Chapter 2
Direct Estimates for Approximation by Linear Combinations

2.1 Direct Estimates in L_p-Norm

The aim of this chapter is to collect the known results for the error of approximation by linear combinations $L_{n,r}$, measured in different norms $L_p(B)$ and usually in terms of Ditzian–Totik moduli of smoothness $\omega_\varphi^r(f,t)_p$, or the ordinary moduli of continuity $\omega^r(f,t)_p$. We will see the importance of the information about central moments of the p.l.o. (respectively of $L_{n,r}$).

We recall some well-known definitions from the book of Ditzian–Totik. The classical modulus of continuity of order r is defined by

$$\omega^r(f,t)_p = \sup_{0<h\le t} ||\Delta_h^r f||_{L_p(B)}, \tag{2.1.1}$$

where $\Delta_h^r f(x) = \sum_{i=0}^r \binom{r}{i}(-1)^i f(x+i.h)$.

The Ditzian–Totik modulus of smoothness of order r is defined by: (see [50, Chapter 1])

$$\omega_\varphi^r(f,t)_p = \sup_{0<h\le t} ||\Delta_{h\varphi}^r f||_{L_p(D)}, \tag{2.1.2}$$

where $1 \le p \le \infty, D$ will be given below for some concrete p.l.o. and $\varphi(x)$ denotes the corresponding weight function. For example, if

- $L_n = B_n, \widehat{B}_n, \varphi^2(x) = x(1-x), D = C[0,1]$ for $B_n f$ and $D = L_p[0,1]$ for $L_n = \widehat{B}_n$;
- $L_n = S_n, \widehat{S}_n, \varphi^2(x) = x, D = C[0,\infty)$ for $S_n f$ and $D = L_p[0,\infty)$ for $L_n = \widehat{S}_n$;
- $L_n = V_n, \widehat{V}_n, \varphi^2(x) = x(1+x), D = C[0,\infty)$ for $V_n f$ and $D = L_p[0,\infty)$ for $L_n = \widehat{V}_n$;

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The first direct theorem is the following:

Theorem 2.1 *Let $1 \leq p < \infty$. For L_n one of the operators mentioned above with the correspondent domain D and weight function φ the following Jackson-type estimate (direct estimate) holds true*

$$||L_{n,r}f - f||_D \leq M\left(\omega_{\varphi}^{2r}(f, n^{-1/2})_D + n^{-r}||f||_D\right). \tag{2.1.3}$$

Proof The full proof of the statement is given in the Ditzian–Totik book [50, Theorem 9.3.2]. For the sake of completeness we give only the sketch of the proof. We use the technique of K-functional. For $r > 0, r \in \mathbb{N}$, the K-functional is defined by

$$K_{r,\varphi}(f, t^r)_p = \inf_g \left\{||f - g||_p + t^r||\varphi^r g^{(r)}||_p : g^{(r-1)} \in A.C._{loc}\right\}, \tag{2.1.4}$$

where $g^{(r-1)} \in A.C._{loc}$ means that g is $r-1$ times differentiable and $g^{(r-1)}$ is absolutely continuous in every finite interval $[a, b] \subset D$. It is known that for $f \in L_p(D)$ and $1 \leq p \leq \infty$ the moduli $\omega_{\varphi}^r(f, t)_p$ and K-functional $K_{r,\varphi}(f, t^r)_p$ are equivalent, i.e.

$$M^{-1}\omega_{\varphi}^r(f, t)_p < K_{r,\varphi}(f, t^r)_p \leq M\omega_{\varphi}^r(f, t)_p, 0 < t \leq t_0 \tag{2.1.5}$$

for some constants $M > 0$ and t_0. The proof relies on the following three estimates:

$$||L_{n,r}f||_D \leq M||f||_D, \tag{2.1.6}$$

$$||\varphi^{2r}L_n^{(2r)}f||_D \leq Mn^r||f||_D \text{ (Bernstein type inequality)}, \tag{2.1.7}$$

$$||L_{n,r}f - f||_D \leq Mn^{-r}\left[||\varphi^{(2r)}f^{(2r)}||_D + ||f||_D\right], \tag{2.1.8}$$

for $f^{(2r-1)} \in A.C._{loc}$. The first estimate (2.1.6) follows from the definition of $L_{n,r}(f, x)$, the property of the coefficients $\alpha_i(n)$ and the boundedness of the operator L_n. The Bernstein type inequality (2.1.7) is proved in Ditzian–Totik book [50, Section 9.4] and the last estimate (2.1.8) is proved in [50, Section 9.5] with E_n instead of D and after this the domain is enlarged from E_n to D as it was shown in the proof in [50, Theorem 9.3.2]. Having at hand the three estimates (2.1.6)–(2.1.8) we proceed as follows: We write

$$f = f - g_n + g_n, g_n^{(2r-1)} \in A.C._{loc}$$

and then

$$||L_{n,r}f - f||_D \leq ||L_{n,r}(f - g_n) - (f - g_n)||_D + ||L_{n,r}g_n - g_n||_D \tag{2.1.9}$$

The estimate (2.1.6) implies

$$||L_{n,r}(f-g_n)-(f-g_n)||_D \le C||f-g_n||_D \tag{2.1.10}$$

The estimate (2.1.8) [the proof of which relies on (2.1.7)] implies

$$||L_{n,r}g_n-g_n||_D \le C.n^{-r}[||\varphi^{(2r)}g_n^{(2r)}||_D+||g_n||_D] \tag{2.1.11}$$

and consequently

$$||g_n||_D \le ||f||_D+||f-g_n||_D.$$

Now we sum up the both estimates (2.1.10) and (2.1.11) and take the infimum over all g_n. Using the equivalence (2.1.4) we end the proof. ■

The most difficult part of the proof of Theorem 2.1 is the inequality (2.1.8). The idea of the proof is the following: we apply $L_{n,r}(f,x)$ on the Taylor expansions with integral remainder

$$f(t)=\sum_{i=0}^{2r-1}\frac{(t-x)^i}{i!}f^{(i)}(x)+R_{2r}(f,t,x),$$

where

$$R_{2r}(f,t,x)=\frac{1}{(2r-1)!}\int_x^t (t-u)^{2r-1}f^{(2r)}(u)du.$$

Hence

$$L_{n,r}(f,x)-f(x)=\sum_{i=1}^{2r-1}\frac{f^{(i)}(x)}{i!}L_{n,r}((t-x)^i,x)+L_{n,r}(R_{2r}(f,t,x),x). \tag{2.1.12}$$

Lemma 1.4 shows that for $L_n=B_n,S_n,V_n$ the first sum in (2.1.12) reduces to the sum $\sum_{i=r+1}^{2r-1}$. Lemma 1.5 implies in the case of $L_n=\widehat{B}_n,\widehat{S}_n,\widehat{V}_n$ that the first sum reduces to $\sum_{i=r}^{2r-1}$. Further we apply the estimate (1.2.12) from Lemma 1.4 and (1.2.17) from Lemma 1.5 for the central moments of $L_{n,r}$. Lastly we apply some upper bounds for the weighted norms of intermediate derivatives of f given in the proof of [50, Theorem 9.5.3]. The upper bound for the remainder $L_{n,r}(R_{2r}(f,t,x),x)$ is given in [50, Lemma 9.5.2]. We point out that the direct estimate in Theorem 2.1 for $L_n=\widehat{B}_n,\widehat{S}_n$ or $\widehat{V}_n$ is not fulfilled when D is replaced by $L_\infty[0,1]$ or $L_\infty[0,\infty)$, respectively, for $\widehat{S}_n$ and $\widehat{V}_n$. Similar is the situation for the Durrmeyer modifications $\overline{B}_n,\overline{S}_n,\overline{V}_n$.

2.2 Direct Results for Linear Combinations

2.2.1 *Post–Widder Operators*

Following Widder [194] an integral representation of the Post–Widder operators for fixed integer p was proposed by R. K. S. Rathore and O. P. Singh in [164] as

$$P_{n,x}\bar{f} = \frac{(n/x)^{n+p+1}}{(n+p)!}\int_0^{\infty} e^{-nu/x}u^{n+p}f(u)du, \;\; x \in (0,\infty) \tag{2.2.1}$$

where $\bar{f}$ denotes the Laplace transform of a function f. Rathore and Singh in [164] considered the mth linear combinations of the operators (2.2.1) denoted by $P_{n,x}^{[m]}\bar{f}$ and defined as

$$P_{n,x}^{[m]}\bar{f} = \frac{1}{\Delta_m}\begin{vmatrix} P_{\alpha_0 n,x}\bar{f} & 1/\alpha_0 & 1/\alpha_0^2 & \dots & 1/\alpha_0^{m-1} \\ P_{\alpha_1 n,x}\bar{f} & 1/\alpha_1 & 1/\alpha_1^2 & \dots & 1/\alpha_0^{m-1} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ P_{\alpha_{m-1} n,x}\bar{f} & 1/\alpha_{m-1} & 1/\alpha_{m-1}^2 & \dots & 1/\alpha_{m-1}^{m-1} \end{vmatrix},$$

where α_i's are fixed positive real numbers and Δ_m is the determinant, which can be obtained by replacing first column by the entries 1. For the above linear combinations Rathore and Singh [164] established the following pointwise convergence and error estimation in simultaneous approximation:

Theorem 2.2 *Let f be such that $\bar{f}$ and $f^{(k+2m)}$ exist for some $x \in \mathbb{R}^+$, then*

$$P_{n,x}^{(k)[m+1]}\bar{f} - f^{(k)}(x) = o(n^{-m}), n \to \infty$$

and

$$P_{n,x}^{(k)[m]}\bar{f} - f^{(k)}(x) = O(n^{-m}), n \to \infty$$

where $P_{n,x}^{(k)[m]}\bar{f}$ are the linear combinations of $P_{n,x}^{(k)}\bar{f}$. Further, if $f^{(k+2m)}$ exists in $\langle a,b\rangle \subset \mathbb{R}^+$ and is continuous at each point $x \in [a,b]$, then the above results hold uniformly in $[a,b]$.

Theorem 2.3 *Let $k \in \mathbb{N}$, m being a non-negative integer and $0 \le p' \le 2m+2$. If f is such that $\bar{f}$ exists for some point in positive real-axis and $f^{(p'+k)}$ exists and is continuous on $\langle a,b\rangle \subset \mathbb{R}^+$, then for all $x \in [a,b]$ and n sufficiently large there holds*

$$\left|P_{n,x}^{(k)[m+1]}\bar{f} - f^{(k)}(x)\right| \le \max\left\{\frac{c_k}{n^{p'/2}}\omega(f^{(p'+k)}; n^{-1/2}), c_k' n^{-m-1}\right\}$$

where $c_k = c_k(m)$ *and* $c'_k = c'_k(m;f)$ *are certain constants.*
Also in [164] inverse and saturation results were proved for such linear combinations.

Obviously the Post–Widder operators constitute real inversion formula for the Laplace transform, Li and Wang in [131] considered the following form of Post–Widder operators

$$P_n(f,x) = \frac{(n/x)^n}{(n-1)!}\int_0^\infty e^{-nu/x}u^{n-1}f(u)du, \; x \in (0,\infty) \tag{2.2.2}$$

For the linear combinations mentioned in Section 1.1, Ditzian and Totik in the book [50] established direct and converse results for the operators (2.2.2) in L_p-norm. Their main results show for $f \in L_p[0,\infty)$, $1 \le p \le \infty$ (with $C[0,\infty)$ for $p = \infty$ and $\varphi(x) = x$, that

$$||P_{n,r}(f,x) - f||_p = O(n^{-\alpha}) \Leftrightarrow \omega_\varphi^{2r}(f,t)_p = O(t^{2\alpha})(0 < \alpha < r)$$

where

$$\omega_\varphi^{2r}(f,t)_p = \sup_{0<h\le t} ||\Delta_{h\varphi}^{2r} f||_{L_p[0,\infty)},$$

and $\Delta_h^{2r} f(x) = \sum_{i=0}^{2r}\binom{2r}{i}(-1)^i f(x+(r-i).h)$ for $x \ge rh$ and zero otherwise (see [50, (9.3.2)]).

Li and Wang in [131] proved direct and converse results on weighted simultaneous approximation by linear combinations of (2.2.2) in L_p, $1 \le p \le \infty$. The following two direct estimates were established in [131]:

Theorem 2.4 *Let* $f, f^{(l)}, \varphi^l f^{(l)} \in L_p[0,\infty)$, $1 \le p \le \infty$ *(with* $C[0,\infty)$, *for* $p = \infty$*)*, $l \in \mathbb{N}_0$, $r \in \mathbb{N}$. *Then*

$$||\varphi^l (P_{n,r}(f,x) - f)^l||_p \le C\left\{\omega_\varphi^{2r}(f^{(l)}, n^{-1/2})_{\varphi^l,p} + n^{-r}||\varphi^l f^{(l)}||_p\right\}$$

where $\omega_\varphi^{2r}(f^{(l)},t)_{\varphi^l,p} = \sup_{0<h\le t}||\varphi^l \Delta_{h\varphi}^{2r} f^{(l)}||_{L_p[0,\infty)}$ *is the weighted Ditzian–Totik modulus of smoothness.*
The above modulus of smoothness is equivalent to the weighted K-functional:

$$K_\varphi^{2r}(f^{(l)}, t^{2r})_{\varphi^l,p} = \inf\left\{||\varphi^l(f^{(l)} - g)||_p + t^{2r}||\varphi^{l+2r}g^{(2r)}||_p \right.$$
$$\left. \varphi^l g, \varphi^{l+2r}g^{(2r)} \in L_p[0,\infty), 1 \le p \le \infty\right\},$$

for details see [50, Chapter 6].

Theorem 2.5 *Let* $f, f^{(l)}, \varphi^l f^{(l)} \in L_p[0,\infty), 1 \le p \le \infty$ *(with* $C[0,\infty)$*, for* $p = \infty$*)*, $l \in \mathbb{N}_0, r \in \mathbb{N}, t > 0$. *Then*

$$K_\varphi^{2r}(f^{(l)}, t^{2r})_{\varphi^l,p} \le ||\varphi^l (P_{n,r}(f,x) - f)^{(l)}||_p + M(nt)^r K_\varphi^{2r}(f^{(l)}, n^{-2r})_{\varphi^l,p}.$$

2.2.2 Durrmeyer Type Operators: $\overline{B}_n, \overline{S}_n, \overline{V}_n$

In the year 1983, B. Wood [195] considered Bernstein–Durrmeyer operators and obtained a direct result in the space $L_p[0,1], p \ge 1$ of pth power Lebesgue integrable functions on the interval [0, 1] by means of the linear combinations due to Rathore [163] and May [145] defined in Section 1.1 and obtained the following result:

Theorem 2.6 *If* $f \in L_p[0,1], 1 \le p < \infty$, *then for n sufficiently large*

$$||\overline{B}_n(f,k,.) - f||_p \le C_{p,k}\left[n^{-(k+1)}||f||_p + \omega_{2k+2}(f,p,n^{-1/2})\right],$$

where the constant $C_{p,k}$ *depends on k and p but is independent of f and n.*
In the year 1989, Agrawal and Gupta [13] considered such combinations of Bernstein–Durrmeyer operators, in simultaneous approximation. By $L_B[0,1]$, we mean the class of bounded and Lebesgue integrable functions on [0, 1]. In [13] the following direct results were obtained:

Theorem 2.7 *Let* $1 \le p \le 2k+2, f \in L_B(0,1]$ *and* $\eta > 0$ *is arbitrary. If* $f^{(p+r)}$ *exists and is continuous on* $(a-\eta, b+\eta) \subset [0,1]$ *having the modulus of continuity* $\omega(f^{(p+r)},\delta)$ *on* $(a-\eta, b+\eta)$, *then for n sufficiently large*

$$||\overline{B}_n^{(r)}(f,k,.) - f^{(r)}|| \le \max\{C_1 n^{-p/2}\omega(f^{(p+r)}, n^{-1/2}), C_2 n^{-(k+1)}\}$$

where $C_1 = C_1(k,p,r), C_2 = C_2(k,p,r,f)$ *and norm* $||.||$ *is the sup-norm on* $[a,b]$.

Let us denote by $I = [0,1]$ for $c = -1$ (correspondent to the Durrmeyer–Bernstein operators $\overline{B}_n$) and $I = [0,\infty)$ for $c = 0$ and $c = 1$ (correspondent to the Durrmeyer modifications $\overline{S}_n$ and $\overline{V}_n$, respectively). The following direct estimate was proved by M. Heilmann in her paper [115, Satz 6.12]:

Theorem 2.8 *Let* $f \in L_p(I), 1 \le p < \infty$, $\varphi(x) = \sqrt{x(1+cx)}$ *and* $n \in \mathbb{N}$. *Then, we have*

$$||\overline{M}_{n,r}^c f - f||_p \le C \begin{cases} \omega_\varphi^{2r}(f, n^{-1/2})_p, & for\ \ c = 0, c = 1 \\ \omega_\varphi^{2r}(f, n^{-1/2})_p + n^{-r}||f||_p, & for\ \ c = -1. \end{cases} \qquad (2.2.3)$$

We point out that the statement in Theorem 2.8 holds true only for $p < \infty$, because the proof relies on the use of Hardy inequality. The proof is similar to the proof of Theorem 2.1 and is based on these inequalities similar to (2.1.6), (2.1.7) and (2.1.8).

For the case of Szász–Mirakjan–Durrmeyer operators the last theorem can be generalized for simultaneous approximation of smooth functions f, such that $\varphi^{2s}f(2s) \in L_p[0,\infty), 1 \le p < \infty, \varphi = \sqrt{x}, s \in \mathbb{N}_0$.

Theorem 2.9 *If f satisfies the conditions mentioned above, then we have*

$$||\varphi^{2s}(\overline{S}_{n,r}f - f)^{(2s)}||_p \le C\omega_{\varphi}^{2r}(f^{(2s)}, n^{-1/2})_{\varphi^{2s},p}, \tag{2.2.4}$$

where the last moduli are the weighted Ditzian–Totik moduli of smoothness of order $2r$ (see [50, Definition 6.1.6]) and given for $f \in L_p(I), \varphi^{2s}f \in L_p(I), 1 \le p \le \infty$ by

$$\omega_{\varphi}^{r}(f,t)_{\varphi^{2s},p} := \sup_{0<h<t} ||\varphi^{2s}\Delta_{h\varphi}^{r}f||_{L_p[t^*,\infty)} + \sup_{0<h\le t^*} ||\varphi^{2s}\overline{\Delta}_{h\varphi}^{r}f||_{L_p[0,12t^*]},$$

with $t^* = r^2t^2$ and $\overline{\Delta}_{h\varphi}^{r}$ is the forward difference.
Li Bingzheng in [31] also considered the linear combinations of generalized Durrmeyer operators containing all the three operators $\overline{B}_n, \overline{S}_n, \overline{V}_n$ as special cases and gave a characterization in terms of the classical modulus of smoothness by means of the pointwise simultaneous approximation. In [31], $C[I]\bigcap L_\infty[I]$ be the set of continuous and bounded functions on I. The norm $||.||_\infty$ denotes the sup-norm and the rth modulus of smoothness in $C[I]\bigcap L_\infty[f]$ is defined by

$$\omega_r(f,t) = \sup_{0\le h<t} ||\Delta_h^r f||_\infty,$$

where $\Delta_h^r f(x) = \sum_{k=0}^{r}(-1)^{r-k}\binom{r}{k}f(x+(k-r/2)h)$, if $x \pm (r/2)h \in I$; $\Delta_h^r f(x) = 0$, otherwise. Also the K-functional considered in [31] by $K_r(f,t^r) = \inf_{g\in D_r}\{||f-g||_\infty + t^r||g||_{D_r}\}$, where the Sobolov space D_r and its norm are defined by $D_r = \{g \in C(I) : g^{(r-1)} \in A.C._{loc}, g^{(r)} \in L_\infty(I)\}$, $||g||_{D_r} = ||g||_\infty + ||g^{(r)}||_\infty$. Obviously $M_0^{-1}\omega_r(f,t) \le K_r(f,t^r) \le M_0\omega_r(f,t)$, with the constant M_0 independent of f and $t > 0$. The following direct estimate was established in [31].

Theorem 2.10 *Let $f^{(s)} \in C(I)\bigcap L_\infty[I], s \in \mathbb{N}_0, r \in \mathbb{N}$, then*

$$|(M_{n,r}^c(f,x) - f)^{(s)}| \le M'\omega_r(f^{(s)}, (\varphi^2(x).n^{-1} + n^{-2})^{1/2}),$$

where M' is a constant independent of $f, n \in \mathbb{N}, x \in I$ and $\varphi(x) = \sqrt{x(1+cx)}$.

2.2.3 Phillips Operators

In the recent paper by M. Heilmann and G. Tachev [119], the following two direct estimates for approximation by $\widetilde{S}_{n,r}$ in $L_p[0,\infty)$-norm, $1 \le p < \infty$ were discussed:

Theorem 2.11 (See [119, Theorem 5.6]) *Let* $\varphi(x) = \sqrt{x}, 1 \leq p < \infty f \in L_{p,0}$ $[0,\infty)$*, where* $L_{p,0}[0,\infty) = \{f \in L_p[0,\infty), \lim_{x\to\infty} f(x) = f_0 < \infty\}$*. Then*

$$||\widetilde{S}_{n,r}f - f||_p \leq C\omega_\varphi^{2(r+1)}(f, n^{-1/2})_p, \tag{2.2.5}$$

where C is an absolute constant independent of n.

Theorem 2.12 (See [119, Theorem 5.8]) *Let* $f \in L_{p,0}[0,\infty), 1 \leq p < \infty, m \in \mathbb{N}$*, such that* $\varphi^{(2m)} f^{(2m)} \in L_p[0,\infty)$*. Then*

$$||\varphi^{2m}(\widetilde{S}_{n,r}f - f)^{2m}||_p \leq C\omega_\varphi^{2(r+1)}(f^{(2m)}, n^{-1/2})_{p,\varphi^{2m}}. \tag{2.2.6}$$

where C is an absolute constant independent of n.

Remark 2.1 We point out that the proofs of both Theorems 2.11 and 2.12, similar to Theorems 2.8 and 2.9 are valid only for $p < \infty$ which follows from the use of Hardy's inequality. The case $p = \infty$ for $L_p[0,\infty)$ and approximation of bounded continuous functions $f \in C_B[0,\infty)$ by $\widetilde{S}_{n,r}$ was considered very recently by Tachev in [183]:

Theorem 2.13 *For* $\varphi(x) = \sqrt{x}, x \in [0,\infty)$ *fixed, we have*

$$|\widetilde{S}_{n,r}(f,x) - f(x)| \leq C\{\omega_\varphi^{2(r+1)}(f, n^{-1/2})_\infty + n^{-(r+1)}||f||_{C_B[0,\infty)}\}, \tag{2.2.7}$$

where $C > 0$ *depends only on r.*

2.2.4 Szász–Mirakjan–Baskakov Operators

In a recent joint paper Gupta and Tachev [105] (see also [98]) obtained the following direct theorem for approximation of $f \in C_B[0,\infty)$ by linear combinations $D_{n,r}$ of Szász–Mirakjan–Baskakov operators:

Theorem 2.14 *Let* $f \in C_B[0,\infty)$*. Then for every* $x \in [0,\infty)$ *and for* $C > 0, n > r$*, we have*

$$|(D_{n,r}(f,x) - f(x)| \leq C \cdot \omega^{r+1}\left(f, \frac{1}{\sqrt{n}}\right), \tag{2.2.8}$$

The following Voronovskaja-type estimate for linear combinations was established in [105], using the moments (1.3.6):

Theorem 2.15 *Let* $f, f', \ldots, f^{(r+2)} \in C_B[0,\infty)$. *Then, if* $r = 2k+1, k = 0, 1, 2, \ldots$ *for* $x \in [0,\infty)$ *it follows*

$$\lim_{n\to\infty} n^{k+1} \cdot [D_{n,2k+1}(f,x) - f(x)] = P_{2k+2}(x) \cdot f^{(2k+2)}(x),$$

where

$$P_{2k+2}(x) = \lim_{n\to\infty} \left(D_{n,2k+1}(\psi_x^{2k+2}(t), x) n^{k+1} \right).$$

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