{D}} |(f \circ \varphi_a)'(z)|^2 K\left(\log \frac{1}{|z|}\right) dA(z). \end{aligned}$$

Since f is not constant, we can find some $a \in \mathbb{D}$ such that the function

$$g(z) = (f \circ \varphi_a)'(z) = -f'(\varphi_a(z)) \frac{1 - |a|^2}{(1 - \bar{a}z)^2}$$

has the property that

$$g(0) = -(1 - |a|^2)f'(a) \neq 0.$$

With this choice of a , we have

$$\begin{aligned} I(a) &= \int_{\mathbb{D}} |g(z)|^2 K \left(\log \frac{1}{|z|} \right) dA(z) \\ &= \frac{1}{\pi} \int_0^1 K \left(\log \frac{1}{r} \right) r dr \int_0^{2\pi} |g(re^{i\theta})|^2 d\theta \\ &\geq 2|g(0)|^2 \int_0^1 K \left(\log \frac{1}{r} \right) r dr. \end{aligned}$$

This proves that condition (a) implies (c), which completes the proof of the theorem.

We see that when $\mathcal{Q}_K$ is nontrivial, it must contain all the polynomials. Combining this with Theorem 1.7, we see that when $\mathcal{Q}_K$ is nontrivial, it must contain the Besov space B_1 . Note that the action of the circle group on $\mathcal{Q}_K$ is not necessarily continuous, although for any $f \in \mathcal{Q}_K$ and $0 < r < 1$, the mapping $e^{i\theta} \mapsto f(re^{i\theta})$ is continuous from the unit circle into $\mathcal{Q}_K$.

We now look at the other extreme of the spectrum: how large can a $\mathcal{Q}_K$ space be? Recall from Theorem 1.13 that any Möbius invariant function space possessing a decent linear functional must be contained in the Bloch space. We will show that each $\mathcal{Q}_K$ space has plenty of decent linear functionals. In particular, each point evaluation of the derivative is a bounded linear functional. Note that the only assumption needed here is that K is not identically zero.

Lemma 2.8 *There exists a positive constant $C = C_K$ such that $|f'(0)| \leq C\|f\|_{\mathcal{Q}_K}$ for all $f \in \mathcal{Q}_K$.*

Proof It is clear that

$$\begin{aligned} \|f\|_{\mathcal{Q}_K}^2 &\geq \int_{\mathbb{D}} |f'(z)|^2 K(g(z, 0)) dA(z) \\ &= \int_{\mathbb{D}} |f'(z)|^2 K \left(\log \frac{1}{|z|} \right) dA(z). \end{aligned}$$

Since K is nondecreasing and not identically zero, there exists some $r \in (0, 1)$ such that $c = K(-\log r) > 0$. We then have

$$\|f\|_{\mathcal{Q}_K}^2 \geq c \int_{|z| \leq r} |f'(z)|^2 dA(z).$$

The desired result then follows from polar coordinates and the subharmonicity of $|f'(z)|^2$.

As a consequence of the lemma above, we now prove the completeness of the semi-norm $\|\cdot\|_{\mathcal{Q}_K}$, so that each space $\mathcal{Q}_K$ is a Banach space.

Lemma 2.9 *The space $\mathcal{Q}_K$ is a Banach space with the norm*

$$\|f\|_K = |f(0)| + \|f\|_{\mathcal{Q}_K}.$$

Proof It is easy to see that $\|\cdot\|_{\mathcal{Q}_K}$ is a semi-norm and $\|\cdot\|_K$ is a norm on $\mathcal{Q}_K$. To prove the completeness of $\|\cdot\|_K$, assume that $\{f_n\}$ is a Cauchy sequence in $\mathcal{Q}_K$. Then $\|f_n - f_m\|_{\mathcal{Q}_K} \rightarrow 0$ and $|f_n(0) - f_m(0)| \rightarrow 0$ as $n, m \rightarrow \infty$. In particular, $\{f_n(0)\}$ is a convergent sequence. Since

$$\|f_n - f_m\|_{\mathcal{Q}_K} = \|(f_n - f_m) \circ \varphi_z\|_{\mathcal{Q}_K}$$

for all $z \in \mathbb{D}$, an application of Lemma 2.8 shows that there exists a positive constant C such that

$$(1 - |z|^2)|f'_n(z) - f'_m(z)| = |(f_n \circ \varphi_z - f_m \circ \varphi_z)'(0)| \leq C\|f_n - f_m\|_{\mathcal{Q}_K}.$$

This shows that $\{f'_n(z)\}$ converges uniformly on compact subsets to an analytic function g on $\mathbb{D}$. This together with the convergence of $\{f_n(0)\}$ shows that $\{f_n(z)\}$ converges uniformly on compact subsets to an analytic function f on $\mathbb{D}$ with $f' = g$.

Given $\varepsilon > 0$, we choose a positive integer N such that

$$\begin{aligned} \varepsilon &> \|f_n - f_m\|_K \\ &= |f_n(0) - f_m(0)| + \left[\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'_n(z) - f'_m(z)|^2 K(g(z, a)) dA(z) \right]^{\frac{1}{2}} \\ &\geq |f_n(0) - f_m(0)| + \left[\int_{\mathbb{D}} |f'_n(z) - f'_m(z)|^2 K(g(z, a)) dA(z) \right]^{\frac{1}{2}} \end{aligned}$$

for all $n, m > N$ and $a \in \mathbb{D}$. Let $m \rightarrow \infty$ and apply Fatou's lemma. We obtain

$$|f_n(0) - f(0)| + \left[\int_{\mathbb{D}} |f'_n(z) - f'(z)|^2 K(g(z, a)) dA(z) \right]^{\frac{1}{2}} \leq \varepsilon$$

for all $n > N$. Since $a \in \mathbb{D}$ is arbitrary, this shows that $f \in \mathcal{Q}_K$ and $\|f_n - f\|_K \leq \varepsilon$ for all $n > N$. Since ε is arbitrary, we conclude that $\|f_n - f\|_K \rightarrow 0$ as $n \rightarrow \infty$, so that the space $\mathcal{Q}_K$ is complete.

As a consequence of Theorem 1.13 and Lemma 2.8, we obtain the following.

Corollary 2.10 *Each space $\mathcal{Q}_K$ is boundedly contained in the Bloch space.*

It is natural to ask the question of exactly when $\mathcal{Q}_K$ is the Bloch space. The answer is provided by the next result.

Theorem 2.11 $\mathcal{Q}_K = \mathcal{B}$ if and only if

$$\int_0^1 K(\log(1/r))(1-r^2)^{-2} r dr < \infty. \quad (2.7)$$

Proof First assume that (2.7) holds. From Corollary 2.10, we always have $\mathcal{Q}_K \subset \mathcal{B}$. To prove that $\mathcal{B} \subset \mathcal{Q}_K$, we assume $f \in \mathcal{B}$. Since $(1 - |z|^2)^{-2} dA(z)$ is Möbius invariant, we have

$$\begin{aligned} \int_{\mathbb{D}} |f'(z)|^2 K(g(z, a)) dA(z) &\leq \|f\|_{\mathcal{B}}^2 \int_{\mathbb{D}} (1 - |z|^2)^{-2} K(g(z, a)) dA(z) \\ &= \|f\|_{\mathcal{B}}^2 \int_{\mathbb{D}} (1 - |z|^2)^{-2} K(g(z, 0)) dA(z) \\ &= 2\|f\|_{\mathcal{B}}^2 \int_0^1 (1 - r^2)^{-2} K(\log(1/r)) r dr \\ &< \infty. \end{aligned}$$

Thus $f \in \mathcal{Q}_K$ and $\mathcal{Q}_K = \mathcal{B}$.

Conversely, we assume $\mathcal{Q}_K = \mathcal{B}$. To prove (2.7), we consider the lacunary series

$$f(z) = \sum_{k=1}^{\infty} z^{2^k},$$

which is well known to be a function in the Bloch space (see Theorem 2.24). For $0 < r < 1$, let us consider

$$L(r) = \int_0^{2\pi} |f'(re^{i\theta})|^2 d\theta = 2\pi \sum_{k=1}^{\infty} 2^{2k} r^{2^{k+1}-2}.$$

It is easy to check that

$$\log r \geq 2(r-1), \quad r \in \left[\frac{1}{2}, 1\right).$$

It follows that

$$\begin{aligned} \log r^{2^{k+1}-2} &\geq 2 \left(r^{2^{k+1}-2} - 1 \right) \\ &= 2(r-1) \left(r^{2^{k+1}-3} + \cdots + 1 \right) \end{aligned}$$

$$\begin{aligned}
&\geq 2(r-1)(2^{k+1}-2) \\
&= 2^{k+2}(r-1)(1-2^{-2k}) \\
&\geq 2^{k+2}(r-1)
\end{aligned}$$

for $1/2 \leq r < 1$. Exponentiating both sides, we obtain

$$r^{2^{k+1}-2} \geq \exp[2^{k+2}(r-1)], \quad r \in \left[\frac{1}{2}, 1\right).$$

Therefore,

$$L(r) \geq 2\pi \sum_{k=1}^{\infty} 2^{2k} e^{2^{k+2}(r-1)} = 2\pi \sum_{k=1}^{\infty} 2^{2k} e^{-4 \cdot 2^k(1-r)}.$$

With $t_k = 2^k(1-r)$, we have

$$L(r) \geq \frac{2\pi}{(1-r)^2} \sum_{k=1}^{\infty} t_k^2 e^{-4t_k}.$$

The function $h(t) = t^2 e^{-4t}$ is decreasing on $[1/2, \infty)$. For any $3/4 \leq r < 1$, we can find a positive integer k such that

$$\frac{1}{2} \leq 2^k(1-r) < 1.$$

With the help of this particular k , we obtain

$$L(r) \geq \frac{2\pi}{(1-r)^2} t_k^2 e^{-4t_k} \geq \frac{2\pi}{e^4(1-r)^2}$$

for all $3/4 \leq r < 1$.

The assumption $\mathcal{Q}_K = \mathcal{B}$ implies that the function f above belongs to $\mathcal{Q}_K$. Thus

$$\begin{aligned}
\infty &> \int_{\mathbb{D}} |f'(z)|^2 K(\log(1/|z|)) dA(z) \\
&= \frac{1}{\pi} \int_0^1 K(\log(1/r)) L(r) r dr \\
&\geq \frac{2}{e^4} \int_{\frac{3}{4}}^1 K(\log(1/r)) (1-r)^{-2} r dr.
\end{aligned}$$

On the other hand, it follows from Theorem 2.7 that

$$\int_0^{\frac{3}{4}} K(\log(1/r)) (1-r)^{-2} r dr < \infty.$$

This proves condition (2.7) and completes the proof of the theorem.

The function K will be called a weight. In the case when $K(t) = t^p$, $0 \leq p < \infty$, the space $\mathcal{Q}_K$ is usually written as $\mathcal{Q}_p$. We briefly mention several simple cases here.

When $K(t) = t$, the space $\mathcal{Q}_K$ coincides with BMOA. See Theorem 1.6.

We see from Corollary 2.10 and Theorem 2.11 that $\mathcal{Q}_p = \mathcal{B}$ when $p > 1$ and $\mathcal{Q}_p \subsetneq \mathcal{B}$ when $0 < p \leq 1$. In the case when $K(t) = 1$ is the constant function, it is clear that $\mathcal{Q}_K$ becomes the Dirichlet space $\mathcal{D}$.

The function theory of $\mathcal{Q}_K$ obviously depends on the properties of K . Given two weight functions K_1 and K_2 , we are going to write $K_1 \lesssim K_2$ if there exists a constant $C > 0$ such that $K_1(t) \leq CK_2(t)$ for all t . The notation $K_1 \gtrsim K_2$ is used in a similar fashion. When $K_1 \lesssim K_2 \lesssim K_1$, we write $K_1 \approx K_2$.

It is clear that $K_1 \lesssim K_2$ implies $\mathcal{Q}_{K_2} \subset \mathcal{Q}_{K_1}$. Consequently, K_1 and K_2 give rise to the same $\mathcal{Q}_K$ space whenever $K_1 \approx K_2$. The converse is false in general, as is demonstrated by the fact that $\mathcal{Q}_p$ equals the Bloch space for all $p > 1$.

2.3 The Subspace $\mathcal{Q}_{K,0}$

The space $\mathcal{Q}_K$ is usually not separable. However, it always contains the following very important separable subspace:

$$\mathcal{Q}_{K,0} = \left\{ f \in H(\mathbb{D}) : \lim_{|a| \rightarrow 1^-} \int_{\mathbb{D}} |f'(z)|^2 K(g(z, a)) dA(z) = 0 \right\}.$$

Recall that the little Bloch space $\mathcal{B}_0$ consists of functions $f \in H(\mathbb{D})$ such that

$$\lim_{|z| \rightarrow 1^-} |f'(z)|(1 - |z|^2) = 0.$$

In many situations, the subspace $\mathcal{Q}_{K,0}$ is simply the closure in $\mathcal{Q}_K$ of the set of polynomials. Examples include BMOA and the Bloch space. There are also cases when the polynomials are dense in $\mathcal{Q}_K$. For example, if $K = 1$, then $\mathcal{Q}_K$ is the Dirichlet space; in this case, the polynomials are dense, but $\mathcal{Q}_{K,0}$ as defined at the beginning of this section consists of just the constant functions.

Theorem 2.12 *We always have $\mathcal{Q}_{K,0} \subset \mathcal{B}_0$.*

Proof Since K is not identically zero, we can find some $t_0 > 0$ such that $K(t_0) > 0$. With $r = e^{-t_0}$, we have

$$K(t_0) \int_{D(a,r)} |f'(z)|^2 dA(z) \leq \int_{\mathbb{D}} |f'(z)|^2 K(g(z, a)) dA(z)$$

for all $a \in \mathbb{D}$. It follows that

$$\lim_{|a| \rightarrow 1^-} \int_{D(a,r)} |f'(z)|^2 dA(z) = 0$$

for all $f \in \mathcal{Q}_{K,0}$. By Lemma 2.6 and the fact that $|D(a, r)| \approx (1 - |a|^2)^2$, we can find a positive constant $C = C_r$ such that

$$(1 - |a|^2)^2 |f'(a)|^2 \leq C \int_{D(a,r)} |f'(z)|^2 dA(z) \quad (2.8)$$

for all $a \in \mathbb{D}$. This shows that every function in $\mathcal{Q}_{K,0}$ belongs to $\mathcal{B}_0$.

Lemma 2.13 *Let K satisfy (2.6) and f be an analytic function on $\mathbb{D}$. Then the following conditions are equivalent:*

- (i) f belongs to $\mathcal{B}_0$.
- (ii) For any $0 < r < 1$, we have

$$\lim_{|a| \rightarrow 1^-} \int_{D(a,r)} |f'(z)|^2 K(g(z, a)) dA(z) = 0.$$

- (iii) There exists an r , $0 < r < 1$, such that

$$\lim_{|a| \rightarrow 1^-} \int_{D(a,r)} |f'(z)|^2 K(g(z, a)) dA(z) = 0.$$

Proof For any $r \in (0, 1)$, we have

$$\begin{aligned} & \int_{D(a,r)} |f'(z)|^2 K(g(z, a)) dA(z) \\ & \leq 2 \sup_{z \in D(a,r)} \{(1 - |z|^2)^2 |f'(z)|^2\} \int_0^r \rho(1 - \rho^2)^{-2} K(\log(1/\rho)) d\rho. \end{aligned}$$

By (2.6), the integral above converges for any fixed $r \in (0, 1)$. Since $D(a, r)$ collapses to the unit circle as $|a| \rightarrow 1^-$, we see that condition (i) implies (ii).

It is obvious that condition (ii) implies (iii).

To prove that condition (iii) implies (i), we remark that if condition (iii) holds for some $r_0 \in (0, 1)$, then it also holds for any $r \in (0, r_0)$. Therefore, since K is nondecreasing and not identically zero, we may assume that the r in condition (iii) is small enough such that

$$K(g(z, a)) = K\left(\log \frac{1}{|\varphi_a(z)|}\right) \geq K\left(\log \frac{1}{r}\right) > 0$$

for all $z \in D(a, r)$. In this case, if the limit of the integral in (iii) is zero, we will have

$$\lim_{|a| \rightarrow 1^-} \int_{D(a, r)} |f'(z)|^2 dA(z) = 0.$$

It follows from (2.8) that f belongs to the little Bloch space.

Theorem 2.14 $\mathcal{Q}_{K,0} = \mathcal{B}_0$ if and only if (2.7) holds.

Proof Let us first assume that (2.7) holds. By Theorem 2.12, it suffices to prove that $\mathcal{B}_0 \subset \mathcal{Q}_{K,0}$. Suppose that $f \in \mathcal{B}_0$. Since (2.7) holds, for any $\varepsilon > 0$, there exists an $r \in (0, 1)$ such that

$$\int_r^1 K(\log(1/\rho))(1 - \rho^2)^{-2} \rho d\rho < \varepsilon.$$

Then

$$\begin{aligned} & \int_{\mathbb{D} \setminus D(a, r)} |f'(z)|^2 K(g(z, a)) dA(z) \\ & \leq \|f\|_{\mathcal{B}}^2 \int_{\mathbb{D} \setminus D(a, r)} (1 - |z|^2)^{-2} K(g(z, a)) dA(z) \\ & = 2\|f\|_{\mathcal{B}}^2 \int_r^1 K(\log(1/\rho))(1 - \rho^2)^{-2} \rho d\rho \\ & < 2\|f\|_{\mathcal{B}}^2 \varepsilon. \end{aligned}$$

Since $f \in \mathcal{B}_0$, we have

$$|f'(\varphi_a(w))|(1 - |\varphi_a(w)|^2) \rightarrow 0$$

uniformly for $|w| < r$ as $|a| \rightarrow 1^-$. Making a change of variables $z = \varphi_a(w)$ and using condition (2.7), we see that there is a positive constant C such that

$$\begin{aligned} & \int_{D(a, r)} |f'(z)|^2 K(g(z, a)) dA(z) \\ & = \int_{|w| < r} |f'(\varphi_a(w))|^2 (1 - |\varphi_a(w)|^2)^2 (1 - |w|^2)^{-2} K(\log(1/|w|)) dA(w) \\ & \leq 2\pi \sup_{|w| < r} |f'(\varphi_a(w))|^2 (1 - |\varphi_a(w)|^2)^2 \int_0^1 K(\log(1/\rho))(1 - \rho^2)^{-2} \rho d\rho \\ & \leq C \sup_{|w| < r} |f'(\varphi_a(w))|^2 (1 - |\varphi_a(w)|^2)^2. \end{aligned}$$

Combining the estimates above, we get

$$\lim_{|a| \rightarrow 1^-} \int_{\mathbb{D}} |f'(z)|^2 K(g(z, a)) dA(z) = 0,$$

which shows that $f \in \mathcal{Q}_{K,0}$. Thus (2.7) is a sufficient condition for $\mathcal{Q}_{K,0} = \mathcal{B}_0$ to hold.

If (2.7) does not hold, we can find a continuous, strictly decreasing function $h : [0, 1) \rightarrow (0, 1]$ tending to zero at 1 such that

$$\int_0^1 h(r) K(\log(1/r)) (1-r^2)^{-2} r dr = \infty.$$

Given $r \in [3/4, 1)$, we find an integer k such that

$$\frac{1}{2} \leq 2^k (1-r) < 1.$$

Rewrite the second inequality above as $1 - 2^{-k} < r$, set

$$a_k = [h(1 - 2^{-k})]^{\frac{1}{2}},$$

and consider the lacunary series

$$f(z) = \sum_{k=1}^{\infty} a_k z^{2^k}.$$

Since $a_k \rightarrow 0$ as $k \rightarrow \infty$, we have $f \in \mathcal{B}_0$ (see Theorem 2.24). Write

$$L_0(r) = \int_0^{2\pi} |f'(re^{i\theta})|^2 d\theta = 2\pi \sum_{k=1}^{\infty} a_k^2 2^{2k} r^{2^{k+1}-2}$$

and argue in the same way as in the second half of the proof of Theorem 2.11. We deduce that

$$L_0(r) \geq 2\pi h(r) (1-r)^{-2} e^{-4} = Ch(r) (1-r)^{-2}, \quad r \in [3/4, 1).$$

It follows that

$$\int_{\mathbb{D}} |f'(z)|^2 K(\log(1/|z|)) dA(z) \geq C \int_{\frac{3}{4}}^1 h(r) K(\log(1/r)) (1-r^2)^{-2} r dr = \infty.$$

This shows $f \in \mathcal{B}_0 \setminus \mathcal{Q}_{K,0}$ and finishes the proof of the theorem.

Recall from Chap. 1 that the Bloch space $\mathcal{B}$ is the largest Möbius invariant function space and the diagonal Besov space B_1 is the smallest. In the context of $\mathcal{Q}_K$ spaces, the result below shows that the smallest natural space is the Dirichlet space $\mathcal{D}$.

Theorem 2.15 *Suppose K is right continuous at 0 and satisfies condition (2.6). Then the Dirichlet space $\mathcal{D}$ is contained in $\mathcal{Q}_{K,0}$ if and only if $K(0) = 0$. Moreover, if $K(0) > 0$, then $\mathcal{Q}_K = \mathcal{D}$ and $\mathcal{Q}_{K,0}$ is just the set of constant functions.*

Proof Let us first assume that $K(0) = 0$ and that $f \in \mathcal{D}$. For $0 < r < 1$, we have

$$\begin{aligned} \int_{\mathbb{D} \setminus D(a,r)} |f'(z)|^2 K(g(z,a)) dA(z) &\leq K(\log(1/r)) \int_{\mathbb{D} \setminus D(a,r)} |f'(z)|^2 dA(z) \\ &\leq \|f\|_{\mathcal{D}}^2 K(\log(1/r)). \end{aligned}$$

Given any $\varepsilon > 0$, it follows from the right continuity of K and $K(0) = 0$ that we can choose an $r_0 \in (0, 1)$ such that

$$\int_{\mathbb{D} \setminus D(a,r_0)} |f'(z)|^2 K(g(z,a)) dA(z) \leq \|f\|_{\mathcal{D}}^2 K(\log(1/r_0)) < \varepsilon$$

for all $a \in \mathbb{D}$.

It is well known that $f \in \mathcal{B}_0$ if $f \in \mathcal{D}$. Thus we can find some $\delta(\varepsilon) > 0$ such that

$$|f'(z)|(1 - |z|^2) < \varepsilon, \quad 1 - |z| < \delta(\varepsilon).$$

For $|a|$ close to 1, we deduce that

$$\begin{aligned} \int_{D(a,r_0)} |f'(z)|^2 K(g(z,a)) dA(z) &\leq \varepsilon^2 \int_{D(a,r_0)} K(g(z,a))(1 - |z|^2)^{-2} dA(z) \\ &= \varepsilon^2 \int_{|w| < r_0} K(\log(1/|w|))(1 - |w|^2)^{-2} dA(w) \\ &= C\varepsilon^2. \end{aligned}$$

Condition (2.6) ensures that the constant C above is finite. This shows that

$$\lim_{|a| \rightarrow 1} \int_{\mathbb{D}} |f'(z)|^2 K(g(z,a)) dA(z) = 0,$$

or $f \in \mathcal{Q}_{K,0}$. Therefore, the condition $K(0) = 0$ implies that $\mathcal{D}$ is contained in $\mathcal{Q}_{K,0}$. Note that the arguments above also show that $\mathcal{D}$ is contained in $\mathcal{Q}_K$, whether $K(0) = 0$ or not.

For the remainder of the proof, we assume that $K(0) > 0$.

Consider the weight function

$$K_1(t) = \inf\{K(t), K(1)\} = \begin{cases} K(t), & 0 \leq t < 1, \\ K(1), & t \geq 1. \end{cases}$$

We have $K_1 \leq K$, so $\mathcal{Q}_K \subset \mathcal{Q}_{K_1}$. But the function K_1 is bounded below by the positive number $K(0)$ and bounded above by the positive number $K(1)$. It follows easily that $\mathcal{Q}_{K_1} = \mathcal{D}$ and $\mathcal{Q}_{K_1,0}$ consists of just the constant functions. Thus the condition $K(0) > 0$ implies that $\mathcal{Q}_{K,0}$ consists of just the constant functions and

$$\mathcal{Q}_K \subset \mathcal{Q}_{K_1} = \mathcal{D} \subset \mathcal{Q}_K,$$

which gives $\mathcal{Q}_K = \mathcal{D}$. In particular, the condition $K(0) > 0$ implies that $\mathcal{D}$ is not contained in $\mathcal{Q}_{K,0}$. This completes the proof of the theorem.

Corollary 2.16 *We have $\mathcal{D} \subset \mathcal{Q}_{p,0}$ for all p , $0 < p < \infty$. Here $\mathcal{Q}_{p,0}$ denotes the space $\mathcal{Q}_{K,0}$ for $K(t) = t^p$.*

2.4 Comparing Two $\mathcal{Q}_K$ Spaces

We first show that the space $\mathcal{Q}_K$ only depends on the behavior of the function $K(t)$ for small t . In other words, we can alter the values of $K(t)$ for large t , and the space $\mathcal{Q}_K$ will remain the same.

Theorem 2.17 *Assume $K(t_0) > 0$ and K satisfies condition (2.6). Define*

$$K_1(t) = \inf\{K(t), K(t_0)\} = \begin{cases} K(t), & 0 \leq t < t_0, \\ K(t_0), & t \geq t_0. \end{cases}$$

Then $\mathcal{Q}_K = \mathcal{Q}_{K_1}$ and $\mathcal{Q}_{K,0} = \mathcal{Q}_{K_1,0}$.

Proof For convenience and transparency, we prove the result for $t_0 = 1$. The general case is handled in exactly the same way.

It is easy to see that the truncated function K_1 is still nondecreasing. It is clear that $K_1 \leq K$, so we must have $\mathcal{Q}_K \subset \mathcal{Q}_{K_1}$ and $\mathcal{Q}_{K,0} \subset \mathcal{Q}_{K_1,0}$.

To prove the inclusions $\mathcal{Q}_{K_1} \subset \mathcal{Q}_K$ and $\mathcal{Q}_{K_1,0} \subset \mathcal{Q}_{K,0}$, we note that

$$g(z, a) > 1, \quad z \in D(a, 1/e),$$

and

$$g(z, a) \leq 1, \quad z \in \mathbb{D} \setminus D(a, 1/e).$$

It follows that

$$K(g(z, a)) = K_1(g(z, a)), \quad z \in \mathbb{D} \setminus D(a, 1/e).$$

So it suffices to deal with integrals over $D(a, 1/e)$. If $f \in \mathcal{Q}_{K_1}$, then f is a Bloch function (see Corollary 2.10) and

$$\begin{aligned} \int_{D(a, 1/e)} |f'(z)|^2 K(g(z, a)) dA(z) &\leq \|f\|_{\mathcal{B}}^2 \int_{D(a, 1/e)} (1 - |z|^2)^{-2} K(g(z, a)) dA(z) \\ &= 2\|f\|_{\mathcal{B}}^2 \int_0^{1/e} r(1 - r^2)^{-2} K(\log(1/r)) dr. \end{aligned}$$

Condition (2.6) ensures that the integral above converges to a constant that is independent of a . This completes the proof of the theorem.

Therefore, when studying $\mathcal{Q}_K$ spaces, we can always assume that $K(t)$ is eventually a positive constant. We will also assume that K is always right continuous.

We say that K satisfies the doubling condition if there exist constants C and M such that

$$K(t) \leq K(2t) \leq CK(t), \quad 0 < t \leq M. \quad (2.9)$$

The doubling condition is essentially a condition about the behavior of K near $t = 0$. In fact, if $K(t) > 0$ for all $t > 0$, then it is easy to check that K satisfies the doubling condition if and only if

$$\limsup_{t \rightarrow 0^+} \frac{K(2t)}{K(t)} < \infty.$$

This is because for $\delta \leq t \leq M$, we always have

$$\frac{K(2t)}{K(t)} \leq \frac{K(2M)}{K(\delta)} < \infty.$$

In particular, if the function $K(t)$ has a one-sided derivative $K'(0^+) \in (0, \infty)$, then K satisfies the doubling condition. Also, for every $0 < p < \infty$, the function $K(t) = t^p$ satisfies the doubling condition.

Our next result shows that the logarithmic Green function used in the definition of $\mathcal{Q}_K$ can be replaced by a more transparent rational function. There are situations in which the Green function is more useful, and there are situations in which this new rational form is more convenient to use.

Theorem 2.18 *Suppose K satisfies (2.6) and (2.9). Then a function $f \in H(\mathbb{D})$ belongs to $\mathcal{Q}_K$ if and only if*

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 K(1 - |\varphi_a(z)|^2) dA(z) < \infty. \quad (2.10)$$

Proof It follows from the elementary inequality

$$1 - x \leq \log \frac{1}{x}, \quad 0 < x < 1,$$

that

$$K(1 - |\varphi_a(z)|^2) \leq K(2g(z, a))$$

for all a and z in $\mathbb{D}$. This together with the doubling condition (2.9) shows that every function $f \in \mathcal{Q}_K$ satisfies the condition in (2.10).

Note that condition (2.10) is equivalent to

$$\sup_{\varphi \in \mathcal{M}} \int_{\mathbb{D}} |f'(z)|^2 K(1 - |\varphi(z)|^2) dA(z) < \infty, \quad (2.11)$$

which is the same as

$$\sup_{\varphi \in \mathcal{M}} \int_{\mathbb{D}} |(f \circ \varphi)'(z)|^2 K(1 - |z|^2) dA(z) < \infty. \quad (2.12)$$

Thus, condition (2.10) defines a Möbius invariant function space as well. In particular, if f satisfies (2.10), then

$$\int_{\mathbb{D}} |f'(z)|^2 K(1 - |z|^2) dA(z) < \infty.$$

By polar coordinates and the sub-mean-value property, there is another positive constant C such that

$$|f'(0)|^2 \leq C \sup_{\varphi \in \mathcal{M}} \int_{\mathbb{D}} |f'(z)|^2 K(1 - |\varphi(z)|^2) dA(z).$$

Replacing f by $f \circ \varphi_a$, we conclude that

$$(1 - |a|^2)^2 |f'(a)|^2 \leq C \sup_{\varphi \in \mathcal{M}} \int_{\mathbb{D}} |f'(z)|^2 K(1 - |\varphi(z)|^2) dA(z)$$

for all $a \in \mathbb{D}$. Thus, condition (2.10) implies that f belongs to the Bloch space.

To prove the other implication, we may as well assume that $K(t) = K(1)$ for all $t > 1$. See Theorem 2.17. In this case, we write the integral

$$I(a) = \int_{\mathbb{D}} |f'(z)|^2 K(g(z, a)) dA(z)$$

as $I_1(a) + I_2(a)$, where

$$\begin{aligned}
 I_1(a) &= \int_{|g(z,a)|>1} |f'(z)|^2 K(g(z,a)) dA(z) \\
 &= K(1) \int_{|g(z,a)|>1} |f'(z)|^2 dA(z) \\
 &= K(1) \int_{|z|<e^{-1}} |f'(\varphi_a(z))|^2 \frac{(1-|a|^2)^2}{|1-\bar{a}z|^4} dA(z) \\
 &\leq C \int_{|z|<e^{-1}} (1-|\varphi_a(z)|^2)^2 |f'(\varphi_a(z))|^2 dA(z).
 \end{aligned}$$

Since f is a Bloch function, there exists a constant $C > 0$ such that $I_1(a) \leq C$ for all $a \in \mathbb{D}$.

On the other hand, it is elementary to check that

$$1 - x^2 \geq \frac{1}{2^2} \log \frac{1}{x}, \quad \frac{1}{e} < x < 1.$$

It follows from the doubling condition (2.9) that

$$\begin{aligned}
 I_2(a) &= \int_{|g(z,a)| \leq 1} |f'(z)|^2 K(g(z,a)) dA(z) \\
 &\leq C \int_{|g(z,a)| \leq 1} |f'(z)|^2 K\left(\frac{1}{2^2} \log \frac{1}{|\varphi_a(z)|}\right) dA(z) \\
 &\leq C \int_{|g(z,a)| \leq 1} |f'(z)|^2 K(1 - |\varphi_a(z)|^2) dA(z) \\
 &\leq C \int_{\mathbb{D}} |f'(z)|^2 K(1 - |\varphi_a(z)|^2) dA(z).
 \end{aligned}$$

So there exists another constant C such that $I_2(a) \leq C$ for all $a \in \mathbb{D}$. Combining this with the conclusion of the previous paragraph, we conclude that condition (2.10) implies that $f \in \mathcal{Q}_K$.

Our next result gives a sufficient condition for two $\mathcal{Q}_K$ spaces to be different, which is useful in the construction of many examples and counterexamples.

Theorem 2.19 *Suppose $K_1 \lesssim K_2$ in $(0, t_0)$ for some $t_0 > 0$ and $K_1(r)/K_2(r) \rightarrow 0$ as $r \rightarrow 0$. If (2.6) holds for both K_1 and K_2 and if the integral in (2.7) is divergent for K_2 , then $\mathcal{Q}_{K_2} \subsetneq \mathcal{Q}_{K_1}$.*

Proof Without loss of generality, we assume that $t_0 = 1$. Recall from Theorem 2.17 that $K_1 \lesssim K_2$ in $(0, 1)$ implies $\mathcal{Q}_{K_2} \subset \mathcal{Q}_{K_1}$, and the inclusion is continuous. We assume $\mathcal{Q}_{K_2} = \mathcal{Q}_{K_1} = \mathcal{Q}$ and show that this will lead to a contradiction.

By the open mapping theorem, the identity map from $\mathcal{Q}_{K_2}$ to $\mathcal{Q}_{K_1}$ is continuous and has a continuous inverse. Thus there exists a constant C such that $\|f\|_{K_2} \leq C\|f\|_{K_1}$ for all $f \in \mathcal{Q}$. Since $K_1(t)/K_2(t) \rightarrow 0$ as $t \rightarrow 0^+$, we can find a positive s_0 such that

$$K_1(t) \leq (2C)^{-1}K_2(t), \quad 0 < t \leq s_0.$$

For $r_0 = e^{-s_0}$ and $f \in \mathcal{Q}$, we have

$$\begin{aligned} \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 K_2(g(z, a)) dA(z) &\leq C \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 K_1(g(z, a)) dA(z) \\ &\leq C \sup_{a \in \mathbb{D}} \int_{D(a, r_0)} |f'(z)|^2 K_1(g(z, a)) dA(z) \\ &\quad + C \sup_{a \in \mathbb{D}} \int_{\mathbb{D} \setminus D(a, r_0)} |f'(z)|^2 K_1(g(z, a)) dA(z) \\ &\leq C \sup_{a \in \mathbb{D}} \int_{D(a, r_0)} |f'(z)|^2 K_1(g(z, a)) dA(z) \\ &\quad + \frac{C}{2} \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 K_2(g(z, a)) dA(z). \end{aligned}$$

Consequently,

$$\int_{\mathbb{D}} |f'(z)|^2 K_2(g(z, a)) dA(z) \leq 2C \sup_{a \in \mathbb{D}} \int_{D(a, r_0)} |f'(z)|^2 K_1(g(z, a)) dA(z).$$

Since $\mathcal{Q} \subset \mathcal{B}$, we must have $f \in \mathcal{B}$, so the right-hand side of the inequality above is dominated by

$$C\|f\|_{\mathcal{B}}^2 \int_0^{r_0} t(1-t^2)^{-2} K_1(\log(1/t)) dt.$$

Thus there exists a constant C' such that

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 K_2(g(z, a)) dA(z) \leq C'\|f\|_{\mathcal{B}}^2, \quad f \in \mathcal{Q}. \quad (2.13)$$

For an arbitrary $h \in \mathcal{B}$, we consider the functions $h_r(z) = h(rz)$, $0 < r < 1$. We have $\|h_r\|_{\mathcal{B}} \leq \|h\|_{\mathcal{B}}$ and $h_r \in \mathcal{Q}$ for each $r \in (0, 1)$. Setting $f = h_r$ in (2.13) and using Fatou's lemma, we deduce that

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |h'(z)|^2 K_2(g(z, a)) dA(z) \leq C'\|h\|_{\mathcal{B}}^2$$

for every $h \in \mathcal{B}$. This proves $\mathcal{B} \subset \mathcal{Q}_{K_2}$. But $\mathcal{Q}_K$ is always contained in the Bloch space. So $\mathcal{Q}_{K_2} = \mathcal{B}$, which is a contradiction to Theorem 2.11.

As consequences of Theorem 2.19 above, we obtain precise containment relations among several distinguished $\mathcal{Q}_K$ spaces.

Corollary 2.20 *Suppose K satisfies condition (2.6). We have*

- (i) $\mathcal{Q}_p \subsetneq \mathcal{Q}_q$ whenever $0 \leq p < q \leq 1$.
- (ii) $\mathcal{D} \subset \mathcal{Q}_K$; and $\mathcal{D} = \mathcal{Q}_K$ if and only if $K(0) > 0$.
- (iii) If

$$K_0(t) = \begin{cases} \frac{t}{|\log t|}, & 0 < t \leq 1/e, \\ t, & t > 1/e, \end{cases}$$

then $BMOA = \mathcal{Q}_1 \subsetneq \mathcal{Q}_{K_0} \subsetneq \mathcal{B}$.

Proof For (i) we choose $K_2(r) = r^p$, $K_1(r) = r^q$, and apply Theorem 2.19. Part (ii) follows from Theorem 2.15.

To prove (iii), we choose $K_1(t) = K_0(t)$, $K_2(t) = t$, and apply Theorem 2.19. The result is that $BMOA \subsetneq \mathcal{Q}_{K_0}$. Recall that we always have $\mathcal{Q}_{K_0} \subset \mathcal{B}$. With K_0 as above, the integral in (2.7) is divergent. So it follows from Theorem 2.11 that $\mathcal{Q}_{K_0} \subsetneq \mathcal{B}$.

2.5 Examples of Functions in $\mathcal{Q}_K$

In this section, we consider some examples of functions in $\mathcal{Q}_K$ spaces. We begin with the membership in $\mathcal{Q}_K$ of the classical borderline example $f(z) = \log(1 - z)$.

Theorem 2.21 *The function $\log(1 - z)$ is in $\mathcal{Q}_K$ if and only if*

$$\int_0^1 (1 - r^2)^{-1} K(\log(1/r)) r dr < \infty. \quad (2.14)$$

Proof By definition, the function $\log(1 - z)$ belongs to $\mathcal{Q}_K$ if and only if there exists a positive constant C such that $I(a) \leq C < \infty$ for all $a \in \mathbb{D}$, where

$$I(a) = \int_{\mathbb{D}} \frac{1}{|1 - z|^2} K\left(\log \frac{1}{|\varphi_a(z)|}\right) dA(z).$$

In particular, membership of $\log(1 - z)$ in $\mathcal{Q}_K$ implies that $I(0) < \infty$. By polar coordinates, we have

$$I(0) = \frac{1}{\pi} \int_0^1 K\left(\log \frac{1}{r}\right) r dr \int_0^{2\pi} \frac{d\theta}{|1 - re^{i\theta}|^2} = 2 \int_0^1 K\left(\log \frac{1}{r}\right) \frac{r dr}{1 - r^2}.$$

So condition (2.14) holds whenever the function $\log(1 - z)$ is in $\mathcal{Q}_K$.

To prove the other direction, we make a change of variables to obtain

$$I(a) = \int_{\mathbb{D}} K\left(\log \frac{1}{|z|}\right) \frac{1}{|1 - \varphi_a(z)|^2} \frac{(1 - |a|^2)^2}{|1 - \bar{a}z|^4} dA(z).$$

Note that

$$1 - \varphi_a(z) = \frac{1 - a + (1 - \bar{a})z}{1 - \bar{a}z} = (1 - a) \frac{1 + \frac{1-\bar{a}}{1-a}z}{1 - \bar{a}z}.$$

We have

$$I(a) = \frac{(1 - |a|^2)^2}{|1 - a|^2} \int_{\mathbb{D}} K\left(\log \frac{1}{|z|}\right) \frac{dA(z)}{|1 + \frac{1-\bar{a}}{1-a}z|^2 |1 - \bar{a}z|^2}.$$

Since the number $(1 - \bar{a})/(1 - a)$ is unimodular and the area measure is rotation invariant, we also have

$$I(a) = \frac{(1 - |a|^2)^2}{|1 - a|^2} \int_{\mathbb{D}} K\left(\log \frac{1}{|z|}\right) \frac{dA(z)}{|1 + z|^2 |1 - \lambda z|^2},$$

where

$$\lambda = \frac{1 - a}{1 - \bar{a}} \bar{a}.$$

By partial fractions,

$$\frac{1}{(1 + z)(1 - \lambda z)} = \frac{1}{1 + \lambda} \left(\frac{1}{1 + z} + \frac{\lambda}{1 - \lambda z} \right).$$

Since

$$1 + \lambda = \frac{1 - |a|^2}{1 - \bar{a}},$$

we arrive at

$$I(a) = \int_{\mathbb{D}} K\left(\log \frac{1}{|z|}\right) \left| \frac{1}{1 + z} + \frac{\lambda}{1 - \lambda z} \right|^2 dA(z).$$

By the triangle inequality,

$$I(a) \leq 2 \int_{\mathbb{D}} K\left(\log \frac{1}{|z|}\right) \left(\frac{1}{|1 + z|^2} + \frac{|\lambda|^2}{|1 - \lambda z|^2} \right) dA(z).$$

It follows from polar coordinates and the identity $|\lambda| = |a|$ that

$$I(a) \leq 4 \int_0^1 K \left(\log \frac{1}{r} \right) \left(\frac{1}{1-r^2} + \frac{|a|^2}{1-|a|^2 r^2} \right) r dr.$$

Since $|a| < 1$ and $1 - |a|^2 r^2 \geq 1 - r^2$, we conclude that

$$I(a) \leq 8 \int_0^1 K \left(\log \frac{1}{r} \right) \frac{r dr}{1-r^2}$$

for all $a \in \mathbb{D}$. Thus, condition (2.14) implies that the function $\log(1-z)$ is in $\mathcal{Q}_K$, which completes the proof of the theorem.

Specializing to $\mathcal{Q}_p$ spaces, we have the following.

Corollary 2.22 *The function $\log(1-z)$ belongs to $\mathcal{Q}_p$ for all $p \in (0, \infty)$. But it is not in the Dirichlet space $\mathcal{D}$.*

Corollary 2.23 *If (2.14) holds, then $\mathcal{Q}_{K,0} \subsetneq \mathcal{Q}_K$.*

Proof By Theorems 2.12 and 2.21, we have

$$\log(1-z) \in \mathcal{Q}_K \setminus \mathcal{B}_0 \subset \mathcal{Q}_K \setminus \mathcal{Q}_{K,0},$$

which proves the desired result.

In the rest of this section, we consider a family of function spaces defined by a growth condition of the derivative. They are similar to the Bloch space but are not Möbius invariant. Functions in these spaces can be used as examples in certain $\mathcal{Q}_K$ spaces, further revealing the size of these $\mathcal{Q}_K$ spaces.

For $\alpha \in (0, \infty)$ we denote by $\mathcal{B}^\alpha$ the space of all functions $f \in H(\mathbb{D})$ for which

$$\|f\|_{\mathcal{B}^\alpha} = |f(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |f'(z)| < \infty.$$

Similarly, the space $\mathcal{B}_0^\alpha$ is defined by

$$\mathcal{B}_0^\alpha = \left\{ f \in H(\mathbb{D}) : \lim_{|z| \rightarrow 1} (1 - |z|^2)^\alpha |f'(z)| = 0 \right\}.$$

It is easy to check that each $\mathcal{B}^\alpha$ is a Banach space and $\mathcal{B}_0^\alpha$ is a closed subspace of $\mathcal{B}^\alpha$.

When $\alpha = 1$, $\mathcal{B}^\alpha$ is simply the Bloch space. For this reason, we are going to call $\mathcal{B}^\alpha$ a Bloch-type space or the α -Bloch space. It is well known that if $\alpha \in (0, 1)$, then $\mathcal{B}^\alpha$ coincides with the classical Lipschitz space $\Lambda_{1-\alpha}$ consisting of functions $f \in H(\mathbb{D})$ satisfying the Lipschitz condition

$$|f(z) - f(w)| \leq C|z - w|^{1-\alpha}$$

for some $C > 0$ (depending on f) and all z and w in $\mathbb{D}$.

We will determine exactly when a $\mathcal{Q}_K$ space contains a $\mathcal{B}^\alpha$ space. Equivalently, we will obtain sufficient conditions for membership of f in $\mathcal{Q}_K$ in terms of an integral property of the weight function and a growth condition of f . Our first step is to construct several families of analytic functions with the prescribed boundary growth.

The following result gives a characterization of $f \in \mathcal{B}^\alpha$ for gap series or lacunary series. This provides a rich family of nontrivial examples of functions in the α -Bloch spaces and certain $\mathcal{Q}_K$ spaces. In a later chapter, under an additional mild condition on the weight, we will obtain a characterization of lacunary series in $\mathcal{Q}_K$.

Theorem 2.24 *Suppose $\alpha > 0$ and*

$$f(z) = \sum_{j=1}^{\infty} a_j z^{n_j}$$

is analytic in $\mathbb{D}$ with

$$\frac{n_{j+1}}{n_j} \geq \lambda > 1, \quad j = 1, 2, \dots.$$

Then $f \in \mathcal{B}^\alpha$ if and only if

$$\limsup_{j \rightarrow \infty} |a_j| n_j^{1-\alpha} < \infty. \quad (2.15)$$

Moreover, $f \in \mathcal{B}_0^\alpha$ if and only if

$$\lim_{j \rightarrow \infty} |a_j| n_j^{1-\alpha} = 0.$$

Proof First assume that $f \in \mathcal{B}^\alpha$. By Cauchy's integral formula, we obtain for $j \geq 1$ and $r \in (0, 1)$ that

$$|a_j| = \frac{1}{2\pi n_j} \left| \int_0^{2\pi} f'(re^{i\theta}) r^{1-n_j} e^{i(1-n_j)\theta} d\theta \right| \leq \frac{Cr^{1-n_j}}{n_j(1-r)^\alpha}.$$

For any $j \geq 1$, we choose $r = 1 - 1/n_j$ to obtain

$$|a_j| \leq C n_j^{\alpha-1} (1 - 1/n_j)^{1-n_j},$$

which clearly shows that condition (2.15) holds.

Next assume that condition (2.15) holds. For an arbitrary $z \in \mathbb{D}$, we have

$$\begin{aligned} \frac{|zf'(z)|}{1-|z|} &\leq (|a_1|n_1|z|^{n_1} + \cdots + |a_k|n_k|z|^{n_k} + \cdots)(1 + |z| + \cdots + |z|^m + \cdots) \\ &\leq \sum_{n=1}^{\infty} \left(\sum_{n_k \leq n} n_k |a_k| \right) |z|^n \lesssim \sum_{n=1}^{\infty} \left(\sum_{n_k \leq n} n_k^\alpha \right) |z|^n. \end{aligned}$$

Fix any positive integer n and let $K = \max\{k : n_k \leq n\}$. Then

$$\begin{aligned} n^{-\alpha} \sum_{n_k \leq n} n_k^\alpha &= \left(\frac{n_K}{n} \right)^\alpha \left[1 + \left(\frac{n_{K-1}}{n_K} \right)^\alpha + \cdots + \left(\frac{n_1}{n_K} \right)^\alpha \right] \\ &\leq 1 + \lambda^{-\alpha} + \lambda^{-2\alpha} + \cdots + \lambda^{-(K-1)\alpha} \leq \frac{\lambda^\alpha}{\lambda^\alpha - 1}. \end{aligned}$$

Therefore,

$$\frac{|zf'(z)|}{1-|z|} \lesssim \sum_{n=1}^{\infty} \left(\sum_{n_k \leq n} n_k^\alpha \right) |z|^n \lesssim \sum_{n=1}^{\infty} (n+1)^\alpha |z|^n \lesssim \frac{|z|}{(1-|z|)^{1+\alpha}},$$

where we used

$$\frac{1}{(1-|z|)^{1+\alpha}} = \sum_{n=0}^{\infty} A_n |z|^n, \quad A_n \approx \frac{n^\alpha}{\Gamma(1+\alpha)}.$$

This shows that $f \in \mathcal{B}^\alpha$.

The proof for the characterization of lacunary series in $\mathcal{B}_0^\alpha$ is similar. We leave the routine details to the interested reader.

Although a single function f in $\mathcal{B}^\alpha$ may not achieve maximal growth rate in *every* radial direction, the next result shows that we can do that with two functions.

Theorem 2.25 *For any $\alpha \in (0, \infty)$ there are functions f and g in $\mathcal{B}^\alpha$ such that*

$$|f'(z)| + |g'(z)| \approx (1 - |z|^2)^{-\alpha}$$

for $z \in \mathbb{D}$.

Proof Fix $\alpha \in (0, \infty)$ and consider the gap series

$$f_\alpha(z) = \sum_{j=0}^{\infty} q^{j(\alpha-1)} z^{q^j}, \quad |z| < 1,$$

where q is a large positive integer. By Theorem 2.24, $f_\alpha \in \mathcal{B}^\alpha$. We are going to show that there exists a positive constant C_1 such that

$$(1 - |z|^2)^\alpha |f'_\alpha(z)| \geq C_1, \quad 1 - q^{-k} \leq |z| \leq 1 - q^{-(k+1/2)}, \quad (2.16)$$

for all $k \geq 1$.

Let k be a positive integer and fix some z such that

$$1 - q^{-k} \leq |z| \leq 1 - q^{-(k+1/2)}.$$

With $x = |z|^{q^k}$, we have

$$(1 - q^{-k})^{q^k} \leq x \leq (1 - q^{-(k+1/2)})^{q^k}.$$

We may assume that q is large enough so that for all $k \geq 1$, we have

$$\frac{1}{3} \leq x \leq \left(\frac{1}{2}\right)^{q^{-1/2}}. \quad (2.17)$$

By the triangle inequality, we have

$$|f'_\alpha(z)| \geq q^{\alpha k} |z|^{q^k} - \sum_{j=0}^{k-1} q^{j\alpha} |z|^{q^j} - \sum_{j=k+1}^{\infty} q^{j\alpha} |z|^{q^j} = T_1 - T_2 - T_3,$$

where

$$T_1 = q^{\alpha k} |z|^{q^k} \geq q^{\alpha k} / 3.$$

An easy computation shows that

$$T_2 = \sum_{j=0}^{k-1} q^{j\alpha} |z|^{q^j} \leq \sum_{j=0}^{k-1} q^{j\alpha} \leq \frac{q^{\alpha k}}{q^\alpha - 1}.$$

By (2.17), we have

$$\begin{aligned} T_3 &= \sum_{j=k+1}^{\infty} q^{j\alpha} |z|^{q^j} \\ &\leq q^{(k+1)\alpha} |z|^{q^{(k+1)}} \sum_{j=0}^{\infty} \left(q^\alpha |z|^{(q^{k+2} - q^{k+1})} \right)^j \end{aligned}$$

$$\begin{aligned}
&= \frac{q^{(k+1)\alpha} |z|^{q^{(k+1)}}}{1 - q^\alpha |z|^{(q^{k+2} - q^{k+1})}} = \frac{q^{(k+1)\alpha} x^q}{1 - q^\alpha x^{(q^2 - q)}} \\
&\leq \frac{q^{(k+1)\alpha} 2^{-q^{1/2}}}{1 - q^\alpha 2^{-(q^{3/2} - q^{1/2})}}.
\end{aligned}$$

Therefore,

$$|f'_\alpha(z)| \geq q^{\alpha k}/3 - \frac{q^{\alpha k}}{q^\alpha - 1} - \frac{q^{(k+1)\alpha} 2^{-q^{1/2}}}{1 - q^\alpha 2^{-(q^{3/2} - q^{1/2})}} \geq \frac{1}{4q^{\alpha/2}(1 - |z|)^\alpha},$$

which shows (2.16) holds.

Similarly, if we consider the gap series

$$g_\alpha(z) = \sum_{j=0}^{\infty} q^{(j+1/2)(\alpha-1)} z^{q^j}, \quad |z| < 1,$$

where q is the same large enough positive integer, then we would have $g_\alpha \in \mathcal{B}^\alpha$ and

$$(1 - |z|^2)^\alpha |g'_\alpha(z)| \geq C_2, \quad 1 - q^{-(k+1/2)} \leq |z| \leq 1 - q^{-(k+1)}, \quad (2.18)$$

for all $k \geq 1$. Combining (2.16) and (2.18), we obtain the desired estimate.

The next result characterizes functions in $\mathcal{B}^\alpha$ whose Taylor coefficients are nonnegative. This gives another construction of nontrivial functions in the space $\mathcal{B}^\alpha$.

Theorem 2.26 *Suppose $\alpha > 0$ and*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

with all $a_n \geq 0$. Then the following conditions are equivalent:

- (i) $f \in \mathcal{B}^\alpha$.
- (ii) For each positive integer m , we have

$$\limsup_{n \rightarrow \infty} \frac{1}{n^{\alpha+m-1}} \sum_{k=m}^n k^m a_k < \infty. \quad (2.19)$$

- (iii) There exists a positive integer m such that

$$\limsup_{n \rightarrow \infty} \frac{1}{n^{\alpha+m-1}} \sum_{k=m}^n k^m a_k < \infty. \quad (2.20)$$

Proof We first assume that $f \in \mathcal{B}^\alpha$. Then there exists a constant C such that

$$(1 - |z|)^\alpha \left| \sum_{k=1}^{\infty} k a_k z^{k-1} \right| \leq C, \quad |z| < 1.$$

For each positive integer $n \geq 2$, we choose $z = 1 - 1/n$ to obtain

$$\frac{1}{n^\alpha} \sum_{k=1}^n k a_k \left(1 - \frac{1}{n}\right)^{k-1} \leq C.$$

Here we used the assumption that all Taylor coefficients of f are nonnegative. When $1 \leq k \leq n$, we have

$$\left(1 - \frac{1}{n}\right)^{k-1} \geq \left(1 - \frac{1}{n}\right)^{n-1} \rightarrow \frac{1}{e}$$

as $n \rightarrow \infty$. It follows that

$$\sum_{k=1}^n k a_k \leq C n^\alpha$$

for $n \geq 2$. If m is any positive integer, we then have

$$\sum_{k=m}^n k^m a_k \leq n^{m-1} \sum_{k=m}^n k a_k \leq C n^{\alpha+m-1}$$

for all $n \geq m$. This shows that condition (i) implies (ii).

It is trivial that condition (ii) implies (iii).

To prove (iii) implies (i), we assume that (2.20) holds for some positive integer m and the left-hand side of (2.20) is C . Since

$$\frac{1}{(1 - |z|)^{1+\alpha}} = \sum_{n=0}^{\infty} A_n |z|^n, \quad A_n \approx n^\alpha, \quad (2.21)$$

it follows from (2.20) and (2.21) that

$$\begin{aligned} |f^{(m)}(z)| &\leq \sum_{n=m}^{\infty} n^m a_n |z|^{n-m} \sum_{n=m}^{\infty} |z|^{n-m} (1 - |z|) \\ &= (1 - |z|) \sum_{n=m}^{\infty} \left(\sum_{k=m}^n k^m a_k \right) |z|^{n-m} \\ &\leq C(1 - |z|) \sum_{n=m}^{\infty} n^{\alpha+m-1} |z|^{n-m} \\ &\leq C(1 - |z|)^{-(\alpha+m-1)}. \end{aligned}$$

Successive integration then shows that f is an α -Bloch function.

Theorem 2.27 Suppose $\alpha > 0$ and

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

with all $a_n \geq 0$. Then the following conditions are equivalent:

- (i) $f \in \mathcal{B}_0^\alpha$.
- (ii) For each positive integer m , we have

$$\lim_{n \rightarrow \infty} \frac{1}{n^{\alpha+m-1}} \sum_{k=m}^n k^m a_k = 0.$$

- (iii) For some positive integer m , we have

$$\lim_{n \rightarrow \infty} \frac{1}{n^{\alpha+m-1}} \sum_{k=m}^n k^m a_k = 0.$$

Proof The proof is similar to that of Theorem 2.26. Details are omitted here.

We proceed to characterize the weight functions K such that the α -Bloch space is contained in $\mathcal{Q}_K$. This, together with Theorems 2.24 and 2.26, will give us additional examples of functions in $\mathcal{Q}_K$.

First notice that if $0 \leq \alpha < 1/2$, we always have $\mathcal{B}^\alpha \subset \mathcal{D} \subset \mathcal{Q}_K$, where the last inclusion is a consequence of Theorem 2.15. If we also have $K(0) = 0$, then Theorem 2.15 tells us that $\mathcal{B}^\alpha \subset \mathcal{Q}_{K,0}$ as well.

Theorem 2.28 Let $1/2 \leq \alpha < 1$. The following conditions are equivalent:

- (i) The weight function K satisfies

$$I_K(\alpha) = \int_0^1 K(\log(1/r))(1-r^2)^{-2\alpha} r dr < \infty. \quad (2.22)$$

- (ii) The space $\mathcal{B}^\alpha$ is contained in $\mathcal{Q}_{K,0}$.
- (iii) The space $\mathcal{B}^\alpha$ is contained in $\mathcal{Q}_K$.

Proof Assume that (2.22) holds. Changing variables in the integral, we see that

$$\begin{aligned} \int_{\mathbb{D}} |f'(z)|^2 K(g(z, a)) dA(z) &\leq \|f\|_{\mathcal{B}^\alpha}^2 \int_{\mathbb{D}} (1-|z|^2)^{-2\alpha} K(g(z, a)) dA(z) \\ &= \frac{1}{\pi} \|f\|_{\mathcal{B}^\alpha}^2 \int_0^1 K(\log(1/r))(1-r^2)^{-2\alpha} I(r, \alpha) r dr, \end{aligned}$$

where

$$I(r, \alpha) = \int_0^{2\pi} \frac{(1 - |a|^2)^{2-2\alpha}}{|1 - \bar{a}re^{i\theta}|^{4-4\alpha}} d\theta.$$

By Lemma 2.5 and the elementary inequality

$$\frac{1}{1 - r^2|a|^2} \leq \frac{1}{1 - |a|^2}, \quad a \in \mathbb{D}, \quad r \in (0, 1),$$

we can find a constant $C > 0$ such that

$$I(r, a) \leq C \begin{cases} (1 - |a|^2)^{2\alpha-1}, & \frac{1}{2} \leq \alpha < \frac{3}{4}, \\ (1 - |a|^2)^{2-2\alpha} \log \frac{2}{1 - |a|^2}, & \alpha = \frac{3}{4}, \\ (1 - |a|^2)^{2-2\alpha}, & \frac{3}{4} < \alpha < 1. \end{cases}$$

In particular, we have $I(r, a) \rightarrow 0$ uniformly for $r \in (0, 1)$ as $|a| \rightarrow 1^-$. This shows that

$$\lim_{|a| \rightarrow 1^-} \int_{\mathbb{D}} |f'(z)|^2 K(g(z, a)) dA(z) = 0,$$

or $f \in \mathcal{Q}_{K,0}$. Thus, condition (i) implies (ii).

It is trivial that (ii) implies (iii).

To prove (iii) implies (i), let us assume that $\mathcal{B}^\alpha \subset \mathcal{Q}_K$. Choose two functions f and g according to Theorem 2.25. Our assumption implies that they also belong to $\mathcal{Q}_K$. Therefore,

$$\begin{aligned} \infty &> 2 \int_{\mathbb{D}} (|f'(z)|^2 + |g'(z)|^2) K(\log(1/|z|)) dA(z) \\ &\geq \int_{\mathbb{D}} (|f'(z)| + |g'(z)|)^2 K(\log(1/|z|)) dA(z) \\ &\geq C \int_0^1 (1 - r^2)^{-2\alpha} K(\log(1/r)) r dr. \end{aligned}$$

This completes the proof of the theorem.

Note that Theorems 2.26 and 2.27 should be compared to a result in [AXZ] which says that the gap series

$$f(z) = \sum_{k=0}^{\infty} a_k z^{n_k}, \quad n_{k+1}/n_k \geq \lambda > 1, \quad k = 0, 1, \dots,$$

is in $\mathcal{Q}_p$, $0 < p \leq 1$, if and only if

$$\sum_{k=1}^{\infty} 2^{k(1-p)} \sum_{n_j \in I(k)} |a_j|^2 < \infty,$$

where

$$I(k) = \{n : 2^k \leq n < 2^{k+1}, n \in \mathbb{N}\}.$$

This result was used in [AXZ] to prove that

$$\mathcal{Q}_s \subsetneq \mathcal{Q}_q, \quad 0 \leq s < q < 1. \quad (2.23)$$

One generalization of (2.23) was given in Theorem 2.19. We will use Theorem 2.28 to give another generalization of this fact, namely, we obtain a simple integral condition on the weight function that enables us to distinguish two different $\mathcal{Q}_K$ spaces.

Recall that if $K_1 \lesssim K_2$ in some interval $(0, t_0)$ then $\mathcal{Q}_{K_2} \subset \mathcal{Q}_{K_1}$. A natural question is this: under what conditions on the weights is this inclusion strict?

Corollary 2.29 *Let $K_1 \lesssim K_2$ in some $(0, t_0)$ and $1/2 \leq \alpha < 1$. If $I_{K_1}(\alpha)$ is finite but $I_{K_2}(\alpha)$ is infinite, then $\mathcal{Q}_{K_2} \subsetneq \mathcal{Q}_{K_1}$.*

Proof It follows from Theorem 2.28 that $\mathcal{B}^\alpha \subset \mathcal{Q}_{K_1}$ and that $\mathcal{B}^\alpha \not\subset \mathcal{Q}_{K_2}$.

To see that Corollary 2.29 is a generalization of (2.23), we note that if $K(t) = K_p(t) = t^p$, $0 \leq p < 1$, then $I_{K_p}(\alpha)$ is finite if $p > 2\alpha - 1$ and infinite if $p \leq 2\alpha - 1$. Choosing $\alpha \in [1/2, 1)$ such that $s < 2\alpha - 1 < q$ and applying Corollary 2.29 to $K_q \lesssim K_s$, we obtain (2.23).

2.6 Notes

All preliminary material in Sect. 2.1, with the exception of Lemma 2.4, was standard and can be found in [Zhu1], for example.

Different versions of Lemma 2.4 exist in the literature. For example, under the assumptions

$$s > -1, r > 0, t > 0, r + t - s > 2, t < s + 2 < r,$$

the result first appeared in [OF] without a proof. Under these assumptions on the parameters, a full proof was given in [Zhaol]. Our version here and its proof appear to be new. The most complete version will appear in the forthcoming paper [LZX], which gives a two-way estimate for the two-parameter integral in various different situations.

The spaces $\mathcal{Q}_K$ were first introduced in [WW1] and [EW]. More substantial study of these spaces began in [EW]. In particular, a weaker version of Theorem 2.7 appeared in [EW]. We show in this book for the first time that condition (2.6) is actually equivalent to the space $\mathcal{Q}_K$ being nontrivial.

Theorems 2.11, 2.14, 2.15, 2.17, 2.19, 2.21, and 2.28 were all proved in [EW]. Theorem 2.19 can be found in [EWX]. Theorem 2.25 was proved in [RU] for the case $\alpha = 1$ and in [Lou2] and [X1] for the general case. The material on α -Bloch spaces can be found in [Ya, Zhu3].

For K_0 in Sect. 2.4, the space $\mathcal{Q}_{K_0}$ is especially interesting. It was actually one of the reasons why the $\mathcal{Q}_K$ spaces were introduced in the first place (see [EW]). In fact, before the $\mathcal{Q}_K$ spaces came about, there was no natural example of a Möbius invariant function space that lies strictly between the BMOA and the Bloch space. Even the theory of $\mathcal{Q}_p$ spaces (see [X1, X2]) was inadequate in this regard. Now the theory of $\mathcal{Q}_K$ spaces provides many examples of such spaces.

2.7 Exercises

1. Suppose $K(t) = 0$ for $0 \leq t < \sigma$ and $K(t) = 1$ for $t \geq \sigma$, where σ is any fixed positive number. Show that $\mathcal{Q}_K$ is the Bloch space.
2. Prove the characterization of lacunary series in $\mathcal{B}_0^\alpha$ given in Theorem 2.24.
3. Prove the characterization of functions in $\mathcal{B}_0^\alpha$ whose Taylor coefficients are nonnegative; see Theorem 2.27.
4. Let $1/2 \leq \alpha < 1$ and

$$f(z) = \sum_{n=1}^{\infty} 2^{-n(1-\alpha)} z^{2^n}.$$

Show that the condition in (2.22) holds if and only if $f \in \mathcal{Q}_{K,0}$.

5. If K is not identically zero, show that for any $r \in (0, 1)$ and $f \in \mathcal{Q}_K$, the mapping $e^{i\theta} \mapsto f_r(e^{i\theta}z)$ is continuous from the unit circle into $\mathcal{Q}_K$.
6. Show that if K is continuous on $[0, \infty)$, then the function $F(a)$ in Theorem 2.7 has a finite limit as a approaches the unit circle. What is this limit?
7. Show that the function $\log(1 - z)$ belongs to $\mathcal{Q}_K$ if and only if the space $\mathcal{B}^{1/2}$ is contained in $\mathcal{Q}_K$.
8. Show that $\mathcal{Q}_K$ is nontrivial if and only if

$$\int_0^\infty K(t)e^{-2t} dt < \infty.$$

9. For a weight function K , we define

$$\rho = \rho(K) = \limsup_{r \rightarrow \infty} \frac{\log^+ \log^+ K(r)}{\log r},$$

and

$$\sigma = \sigma(K) = \limsup_{r \rightarrow \infty} \frac{\log^+ K(r)}{r^\rho}.$$

Show that $\mathcal{Q}_K$ is trivial whenever $\rho > 1$.

10. Show that $\mathcal{Q}_K$ is trivial if $\rho(K) = 1$ and $\sigma(K) > 2$. See [WW1] for this and the previous problem.

11. Show by examples that, in the critical case $\rho(K) = 1$ and $\sigma(K) = 2$, the space $\mathcal{Q}_K$ may be trivial or nontrivial.

12. Is it possible to characterize the weight functions K such that $\mathcal{Q}_{K,0}$ is the closure of the polynomials?

13. Is it true that $\mathcal{Q}_{K,0}$ is the closure of the polynomials if and only if every function in $\mathcal{Q}_{K,0}$ can be approximated in norm by its dilations?

14. Open problem: identify the Möbius invariant dual and/or predual of the spaces $\mathcal{Q}_K$ and $\mathcal{Q}_{K,0}$.

15. Open problem: give a description of K such that $\mathcal{Q}_K = \text{BMOA}$.



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