

Preface

The notion of group actions is a central idea in mathematics. The most important group in complex analysis is probably the Möbius group of the unit disk $\mathbb{D}$, denoted by $\mathcal{M}$, which acts naturally on numerous spaces X of analytic functions on $\mathbb{D}$ by composition: $(\varphi, f) \mapsto f \circ \varphi$, where $f \in X$ and $\varphi \in \mathcal{M}$.

Let X be a linear space of analytic functions on $\mathbb{D}$ equipped with a complete semi-norm $\| \cdot \|$. If $\mathcal{M}$ acts on X isometrically, that is, $\|f \circ \varphi\| = \|f\|$ for all $f \in X$ and $\varphi \in \mathcal{M}$, we say that X is a Möbius invariant function space.

Möbius invariant function spaces constitute a large family of important spaces in complex analysis. Familiar examples include H^∞ , the space of bounded analytic functions on $\mathbb{D}$ equipped with the sup-norm, the Dirichlet space $\mathcal{D}$, consisting of analytic functions f on $\mathbb{D}$ such that

$$\|f\|_{\mathcal{D}}^2 = \int_{\mathbb{D}} |f'(z)|^2 dA(z) < \infty,$$

where dA is area measure, and the Bloch space $\mathcal{B}$, consisting of analytic functions f on $\mathbb{D}$ with

$$\|f\|_{\mathcal{B}} = \sup\{(1 - |z|^2)|f'(z)| : z \in \mathbb{D}\} < \infty.$$

In this monograph, we study a large family of Möbius invariant function spaces that have become known as $\mathcal{Q}_K$ spaces. Each $\mathcal{Q}_K$ space is induced by a non-decreasing function $K : [0, \infty) \rightarrow [0, \infty)$ and is equipped with the semi-norm

$$\|f\|_{\mathcal{Q}_K}^2 = \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 K(g(z, a)) dA(z),$$

where $g(z, a)$ is the Green function of $\mathbb{D}$ with a simple pole at a . The Möbius invariance of $\mathcal{Q}_K$ follows easily from the particular structure of the Green function.

When $K(t) = t^p$, the resulting $\mathcal{Q}_K$ space is usually written as $\mathcal{Q}_p$. Xiao's monographs [X1, X2] contain most of the results about $\mathcal{Q}_p$ spaces. In particular,

it is well known that $\mathcal{Q}_p$ coincides with the Bloch space when $p > 1$, and when $p = 1$, $\mathcal{Q}_p$ is the space BMOA. Thus, the most interesting case for the $\mathcal{Q}_p$ theory is when $0 < p < 1$.

The spaces $\mathcal{Q}_K$ were introduced by the first-named author and his collaborators at the beginning of this century. One of the main motivations for the introduction of $\mathcal{Q}_K$ spaces is to understand the gap between the spaces BMOA and the Bloch space. For example, a natural question is whether or not there exist Möbius invariant function spaces between BMOA and the Bloch space. The $\mathcal{Q}_p$ theory is clearly not fine enough for this. But the theory of $\mathcal{Q}_K$ spaces answers this question in the affirmative and provides a large number of examples.

There have been considerable activity and developments since $\mathcal{Q}_K$ spaces were introduced, and in less than two decades, the theory has reached a certain level of maturity. We feel that a monograph summarizing the major achievements so far will be helpful to graduate students and new researchers interested in these spaces. We also hope that the monograph will attract the attention of more analysts and stimulate further research in this area of complex analysis.

On a few rare occasions, the book makes use of results from such classics as Duren's book [Du] and other sources. However, anyone with a standard graduate background in real and complex analysis will be able to go through most of the book without much difficulty. We have tried our best to construct a comprehensive and up-to-date bibliography. But we realize that this is always a difficult task. We apologize in advance if any significant references somehow escaped our attention.

Our thoughts are with Rauno Aulaskari and Matts Essén who passed away during the time when this book was written. They were two of the pioneers in the theory of $\mathcal{Q}_p$ and $\mathcal{Q}_K$ spaces and had been mentors and friends with us for many years. We also wish to thank Guanlong Bao and Fanglei Wu for carefully reading early versions of the book and catching numerous issues with the presentation of the book.

Finally, we want to thank our families for their constant and continuous support of our study and research. This book is dedicated to them.

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