Chapter 2
Units and Orders of Magnitude

Especially mathematics, but also theoretical physics works basically with structures. The experimental verification or falsification of the physical structures employs units, which allow, in addition to a quantification by numbers, also a qualitative differentiation — a length, for example three meter, is different from a mass, for example three kilogram. But where is the decisive structural reason to differentiate between mass and length?

The three life-important elements *sun*, *earth*, and *water* were used for a first definition of *human or anthropomorphic* units for time, length and mass, respectively, for example in the MKS-system with, respectively, the second (s) as the 86,400th part of the averaged sun day, the meter (m) as the 10 millionth part of a quarter of the circumference of the earth, and the kilogram (kg) as mass of one liter (10^{-3}m^3) water. In the course of a scientific penetration of nature, such anthropocentric ("daily life") measures have been specified and replaced by *natural* measures, inherent to physical laws.

The heaven- and sky-oriented sexagesimally prone system of the Sumerians and Babylonians with about $360 = 6 \times 60 = 12 \times 30$ days a year, used for the angle number 360 of a full circle, and about $30 = \frac{1}{2} \times 60$ days a month and $86{,}400 = 2 \times 12 \times 60^2$ s a day, was replaced by the human-oriented decimal system reflecting our ten fingers. Our feeling for order of magnitudes seems to be not hexal or decimal, but logarithmic with $\log 10 \sim 2.3$, as expressed by the Weber–Fechner[1] law.

Because of our biological conditions, our physical argumentation starts from anthropomorphic concepts and measures. Empedocles[2] used a distinction between "matter," given by fire, earth, water, and air, and "interactions," presented by strife and love. The original division of physics in fields like mechanics, optics and acoustics — for working, seeing, and hearing — shows a human related conception of physics.

[1]Ernst Heinrich Weber (1795–1878), Gustav Theodor Fechner (1801–1887).

[2]Empedocles of Acragas, around -(490–430).

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In modern times, it was gradually given up in favor of more structural concepts like particles and interactions, and, still more abstract, symmetries and operations. Basically, the physical nature does not have to be anthropomorph; however, trivial to say, experiments are necessarily made or registered by human beings.

What is the origin of natural constants, or, as formulated in the following, of "natural units" like Planck's[3] constant $\hbar$, to express a distinction to "natural numbers" like the perimeter number π? Does it make sense to distinguish a finite number of units as basic, and, if yes, what does "basic" mean in this context? Microscopic and macroscopic fields, which differ from our mesoscopic orders of magnitude, elude our evolution-determined familiarization. The phenomena of relativity and the contact with nature in quantum theory balk at a naive understanding and a basic interpretation with anthropomorphic concepts.

Since natural units, related to deeper insights, do not necessarily directly reproduce, neither quantitatively nor qualitatively, our familiar human measures, there arise, already by a simple dimensional analysis, interesting and naively unexpected conversions. Related conclusions from units to physical processes use, in general, only simple power laws. Additional numerical factors like 4π or dimensionless ratios of dimensioned quantities, e.g., mass ratios, and, with them, natural logarithms and exponentials or other functions, cannot be conjectured so easily. In addition, special initial or boundary conditions of a specific problem are not taken into account. Dimension-analytic conclusions are rough and superficial projections like the characterization of a function by the value of its integral. They try to cherry-pick from the cake and are, on the one hand, simple and attractive, but, on the other hand, by their crudeness, unreliable or prove even wrong, if one knows, by other, in general more reliable considerations in "complete theories" with functional laws, what one has to look for. To avoid a naive numerology, a playing around with units and dimensions, as also done in the following, has to be performed with due care. It can serve only as a motivation for a further elaboration of the suggested structures and may be used as mnemonics.

2.1 External Interactions and Newton's Unit

Newton's[4] law for gravity is probably the first rather clearly formalized physical interaction with a basic universal importance. Presumably, the depth of this law is not yet exhausted: Today, after its general relativistic embedding and its interpretation in terms of spacetime metric and curvature, classical gravity waits for a connection with and an understanding in the framework of quantum theory.

Gravity is determined quantitatively by *Newton's unit* G (for *G*ravity) as given in the Newton potential of two masses $m_{1,2}$ with distance r:

[3]Max Planck (1858–1947).

[4]Isaac Newton (1642–1727).

$$V_{\text{Newt}}(r) = -G\frac{m_1 m_2}{r},$$
$$G \simeq 6.7 \times 10^{-11}\ \frac{\text{m}^3}{\text{kg s}^2}(\text{minimal(?)})$$

The Newton potential, determined up to an additive constant, contains the *heavy* masses $V_{\text{Newt}}(r) = -\frac{M_1 M_2}{r}$, characterizable as "eigenvalues" M for gravitational interactions, with the remarkable universal proportionality $(\frac{M}{m})^2 = G$ to the *inert* masses, used in the equations of motion as invariants m^2 for spacetime translations. G is no number, it has the dimension of a cubic length (volume) divided by the product of a mass with a time square, therefore Newton's "unit." Because of the rotation symmetric potential, a replacement $G = \frac{G_0}{4\pi}$ with the area $|\Omega^2| = 4\pi$ of the unit sphere would have been somewhat more appropriate.

Perhaps, the numerical value of G is minimal for all physically relevant quantities with this dimension. The soft concepts "maximal" and "minimal" will be used only up to factors like 12, $\sqrt{5}$, $\frac{1}{2\pi}, \ldots$, which do not change the order of magnitude — not however factors like $10^{\pm 4}$. The speed of light may serve as a familiar example for the concept "maximal" in the case of velocities for physical actions.

Why seems Newton's unit small for us human beings, how can it be reproduced from familiar orders of magnitude? It is related to anthropomorphic measures by the gravity acceleration $g_\bullet \simeq 9.8\frac{\text{m}}{\text{s}^2}$ (gravity field strength on the surface of the earth), the average earth density $\rho_\bullet \simeq 5.5 \times 10^3\ \frac{\text{kg}}{\text{m}^3}$ (water $10^3\ \frac{\text{kg}}{\text{m}^3}$, stones about $3 \times 10^3\ \frac{\text{kg}}{\text{m}^3}$, iron $7.9 \times 10^3\ \frac{\text{kg}}{\text{m}^3}$, platinum $21.5 \times 10^3\ \frac{\text{kg}}{\text{m}^3}$) and by the circumference of the earth $2\pi r_\bullet \simeq 4 \times 10^7$ m, with $mg_\bullet = G\frac{mm_\bullet}{r_\bullet^2}$ and $m_\bullet = \frac{4\pi}{3}r_\bullet^3\rho_\bullet \sim 6 \times 10^{24}$kg, therefore $G \simeq \frac{3g_\bullet}{4\pi r_\bullet \rho_\bullet}$. Its MKS-order of magnitude is the inverse of the number of millimeters of the earth's circumference.

The *escape velocity* from a distance r from a centrally symmetrically distributed mass m equalizes kinetic and potential energy:

$$v_{\text{esc}} = \sqrt{\frac{2Gm}{r}},$$

e.g., the escape velocity from the earth surface is $v_{\text{esc},\bullet} = \sqrt{2g_\bullet r_\bullet} \sim 11.2 \times 10^3\frac{\text{m}}{\text{s}}$.

The remarkable universality of physical laws in time and space was realized by Newton. Newton's potential determines the fall of an apple in about one second from a five meter high tree to the earth with mass $m_\bullet \simeq 5.8 \times 10^{24}$ kg as well as the one year long orbit of this earth around the sun with the mass $m_\odot \simeq 2 \times 10^{30}$ kg.

$\frac{1}{G}$ carries the dimension of a quadratic time, multiplied with a mass density. This dimensional analysis allows a connection between a density ρ and its gravitational collapse time:

$$t_{\text{collapse}}(\rho) = \sqrt{\frac{3}{4\pi G\rho}} = \sqrt{\frac{r^3}{Gm}} \text{ for sphere } \rho = \frac{3m}{4\pi r^3}.$$

According to such a collapse time, only to be used for an order of magnitude and neglecting all other interactions, all mass, contributing to a density ρ, would be unified in one point. The collapse time of the Earth is given by $t_{\text{collapse}}(\rho_\bullet) \simeq \sqrt{\frac{r_\bullet}{g_\bullet}} \simeq 800\,s$.

Interpreting the relation between mass density and time in the inverse form as a flying apart with the daring parametrization[5] of a cosmic matter density:

$$\rho_{\cos,p}(t) = \frac{3}{4\pi G}\frac{1}{t^2},$$

thinning out with an inverse quadratic time dependence for constant G, then one obtains for the Hubble[6] time $t_{\text{Hubl}} \simeq 10^{10}$ years $\simeq 3.1 \times 10^{17}$ s, obtained from the red shifts of the spectral lines of the stars, the cosmic density $\rho_{\cos,p}(t_{\text{Hubl}}) \simeq 4 \times 10^{-26}\,\frac{\text{kg}}{\text{m}^3}$. This yields, with the proton mass $m_p \simeq 1.67 \times 10^{-27}$ kg a baryon density $n_{\cos,p}(t_{\text{Hubl}}) \simeq 24\frac{1}{\text{m}^3}$.

The Hubble time has to be used carefully, especially when it leads to the very far past, e.g., to the "first" three nanoseconds. Probably, it can give only the order of magnitude of the time span, where, backwards, the familiar pictures for space and time give physically meaningful statements, e.g., for the absolute time and the absolute space of Newton — space and time are inert boxes, wherein mass points may perform some dynamics — or for the relational space and time of Leibniz[7] — space and time are translations as operational relations of physical events.

If one calculates from the Newton force $\vec{K}_{\text{Newt}}(r) = \frac{\partial}{\partial \vec{x}} V_{\text{Newt}}(r) = G\frac{mm'}{r^3}\vec{x}$ the gravitational field strength $\frac{\vec{K}_{\text{Newt}}(r)}{m'}$ and collects the $\frac{1}{r^2}$-proportional flux on a closed surface around a mass m, e.g., for a sphere, one obtains the linearly mass-dependent[8] Kepler quotient — here, the exact $\frac{1}{r}$-behavior is important:

$$q_{\text{Kepl}}(m) = \frac{1}{4\pi}\int d\vec{F}\,\frac{\vec{K}_{\text{Newt}}(r)}{m'} = Gm = \begin{cases} Gm_\bullet \simeq 3.9 \times 10^{14}\,\frac{\text{m}^3}{\text{s}^2} & \text{for earth,} \\ Gm_\odot \simeq 1.5 \times 10^{20}\,\frac{\text{m}^3}{\text{s}^2} & \text{for sun.} \end{cases}$$

In a closed system with gravitational interaction only and one "dominating" large mass, e.g., in a quite good approximation, our planetary system or the earth–moon system, Gm is a "universal" intrinsic unit. This yields, in the example with the sun mass, the Kepler ratio $\frac{q_{\text{Kepl}}(m_\odot)}{(2\pi)^2} = \frac{R_P^3}{T_P^2} \simeq 3.8 \times 10^{18}\,\frac{\text{m}^3}{\text{s}^2}$ of the cubic averaged planetary distances R_P to the quadratic orbit times T_P for all planets. In the form

[5] Arthur Eddington (1882–1944).

[6] Edwin Hubble (1889–1953).

[7] Gottfried Wilhelm Leibniz (1646–1716).

[8] Johannes Kepler (1571–1630).

$$\frac{Gm_\odot}{R_P} = \left(\frac{2\pi R_P}{T_P}\right)^2,$$

it can be related to the virial theorem, which relates to each other the averages of potential and kinetic energy $\overline{V(x)} = \overline{E_{\text{kin}}}$.

Kepler's third law uses as intrinsic unit a ratio with orbit radius and time — not a position unit and not a time unit. It allows to estimate the moon-earth distance via $\frac{R_m^3}{T_m^2} \sim \frac{Gm_\bullet}{(2\pi)^2} = \frac{g_\bullet r_\bullet^2}{(2\pi)^2}$ and the month length $T_m \sim 28$ days, leading to $R_m \sim 3.8 \times 10^8$m and for $T_s \sim 1$ day the radius $R_s \sim 4.1 \times 10^7$m of a geostationary orbit. It is not difficult to estimate or to measure, on earth, Newton's constant G, at least the right order of magnitude, and the earth's orbit time (year). One non-terrestric measurement is necessary to quantify astronomical consequences "over the moon," e.g., the not so easy measurement of the orbit radius of the Earth which, then, allows the computation of the sun mass.

2.2 Speed of Light as Space-Time Hinge

The *speed of light c* (for *c*eleritas) as probably *maximal velocity of an action* is about seven and a half circumferences of the earth per second — "straight" light rays, of course. It defines, for an otherwise determined time unit, the associated natural length unit, so, since 1983 the meter for the second:

$$c = 299\,792\,458 \frac{\text{m}}{\text{s}} \simeq 3 \times 10^8\ \frac{\text{m}}{\text{s}}\ \text{(maximal)}$$

More light — away with the Parisian ur-meter! The huge ratio of the action velocity of gravity and electromagnetism to human-related velocities makes it understandable that the first physical theories assumed instantaneous long-distance interactions. Newton was not comfortable with that. The limit of nonrelativistic space–time is reached from relativistic spacetime for an infinite speed of action in the contraction:

$$\text{spacetime} \xrightarrow{\frac{1}{c}\to 0} \text{space}-\text{time}.$$

The maximal distance for the speed of light, is, for the Hubble time, about 10 billion light years, $ct_{\text{Hubl}} \simeq 0.9 \times 10^{26}$ m.

Einstein's[9] *energy equivalent of a mass m* as the trivial momentum value of the relativistic energy-momentum relation $E^2 = m^2c^4 + \vec{p}^2c^2$:

$$E_{\text{Einst}}(m) = mc^2,$$

[9] Albert Einstein (1879–1955).

gives for a hydrogen atom or the proton mass $m_p c^2 \simeq 1.5 \times 10^{-10}$ J with the energy-unit Joule[10] $\mathrm{J} = \frac{\mathrm{m^2 kg}}{\mathrm{s^2}}$. The first order nonrelativistic correction, for the contraction $\frac{1}{c} \to 0$, is the kinetic mass point energy:

$$E = \sqrt{m^2c^4 + \vec{p}^2c^2} = mc^2\sqrt{1 + \frac{\vec{p}^2}{m^2c^2}} = mc^2 + \frac{\vec{p}^2}{2m} + \dots,$$
$$\lim_{c\to\infty}(E - mc^2) = \frac{\vec{p}^2}{2m}.$$

Good to know: Vectors with an arrow, here $\vec{p}$, are assumed to be from a Euclidean space with scalar product, here $\vec{p}^2 \geq 0$.

The, without concessions, maximal action velocity is, simultaneously, the *natural velocity unit* and the *natural conversion factor* between time and length. According to special relativity, there is no absolute reference frame for separate space and time translations. However, there exists an absolutely highest action velocity. Not a static position length or a time period, but the space and time connecting velocity c with dynamical implications seems to be a fundamental unit. The related lightcones in spacetime are "absolute."

The "unification" of time and space in spacetime has to be qualified: Although time and space are connected with each other by relativity to spacetime, forever and everywhere, shown by the existence of an universal velocity, both structures stay clearly separated — now in a modified form with the relativistically meaningful concepts "timelike" and "spacelike." However, the causality structure (future, past and also a generalized presence as causally not influenceable spacelike region) is no longer absolute, but observer-dependent. In a mathematical terminology, there does not exist a total order structure, but only a partial one. Completely new and relativity-characteristic is the lightlike region where nontrivial time distances coincide with nontrivial space distances, i.e., where spacetime distances vanish. The mathematical designation *singular* for nontrivial distances with vanishing "length" is, also for the physics of spacetime, very appropriate: Light is really a singular structure.

With the embedding of a three-dimensional space and a one-dimensional time into four-dimensional relativistic Minkowski[11] spacetime or, expressed otherwise, with the blowing up of a pointlike presence in time to a causally not affected and not affecting four-dimensional spacelike manifold, the classical time-dependent point-particle physics is replaced by a description with spacetime fields. Although there exist hybrid theories with relativistic point particles, which move, e.g., in gravitational and electromagnetic fields, mathematically formalizable with the eigentime, the development of physics, not at least of quantum theory with probability amplitudes and densities, shows that the concept of a point particle, perhaps even the concept of a particle, is a pragmatically useful way for the interpretation of experiments

[10] James Prescott Joule (1818–1889).
[11] Hermann Minkowski (1864–1909).

in terms of interaction-free structures, but possibly not appropriate for a fundamental formulation of dynamics and interactions.

Good to know: In mathematics, space is used as a very general concept — vector space, topological space, etc. To distinguish the physical "space" in spacetime, it will be often called "position." Inconsequentially, "spacelike" will not be changed to "positionlike." In the literature, "Minkowski spacetime" is sometimes generally used for a Riemannian manifold with $(1, 3)$-signature metric. In the following, "Minkowski spacetime" is used only for a vector space with $(1, 3)$-signature metric.

With a transition from time and space (position) to spacetime, there come some new features, often not appreciated enough. For example, it is a mortal sin against the special relativistic spirit to distinguish a basis with one time and three position vectors or also a class of bases, e.g., orthonormal bases where the Lorentz[12] metric has the matrix $\eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ for the spacetime "length" $x^2 = x_j \eta^{jk} x_k = x_0^2 - \vec{x}^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2$, without giving a physical reason for this choice. One such reason may be a particle with nonvanishing mass, e.g., a human being, where there exist rest systems. This must not be absolutized. For example, a lightlike object, e.g., a particle with vanishing mass, distinguishes completely different spacetime bases, e.g., ligthlike tetraeder bases where the Lorentz metric has a 0-diagonal symmetric matrix $\zeta = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$ for $x^2 = x_j \zeta^{jk} x_k = 2(x_0x_1 + x_0x_2 + x_0x_3 + x_1x_2 + x_1x_3 + x_2x_3)$ without manifest distinction for time and space.

Good to know: Nonrelativistic space-time is the direct sum of two vector spaces $\mathbb{R} \oplus \mathbb{R}^3$ for time and position. There is no vector space decomposition of special relativistic Minkowski spacetime $\mathbb{R}^4$, compatible with the Lorentz transformations. The Lorentz-compatible decomposition into two four-dimensional manifolds $\mathbb{R}^4_{\text{caus}} \uplus \mathbb{R}^4_{\text{position}}$ contains, in addition to the spacelike vectors $\mathbb{R}^4_{\text{position}}$ with $x^2 < 0$ the causal vectors $\mathbb{R}^4_{\text{caus}} = \mathbb{R}^4_{\text{time}} \uplus \{0\} \uplus \mathbb{R}^4_{\text{light}}$, with the proper timelike vectors $x^2 > 0$, the trivial vector, and the lightlike vectors $x^2 = 0$ for $x \neq 0$. The timelike vectors are the union of the future cone $\mathbb{R}^4_+$ and the past cone $\mathbb{R}^4_-$, both four-dimensional and characterized by $x^2 \geq 0$ and either $x_0 > 0$ or $x_0 < 0$. Cones are not vector spaces: They are closed under vector addition $x + y$, but under multiplication only with positive scalars, αx for $\alpha \geq 0$. The lightlike vectors $\mathbb{R}^4_{\text{light}}$ constitute a three-dimensional manifold. The order conditions $x_0 > 0$ and $x_0 < 0$ for $x^2 \geq 0$ are compatible with the orthochronous Lorentz group transformations.

Connected with all this, there is a shift of fundamental concepts: For example, the concept "velocity," dependent one a space-time decomposition, is not good for a relativistic theory. Vectors x, i.e., spacetime translations, and their dual vectors p, i.e., energy-momenta, are decomposable by $x = (x_0, \vec{x})$ and $p = (p^0, \vec{p})$ only with the distinction of a time direction. Then, the quotient yields a velocity $\frac{\vec{v}}{c} = \frac{\vec{p}}{p^0}$ in this system. It is interesting, that both relativity and also quantum theory with $[i\mathbf{p}, \mathbf{x}] = \hbar\mathbf{1}$ replace the velocity concept, familiar from daily life, by the more fundamental and somewhat more abstract momentum concept. Energy–momentum values and time-space translations are closely connected as dual partners, e.g., no time concept without the energy concept.

Starting from special relativity, characterized by the unit c, two ways open up: The macroscopic relativistic gravity road and the microscopic relativistic quantum road, which both will be pursued in the following.

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[12] Hendrik Antoon Lorentz (1853–1928).

2.3 Gravity and Maximal Speed of Action

In Newton's gravity with the, already for Newton himself, awkward action at a distance, there is no maximal speed of action, only in Einstein's relativistic field theories. The spacetime unit c together with Newton's unit give the *natural gravity conversion factors* for time and length to mass:

$$\frac{G}{c^3} \simeq 2.5 \times 10^{-36} \, \frac{\mathrm{s}}{\mathrm{kg}} \text{ (minimal(?)),}$$
$$\frac{G}{c^2} \simeq 7.4 \times 10^{-28} \, \frac{\mathrm{m}}{\mathrm{kg}} \text{ (minimal(?)).}$$

If $\frac{G}{c^3}$ would be easier experimentally accessible, the kilogram could be defined for a given time unit.

The parametrization of a mass increasing with time, called cosmic mass:

$$m_{\cos,p}(t) = \frac{c^3}{G} t,$$

(where does this mass come from?) yields for the Hubble-time $m_{\cos,p}(t_{\mathrm{Hubl}}) \simeq 1.2 \times 10^{53}$ kg. With the proton mass, one obtains, according to Eddington, the cosmic "baryon number:"

$$N_{\cos,p}(t) = \frac{m_{\cos,p}(t)}{m_p} = \frac{c^3}{G m_p} t,$$

e.g., $N_{\cos,p}(t_{\mathrm{Hubl}}) \simeq 7.1 \times 10^{79}$ for the Hubble-time. Here, the speed of light is assumed to be constant in time.

The linearly mass-dependent *Schwarzschild*[13] *length:*

$$l_{\mathrm{Schwarz}}(m) = \frac{q_{\mathrm{Kepl}}(m)}{c^2} = \frac{G}{c^2} m$$

characterizes a black hole and leads, e.g., to the Schwarzschild radius of the sun $2 l_{\mathrm{Schwarz}}(m_\odot) \simeq 3 \times 10^3$ m. At twice the Schwarzschild length, the escape velocity is the highest action velocity $v_{\mathrm{esc}} = c = \sqrt{\frac{2Gm}{l_{\mathrm{Schwarz}}}}$, no action can come over the horizon of such a sperically symmetric black hole.

Reversing this relation, each length can be associated with its *Newton mass*:

$$m_{\mathrm{Newt}}(l) = \frac{c^2}{G} l.$$

[13] Karl Schwarzschild (1873–1916).

The Newton mass characterizes the equilibrium of the gravitative energy and the mass energy of two masses with distance l:

$$m_1 c^2 + m_2 c^2 = G \frac{m_1 m_2}{l} \Rightarrow \frac{1}{m_{\rm red}} = \frac{1}{m_1} + \frac{1}{m_2} = \frac{G}{c^2 l}.$$

Here $m_{\rm red}$ is the reduced mass of the system, l its Schwarzschild length.

The ratio of the Schwarzschild length of a mass to its radius r for a rotation symmetric mass distribution gives, up to a factor, understandable in general relativity, the deflection angle for a tangent light ray:

$$\delta(m,r) = 4 \frac{l_{\rm Schwarz}(m)}{r} = 4 \frac{G}{c^2} \frac{m}{r}.$$

This relation is approximately valid for a small angle. An example is a light ray from a star, tangent at the sun with radius $r_\odot \simeq 0.7 \times 10^9$ m, leading to a deflection of about 1.74 arc second. With the same general relativistic basic structures, the ratio of the Schwarzschild length of the sun to the radius of a planetary orbit R_P gives the order of magnitude for the perihel rotation of the planet for one orbit:

$$\Delta(m_\odot, R_P) = 6\pi \frac{l_{\rm Schwarz}(m_\odot)}{R_P} = 6\pi \frac{q_{\rm Kepl}(m_\odot)}{c^2 R_P} = 6\pi \frac{G}{c^2} \frac{m_\odot}{R_P}.$$

The radius 57.9×10^6 km of the Mercury orbit and the orbit time of 88 earth days gives as perihel rotation of the Mercury per century about 43 arc seconds.

This example shows, how a dimensional analysis is, on the one hand, inspiring, but also, on the other hand, restricted: Before Einstein, the phenomenon of the perihel rotation for Mercury was well known, as well as the values for the speed of light, the gravity constant, the Sun mass, and the Mercury orbit radius. Therefore, in principle, already in the year 1870 one could have played around with the dimensionless ratio above. For its understanding, the idea would not have been too far fetched, that the perihel rotation has to do with gravity. To have the insight and even a theory, that also the speed of light plays an essential role, needed the genius Einstein, who formulated gravity in a framework compatible with a highest speed of action and independent of a special reference frame (theory of general relativity). At this time, some theoreticians tinkered with the integer power $\frac{1}{r^{1+\epsilon}}$ in Newton's law, and some astronomers tried to understand the perihel rotation by a slowdown or by an additional planet ("Vulcanus") nearer to the Sun, which, remarkably, was observed rather often before Einstein's explanation. The modifications by Einstein's relativity lead to the effective potential with an individual correction from the planet's angular momentum:

$$V_{\rm eff}(r) = \left(1 - \frac{\ell}{r}\right)\left(1 + \frac{\ell R}{2r^2}\right) + \cdots, \quad \ell = \frac{G m_\odot}{c^2}, \quad R = R_P.$$

2.4 Quantum Structure and Planck's Unit

On the quantum road, the *Avogadro (Loschmidt)*[14] *number* N_{Avo}, which gives the number of molecules in one kilomol, serves as a bridge from the anthropomorphic to the atomic order of magnitudes:

$$N_{\text{Avo}} \simeq 6.03 \times 10^{26} \frac{1}{\text{kilomol}}.$$

That is about the number of poppy seeds that can be distributed on one million earth surfaces — one seed on each square millimeter. Or: To exhaust the Hubble radius $r_{\cos,p}(t_{\text{Hubl}}) \simeq 0.9 \times 10^{26}$ m, one needs about as many meter sticks as there are molecules in one kilomol.

One kilomol of one-atomic hydrogen gas has the mass of about one kilogram, i.e., one hydrogen atom has the mass $m_H \simeq 1.67 \times 10^{-27}$ kg.

To reproduce the intuitively not graspable Avogadro number, at least its order of magnitude, from daily life experiences one can measure the extension of a drop of oil on water[15]: A drop of oil (assume a cube of 1 mm^3) covers maximally, without teared up, about 3 m^2 water. Assuming the layer thickness $\frac{10^{-9}}{3}$m for a molecular cube, the oil drop contains about 2.7×10^{19} oil molecules, and 1 m^3 oil about 2.7×10^{28} molecules.

With the relative molecule mass numbers $Z(M)$ for a molecule M, normalized by carbon $Z(\text{C}) = 12$, and the density $\rho(T, P)$ for temperature T and pressure P, one obtains for the volume of a kilomol:

$$V_{\text{kilomol}}(M; T, P) = \frac{Z(M)}{\rho(T, P)} \frac{\text{kg}}{\text{kilomol}},$$

and, after division with N_{Avo}, the volume for one molecule. For example, for iron with $Z(\text{Fe}) \simeq 55.85$ and the density $7.87 \times 10^3 \frac{\text{kg}}{\text{m}^3}$ at normal conditions $T_0 = 273.15$ K (for Kelvin[16]) and $P_0 = 1 \text{ atm} = 1.01325 \times 10^5 \frac{\text{kg}}{\text{s}^2\text{m}}$, the volume of one kiloatom is around 7.1×10^{-3} m^3 — one iron atom has at its disposal around $(2.3 \times 10^{-10} \text{ m})^3$. In the volume of about $22.4 \frac{\text{m}^3}{\text{kilomol}}$ at normal conditions, valid for all pure gases, a molecule has as space about $(30 \times 10^{-10} \text{ m})^3$, thousand times as much as iron.

The macroscopic pressure of a gas is related to the density and the squared velocity average of the microscopic gas molecules by $P = \frac{\rho}{2}\langle v^2 \rangle$, where $\rho = \frac{Nm}{V}$ with the volume, the number and the mass of the molecules. The *Boltzmann*[17] *conversion factor* connects the holistic-statistical temperature concept with the energy. For the kinetic energy, the temperature is proportional to the mass:

[14] Amedeo Avogadro (1776–1856), Joseph Loschmidt (1821–1895).

[15] George Gamow (1904–1968).

[16] William Thomson Kelvin (1824–1907).

[17] Ludwig Boltzmann (1844–1906).

$$E = \frac{m}{2}\langle v^2\rangle = k_{\text{Boltz}}T.$$

With the kilomol volume of ideal gases, one obtains the equation for ideal gases:

$$PV_{\text{kilomol}} = N_{\text{Avo}}k_{\text{Boltz}}T \Rightarrow \begin{cases} k_{\text{Boltz}} \simeq 1.4 \times 10^{-23}\ \frac{\text{J}}{\text{K}}, \\ \frac{k_{\text{Boltz}}}{c^2} \simeq 1.6 \times 10^{-40}\ \frac{\text{kg}}{\text{K}}, \end{cases}$$

Hydrogen atoms with $m_p \simeq 1.7 \times 10^{-27}$ kg need an average speed $\sqrt{\langle v^2\rangle} \simeq 2.2 \times 10^3\ \frac{\text{m}}{\text{s}}$ for the temperature $T_0 = 273.15$ K. If there were ideal gases, the temperature unit could be defined theoretically by fixing the Boltzmann factor.

From atoms one has recognized *Planck's unit* $\hbar$ as the *minimal quantum of an action* with the dimension of a time times an energy or, what is the same, the dimension of an *angular momentum*, i.e., position times momentum:

$$\hbar = \frac{h}{2\pi} \simeq 1.05 \times 10^{-34}\ \frac{\text{m}^2\ \text{kg}}{\text{s}}\ \text{(minimal)}.$$

Perhaps qualitatively, but by no means quantitatively the word "quantum leap", loved by politicians, makes sense. There is work going on to define the kilogram, with a Watt[18] balance, via a fixed value for Planck's unit. Since 2011, the units kilogram and Kelvin are coupled to the values of the Planck and Boltzmann constant, respectively. The letter h for Planck's unit comes from the capital Greek H (eta) used for the state sum; it has nothing to do with $\hbar$eisenberg. The action quantum or angular momentum quantum is the commutator value in the commutation relations of Born[19] and Heisenberg[20] for position and momentum, i.e., a quantitative measure for the noncommutativity of the quantum structure (Heisenberg's uncertainty relation).

Without special relativity, it is not self-evident that both the product of position with momentum and the product of time with energy are equally dimensioned $\text{m} \cdot \frac{\text{kg m}}{\text{s}} = \text{s} \cdot \frac{\text{kg m}^2}{\text{s}^2}$ by an action unit. The related conclusion to postulate a nontrivial commutation for time and energy operators as for position and momentum operators $[\mathbf{x}, \mathbf{p}] = i\hbar\mathbf{1} = -[\mathbf{t}, \mathbf{E}]$ is not justified. Energy is "quantized" in the Hamilton[21] element (Hamiltonian) $\mathbf{H}(\mathbf{x}, \mathbf{p})$ via the time-dependent position and momentum $t \longmapsto \mathbf{x}(t), \mathbf{p}(t)$. The naively expected special relativistic extension $(\mathbf{x}_a)_{a=1,2,3} \hookrightarrow (\mathbf{x}_j)_{j=0,1,2,3}$ where not only position, but also time is "quantized," both as operators in $[\mathbf{x}_j, \mathbf{p}_k] = -i\hbar\eta_{jk}\mathbf{1}$ with the Lorentz metric η_{jk}, seems not to be appropriate. In canonically quantized special relativistic field theories with fields $x \longmapsto \mathbf{\Phi}(x)$, all spacetime coordinates, for time and also for position, are used not as operators, but as number valued parameters. The quantization related nontrivial (anti-)commutators involve the field degrees of freedom, e.g., for

[18] James Watt (1736–1819).

[19] Max Born (1882–1970).

[20] Werner Heisenberg (1901–1976).

[21] William Hamilton (1805–1865).

a scalar field $[\mathbf{\Phi}(x_0,\vec{x}),\frac{\partial}{\partial x_0}\mathbf{\Phi}(x_0,\vec{y})]=i\hbar\delta(\vec{x}-\vec{y})\mathbf{1}$ with the Dirac[22] space distribution. This sheds also some light on the attempts to "quantize spacetime."

Planck's unit with the Boltzmann factor allow the conversion of time (frequency) in temperature:

$$\frac{\hbar}{k_{\text{Boltz}}}\simeq 7.5\times 10^{-12}\ \text{s K}.$$

Avogadro's number and Planck's unit are two bridges from the anthropomorphic measures to the atoms — do they condition each other? If each molecule in one kilomol would carry an equally directed spin the total spin would be still tiny in anthropomorphic units:

$$N_{\text{Avo}}\hbar\simeq 0.6\times 10^{-7}\ .$$

Therefore, this product is not appropriate to relate both bridges.

2.5 Minimal Quantum of Action and Maximal Speed of Action

The maximal action speed and the minimal action quantum define the *natural quantum conversion factors* from time and length to mass:

$$\begin{aligned}\frac{\hbar}{c^2}&\simeq 1.2\times 10^{-51}\ \text{s kg (minimal)},\\ \frac{\hbar}{c}&\simeq 3.5\times 10^{-43}\ \text{m kg (minimal)}.\end{aligned}$$

Both values should be, up to factors not changing the order of magnitude, minimal physical values with this dimensions. Also $\frac{\hbar}{c^2}$ could be used in principle, to define a mass unit for a given time unit.

The parametrization with the inverse linearly mass-dependent *Compton*[23] *length*:

$$l_{\text{Compt}}(m)=\frac{\hbar}{c}\frac{1}{m}$$

give, e.g., for the proton $l_{\text{Compt}}(m_p)\simeq 2.1\times 10^{-16}$ m. Conversely, each length can be associated with its *Yukawa*[24] *mass*:

[22] Paul Dirac (1902–1984).

[23] Arthur Compton (1892–1962).

[24] Hideki Yukawa (1907–1981).

$$m_{\text{Yuk}}(l) = \frac{\hbar}{c}\frac{1}{l},$$

which played a role as range, e.g., for the nuclear forces in a Yukawa potential $V_{\text{Yuk}}(r) = g_0^2 \frac{e^{-\frac{r}{l}}}{r}$ with around 10^{-15} m, leading to the discovery of the π-meson, around 0.1 proton masses.

The *Lummer*[25] *conversion factor* between temperature and length:

$$\frac{\hbar c}{k_{\text{Boltz}}} \simeq 2.25 \times 10^{-3} \text{ m K},$$

allows, up to a factor of about 1.3, for the black body radiation the transition from the maximally radiated wavelength to its surface temperature by *Wien's*[26] *law*. For example, the sun light with $\lambda_{\max,\odot} \simeq 5 \times 10^{-7}$ m shows us the surface temperature $T_\odot \simeq 5800$ K of the sun. Here, at the temperature distribution of the black body radiation, was Planck's starting point, leading to quantum theory. The Schwarzschild length of a mass gives the *Hawking*[27] *temperature* which goes with the inverse mass:

$$8\pi T_{\text{Hawk}}(m) = \frac{\hbar c}{k_{\text{Boltz}}}\frac{1}{l_{\text{Schwarz}}(m)} = \frac{\hbar c^3}{G k_{\text{Boltz}}}\frac{1}{m},$$

e.g., $T_{\text{Hawk}}(m_\odot) \simeq 1.8 \times 10^{-7}$ K for the sun.

With Lummer's conversion factor, a photon density can be defined for the cosmic background radiation:

$$n_{\text{cos,G}}(T) = \left(\frac{k_{\text{Boltz}}}{\hbar c}\right)^3 T^3,$$

whose present order of magnitude $n_{\text{cos,G}}(T_{\text{Hubl}}) \simeq 1.7 \times 10^9 \frac{1}{\text{m}^3}$ with $T_{\text{Hubl}} \sim 2.7$ K can be compared with the present value $n_{\text{cos,p}}(t_{\text{Hubl}}) \simeq 24\frac{1}{\text{m}^3}$ of the baryon density.

2.6 Intrinsic Units

A definition of units uses, more or less aware, some physical laws. For example, the original definition of the second relies on the regularity of the earth's motion and, with today's precision of time measurement, is no longer satisfactory. Special dynamical problems involve intrinsic units. If a nonrelativistic mechanical problem is characterized by its total energy via a Hamiltonian, i.e., by the sum of kinetic and potential energy, e.g., for a homogeneous potential $V^z(\underline{x}) = \epsilon \frac{g_0^2}{z} \underline{x}^z$ with position $\underline{x}$-dependence of order z:

[25] Otto Lummer (1860–1925).

[26] Wilhelm Wien (1864–1928).

[27] Stephen Hawking (1942–).

$$\underline{H} = \frac{\underline{p}^2}{2m_0} + \frac{\epsilon g_0^2}{z}\underline{x}^z, \quad m_0, g_0^2 > 0, \quad \epsilon = \pm 1, \quad z \in \mathbb{Z}, \quad z \neq 0,$$

the arising variables have the dimensions [. . .] of energy, length and momentum, in the MKS-system:

$$[\underline{H}] = \frac{\mathrm{m}^2\mathrm{kg}}{\mathrm{s}^2}, \quad [\underline{x}] = \mathrm{m}, \quad [\underline{p}] = \frac{\mathrm{m\ kg}}{\mathrm{s}}.$$

With that, the inert mass m_0 and the coupling constant g_0^2 can be used, obviously up to order of magnitude-compatible factors, as the intrinsic units of this dynamics with the MKS-dimension:

$$[m_0] = \mathrm{kg}, \quad [g_0^2] = \frac{\mathrm{m}^{2-z}\ \mathrm{kg}}{\mathrm{s}^2}.$$

A homogeneous potential $V^z(\underline{x})$ describes for order $z = 1$ the fall of "mass points" near the earth's surface. For order $z = -1$, it is characteristic for a central potential, e.g., the gravitational Newton potential $G\frac{m_1 m_2}{|\vec{x}_1 - \vec{x}_2|}$, and for order $z = 2$ for the oscillator potential (pendulum with small amplitudes).

Galileo[28] was the first to check the suggestion, which was later Einstein's main starting point for the foundation of his general theory of relativity, that, for all bodies, freely falling at the earth's surface, the intrinsic units g_0^2 (weight, heavy mass) and m_0 (inert mass) have the same ratio $g_\bullet$:

$$V_{\mathrm{Newt}}(\vec{x}_\bullet + \vec{\underline{x}}) = G\frac{m_\bullet m_0}{|\vec{x}_\bullet + \vec{\underline{x}}|} = V_{\mathrm{Gal}}(\underline{x}) + \ldots,$$
$$V_{\mathrm{Gal}}(\underline{x}) = \frac{Gm_\bullet}{r_\bullet^2} m_0 \underline{x} = g_\bullet m_0 \underline{x} = g_0^2 \underline{x}.$$

Therefore, for the Hamiltonian with the Galileo potential, the inert mass m_0 (individual) and the acceleration $g_\bullet = \frac{Gm_\bullet}{r_\bullet^2}$, caused by the earth (universal), are the intrinsic units. If, and in an airless space, you let fall any mass from your height (about one meter), it hits the ground after about $\frac{1}{2}$ second. That the human time unit second, probably taken from such experiences, is approximately equal to our heartbeat time unit—not that of mice or elephants — is presumably a coincidence.

The Hamiltonian of a special problem effects, physically as energy as well as mathematically by implementation as transformation, the time development of all quantities, which can be built by position $\underline{x}$ und momentum $\underline{p}$. For explicitly time-independent problems, $\frac{\partial \underline{H}}{\partial t} = 0$, as exemplified earlier, the Hamiltonian gives the energy, conserved in time:

[28] Galileo Galilei (1564–1642).

$$\underline{H} = \frac{\underline{p}^2}{2m_0} + V(\underline{x}) \stackrel{\text{e.g.,}}{=} \frac{\underline{p}^2}{2m_0} + \frac{\epsilon g_0^2}{z}\underline{x}^z,$$
$$\frac{d\underline{x}}{d\underline{t}} = \frac{\partial \underline{H}}{\partial \underline{p}} = \frac{\underline{p}}{m_0}, \quad \frac{d\underline{p}}{d\underline{t}} = -\frac{\partial \underline{H}}{\partial \underline{x}} = -\frac{\partial V}{\partial \underline{x}} = -\epsilon g_0^2 \underline{x}^{z-1} \Rightarrow \frac{d\underline{H}}{d\underline{t}} = 0.$$

The equations of motion can be classically characterized by a Lagrange[29] function (Lagrangian), which connects the Hamiltonian with the time-dependence of the *canonical pair* "position-momentum" $(\underline{x}, \underline{p})$:

$$\underline{L} = \frac{1}{2}\left(\underline{p}\frac{d\underline{x}}{d\underline{t}} - \underline{x}\frac{d\underline{p}}{d\underline{t}}\right) - \underline{H}.$$

In the equation of motion of any "sufficiently smooth" position-momentum dependent function $(x, p) \longmapsto F(x, p)$, the time derivative is effected by the position and momentum derivatives via the Poisson[30] bracket with the Hamiltonian:

$$\frac{dF}{dt} = [H, F]_P = \frac{\partial H}{\partial p}\frac{\partial F}{\partial x} - \frac{\partial H}{\partial x}\frac{\partial F}{\partial p}.$$

Good to know: The bilinear Poisson bracket is a Lie[31] algebra bracket (no commutator for the pointwise product!) for such functions. It is antisymmetric and obeys Leibniz's rule, i.e., the product rule for derivations $D(f_2 f_3) = (Df_2)f_3 + f_2(Df_3)$, equivalent to the Jacobi[32] identity:

$$[F_1, F_2]_P = \frac{\partial F_1}{\partial p}\frac{\partial F_2}{\partial x} - \frac{\partial F_2}{\partial p}\frac{\partial F_1}{\partial x} : \begin{cases} [F, F]_P = 0 \Rightarrow [F_1, F_2]_P = -[F_2, F_1]_P, \\ [F_1, [F_2, F_3]_P]_P = [[F_1, F_2]_P, F_3]_P + [F_2, [F_1, F_3]_P]_P. \end{cases}$$

The related Lie algebra is infinite-dimensional.

In a quantum mechanical theory, the dimension of the noncommuting canonical position–momentum pair is determined by Planck's action quantum $\hbar$:

$$[\underline{H} \cdot \underline{t}] = [\underline{x} \cdot \underline{p}] = [\hbar] = \frac{\text{m}^2\text{kg}}{\text{s}},$$
$$\underline{x}\,\underline{p} - \underline{p}\,\underline{x} = [\underline{x}, \underline{p}] = i\hbar\mathbf{1}.$$

For the Hamiltonians earlier with $z \neq -2$, the arbitrary MKS-system can be replaced by an intrinsic $(m_0, g_0^2, \hbar)$-system, specific for the given problem. It gives for the intrinsic length and time unit:

$$l_0 = \left(\frac{\hbar^2}{g_0^2 m_0}\right)^{\frac{1}{2+z}}, \quad t_0 = \left(\frac{\hbar^{2-z} m_0^z}{g_0^4}\right)^{\frac{1}{2+z}}, \quad [l_0] = \text{m}, \quad [t_0] = \text{s}.$$

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[29] Joseph-Louis Lagrange (1736–1813).

[30] Simeon Poisson (1781–1840).

[31] Sophus Lie (1842–1899).

[32] Carl Gustav Jacob Jacobi (1804–1851).

Also the intrinsic energy unit E_0 reflects the position power z in the potential:

$$E_0 = \left(\frac{\hbar^{2z} g_0^4}{m_0^z}\right)^{\frac{1}{2+z}} = \begin{cases} \frac{g_0^4 m_0}{\hbar^2}, & z = -1, \\ \hbar\sqrt{\frac{g_0^2}{m_0}}, & z = 2. \end{cases}$$

If, for the computation of the energy only natural multiples of the action quantum are admitted, i.e., $n\hbar$ with $n = 1, 2 \ldots$, then there follow the energy levels $\frac{E_0}{n^2}$ for $z = -1$, Kepler potential, e.g., the hydrogen atom, and nE_0 for $z = 2$, harmonic oscillator.

Using intrinsic units $(E_0, l_0, \hbar)$, the dynamics can be formulated in dimensionless variables:

$$\underline{H} = E_0 H, \quad \underline{x} = l_0 x, \quad \underline{p} = \frac{\hbar}{l_0} p \Rightarrow H = \frac{p^2}{2} + \epsilon \frac{x^z}{z}.$$

There remain "purely mathematical" forms with dimensionless parameters, which, in basically important cases can be characterized by symmetry considerations and representation properties of operation groups, e.g., in quantum theories, as eigenvalue equations for representation characterizing invariants. For instance, the gravitative Newton and the electromagnetic Coulomb[33] potential have equal dimensionless mathematical forms. Mathematics takes over, e.g., by a mathematically intrinsic characterization of a Hamiltonian as an invariant of an operation group.

If there occur more than three independent intrinsic units, pure numbers are physically relevant. An example is the light deflection at the sun boundary (earlier).

A analogue situation with respect to intrinsic units and canonical pairs occurs in field theories. A space or a spacetime field (for generality also a time field) is defined as a function of the corresponding time, position and spacetime parameters — in the basically important cases valued as a vector in a space with time, space and spacetime representing operations. An example in a relativistic field theory is a Lorentz-scalar spacetime field $\underline{\mathbf{\Phi}}$, where each spacetime translation $\underline{x}$ with equally dimensioned spacetime coordinates $(\underline{x}^j)_{j=0}^3 = (c\underline{t}, \underline{x}, \underline{y}, \underline{z})$ is associated with a value $\underline{\mathbf{\Phi}}(\underline{x})$, measuring, e.g., a charge, a current strength, etc. For a classical field theory, the dynamics can be given by a Lagrangian density $\underline{\mathcal{L}}$, e.g., with a power-"potential $V^n(\underline{\mathbf{\Phi}}) = g_0^2 \underline{\mathbf{\Phi}}^n$:"

$$\underline{\mathcal{L}} = \frac{1}{2} \frac{\partial \underline{\mathbf{\Phi}}}{\partial \underline{x}_j} \frac{\partial \underline{\mathbf{\Phi}}}{\partial \underline{x}^j} - g_0^2 \underline{\mathbf{\Phi}}^n, \quad n \in \mathbb{N}.$$

Good to know: Einstein made the mathematical notation more transparent by his shorthand sum convention: Doubly occurring indices involve a summation, here: $\frac{\partial \underline{\mathbf{\Phi}}}{\partial \underline{x}_j} \frac{\partial \underline{\mathbf{\Phi}}}{\partial \underline{x}^j} = \sum_{j=0}^{3} \frac{\partial \underline{\mathbf{\Phi}}}{\partial \underline{x}_j} \frac{\partial \underline{\mathbf{\Phi}}}{\partial \underline{x}^j}$.

[33] Charles Augustin de Coulomb (1736–1806).

The spacetime-integrated Lagrange density $\int d^4\underline{x}\ \underline{\mathcal{L}}(\underline{x})$ has the dimension of an action. The *canonical field pair* $(\underline{\boldsymbol{\Phi}}, \frac{\partial \underline{\boldsymbol{\Phi}}}{\partial \underline{x}_0})$, comprising the field and its time derivative, has the dimension of an action position density. It involves two distributions in position space, which have broken MKS-units, here square roots:

$$\left.\begin{array}{r}[\underline{x}_j] = \text{m}, \\ [\underline{\mathcal{L}} \cdot \underline{x}_j] = [\underline{\boldsymbol{\Phi}} \cdot \frac{\partial \underline{\boldsymbol{\Phi}}}{\partial \underline{x}_j}] = \frac{[\hbar]}{\text{m}^3}\end{array}\right\} \Rightarrow, [\underline{\boldsymbol{\Phi}}] = \sqrt{\frac{\text{kg}}{\text{s}}}.$$

Therefore, in addition to the universal intrinsic speed of light unit c, one has the special intrinsic unit g_0^2:

$$[c] = \frac{\text{m}}{\text{s}}, \quad [g_0^4] = \frac{\text{kg}^{2-n}}{\text{m}^4\text{s}^{2-n}}.$$

In a relativistic quantum field theory with $(\hbar, c, g_0^2)$ all units are intrinsic.

2.7 The $(\hbar, c, G)$-System — Universal Units?

For a basic understanding and definition of units, it is unsatisfactory to use the intrinsic units of a special problem with a special dynamics (Hamiltonian) — one would prefer to start from a universal dynamics, or, more general, from structures that underly all physical dynamics and lead to units, valid "for all times…and cultures" (Planck 1900).

Physics describes *causal actions in space*. To quantify this, one needs three basic units: a unit for time (causality), a unit for space (position), and a unit for actions. Obviously, by phenomenological and convenience reasons, one may employ more units, e.g., in addition, an electric charge unit and a temperature unit. Interesting is the question, always again leading to long and heated discussions, if a complete dimensionalization is possible by three units and if this number makes sense as a minimal one with a deeper significance. Thus, one could think, on a first view, that the completely different and, in the historical development after gravity, new electromagnetic actions would require also a new unit, e.g., for electric charge. That, however, is not the case: Electromagnetism needs no new units. Analogously, one could think of new units for nuclear binding or decay forces (strong and weak interactions) — also that is not necessary. If one takes, in the other direction, the extreme point of view, that one considers a dynamics, formulated only in dimensionless variables — like with a dimensionless Hamiltonian H earlier, where there occur no units at all, one can ask the question, if there exist dynamical features, which can be quantified by one, two, three, or more units to introduce dimensions as quantitative markers for qualitatively different structures, e.g., for time and space and for different operation groups.

If one looks into a physics textbook with respect to the definition of units systems, two types of definitions can be distinguished: Thus, in the MKS-system (m, kg, s), the definition of meter, kilogram, and second used, originally, the simple reference to prototypes, e.g., to the ur-meter and the ur-kilogram in Paris, and to the average sun-day. Later on, the definitions were refined; e.g., the meter was defined via the speed of light and the otherwise defined second. For a "good," i.e., reliable definition of those basic units, the prototypes must have certain properties, e.g., a constant behavior in time; i.e., the validity of, possibly still unknown, physical laws is presupposed.

These three basic units are supplemented by "dynamical units," whose definition uses directly a physical law. An example is the definition of the temperature unit Kelvin via the Boltzmann conversion factor, which uses the ideal gas equation, or the definition of the charge unit Coulomb via the Coulomb law, and, then, the definition of the current strength unit Ampère.[34]

Unit systems are distinguished quantitatively and qualitatively. Instead of meter, kilogram, and second one can use handbreadth, stone, and heartbeat time — here is only a quantitative difference, effected by recalibrations (gauging, dilations). Two such unit systems are related to each other by three real dilation factors. For different dimensions of the three basic units, e.g., time, energy, and mass instead of length, momentum, and velocity, there exists a qualitative difference in addition to the three scale transformations.

A minimal *universal* unit system could be distinguished by a universal dynamics and an all physics-underlying law, whose structure determines both the number and the dimensions of the basic units. $(\hbar, c, G)$ with the quantum unit (Planck's unit) $\hbar$, the field unit (Einstein's unit) c and the gravity unit (Newton's unit) G could possibly be such a unit system. Deep structural insights are connected with the maximal Einstein unit from special relativity, as well as with the minimal Planck unit from quantum theory. It is strongly suggestive, that an understanding of the third unit — perhaps Newton's unit — will arise from a qualitative progress in the unification of the spacetime interactions, possibly by a common understanding of the geometrical structures of general relativity and the probability oriented information structures of quantum theory.

Natural equivalences for time and length, on the one side, and mass, on the other side, arise from products on the "quantum road" (Compton length), and from quotients on the "gravity road" (Schwarzschild length). If both roads merge, there arise *natural units for time and length*, perhaps minimal, if Newton's unit G is minimal:

$$t_{\min} = \sqrt{\frac{\hbar G}{c^5}} \simeq 5.5 \times 10^{-44}\ \text{s} \qquad \text{(minimal(?))},$$
$$l_{\min} = \sqrt{\frac{\hbar G}{c^3}} \simeq 1.6 \times 10^{-35}\ \text{m} \qquad \text{(minimal(?))}.$$

[34] André Marie Ampère (1775–1836).

The also arising *Planck mass* is neither minimal nor maximal:

$$M_{\text{Planck}} = \sqrt{\frac{\hbar c}{G}} \simeq 2.2 \times 10^{-8}\ \text{kg}.$$

It is about 20 µg — a bacterium or almost a little flea: This magnitude is not totally exotic for us. With this mass and the easily rememberable connection of G and c to the anthropomorphic measures and the magnitude of the earth, one can compute the value of Planck's quantum unit $\hbar$ without using data books.

The real factors for the conversion of $(l_{\text{min}}, M_{\text{Planck}}, t_{\text{min}})$ to the MKS-system, basically uninteresting but characteristic for us human beings, constitute an example for the scale transformations, mentioned earlier.

It is not obvious, if the minimal(?) time and length have really an operational minimal meaning — perhaps they are nothing more than a naive combination of units. It is at least conceivable, if Newton's unit G is minimal.

The "mesoscopic" Planck mass, neither minimal nor maximal, is a borderline between macroscopic gravity and microscopic quantum structure: The product of Schwarzschild and Compton length, which are, respectively, directly and inversely directly proportional to the mass, are, for all masses, the square of the minimal(?) length. Both lengths are equal for the Planck mass:

$$\begin{aligned} l_{\text{Schwarz}}(m)\, l_{\text{Compt}}(m) &= l_{\text{min}}^2 = \frac{\hbar G}{c^3}, \\ l_{\text{Schwarz}}(M_{\text{Planck}}) &= l_{\text{Compt}}(M_{\text{Planck}}) = l_{\text{min}}, \\ \frac{l_{\text{Schwarz}}(m)}{l_{\text{min}}} &= \frac{l_{\text{min}}}{l_{\text{Compt}}(m)} = \frac{m}{M_{\text{Planck}}}. \end{aligned}$$

Then, for the minimal(?) length, the Newton and Yukawa mass are equal to Planck's mass:

$$\begin{aligned} m_{\text{Newt}}(l)\, m_{\text{Yuk}}(l) &= M_{\text{Planck}}^2 = \frac{\hbar c}{G}, \\ m_{\text{Newt}}(l_{\text{min}}) &= m_{\text{Yuk}}(l_{\text{min}}) = M_{\text{Planck}}. \end{aligned}$$

With the natural *mass logarithm*, defined by the mass, measured in Planck mass units:

$$\psi(m^2) = \log \frac{m^2}{M_{\text{Planck}}^2}, \quad m^2 = M_{\text{Planck}}^2 e^{\psi(m^2)},$$

the quantum-gravity borderline (Planck's mass) can be put to zero — smaller masses in the "quantum region" have negative, larger masses in the "gravity region" have positive mass logarithms. As an example, two masses are considered, which, with respect to the geometrical mean, have about equal distances to the human measures:

$$\text{electron: } m_e \simeq 0.91 \times 10^{-30}\text{ kg} \quad \Rightarrow \begin{cases} l_{\text{Compt}}(m_e) \simeq 3.9 \times 10^{-13}\text{ m}, \\ l_{\text{Schwarz}}(m_e) \simeq 7 \times 10^{-58}\text{ m}, \\ \psi(m_e^2) \simeq -103, \end{cases}$$

$$\text{sun: } m_\odot \simeq 2 \times 10^{30}\text{ kg} \quad \Rightarrow \begin{cases} l_{\text{Compt}}(m_\odot) \simeq 1.7 \times 10^{-73}\text{ m}, \\ l_{\text{Schwarz}}(m_\odot) \simeq 1.5 \times 10^{3}\text{ m}, \\ \psi(m_\odot^2) \simeq 175, \end{cases}$$

$$\sqrt{m_e m_\odot} \simeq 1.35\text{ kg}.$$

In a "wild argumentation" with the $(\hbar, G, c)$ units, one can conceive a science fiction scenario, not to be taken too seriously: At the minimal(?) time, there was the maximal(?) cosmic density. The cosmic mass at the minimal(?) time was the Planck mass:

$$\rho_{\cos,p}(t_{\min}) = \frac{3}{8\pi}\frac{c^5}{\hbar G^2} \simeq 6 \times 10^{95}\,\frac{\text{kg}}{\text{m}^3}, \quad m_{\cos,p}(t_{\min}) = M_{\text{Planck}}.$$

The qualitative conversion from one unit system, e.g., from the anthropomorphic MKS-system, to another one, e.g., to the possibly universally intrinsic $(\hbar, c, G)$-system, can be linearized in a dimension space — here a three-dimensional rational space $\mathbb{Q}^3$. In a *dimensional grading*, each physical quantity $a \in \mathcal{A}$ obtains its dimension powers:

$$\dim : \mathcal{A} \longrightarrow \mathbb{Q}^3, a \longmapsto \dim(a).$$

Any unit system defines a basis of the dimension space:

$$\begin{matrix}\text{grading}\\ \text{in MKS-basis:}\end{matrix} \begin{cases} \dim(\text{m}) \cong \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad \dim(\text{kg}) \cong \begin{pmatrix}0\\1\\0\end{pmatrix}, \quad \dim(\text{s}) \cong \begin{pmatrix}0\\0\\1\end{pmatrix}, \\ \dim(\hbar) \cong \begin{pmatrix}2\\1\\-1\end{pmatrix}, \quad \dim(c) \cong \begin{pmatrix}1\\0\\-1\end{pmatrix}, \dim(G) \cong \begin{pmatrix}3\\-1\\-2\end{pmatrix}, \end{cases}$$

$$\begin{matrix}\text{grading}\\ \text{in}(\hbar, c, G) - \text{basis:}\end{matrix} \begin{cases} \dim(\hbar) \cong \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad \dim(c) \cong \begin{pmatrix}0\\1\\0\end{pmatrix}, \quad \dim(G) \cong \begin{pmatrix}0\\0\\1\end{pmatrix}, \\ \dim(\text{m}) \cong \frac{1}{2}\begin{pmatrix}1\\-3\\1\end{pmatrix}, \dim(\text{kg}) \cong \frac{1}{2}\begin{pmatrix}1\\1\\-1\end{pmatrix}, \quad \dim(\text{s}) \cong \frac{1}{2}\begin{pmatrix}1\\-5\\1\end{pmatrix}. \end{cases}$$

Since the association of the dimensions powers is a grading, i.e., one has for the multiplication of physical quantities the logarithmic properties:

$$\dim(a_1 a_2) = \dim a_1 + \dim a_2, \quad \dim(a^k) = k \dim(a), \quad k \in \mathbb{Q},$$

one obtains, in this formalismus, the dimension of a product by vector addition of the dimensions of the factors. This homomorphism formalizes mathematically the dimension analysis, mentioned at the beginning of this chapter. A correct calculation with the complete physical quantities is projected to a correct calculation for the dimensions, obviously not vice versa.

Good to know: Sets S can have a structure, e.g., an order structure, an algebraic structure, or a topological structure, etc. Mappings for sets $f : S_1 \longrightarrow S_2$ with the same structure are called *morphisms*, if they are compatible with this structure. For instance, vector space morphisms $f : V_1 \longrightarrow V_2$ are compatible with the vector space defining structures — the vector addition $f(v+w) = f(v) + f(w)$ and the scalar multiplication $f(\alpha v) = \alpha f(v)$. Order morphisms are monotonous mappings, topological morphisms are continuous mappings.
The morphisms sets $\mathbf{kat}(S_1, S_2)$, e.g., all linear mappings $\mathbf{vec}_{\mathbb{R}}(V_1, V_2) = \{f : V_1 \longrightarrow V_2,\ \text{real linear}\}$ for two real vector spaces, can be collected with the sets in a (little) *category* $\underline{\mathbf{kat}}$, e.g., the category $\underline{\mathbf{vec}}_{\mathbb{R}}$ for the real vector spaces and their linear mappings, or the category $\underline{\mathbf{top}}$ for topological spaces with continuous mappings, etc.
Categories formalize mathematical structures. For a (little) category, e.g., complex Hilbert spaces for quantum theories, one is interested especially in its (endo-, iso, auto-)morphisms, in its product structures, and in its quotient (equivalence) structures.
Relations between different (little) categories, e.g., between topological spaces $\underline{\mathbf{top}}$ and complex vector spaces $\underline{\mathbf{vec}}_{\mathbb{C}}$, are formalized by *functors*.

Complete systems with basic units are related to each other by nonsingular transformations $\mathbf{E}$ of the unit space, here 3×3-matrices with half-integer entries, where the powers of the basic units stand in the columns, e.g., the connection transformations of $(\hbar, c, G)$- and MKS-system:

$$\dim_{(\hbar,c,G)}(a) = \mathbf{E}^{(\mathrm{M,K,S})}_{(\hbar,c,G)} \dim_{(\mathrm{M,K,S})}(a), \qquad \text{with } \mathbf{E}^{(\mathrm{M,K,S})}_{(\hbar,c,G)} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 \\ -3 & 1 & -5 \\ 1 & -1 & 1 \end{pmatrix},$$

$$\dim_{(\mathrm{M,K,S})}(a) = \mathbf{E}^{(\hbar,c,G)}_{(\mathrm{M,K,S})} \dim_{(\hbar,c,G)}(a), \qquad \text{with } \mathbf{E}^{(\hbar,c,G)}_{(\mathrm{M,K,S})} = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{pmatrix},$$

e.g., for energy:

$$\dim_{(\mathrm{M,K,S})}(E) = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} = \mathbf{E}^{(\hbar,c,G)}_{(\mathrm{M,K,S})} \begin{pmatrix} \frac{1}{2} \\ \frac{5}{2} \\ -\frac{1}{2} \end{pmatrix}, \quad \frac{\mathrm{m}^2\,\mathrm{kg}}{\mathrm{s}^2} \leftrightarrow \sqrt{\frac{\hbar c^5}{G}}.$$

In the opposite direction, the inverse transformation has to be taken with $\mathbf{E}^{(\mathrm{M,K,S})}_{(\hbar,c,G)} \mathbf{E}^{(\hbar,c,G)}_{(\mathrm{M,K,S})} = \mathbf{1}_3$.

Good to know: An (associative) algebra is a real or complex vector space with a bilinear (associative) product $\bullet$ of the vectors. If an associative algebra is a direct sum $\mathcal{A} = \bigoplus_{j \in H} V_j$ of vector spaces V_j, with the indices taken from an additive semigroup, and if, in addition, the multiplication is compatible with the H-addition, $V_j \bullet V_k \subseteq V_{j+k}$, and if, finally, the vector space V_0 with the neutral element $0 \in H$ is given by the scalar field $\mathbb{R}$ or $\mathbb{C}$, then $\mathcal{A}$ is *graded* by the semigroup H. The grading is the monoid morphism grad: $\mathcal{A} \longrightarrow H$. Examples are tensor, Grassmann,[35] or polynomial algebras; they are graded with the natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$.

The dimension of the dimension-grading space, possibly exactly three, e.g., for a vector space $\mathbb{R}^3$ or for the additive groups $\mathbb{Z}^3$, $(\frac{1}{2}\mathbb{Z})^3$, and $\mathbb{Q}^3$, gives the number of the fundamental units. An integer $\mathbb{Z}^3$- and a half-integer $(\frac{1}{2}\mathbb{Z})^3$-grading uses only integer and half-integer dimensions, respectively, of exactly three basic units.

[35] Hermann Grassmann (1809–1877).

2.8 Electrodynamics and Sommerfeld's Fine-Structure Constant

The electromagnetic unit Coulomb C can be defined in a human-related way by the potential for the Coulomb force, which two resting point charges Q, Q' in the distance r exert on each other:

$$V_{\text{Coul}}(r) = \frac{1}{4\pi\epsilon_0}\frac{QQ'}{r} = \frac{\mu_0 c^2}{4\pi}\frac{QQ'}{r},$$

$$\text{definition of the unit Coulomb: } \frac{\mu_0}{4\pi} = 10^{-7}\,\frac{\text{mkg}}{\text{C}^2},$$

$$\frac{1}{c} = \sqrt{\epsilon_0\mu_0}.$$

ϵ_0 is the electric field constant (vacuum-influence constant), μ_0 is the magnetic field constant (vacuum permeability); their geometrical mean is the inverse maximal action velocity c.

Good to know: Electron is Greek for amber; Magnesia was an ancient Greek town in Minor Asia.

It is highly remarkable, that, apparently, proton and positron have precisely the same electric charge[36]:

$$e \simeq 1.6 \times 10^{-19}\text{ C}.$$

If this charge goes through the potential difference of one Volt[37] V, it obtains with $\text{V} = \frac{\text{J}}{\text{C}}$ the energy with its mass and temperature equivalents:

$$1\text{ eV} \simeq 1.6 \times 10^{-19}\text{ J},\quad 1\frac{\text{eV}}{c^2} \simeq 1.8 \times 10^{-36}\text{ kg},\quad 1\frac{\text{eV}}{k_{\text{Boltz}}} \simeq 1.15 \times 10^{4}\text{ K}.$$

Then, the kinetic energy $\frac{m_e v^2}{2}$ of an electron comes with about two thousands of the speed of light $\frac{v}{c} \sim 2 \times 10^{-3}$.

The charge of particles proves to be discretized by integer multiples of this smallest *elementary charge* e, with individual *charge numbers* z:

$$Q = ez,\quad z \in \mathbb{Z}.$$

Therefore, the definition of the unit Coulomb is the arbitrary fixing of the hopefully integer $\frac{\text{C}}{e} \simeq 6 \times 10^{18}$.

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[36] Hermann von Helmholtz (1821–1894), Robert Andrews Millikan (1868–1953).

[37] Alessandro Volta (1745–1827).

A strong hint for the discretization of charge was historically given by the common electric charge of one kilomol of simply charged ions, e.g., of hydrogen ions, determining the *Faraday*[38] *charge*:

$$Q_{\text{Faraday}} = N_{\text{Avo}} e \simeq 9.648 \times 10^7 \ \frac{\text{C}}{\text{kilomol}}.$$

One millimol has as charge about 96 C; this was the starting point for the "anthropomorphic" Coulomb definition, which comes from — not very accurate — electrolytic experiments with molecules containing silver with a relative atom mass $Z(\text{Ag}) \simeq 108$.

For atoms and elementary particles, the elementary charge serves as unit:

$$\frac{\mu_0 e^2}{4\pi} = \frac{e^2}{4\pi\epsilon_0 c^2} \simeq 2.6 \times 10^{-45} \text{ mkg}.$$

This yields the *intrinsic action unit of electrodynamics*:

$$\hbar_e = e^2 \sqrt{\frac{\mu_0}{\epsilon_0}} = e^2 \mu_0 c = \frac{e^2}{\epsilon_0 c} \simeq 10^{-35} \ \frac{\text{m}^2\text{kg}}{\text{s}}.$$

Here, $\sqrt{\frac{\mu_0}{\epsilon_0}}$ is the wave resistance of the vacuum, with the unit Coulomb about 377 Ω (Ohm[39] $\Omega = \frac{\text{V}}{\text{A}}$).

The electromagnetic action strength is defined as the ratio of the action unit of electrodynamics $\hbar_e$ to Planck's action unit $\hbar$, which gives up to 4π the dimensionless *Sommerfeld*[40] *number*:

$$g_e^2 = \frac{\hbar_e}{\hbar} \simeq \frac{1}{10.9}, \quad \alpha_{\text{Som}} = \frac{g_e^2}{4\pi} \simeq \frac{1}{137.04}.$$

Neither ϵ_0 nor μ_0 have a fundamental meaning, even not the elementary charge e. A basic importance has only the product and the quotient of $\mu_0 e^2$ and $\frac{\epsilon_0}{e^2}$, i.e., $\frac{1}{c^2}$ and $\hbar_e^2$, or, with Planck's unit, the pure number $g_e^2 = \alpha_{\text{Som}} 4\pi$. Electrodynamics has, in addition to c and $\hbar_e$, no third intrinsic unit, e.g., no intrinsic length.

The wave resistance of the vacuum $\frac{\hbar_e}{e^2} \simeq 377\ \Omega$ corresponds to the Hall[41] resistance $\frac{\hbar}{e^2} \simeq 4100\ \Omega$.

The electromagnetic actions are, compared with the gravitative actions, a completely new phenomenon (What was Newton's explanation of lightnings?). However, if one starts with three "mechanical" units, they do not require an extension of the dimensionality of the unit space — as intrinsic units, the maximal action

[38]Michael Faraday (1791–1867).

[39]Georg Simon Ohm (1789–1854).

[40]Arnold Sommerfeld (1868–1951).

[41]Edwin Hall (1855–1938).

velocity c and the electromagnetic action unit $\hbar_e$ have to be added, and integer individual charge numbers $z \in \mathbb{Z}$:

$$\begin{array}{rl} \text{Coulomb potential:} & V_{\text{Coul}}(r) = \frac{1}{4\pi\epsilon_0}\frac{QQ'}{r} = \frac{QQ'}{e^2} g_e^2 \frac{c\hbar}{4\pi r} = zz'\alpha_{\text{Som}}\ \frac{c\hbar}{r}, \\ \text{Newton potential:} & V_{\text{Newt}}(r) = -G\frac{mm'}{r} = -\frac{mm'}{M_{\text{Planck}}^2}\frac{c\hbar}{r}. \end{array}$$

Both potentials have a precise $\frac{1}{r}$-dependence — not, e.g., $\frac{1}{r^{1.0024}}$. Under the Coulomb potential for the electromagnetic "internal" interaction stands the analogously structurized Newton potential for an "external" interaction, where the masses enter with dimensionless positive real ratios $\frac{m}{M_{\text{Planck}}} \in \mathbb{R}_+$ in units of the Planck mass, in analogy to the integer charge numbers $z = \frac{Q}{e} \in \mathbb{Z}$.

Good to know: The precise $\frac{1}{r}$-dependence is required for the solution of the inhomogeneous Laplace[42] equation $-\vec{\partial}^2\frac{1}{r} = 4\pi\delta(\vec{x})$ with the rotation invariant translation action $\vec{\partial}^2 = \partial_1^2 + \partial_2^2 + \partial_3^2$ for flat 3-position and Dirac's delta-distribution. Its special relativistic embedding with Lorentz invariant spacetime translation action $\partial^2 = \partial_0^2 - \vec{\partial}^2$ leads to Feynman[43] propagators for massless fields with the lightcone-supported Dirac distribution $\frac{1}{r} = \int \frac{d^3q}{2\pi^2}\frac{1}{\vec{q}^2}e^{i\vec{q}\vec{x}} \hookrightarrow \delta(x^2)$:

$$\int \frac{d^4q}{4\pi^3}\frac{1}{-q^2 - io}e^{iqx} = \frac{i}{\pi(-x^2 + io)}, \quad \partial^2 \int \frac{d^4q}{4\pi^3}\frac{1}{-q^2 - io}e^{iqx} = \partial^2\delta(x^2) = 4\pi\delta(x),$$

as used, for example, for the electromagnetic field. The Kepler (Newton, Coulomb) potential $\frac{1}{r}$ is embedded as off-shell (non-particle) contribution of the corresponding Feynman propagators.

The classical spacetime theory for the electrostatic Coulomb potential is Maxwell's[44] electrodynamics whereas Newton's gravitostatic potential is the entrance door to Einstein's general relativity and nonflat classical spacetime.

Maxwell's equations in the vacuum describe the spacetime source and vortex structure of the electric field strengths $\vec{\mathbf{E}}$ and the magnetic flux density $\vec{\mathbf{B}}$ with the aid of the charge density $\boldsymbol{\rho}$ and the current density $\vec{\mathbf{j}}$: The electric field swells from the charge density, it whirls around the time-dependent magnetic flux (density); the vortices of the sourcefree magnetic flux (density) are caused by the current (density) or an electric field that changes in time:

$$\begin{array}{rll} \text{homogeneous:} & \text{div}\,\vec{\mathbf{B}} = 0, & \text{rot}\,\vec{\mathbf{E}} + \frac{\partial\vec{\mathbf{B}}}{\partial t} = 0, \\ \updownarrow\ \text{duality} & -\frac{\vec{\mathbf{E}}}{c} \leftrightarrow \vec{\mathbf{B}}, & \frac{1}{c^2} = \epsilon_0\mu_0, \\ \text{inhomogeneous:} & \epsilon_0\text{div}\,\vec{\mathbf{E}} = \boldsymbol{\rho}, & \frac{1}{\mu_0}\text{rot}\,\vec{\mathbf{B}} - \epsilon_0\frac{\partial\vec{\mathbf{E}}}{\partial t} = \vec{\mathbf{j}}. \end{array}$$

There are additional equations of motion for the constituents of charge and current density ("matter" equations).

Introducing for the solution of the two homogeneous equations a vector potential $\vec{\mathbf{A}}$ and a scalar potential $\mathbf{U}$, determined up to gauging (ahead):

[42]Pierre-Simon Laplace (1749–1827).

[43]Richard Feynman (1918–1988).

[44]James Clerc Maxwell (1831–1879).

$$\vec{\mathbf{B}} = \text{rot}\, \vec{\mathbf{A}} \text{ and } \vec{\mathbf{E}} + \frac{\partial \vec{\mathbf{A}}}{\partial t} = -\text{grad}\, \mathbf{U},$$

the Maxwell equations can be written with the intrinsic units in the form

$$\left.\begin{array}{r} \partial_j \mathbf{A}_k - \partial_k \mathbf{A}_j = g_e^2 \mathbf{F}_{kj}, \\ \partial^k \mathbf{F}_{kj} = \mathbf{J}_j \end{array}\right\} \Rightarrow \partial^k(\partial_j \mathbf{A}_k - \partial_k \mathbf{A}_j) = \frac{1}{g_e^2}\mathbf{J}_j.$$

This form is manifestly compatible with the transformation of the Lorentz group (ahead), which were discovered exactly here. Either the Galileo invariance, valid for point mechanics, or the Lorentz invariance, valid for electromagnetism, can be used as basic symmetry. Point mechanics had to be reformulated in an "improved" Lorentz compatible form.

The Lorentz transformations act not only on the four dimensional spacetime translations and the spacetime derivations ("four-vectors"):

$$\partial_j \cong \left(\frac{\partial}{c\partial t}, \frac{\partial}{\partial \vec{x}}\right), \quad [\partial_j] = \frac{1}{\text{m}}.$$

Also the charge-current densities and the potentials constitute Lorentz "4-vectors"; the field strengths as a Lorentz "6-vector" arise as the nontrivial components of an antisymmetric (4×4)-matrix:

$$\mathbf{J}_k \cong \frac{\sqrt{\hbar}}{e}\left(\boldsymbol{\rho}, \frac{\vec{\mathbf{j}}}{c}\right), \quad \mathbf{A}_k \cong \frac{e}{\sqrt{\hbar}}\left(\frac{\mathbf{U}}{c}, \vec{\mathbf{A}}\right),$$

$$\mathbf{F}_{kj} \cong \frac{\hbar}{\hbar_e}\frac{e}{\sqrt{\hbar}}\begin{pmatrix} 0 & -\frac{\mathbf{E}_1}{c} & -\frac{\mathbf{E}_2}{c} & -\frac{\mathbf{E}_3}{c} \\ \frac{\mathbf{E}_1}{c} & 0 & \mathbf{B}_3 & -\mathbf{B}_2 \\ \frac{\mathbf{E}_2}{c} & -\mathbf{B}_3 & 0 & \mathbf{B}_1 \\ \frac{\mathbf{E}_3}{c} & \mathbf{B}_2 & -\mathbf{B}_1 & 0 \end{pmatrix}.$$

To obtain the dual 6-vector $\epsilon^{lmkj}\mathbf{F}_{kj}$, electric and magnetic fields have to be exchanged via $-\frac{\vec{\mathbf{E}}}{c} \leftrightarrow \vec{\mathbf{B}}$.

The potentials are gauge-dependent, i.e., determined up to a spacetime dependent Lorentz-scalar function $x \longmapsto \alpha(x)$:

$$\mathbf{A}_k \longmapsto \mathbf{A}_k + \partial_k \alpha.$$

The dimensions are chosen in such a way, that the potentials $\mathbf{A}$ and the field strengths $\mathbf{F}$ constitute space densities for a canonical field pair; i.e., their product has the dimension of an action density:

$$[\mathbf{A}_k \cdot \mathbf{F}_{jk}] = \frac{[\hbar]}{\mathrm{m}^3} = \mathrm{m}^{-1}\frac{\mathrm{kg}}{\mathrm{s}} \Rightarrow \begin{cases} [\mathbf{A}_k] = \sqrt{\frac{\mathrm{kg}}{\mathrm{s}}}, \\ [\mathbf{F}_{jk}] = \mathrm{m}^{-1}\sqrt{\frac{\mathrm{kg}}{\mathrm{s}}}, \\ [\mathbf{J}_k] = \mathrm{m}^{-2}\sqrt{\frac{\mathrm{kg}}{\mathrm{s}}}. \end{cases}$$

As seen at the units, the electromagnetic vectors ($\vec{\mathbf{E}}$, $\vec{\mathbf{B}}$, $\vec{\mathbf{A}}$) are not valued like position $\vec{x}$, e.g., they do not oscillate in position. However, they transform equivalently under rotations.

Good to know: Loosely speaking, the Maxwell equations formulate the simplest case of a special relativistic theory, i.e., compatible with the Lorentz group action, for a locally conserved quantity (charge): The current $\mathbf{I} = -\frac{d\mathbf{Q}}{dt}$ as the temporal change of a charge, localized with current and charge densities $\int_{\partial V} d\vec{^2x}\, \vec{\mathbf{j}}(t,\vec{x}) = -\frac{d}{dt}\int_V d^3x\, \boldsymbol{\rho}(t,\vec{x})$ with the volume element $d^3x = dx_1dx_2dx_3$ and the three area elements $d\vec{^2x} = (dx_2dx_3, dx_3dx_1, dx_1dx_2)$, and assumed to be valid for any volume V with surface ∂V, leads to the local conservation $\dot{\boldsymbol{\rho}} + \mathrm{div}\,\vec{\mathbf{j}} = 0$. Expressing the source structure of the charge by the electric field $\boldsymbol{\rho} = \mathrm{div}\,\vec{\mathbf{D}}$, $\vec{\mathbf{D}} = \epsilon_0\vec{\mathbf{E}}$, allows the "solution" of the local conservation by the magnetic field $\mathrm{rot}\,\vec{\mathbf{H}} = \partial_t\vec{\mathbf{D}} + \vec{\mathbf{j}}$, $\vec{\mathbf{B}} = \mu_0\vec{\mathbf{H}}$. The special relativistic formulation with the Lorentz 4-vector $\mathbf{J}$ and the adjoint antisymmetric Lorentz 6-vector $\mathbf{F}$ as defined above gives the inhomogeneous equations $\partial^k\mathbf{F}_{kj} = \mathbf{J}_j$. They can be solved by the "four-dimensional rotation" $\mathbf{F}_{kj} = \frac{1}{g_e^2}(\partial_j\mathbf{A}_k - \partial_k\mathbf{A}_j)$ of a Lorentz 4-vector $\mathbf{A}$ which involves the homogeneous equations for the dual adjoint Lorentz 6-vector $\partial_m\epsilon^{lmkj}\mathbf{F}_{kj} = 0$.

2.9 Binding Energies and Couplings

The Kepler dynamics with the Hamiltonian $H = \frac{\vec{p}^2}{2m} + \frac{\epsilon g_0^2}{r}$, $\epsilon = \pm 1$, has, in the classical treatment, two intrinsic units, the mass $[m] = \mathrm{kg}$ and the coupling constant $[g_0^2] = \frac{\mathrm{m}^3\mathrm{kg}}{\mathrm{s}^2}$, i.e., no intrinsic length. For instance, the gravitational Newton interaction can be used for apples, planets and galaxies. In the quantum framework, Planck's unit $[\hbar] = \frac{\mathrm{m}^2\mathrm{kg}}{\mathrm{s}}$ leads to an intrinsic length unit. In nonrelativistic atomic physics with the Coulomb potential, e.g., for the states of the hydrogen atom, there does not occur the maximal action velocity c; there arises the smaller electromagnetically normalized velocity $\alpha_{\mathrm{Som}}c \simeq 2.2 \times 10^6\ \frac{\mathrm{m}}{\mathrm{s}}$, e.g., in the *Bohr*[45] *length*:

$$l_{\mathrm{Bohr}}(m,n) = \frac{\hbar}{\alpha_{\mathrm{Som}}c}\frac{1}{nm} = \frac{\hbar^2}{4\pi c\hbar_e}\frac{1}{nm} = \frac{1}{n\alpha_{\mathrm{Som}}}l_{\mathrm{Compt}}(m),$$

which depends on a mass m and a natural number n. For $n < \frac{1}{\alpha_{\mathrm{Som}}} \sim 137$, the Bohr length is larger than the Compton length. For the Coulomb interaction of two equally charged mass points, one uses, in the related Bohr length, their reduced mass and their negative charge product:

$$\frac{1}{m} = \frac{1}{m_1} + \frac{1}{m_2}, \quad n = -z_1z_2.$$

[45]Niels Bohr (1885–1962).

For charges with equal sign $z_1 z_2 > 0$ and repulsive Coulomb interaction, the Bohr length makes no sense since a binding is impossible without additional forces. The Compton length depends linearly on the action quantum $\hbar$, the Bohr length quadratically. The intrinsic units are the reduced mass, Planck's unit and the "electromagnetically reduced" velocity, i.e., $(m, \hbar, \alpha_{\text{Som}} c)$.

The Bohr length of the hydrogen atom — in a good approximation with the electron mass as reduced mass — gives the order of magnitude for the "size" of an atom:

$$l_{\text{Bohr}}(m_e, 1) = \frac{l_{\text{Compt}}(m_e)}{\alpha_{\text{Som}}} \simeq 0.53 \times 10^{-10} \text{ m}, \quad \frac{m_p}{m_e} \simeq 1836.$$

The product of the corresponding $\hbar^6$-proportional Bohr volume with the Avogadro number describes "densely packed" hydrogen and relates the two bridges $\hbar$ and N_{Avo} from the microscopic to the human order of magnitude phenomena:

$$\frac{4\pi}{3}[l_{\text{Bohr}}(m_e, 1)]^3 N_{\text{Avo}} \simeq 0.4 \times 10^{-3} \frac{\text{m}^3}{\text{kilomol}} \simeq 0.4 \frac{\text{cm}^3}{\text{Mol}}.$$

The product of the electron mass and a charge number $n = 0, 1, \ldots$ in

$$E_{\text{Ryd}}(m_e, n^2) = n^2 m_e (\alpha_{\text{Som}} c)^2 = n^2 m_e \left(\frac{e^2}{4\pi\hbar\epsilon_0}\right)^2,$$

gives, e.g., with $n = 1$ for the proton, the ionization energy $E_{\text{Ryd}}(m_e, 1) \simeq 27.2$ eV of the hydrogen atom. The binding energies contain a factor $\alpha_{\text{Som}}^2 \sim 6 \times 10^{-5}$ that reduces the electron mass-energy $m_e c^2 \sim 0.5$ MeV to the Rydberg[46] energy.

To compare: The binding energy of a nucleon in an atom is about 7 bis 9 MeV, yielding as the ratio of nuclear energy to atomic energy an order of magnitude $\frac{\text{MeV}}{\text{eV}}$, i.e., about one million. If one parametrizes, in analogy to the atomic binding energy the energy for nuclear binding with a strong coupling constant α_{strong} by $E_{\text{strong}}(m) = m(\alpha_{\text{strong}} c)^2$, there follows with the nucleon and electron mass: $\frac{E_{\text{strong}}(m_p)}{E_{\text{Ryd}}(m_e, 1)} = \frac{m_p}{m_e}\left(\frac{\alpha_{\text{Som}}}{\alpha_{\text{strong}}}\right)^2$. If this ratio is assumed about 10^6, the strong coupling constant has as order of magnitude: $4\pi\alpha_{\text{strong}} = g_{\text{strong}}^2 \simeq 1$.

Writing down the binding energies arising "from a fire with wood to a fire with nucleons" (nuclear reactor), one obtains the following table:

[46] Johannes Rydberg (1854–1919).

Process (matter level)	**Binding** $\left[\frac{\text{J}}{\text{kilomol}}\right]$	**Binding** $\left[\frac{\text{MeV}}{\text{particle}}\right]$	**Rest mass** [MeV]	**Coupling** k^2
Water from evaporation* (liquids)	4×10^7	4×10^{-7}	$m_{H_2O}c^2 \simeq 2 \times 10^4$	4×10^{-11}
Water binding from H_2 and O (molecules)	3×10^8	3×10^{-6}	$m_{H_2O}c^2 \simeq 2 \times 10^4$	3×10^{-10}
Hydrogen binding from p^+ and e^- (atoms)		13.6×10^{-6}	$m_e c^2 \simeq 0.5$	6×10^{-5}
Nuclear binding from nucleons (nuclei)		8	$m_p c^2 \simeq 10^3$	1.6×10^{-2}
Nucleon binding with resonances (hadrons)		5×10^2	$m_p c^2 \simeq 10^3$	1

*for normal conditions

Matter levels

As measure for the nucleon binding, the mass difference $(1440 - 940)\frac{\text{MeV}}{c^2}$ is taken between an excited state (resonance) of the nucleon and the nucleon itself. The coupling k^2 (binding coefficient) is defined as twice the ratio of binding energy and rest mass. One obtains for the hydrogen binding with the intrinsic energy unit $m_e(\alpha_{\text{Som}}c)^2 \simeq 27.2$ eV:

$$\text{hydrogen binding coefficient: } k^2 = \frac{m_e(\alpha_{\text{Som}}c)^2}{m_e c^2} = (\alpha_{\text{Som}})^2 \simeq 6 \times 10^{-5},$$

i.e., the binding coefficient is a measure for the involved coupling constant of the interaction.

Weisskopf[47] collected the matter (energy, mass) levels in a *quantum ladder* with the rungs molecules, atoms, nuclei, hadrons, quarks. For elementary particles (nucleon binding) the coupling k has 1 as its order of magnitude. To talk of "parts in a whole" makes sense only, if the binding energy is small in comparison with the energy, stored in the constituents. For the gravitationally bound sun-earth state one has the potential and mass-related energies with the ratio of Schwarzschild length to orbit radius $\frac{Gm_\bullet}{c^2 R_\bullet} \sim 10^{-12}$:

$$\overline{E}_\bullet = G\frac{m_\odot m_\bullet}{R_\bullet} \sim 10^{33}\text{J}, \quad (m_\odot + m_\bullet)c^2 \sim 2 \times 10^{45}\text{J}.$$

It looks as if for elementary particles one has reached the level, where the *particle description for an adequate formulation of a dynamics (interaction) finds its limit*. For instance, to talk about hadrons (protons, pions, etc.) as bound states of quark *particles* may be as slippery as the position concept for an "electron in an atom."

In this sense, it is satisfactory, that quarks are structures, which shape the hadronic dynamics, but which are, up to today, not observed as free particles — although a

[47] Victor F. Weisskopf (1908–2002).

mathematical proof of their confinement in a fundamental theory (quantum chromodynamics), e.g., by extending the flux-tube "arguments," is still missing. A geometric analogy to the theory of the quarks and their missing particle property: Six squares shape, as faces, a cube; however, it does not make sense to call the faces "parts of the cube." No carpenter or no goldsmith has ever manufactured an isolated square — the square lacks the property of three-dimensionality. According to the theory today, quarks shape protons: Quarks come with the homogeneous properties hypercharge, isospin, color and spin; however they lack the translation invariant mass as a free particle property.

2.10 Electroweak Interactions

The electromagnetic interaction connects with each other charged particles. It is the classical, macroscopically easily detectable "tail" of a branched charge-related structure: In addition to the long-range electromagnetic interactions, there exist *three short-ranged weak interactions* — a neutral one, which, like the electromagnetic one, does not exchange electric charges, and two oppositely charged interactions, which rearrange the electric charges, as exemplified by the radioactive decay $n \to p + e + \overline{\nu}_e$ of the neutrons (charge number $z_n = 0$)into a proton ($z_p = 1$) and an electron ($z_e = -1$) with its antineutrino ($z_{\overline{\nu}_e} = 0$), $z_n = z_p + z_e + z_{\overline{\nu}_e}$.

This charge structure with four interactions reflects four *internal (chargelike) degrees of freedom*. Analogously to *time and position*, denoting the four external (spacetimelike) coordinates, there exist four internal "coordinates," called *hypercharge and isospin*. To describe the additional strong interaction, the internal property *color* is used with eight degrees of freedom. If in a simplifying picture, time and space translations are given by one point in a corresponding Minkowski vector space, the internal properties can endow this point — think of a spacetime point with an icecream parlour — with different flavors and colors — the icecream may taste of different fruits, and may come in different colors.

The photon with charge number $z = 0$ and the three weak bosons $Z, W^{\pm}$ with charge numbers $z = 0, \pm 1$ are the particle aspect of the electroweak interactions. From this fourfold particle basis, there leads an axial rotation with the *Weinberg*[48] *angle* to an interaction basis with three isospin-vector fields and one isospin-scalar field — in the flavor picture: two mixes of cream and strawberry tastes are decomposed into the basic ingredients.

The Weinberg angle defines the rectangular *electroweak triangle*: The square of its hypotenusis can be normalized by Sommerfeld's number $\ell_e^2 = \frac{1}{g_e^2} = \frac{1}{4\pi\alpha_{\text{Som}}}$. With Pythagoras'[49] theorem, ℓ_e^2 is distributed to the orthogonal sides, which, in this normalization, give the inverse coupling constants for the isospin and hypercharge interactions. The experimental values for the electroweak length triangle are

[48] Steven Weinberg (1933–).

[49] Pythagoras of Samos, around -530.

$$\tan^2\theta_{\text{Wein}} = \frac{\ell_W^2}{\ell_Y^2} \simeq 0.3, \quad \begin{cases} \ell_e^2 = \ell_W^2 + \ell_Y^2 \simeq 10.9, \\ \frac{1}{\ell_Z^2} = \frac{1}{\ell_W^2} + \frac{1}{\ell_Y^2} \simeq 0.5 \end{cases}$$

$$\Rightarrow \begin{cases} \text{charge hypotenusis: } \ell_e &= \frac{1}{g_e} \simeq 3.3, \\ \text{isospin side: } \ell_W &= \frac{1}{g_W} \simeq 1.6, \\ \text{hypercharge side: } \ell_Y &= \frac{1}{g_Y} \simeq 2.9, \\ \text{height: } \ell_Z &= \frac{1}{g_Z} \simeq 1.4. \end{cases}$$

If one had a symmetry argument for a Platonic[50]"beautiful" triangle, i.e., for the half of an equilateral one with the angles $(90^\circ, 60^\circ, \theta_{\text{Wein}} = 30^\circ)$, the side ratios were $\ell_e : \ell_W : \ell_Y = 2 : 1 : \sqrt{3}$. In such a case, the Weinberg angle would display the distribution of the involved isospin degrees of freedom of the electroweak field:

$$\text{beautiful triangle: } \tan^2\theta_{\text{Wein}} = \frac{1}{3} = \frac{\text{number of isospin scalars}}{\text{number of isospin vectors}}.$$

Since a closer analysis shows the angle to be dependent on the energy of the interaction, there could, in principle, arise such a "beautiful" triangle for some distinguished energy.

The experiments determine not only the form of the electroweak triangle, they give also its normalization by an intrinsic length and mass unit, respectively: The relevant unit for the electroweak interaction is the *Fermi*[51] *unit*:

$$\frac{G_{\text{Fermi}}}{(\hbar c)^3} \simeq \frac{1}{(292.8\text{GeV})^2}, \text{ with GeV } = 10^9\text{eV},$$

associated to the Fermi mass and *weak length (range)*:

$$\frac{G_{\text{Fermi}}}{(\hbar c)^3} = \frac{1}{4\sqrt{2}(M_{\text{Fermi}}c^2)^2} \Rightarrow \begin{cases} M_{\text{Fermi}} \simeq 123\frac{\text{GeV}}{c^2}, \\ l_{\text{Compt}}(M_{\text{Fermi}}) \simeq 1.6 \times 10^{-18}\text{ m}. \end{cases}$$

The electroweak length triangle has the Compton length of the charged weak bosons $W_\pm$ as isospin side, the height is related to the neutral weak boson Z:

$$\ell_W = \frac{M_{\text{Fermi}}}{m_W} \simeq 1.6 \Rightarrow l_{\text{Compt}}(m_W) = 2.6 \times 10^{-18}\text{ m},$$
$$\ell_Z = \frac{\ell_W \ell_Y}{\ell_e} = \frac{M_{\text{Fermi}}}{m_Z} \simeq 1.4 \Rightarrow l_{\text{Compt}}(m_Z) = 2.2 \times 10^{-18}\text{ m}.$$

The mass ratio of the weak bosons does not depend on the Fermi mass:

[50]Platon -(428–348).

[51]Enrico Fermi (1901–1954).

$$\frac{m_W^2}{m_Z^2} = \frac{\ell_Y^2}{\ell_e^2} = \frac{\ell_Z^2}{\ell_W^2} = \cos^2\theta_{\mathrm{Wein}} \begin{cases} \simeq 0.77 \text{ (experiments)}, \\ = 0.75 \text{ (“beautiful” triangle.)} \end{cases}$$

The electroweak triangles for masses or coupling constants, dual to the electroweak length triangle, have the coupling constants $g^2 = \frac{1}{\ell^2}$ as squared sides, the height g_e is the elementary charge. The hypotenusis is the Z-mass, one orthogonal side the W-mass. The masses, associated with the other orthogonal side and with the height, are not particle masses:

$$g_Z^2 = g_W^2 + g_Y^2, \quad \frac{1}{g_e^2} = \frac{1}{g_W^2} + \frac{1}{g_Y^2},$$

$$(m_Z, m_W, m_Y, m_e) = (g_Z, g_W, g_Y, g_e) M_{\mathrm{Fermi}} \simeq (91.2,\ 80.2,\ 43.4,\ 38.2)\frac{\mathrm{GeV}}{c^2}.$$

Good to know: For a given angle $\theta < \frac{\pi}{2}$, there are two dual rectangular triangles, similar to each other, with θ as a base angle, $\tan\theta = \frac{b}{a}$. The first one with doubled area $2F = e^{\psi} = ab$ has sides $a^2 + b^2 = c^2$ and height in $ab = ch$. The dual partner triangle with $\frac{1}{b^2} + \frac{1}{a^2} = \frac{1}{h^2}$ and $\frac{1}{b}\frac{1}{a} = \frac{1}{h}\frac{1}{c}$ has the doubled area $e^{-\psi} = \frac{1}{ab}$. The duality relies on the two orthogonal angles — between the orthogonal sides and between hypotenusis and height. The dilation transformation for all corresponding lengths of the dual triangles multiplies by the doubled triangle area e^{ψ} and $e^{-\psi}$, respectively.

The symmetry (interaction-)basis for the internal isospin and hypercharge coordinates with the hypercharge gauge field $\mathbf{A}_0$ and the third component $\mathbf{A}_3$ of an isospin triplet gauge field is axially rotated to a particle basis with a photon field $\mathbf{A}$ and a neutral weak boson field $\mathbf{Z}$:

$$\begin{pmatrix} \frac{1}{g_e}\mathbf{A} \\ \frac{1}{g_Z}\mathbf{Z} \end{pmatrix} = D(\theta)\begin{pmatrix} \frac{1}{g_Y}\mathbf{A}_0 \\ \frac{1}{g_W}\mathbf{A}_3 \end{pmatrix} \Rightarrow \begin{cases} \mathbf{A} = \cos^2\theta\, \mathbf{A}_0 + \sin^2\theta\, \mathbf{A}_3, \\ \mathbf{Z} = -\mathbf{A}_0 + \mathbf{A}_3, \end{cases}$$

$$\text{with Weinberg rotation: } D(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

The electroweak rotation has a mechanical analogue in the transformations — for two mass points — from individual positions and momenta $(\mathbf{x}_i, \mathbf{p}_i)_{i=1,2}$ to center-of-mass $(\mathbf{X}, \mathbf{P})$ and relative $(\mathbf{x}, \mathbf{p})$ positions and momenta, e.g., for a sympathetic pendulum to one free component with trivial frequency Ω and one oscillator with the frequency ω, or for the gravitational attraction of two masses, as given by the Hamiltonians:

$$\begin{aligned}
\mathbf{H}_{\text{osc}} &= \frac{\mathbf{p}_1^2}{2m_1} + \frac{\mathbf{p}_2^2}{2m_2} + \frac{g_0^2}{2}(\mathbf{x}_1 - \mathbf{x}_2)^2 \\
&= \frac{\mathbf{P}^2}{2M} + \frac{\mathbf{p}^2}{2m} + \frac{g_0^2}{2}\mathbf{x}^2, \quad \Omega = 0, \quad \omega^2 = \frac{g_0^2}{m}, \\
\mathbf{H}_{\text{grav}} &= \frac{\mathbf{p}_1^2}{2m_1} + \frac{\mathbf{p}_2^2}{2m_2} - \frac{Gm_1m_2}{|\mathbf{x}_1 - \mathbf{x}_2|} \\
&= \frac{\mathbf{P}^2}{2M} + \frac{\mathbf{p}^2}{2m} - \frac{GMm}{|\mathbf{x}|}.
\end{aligned}$$

The associated rectangular mass triangle has the masses $m_{1,2}$ as the squares of the orthogonal sides, the mass sum $M = m_1 + m_2$ as hypotenusis square, and the reduced mass $m = \frac{m_1m_2}{M}$ as the square of the height. The corresponding dual frequency triangle has the side squares $(\frac{g_0^2}{m_2}, \frac{g_0^2}{m_1}; \frac{g_0^2}{m})$ and $\frac{g_0^2}{M}$ as squared height. The angle θ for the axial rotation is the basis angle of the rectangular triangle; it is determined by the mass ratio:

$$\begin{aligned}
&\text{center of mass rotation: } D(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \\
&\qquad \text{where } \tan^2\theta = \frac{m_2}{m_1}, \ \cos^2\theta = \frac{m_1}{M} = \frac{m}{m_2}, \ \sin^2\theta = \frac{m_2}{M} = \frac{m}{m_1}, \\
&\qquad \text{with } m_1 + m_2 = M, \ \frac{1}{m_1} + \frac{1}{m_2} = \frac{1}{m},
\end{aligned}$$

and used for the positions in

$$\begin{pmatrix} \sqrt{M}\mathbf{X} \\ \sqrt{m}\mathbf{x} \end{pmatrix} = D(\theta)\begin{pmatrix} \sqrt{m_1}\mathbf{x}_1 \\ \sqrt{m_2}\mathbf{x}_2 \end{pmatrix} \Rightarrow \begin{cases} m_1\mathbf{x}_1^2 + m_2\mathbf{x}_2^2 = M\mathbf{X}^2 + m\mathbf{x}^2, \\ \mathbf{X} = \cos^2\theta\,\mathbf{x}_1 + \sin^2\theta\,\mathbf{x}_2, \\ \mathbf{x} = -\mathbf{x}_1 + \mathbf{x}_2, \end{cases}$$

and with inverse normalization for the momenta in

$$\begin{pmatrix} \frac{1}{\sqrt{M}}\mathbf{P} \\ \frac{1}{\sqrt{m}}\mathbf{p} \end{pmatrix} = D(\theta)\begin{pmatrix} \frac{1}{\sqrt{m_1}}\mathbf{p}_1 \\ \frac{1}{\sqrt{m_2}}\mathbf{p}_2 \end{pmatrix} \Rightarrow \begin{cases} \frac{\mathbf{p}_1^2}{m_1} + \frac{\mathbf{p}_2^2}{m_2} = \frac{\mathbf{P}^2}{M} + \frac{\mathbf{p}^2}{m}, \\ \mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2, \\ \mathbf{p} = -\sin^2\theta\,\mathbf{p}_1 + \cos^2\theta\,\mathbf{p}_2. \end{cases}$$

For example, the center of mass angle in $\tan^2\theta = \frac{m_p}{m_e} \simeq 1836$, $\theta \simeq 88.7°$ for the hydrogen atom involves the electron and proton mass.

In analogy to the sympathetic pendulum in the mechanical analogue, the electromagnetic and the charge neutral weak interactions, related to the massless photon and the massive Z-boson, can be called the interaction for the *center of charge* and for the *relative charge*, respectively.

2.11 Units and Symmetry Normalizations

A dimensional grading of all physical quantities by three fundamental units, perhaps by the Planck, Einstein, and Newton unit $(\hbar, c, G)$, may hint to three basic metrical structures that are connected with the normalizations and invariants of operations for time and space, and, ultimately, to the action of the dilation group. Here, the time independence of the units is assumed. Dirac has speculated about a possible time dependence — so far without any experimental substantiation.

Reversing the historical order of the arguments, one may ponder the question, which properties of perhaps existing fundamental structures in the physical description of nature could cause the introduction of units for their quantitative characterization.

First for the *field unit* c: The maximal action velocity has its operational origin in the noncompact structure of the spacetime Lorentz group $\mathbf{O}(1,3)$, visible in the proper Lorentz transformations (boosts), which arise by an *expansion* of the Galilei transformations for finite action velocity $\frac{1}{c} > 0$. In a simplified two-dimensional spacetime, the Lorentz transformations constitute a real 1-parametric group, affecting the space translations and the time translations by hyperbolic transformations"):

$$\begin{aligned}
\begin{pmatrix} x_0 \\ x_1 \end{pmatrix} &\longmapsto \begin{pmatrix} \cosh\psi & \sinh\psi \\ \sinh\psi & \cosh\psi \end{pmatrix}\begin{pmatrix} x_0 \\ x_1 \end{pmatrix}, \quad \text{invariant: } x_0^2 - x_1^2, \\
\mathbf{SO}_0(1,1) &= \{D_2(\psi) = \begin{pmatrix} \cosh\psi & \sinh\psi \\ \sinh\psi & \cosh\psi \end{pmatrix} \mid \psi \in \mathbb{R}\}, \\
\text{group:}\quad & D_2(\psi_1) \circ D_2(\psi_2) = D_2(\psi_1 + \psi_2).
\end{aligned}$$

Only for a noncompact structure, lightlike vectors with $x_0^2 - \vec{x}^2 = 0$ can be nontrivial. Light separates time with $x_0^2 - \vec{x}^2 > 0$ from position with $x_0^2 - \vec{x}^2 < 0$ ($\vec{x}^2 = x_1^2$ for the one-dimensional space model). Only in such a case, a nontrivial future and past, compatible with orthochronous Lorentz transformations, and, therefore, a nontrivial causality can be defined.

The 2×2-matrix representation $D_2(\psi)$ of the Lorentz transformations in a spacetime basis with the diagonal matrix $g \cong \eta_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ for the symmetric bilinear Lorentz metric (length squares), i.e., $g(x,x) = x_0^2 - x_1^2$, is equivalent to a diagonal matrix with two scale transformations, inverse to each other, for a light basis. In such a basis, the bilinear Lorentz metric comes in the "skew-diagonal" matrix $g \cong \zeta_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, i.e., $g(x,x) = 2x_+x_-$, with $x_\pm = \frac{x_0 \pm x_1}{\sqrt{2}}$:

$$(\cosh\psi - D_\pm)^2 - \sinh^2\psi = 0 \iff D_\pm(\psi) = e^{\pm\psi}, \quad D_2(\psi) \cong \begin{pmatrix} e^{\psi} & 0 \\ 0 & e^{-\psi} \end{pmatrix}.$$

The Lorentz transformation is a self-dual dilation ("Procrustes transformation"): One of the lightlike eigenvectors $x_\pm = \frac{x_0 \pm x_1}{\sqrt{2}}$ is shortened, the other one is stretched with the inverse factor. The invariance of a hyperbola $x_+x_- = \ell^2$ under dilations

$x_\pm \longmapsto e^{\pm\psi}x_\pm$ is well known from the high-school discussions of the conic sections. The dilation structures formalize the "paradoxical" contractions of special relativity.

In the space-time basis representation, the proper Lorentz transformations display the dilation structure via the hyperbolic tangens:

$$D_2(\psi) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} e^{\psi} & 0 \\ 0 & e^{-\psi} \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \cosh\psi\begin{pmatrix} 1 & \tanh\psi \\ \tanh\psi & 1 \end{pmatrix},$$
$$|\tanh\psi| = |\frac{e^{\psi} - e^{-\psi}}{e^{\psi} + e^{-\psi}}| < 1.$$

Its upper limit is parametrizable by a normalization with a constant maximal parameter $c > 0$:

$$\frac{v}{c} = \tanh\psi,\ \ \psi = \frac{1}{2}\log\frac{1+\frac{v}{c}}{1-\frac{v}{c}} \ \text{ with } |v| < c \Rightarrow D_2(\psi) = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\begin{pmatrix} 1 & \frac{v}{c} \\ \frac{v}{c} & 1 \end{pmatrix} = D_2\left(\frac{v}{c}\right),$$
$$\text{or } D_2(\psi) = \frac{1}{mc}\begin{pmatrix} \frac{E}{c} & p \\ p & \frac{E}{c} \end{pmatrix}, \ \text{ with energy-momentum } (E, p)$$
$$\text{and invariant } m^2c^2 = \frac{E^2}{c^2} - p^2,\ \ \det D_2(\psi) = 1.$$

The *rapidity*, as normalized velocity $\psi = \frac{v}{c} + \dots$ for nonrelativistic small $\frac{v}{c}$, measures the length on a hyperbola, e.g.,

$$x_0^2 - x_1^2 = -1,\ \ (dx_0, dx_1) = (\cosh\psi, \sinh\psi)d\psi,$$
$$\int ds = \int\sqrt{dx_0^2 - dx_1^2} = \int dx_0\sqrt{1-\frac{v^2}{c^2}} = \int d\psi.$$

The velocity parametrization, suggested by nonrelativistic point mechanics, has some pitfalls: There does not exist a Lorentz transformation with $|v| = c$, e.g., from Einstein's rest system to a photon. In contrast to the rapidity parametrization $-\infty < \psi < \infty$, the velocity parametrization may suggest a tachyonic extension to velocities $|v| > c$, larger than the maximal value, where, with the imaginary value of $\sqrt{1-\frac{v^2}{c^2}}$, the group structure is lost. The simple linear addition of an hyperbolic "angle" ψ (rapidity) becomes the complicated combination law of its "projection" (addition theorem of the hyperbolic tangens):

$$\tanh(\psi_1 + \psi_2) = \frac{\tanh\psi_1 + \tanh\psi_2}{1 + \tanh\psi_1\tanh\psi_2} \iff v(\psi_1 + \psi_2) = \frac{v(\psi_1) + v(\psi_2)}{1 + \frac{v(\psi_1)v(\psi_2)}{c^2}}.$$

Good to know: If an axial rotation $D(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ from the group $\mathbf{SO}(2)$ would be parametrized not by the arc θ (rotation angle) of the unit circle, but by its projection $s = \sin\theta$ on the y-axis direction, the angle for the product of two rotation would be characterized, not by the angle sum $\theta_1 + \theta_2$ for $D(\theta_1 + \theta_2) = D(\theta_1) \circ D(\theta_2)$, but by the complicated expression $s(\theta_1 + \theta_2) = s_1\sqrt{1 - s_2^2} + s_2\sqrt{1 - s_1^2}$ for the projected arc. An s-parametrization may suggest a useless extension to the region with $s > 1$.

Because of the nontrivial signature of the Lorentz metric, for $\mathbf{O}(1, 1)$ given by $g \cong \eta_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (space-time basis) or by $g \cong \zeta_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (light basis), the Lorentz group structure involves a physically relevant maximal signature parameter c, which can be taken as *relative normalization of the time and position translations*. It can be used for a renormalization of the coordinates (upper and lower components in the vectors with the action of $D_2(\psi)$):

$$\text{spacetime action: } \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \longmapsto D_2(\frac{v}{c})\begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \Rightarrow \begin{cases} \frac{x_0}{c} = t \longmapsto \frac{t + \frac{v}{c^2}x_1}{\sqrt{1 - \frac{v^2}{c^2}}}, \\ x_1 \longmapsto \frac{x_1 + vt}{\sqrt{1 - \frac{v^2}{c^2}}}. \end{cases}$$

The limit of an infinite action velocity $c \to \infty$ describes the transition from the special relativistic spacetime to classical space and time, formalized by the *Inönü*[52]—Wigner[53] *contraction* from – for three position dimensions $s = 3$ — the orthochronous Lorentz group $\mathbf{SO}_0(1, s)$ to the semidirect Galileo group $\mathbf{SO}(s) \vec{\times} \mathbb{R}^s$ with position rotations $\mathbf{SO}(s)$ and velocity transformations $\vec{v} \in \mathbb{R}^s$. The limit is exemplified in a two-dimensional rotation-free spacetime, i.e., with $s = 1$:

$$\begin{pmatrix} \frac{1}{c} & 0 \\ 0 & 1 \end{pmatrix} D_2(\frac{v}{c}) \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \begin{pmatrix} 1 & \frac{v}{c^2} \\ v & 1 \end{pmatrix} \xrightarrow{c \to \infty} \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix},$$

$$\text{i.e., } \begin{cases} t \longmapsto t, \\ x_1 \longmapsto x_1 + vt, \end{cases} \quad \text{for } s = 3\text{: } \begin{cases} t \longmapsto t, \\ \vec{x} \longmapsto \vec{x} + \vec{v}t. \end{cases}$$

If the basic law of physics — in the case it exists — is characterized by a Lorentz symmetry structure, then the unit c makes sense for the relative normalization of the positive and negative sector in the Lorentz metric $g \cong \begin{pmatrix} c^2 & 0 \\ 0 & -\mathbf{1}_3 \end{pmatrix}$.

Good to know: In general for $t, s \geq 1$, there is a contraction $\mathbf{SO}_0(t, s) \xrightarrow{\frac{1}{c} \to 0} \mathbf{SO}_0(t - 1, s) \vec{\times} \mathbb{R}^{t-1+s}$ from a (pseudo-)orthogonal group to a semidirect product group, formalized in the corresponding Lie algebras by $\left(\begin{array}{c|c} 0 & \frac{1}{c}\frac{v}{c} \\ \hline c\frac{v}{c} & \mathcal{L} \end{array}\right) \xrightarrow{\frac{1}{c} \to 0} \left(\begin{array}{c|c} 0 & 0 \\ \hline v & \mathcal{L} \end{array}\right)$ with $\mathcal{L} \in \log \mathbf{SO}_0(t - 1, s)$ and representatives $\frac{v}{c} \in \mathbb{R}^{t-1+s}$ of the cosets $\log \mathbf{SO}_0(t, s)/\log \mathbf{SO}_0(t - 1, s)$.

Now for the *action unit* $\hbar$, experimentally found and interpretable as the quantum of action or as the quantum of angular momentum: As connected with the name of Noether,[54] the invariances of a dynamics and its conservation laws, i.e., its time-independent invariants, condition each other. In the quantum theoretical formalism,

[52] Erdal Inönü (1926–2007).

[53] Eugene Wigner (1902–1995).

[54] Emmy Noether (1882–1935).

this connection proves itself as a simple identity: Dynamically relevant operations, like the time translations or the "infinitesimal" rotations, are described by quantum operators — in the examples by a Hamiltonian $\mathbf{H}$ and by the three angular momenta $\vec{\mathbf{L}}$, respectively. The invariance with respect to an operation is characterized by a trivial commutator (adjoint action) with the corresponding quantum operator. For example, angular momentum is time translation invariant, i.e., conserved, for

$$\mathrm{ad}\,\mathbf{H}(\vec{\mathbf{L}}) = [\mathbf{H}, \vec{\mathbf{L}}] = \mathbf{H}\vec{\mathbf{L}} - \vec{\mathbf{L}}\mathbf{H} = 0 \iff \frac{d\vec{\mathbf{L}}}{dt} = 0;$$

and the time development (dynamics) is rotation invariant (with three rotation angles in $\vec{\theta}\vec{\mathbf{L}}$) with the identical condition, read in the reversed form:

$$\mathrm{ad}\,\vec{\mathbf{L}}(\mathbf{H}) = [\vec{\mathbf{L}}, \mathbf{H}] = \vec{\mathbf{L}}\mathbf{H} - \mathbf{H}\vec{\mathbf{L}} = 0 \iff \frac{\partial \mathbf{H}}{\partial \vec{\theta}} = 0.$$

In an operational framework, the invariants determine the representations of the corresponding action groups and Lie algebras.

The normalization of a conserved angular momentum square by a unit $\hbar^2$ can be transferred, in quantum mechanics via $\vec{\mathbf{L}} = i\vec{\mathbf{x}} \times \vec{\mathbf{p}}$ and $[i\mathbf{p}^a, \mathbf{x}^b] = \delta^{ab}\mathbf{1}$, to a normalization of the noncommutative canonical (dual) position–momentum pair, where $\hbar$ is the product of position and momentum units l_0 and π_0:

$$\underline{\vec{\mathbf{x}}} = l_0\vec{\mathbf{x}},\ \ \underline{\vec{\mathbf{p}}} = \pi_0\vec{\mathbf{p}},\ \text{ with } l_0\pi_0 = \hbar \Rightarrow \begin{cases} [i\underline{\mathbf{p}}^a, \underline{\mathbf{x}}^b] = \delta^{ab}\hbar\mathbf{1}, \\ \ \ i\underline{\vec{\mathbf{x}}} \times \underline{\vec{\mathbf{p}}} = \hbar\vec{\mathbf{L}} = \underline{\vec{\mathbf{L}}}. \end{cases}$$

After two units c (spacetime or light unit) and $\hbar$ (dual or angular momentum unit), related rather satisfactorily with the fundamental relevance of the Lorentz group and quantum noncommutativity (duality), one has no clear qualitative understanding of a third universal intrinsic unit.

Perhaps a representation of the *causality structure* requires at least two real normalizations: Also in our daily life we describe and measure time by one always increasing number, e.g., by piling up the leaves of a tear-off calendar, and by a cyclic number, e.g., on an analogue watch with a circular dial. Take as example the imperial coronation of Charlemagne[55] in the year 800 — this is one linear noncompact date — at the 25th of December, 1 p.m. (?) — this involves three cyclic compact dates. As a mathematical model of a time, measured with a noncompact and compact invariant, one may take a screw thread: Both the noncompact and compact structure have a unit, e.g., year and day:

[55] Carolus Magnus (768–814).

$$\begin{aligned}
\text{screw line: } (x(t), y(t), z(t)) &= R_0(\cos \omega t, \sin \omega t, \Omega t), \\
\text{screw turning time: } &\frac{2\pi}{\omega}, \\
\text{screw raising time: } &\frac{1}{\Omega}.
\end{aligned}$$

A twofold time normalization gives rise to one number like $\frac{\Omega}{\omega}$; for instance, year and day determines the number $365, 2422\ldots = 365 + \frac{1}{4} - \frac{3}{400} + \ldots$ (leap year expansion: add one day for all four years, and, subtract three days for all four hundred years, etc.). With respect to a compact and a noncompact aspect of spacetime, Newton's unit G could be related to the normalization of the external interactions (gravitative) in the noncompact real numbers $\mathbb{R}$ as time model. Its ratio to the normalization of the compact quotient group $\mathbf{U}(1) \cong \mathbb{R}/\mathbb{Z}$ could be related to the coupling constants of the internal unitary interactions (electromagnetic, weak, and strong).

All this is rather vague and, obviously, not a theory. In order not to stay completely with "empty" words, there will be given a numerical illustration — also that probably irrelevant with respect to the concrete details — which relates the Fermi and Planck mass as compact and noncompact unit, respectively, to the order of magnitude of Sommerfeld's number:

$$\frac{M_{\text{Planck}}}{M_{\text{Fermi}}} \simeq 10^{17}, \quad \log \frac{M_{\text{Planck}}}{M_{\text{Fermi}}} \simeq 39, \quad 4\frac{\hbar}{\hbar_e} = \frac{1}{\pi \alpha_{\text{Som}}} \simeq 43.6.$$

In a fundamental law of physics, if it exists, such a twofold normalization of the spacetime structure could be experimentally observable, on the one hand, in the particle and their masses, and, on the other hand, in the interactions and their normalizations (coupling constants), e.g., in the electromagnetic Coulomb or gravitative Newton interaction.

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