Preface

Approximation of functions by positive linear operators is an important branch of the approximation theory. To increase the order of approximation a useful tool is the method of linear combinations of positive linear operators (p.l.o.). The most known example of p.l.o. is the famous Bernstein operators introduced by S. Bernstein [26, 27], which for $f \in C[0,1]$ is given by

$$B_n(f,x) = \sum_{k=0}^{n} p_{n,k}(x) f(k/n), \quad p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}.$$

In 1932 Elena Voronovskaja [193]—a doctoral student of S. Bernstein—proved that if f is bounded on $[0,1]$, differentiable in some neighbourhood of x and has second derivative f'' for some $x \in [0,1]$ then

$$\lim_{n\to\infty} n[B_n(f,x) - f(x)] = \frac{x(1-x)}{2} f''(x).$$

If $f \in C^2[0,1]$ the convergence is uniform. This result shows that the rate of convergence $B_n(f,x) - f(x) \to 0$ as $n \to \infty$ is of order not better than $1/n$ if $f''(x) \neq 0$. In this sense the theorem of E. Voronovskaja is a first example of saturation, i.e. the optimal rate of convergence is $1/n$. To increase the rate of convergence, it was P. L. Butzer [34] who introduced in 1953 the following linear combinations:

$$(2^r - 1)B_{n,r}(f,x) = 2^r B_{2n,r-1}(f,x) - B_{n,r-1}(f,x), B_{n,0}(f,x) \equiv B_n(f,x).$$

Butzer showed that, for smooth functions f, $B_{n,r}(f,x) - f(x)$ tends to 0 faster than $B_n(f,x) - f(x)$. More general combinations are considered by Rathore [163] and

C. P. May in their PhD thesis and in the year 1976 in [145]. The kth order linear combinations $L_n(f,k,x)$ of the operators $L_{d_jn}(f,x)$, discussed by [145], are given as

$$L_n(f,k,x) = \sum_{j=0}^{k} C(j,k) L_{d_jn}(f,x),$$

where

$$C(j,k) = \prod_{\substack{i=0 \\ i\neq j}}^{k} \frac{d_j}{d_j - d_i}, k \neq 0; C(0,0) = 1$$

and $d_0, d_1, \ldots\ldots, d_k$ are $k+1$ distinct arbitrary and fixed positive integers.

Z. Ditzian, the scientific advisor to C. P. May, in the famous book *Moduli of Smoothness* [50] written jointly with V. Totik in 1987 generalized the known methods of linear combinations (see Chapter 9 in the book). The linear combinations $L_{n,r}$ for $r \in \mathbb{N}$ of the operators L_{n_i} are given by

$$L_{n,r}(f,x) = \sum_{i=0}^{r-1} \alpha_i(n) L_{n_i}(f,x)$$

where the numbers n_i and coefficients $\alpha_i(n)$ satisfy the following four conditions:

(a) $an = n_0 < n_1 < \cdots n_{r-1} \leq An$,
(b) $\sum_{i=0}^{r-1} |\alpha_i(n)| < C$,
(c) $\sum_{i=0}^{r-1} \alpha_i(n) = 1$,
(d) $\sum_{i=0}^{r-1} \alpha_i(n) n_i^{-\rho} = 0, \rho = 1, 2, \cdots r-1$.

The last two conditions represent a linear system for the coefficient $\alpha_i(n)$ with unique solution

$$\alpha_i(n) = \prod_{k=0, k\neq i}^{r-1} \frac{n_i}{n_i - n_k}.$$

Note that $L_{n,0} = L_n$ (for $a = 1$). In our book we follow this more general framework of linear combinations. In Chapter 9 in [50], Ditzian and Totik among others proved the following equivalence result (see Theorem 9.3.2)

$$||L_{n,r}f - f||_B = O(n^{-\alpha/2}) \Leftrightarrow \omega_\varphi^{2r}(f,h)_B = O(h^\alpha),$$

$0 < \alpha < 2r$, where the space B and weight function φ are defined as follows: for $L_n = B_n$—Bernstein operator, $B = C[0,1], \varphi(x) = \sqrt{x(1-x)}$; for $L_n = S_n$—Szász–Mirakjan operator $B = C[0,\infty), \varphi(x) = \sqrt{x}$; for $L_n = V_n$—Baskakov operator $B = C[0,\infty), \varphi(x) = \sqrt{x(1+x)}$ and lastly for their Kantorovich variants the weight functions remain the same, where $B = L_p[0,1]$ for $\hat{B}_n$ and $B = L_p(0,\infty), 1 \leq p < \infty$ for $\hat{S}_n, \hat{V}_n$ and $L_{n,r}$ are the linear combinations of these classical p.l.o.—Bernstein, Szász–Mirakjan and Baskakov operators and their Kantorovich [126] modifications. The case of Post–Widder operators was also considered. Since then in the last three decades, hundreds of papers have appeared and considering different problems connected with the methods of linear combinations of p.l.o. We only mention the dissertation of M. Heilmann [115] published in 1992 which may be considered as a second systematical study of linear combinations attached to Durrmeyer modifications of three classical operators mentioned above. It is hardly possible to mention all results on this topic. Together with known results in the past we include also the new results obtained very recently in our joint papers and also results obtained by many other mathematicians in the past 5–10 years. Some of the results are formulated and the reader may find the proofs in the references given at the end of the book.

The book consists of eight chapters. In the first two chapters, we give the known results about the closed expressions (when it is possible) of the moments and the central moments of the operators L_n, two expressions which are crucial tools for further investigation of approximation by linear combinations. Direct and inverse estimates for a broad class of p.l.o. are considered in the next chapters. The cases of finite and unbounded intervals of the real-valued and complex-valued functions are considered. We list also the results for approximation by linear combinations in a pointwise form, obtained very recently. The known strong converse inequalities of type A in the terminology of Ditzian–Ivanov [51] for linear combinations of Bernstein and Bernstein–Kantorovich operators are also included. We represent also various Voronovskaja-type estimates for some linear combinations. Some open problems are also outlined, concerning the approximation by linear combinations of p.l.o.

Quantitative estimates for the sequences of p.l.o. play an important role not merely in approximating the functions, but also in finding the error of approximation. One of the most important convergence results in the theory of approximation is the Voronovskaja-type theorem, which describes the rate of pointwise convergence. The quantitative version of the Voronovskaja theorem for any p.l.o. acting on compact intervals was obtained in [80]. Also Acar–Aral–Rasa in [7] established quantitative results for weighted modulus of continuity in the recent years. Păltănea in [156, 157] introduced the weighted modulus of continuity. Here we discuss some of the results appeared in the recent years on such problems. Also in the last 3 years some papers on new hybrid operators appeared; we also discuss some of them.

In the recent years R. Păltănea in [155] proposed the generalization of Phillips operators based on certain parameter $\rho > 0$, which has a link to the well-known Szász–Mirakjan operators in limiting case. After that also many such operators have been appropriately modified so that they depend on certain parameters and in the limiting case they reduce to the well-known operators available in the literature. We also discuss some of the papers in this direction.

It is our goal in this book to describe the most interesting features connected with approximation by linear combinations of p.l.o. We hope our book may not only be considered as a systematic overview but also be served as a basis for future study and development of this method.

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