Chapter 2
Time, Space and Laws in Newtonian Mechanics

2.1 Newton's Laws of Mechanics

Galileo conceived that all uniform motions are equally simple, as opposed to rest being simpler. This perspective enabled a major breakthrough concerning force as a notion of a body's departure from its 'natural state' due to the action of other physical entities. The particular concept of force introduced to this end by Sir Isaac Newton involves acceleration rather than velocity, in contrast to the latter having been the prevalent conceptualization since Aristotle. Galileo and Newton's conceptions were not priorly obvious due to friction being common in Nature: the rolling stone comes to rest. Moreover, rest was also associated with 'things having their place'. Feudal powers may have favoured such a concept due to its counterpart of 'people having their place', in contrast to social mobility, by which hegemonies can be challenged.

The above perspectives of Galileo and Newton can be further formalized into the first two of Newton's Laws of Mechanics, as follows.[1]

Newton's First Law. Every body continues in its state of rest, or of uniform motion in a right line unless it is compelled to change that state by forces impressed upon it.

Newton's Second Law. The change of motion is proportional to the motive force impressed; it is made in the direction of the right line in which that force is impressed.

Newton's Third Law. To every action there is always an opposing equal reaction. A further alias for this is, consequently, *Action–Reaction Principle*.

[1] This book capitalizes subject areas, specific laws of nature, specific theorems, lemmas and similar, specific principles, and the names of the Problem of Time facets, underlying Background Independence aspects and strategies for resolving these which form the main topic of this book. This is used to keep track of which phrasings refer to specific concepts that have already been introduced in the book, rather than being merely colloquial uses of the words in question.

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In the joint modern formulations of Calculus and vectors,[2] Newton's Second Law can furthermore be expressed as follows. Firstly, let $\underline{x}$ be the position of a Newtonian particle with velocity $\dot{\underline{x}} := \mathrm{d}\underline{x}/\mathrm{d}t$, where t is Newton's notion of time (see below). Next, Newton's notion of momentum is

$$\underline{p} := m\,\frac{\mathrm{d}\underline{x}}{\mathrm{d}t}; \tag{2.1}$$

in most applications the mass m of the particle is taken to be constant. Finally, Newton's Second Law now reads

$$\text{(Impressed force)}, \quad \underline{F} := \frac{\mathrm{d}\underline{p}}{\mathrm{d}t}. \tag{2.2}$$

As one consequence of this, in the absence of external impressed forces, the momentum of a body is *conserved*: $\underline{p} = \text{constant}$.

2.2 Impact of Newtonian Mechanics

Since the inception of civilization, there has been practical demand for 'Terrestrial Mechanics'—in the form of Engineering—and for 'Celestial Mechanics': due to its timekeeping. The underlying laws for these, however, were largely not understood prior to Newton, especially as regards a unified theoretical Paradigm. Indeed, Newton's Laws of Mechanics—alongside Newton's Universal Law of Gravitation, outlined in Sect. 2.7—unified the previously separate subjects of Terrestrial and Celestial Mechanics. This Newtonian Paradigm also provided the practical means of further understanding and predicting a very wide range of phenomena.

More generally, *dynamical laws*—of which Newton's Second Law is an example—have been found to be capable of underlying substantial predictions. In the particular case of Newton's Second Law, the corresponding predictions were experimentally vindicated for over two centuries with essentially no contradictions. Substantial examples include accounting for the following.

i) Galileo's constant-acceleration model for free fall and projectiles.
ii) Uniform circular motion.
iii) Angular momentum conservation under central forces, an subcase of which recovers polymath Johannes Kepler's Second Law of planetary motion: 'equal areas swept out in equal times'. Here *angular momentum* is $\underline{L} := \underline{r} \times \underline{p}$; *total angular momentum* is $L_{\mathrm{Tot}} := L^2$.

Before briefly considering further examples of successes based on Force Laws, let us first turn to how Newton considered his Laws should be interpreted in the context of his absolute notions of time and space.

[2] Calculus was also founded by Newton, and concurrently by Leibniz. However, in the great treatise *Principia Mathematica* [676], Newton himself proved each Mechanics proposition by pictorially laid out rigorous Euclidean Geometry (including use of limiting processes).

2.3 Newtonian Absolute Space

Newton conceived of this as follows [676], presenting it in contrast with his notion of relative space. "*Absolute space, in its own nature, without relation to anything external, remains always similar and immovable. Relative space is some movable dimension or measure of the absolute spaces; which our senses determine by its position to bodies; and which is vulgarly taken for immovable space... Absolute motion is the translation of a body from one absolute place into another: and relative motion, the translation from one relative place into another.*" Newton's absolute space is continuous, infinite, imperceptible (a generalization of invisible to all senses and sensors) and cannot be acted upon. It is mathematically modelled by Euclidean $\mathbb{R}^3$ (with fixed origin and fixed axes); this also amounts to assuming well-definedness globally in space.

2.4 Newtonian Absolute Time

Newton also considered motion to occur *in* time, his principal conception of which was absolute [676]. He explained this, in contrast with his notion of relative time, as follows. "*Absolute, true and mathematical time, of itself, and from its own nature flows equably without relation to anything external, and by another name is called duration: relative, apparent and common time, is some sensible and external (whether accurate or unequable) measure of duration by the means of motion, which is commonly used instead of true time.*" Here 'equably' means 'uniformly'. Also note the use of Newton's concept of duration rather than Chap. 1's paradigm-free version. 'External' is in the sense of external to the physical entities under consideration. This includes Newtonian time being an external parameter rather than a (dependent) dynamical variable. This complies with the parametrization feature of time, but not with reparametrizability.

Newton's absolute time is likewise continuous, infinite, imperceptible and cannot be acted upon. Its infiniteness is mathematically modelled by $\mathbb{R}$ and amounts to assuming well-definedness holds globally in time itself. The last two features run against operational meaningfulness. Newtonian time is also unique enough to avoid multiplicity of times, in fact for now too strongly so, out of contravening freedom of choice of calendar year zero and of tick-duration. It is however straightforward to incorporate these features into one's practical physical calculations.

In the Newtonian Paradigm, absolute time is used to transform *kinematic* geometry into far more physically predictive *Dynamics* [96]. Within the Newtonian Paradigm, this is taken to be a universal time—one time for all the bodies and all the Laws of Physics; this precludes t being position-dependent. In fact, much of this conception of time preceded Newton, being used in the mid 1600s by Isaac Barrow, and Pierre Gassendi, and even as far back as the second century astronomer Ptolemy [521].

The Newtonian Paradigm also possesses a notion of change in time. It also possesses a notion of time as a container: a parameter of choice with respect to which

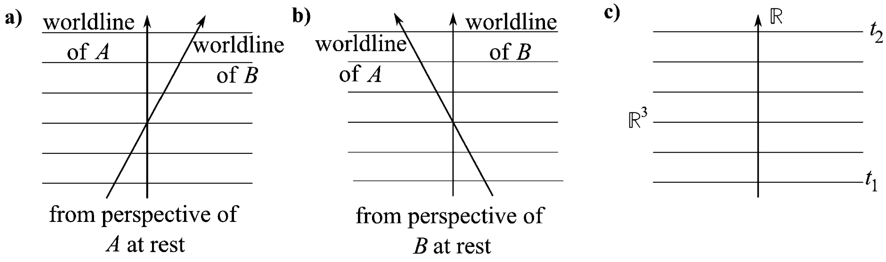

Fig. 2.1 The Aristotelian Paradigm considers **a)** and **b)** to be distinct worlds, whereas in the Galilean Paradigm they are one and the same. This is not in accord with one of **a)** and **b)** being privileged by its further identification with being at rest with respect to absolute space. **c)** contrasts the structure of Newtonian space-time. Each instant, now, or simultaneity is labelled by a value of Newtonian absolute time

change is manifest. Newton's Second Law subsequently plays the corresponding role of Dynamical Law.

Time features 1) to 11) of Sect. 1.6 are straightforwardly realized in the Newtonian Paradigm. The following four statements about these aspects of time in the Newtonian Paradigm are made to subsequently contrast with other Paradigms of Physics departing significantly from these.

1) Absolute time is taken to define a *sequence of simultaneities* representing Nature at each of its instants. Each simultaneity is here a copy of the apparent 3-d Euclidean Geometry of the corresponding space, containing a collection of particles (which possibly constitute extended objects).
2) *Dating procedures* are straightforward in the Newtonian Paradigm and enable the establishing of a *chronological ordering*.
3) *Causal ordering* coincides with chronological ordering here.
4) *Duration* is here indeed just the 'intuitive' *difference of datings*: $|t_2 - t_1|$.

2.5 Aristotelian, Galilean and Newtonian Paradigms Compared

Each of Aristotle and Newton put forward distinct absolute concepts for space and time (Fig. 2.1). Galileo, despite preceding Newton, made a different advance: contrast Fig. 2.1) with Newton's *unique* absolute space. In this way, the Newtonian Paradigm also involves velocity relative to absolute space, V_{abs}. On the other hand, the Galilean Paradigm is free from this, through involving instead a *privileged family* of frames moving at constant velocity v relative to one another. Note that this is a *trading* of one Absolute Paradigm for another: a unique absolute space and an absolute velocity V_{abs} for a non-unique notion of absolute space. Galileo's position did become the widely accepted one, modulo the caveat presented in Sect. 3.5.

The *Galilean transformations* are of the form

$$x \to x' = x - vt \tag{2.3}$$

for constant velocity v. One may adjoin

$$t \rightarrow t' = t \tag{2.4}$$

to this, i.e. there is just the one t, in contrast with other standard Paradigms of Physics' multiplicities. The Galilean transformations interrelate the privileged frames of reference in which Newton's First Law holds, which are termed *inertial frames*. These are at rest in absolute space or moving uniformly through it along a straight line. The Galilean transformations are the basis of Galilean Relativity. Frames related by this transformation can be envisaged as 'boats' in relative motion with constant velocity with respect to each other; these are as good as each other for the formulation of equally simple Laws of Physics. Indeed, Newton's Laws of Mechanics obey Galilean Relativity: they are invariant under Galilean transformations between inertial frames. On the other hand, in non-inertial frames, additional *fictitious forces* are perceived. Finally, contrast how Aristotle did not have any widely applicable law, without which considering simplifications in certain frames is moot.

We additionally consider additive transformations: spatial and temporal translations

$$\underline{x} \longrightarrow \underline{x}' = \underline{x} + \underline{k}, \qquad t \longrightarrow t' = t + t_0. \tag{2.5}$$

This does not incur any further complications. The second of these incorporates the desirable freedom of choice of calendar year zero. All in all, a minor modification of Newtonian time has Sect. 1.8's timefunction properties 1), 4) and 6), but not 2) or 3). As regards property 5)—operational meaningfulness—Newtonian time technically does not possess this, but via the rotation of the earth ('sidereal time') being identified in practice with Newtonian time, this property is in effect acquired.

The second Eq. (2.5) in is the corresponding freedom of choice of origin for absolute space. In fact, Newton himself identified absolute space in terms of the centre of mass of the solar system being at rest.

The issue of spatial rotations is, however, more involved; the transformation here involves a unit-determinant orthogonal matrix

$$\underline{x} \longrightarrow \underline{x}' = \underline{\underline{R}}\underline{x}. \tag{2.6}$$

The full set of translations and rotations constitute the *kinematical group* [814] of transformations between pairs of frames. Two particular cases are constant $\underline{k}$ and $\underline{\underline{R}}$ in the Newtonian kinematical group and time-dependent ones in the 'Leibnizian kinematical group' [278]. In the current setting, translations and rotations originate from modelling space as $\mathbb{R}^3$, for which these are rigid symmetries; see Sect. 2.12 for further discussion. The (infinitesimal) actions of the generators of these on a velocity vector are

$$T_V : \underline{\dot{x}} \longrightarrow \underline{\dot{x}} + \underline{V}, \qquad R_\Omega : \underline{\dot{x}} \longrightarrow \underline{\dot{x}} - \underline{\Omega} \times \underline{x}. \tag{2.7}$$

Using the above infinitesimal rotational action twice, the *real acceleration* relative to an inertial frame is related to the *apparent acceleration* in a rotating frame by

$$\underline{\ddot{x}} = \underline{\ddot{x}}_{\text{apparent}} + 2\,\underline{\Omega} \times \underline{\dot{x}} + \underline{\dot{\Omega}} \times \underline{x} + \underline{\Omega} \times \{\underline{\Omega} \times \underline{x}\}. \tag{2.8}$$

The fictitious terms here are, respectively, the *Coriolis*, *Euler*, and *centripetal* accelerations; see Ex I.4 for more.

2.6 Newton's Bucket

Rotations are less straightforward to handle and indeed led historically to complications. In particular, Newton used rotations in the argument in his Scholium [676] by which he became convinced to the reality of absolute space. *"If a vessel, hung by a long cord, is so often turned about that the cord is strongly twisted, then filled with water, and held at rest together with the water; thereupon, by the sudden action of another force, it is whirled about the contrary way, and while the cord is untwisting itself, the vessel continues for some time in this motion; the surface of the water will at first be plain, as before the vessel began to move; but after that, the vessel, by gradually communicating its motion to the water, will make it begin sensibly to revolve, and recede by little and little from the middle, and ascent to the sides of the vessel, forming itself into a concave figure (as I have experienced), and the swifter the motion becomes, the higher will the water rise, till at last, performing its revolutions in the same times with the vessel, it becomes relatively at rest in it. This ascent of the water shows its endeavor to recede from the axis of its motion; and the true and absolute circular motion of the water, which is here directly contrary to the relative, becomes known, and may be measured by this endeavor. ... There is only one real circular motion of any one revolving body, corresponding to only one power of endeavoring to recede from its axis of motion... And therefore in their system who suppose that our heavens, revolving below the sphere of the fixed stars, carry the planets along with them; the several parts of those heavens, and the planets, which are indeed relatively at rest in their heavens, do yet really move."* We return to this analysis in Chap. 3.1 with some historically-posterior arguments.

We next consider Force Laws within the Newtonian Paradigm.

2.7 Newtonian Gravity

Newton's Universal Law of Gravitation. The gravitational force between two particles with masses m_I at positions $\underline{x}_I$ is[3]

$$\underline{\mathrm{F}}^{\mathrm{g}}_{12} = -\frac{Gm_1m_2}{r_{12}^2}\hat{\underline{r}}_{12}, \tag{2.9}$$

where G is *Newton's universal gravitational constant*. Combining (2.9) with Newton's Second Law, gives the equation of motion for a particle in a gravitational field. In particular, this framework accounts for the following.

[3]Let us use I-indices to run over particle labels, currently 1 and 2, and $\underline{\mathrm{r}}_{IJ} := \underline{x}_I - \underline{x}_J$, and small hats for unit vectors.

1) Kepler's other two Laws of Plantary Motion: that the planets move on ellipses with the sun at one focus and with (orbital period) $\propto$ (semi-major axis)$^{3/2}$.
2) Gravitation near the surface of the Earth. Thereby, Newton unified Terrestrial and Celestial Mechanics. Indeed, Newtonian Gravitation has considerable further success at accounting for Solar System motions, e.g. in modelling perturbations due to interactions between planets.

By the 19th century, physicists began to favour the description of forces in terms of fields pervading space. In other words, they began to consider Field Theories. From this perspective, the Newtonian gravitational potential is a scalar field: the *gravitational potential* $\phi_{12} := m_2/|\underline{\mathrm{r}}_{12}|$ at $\underline{x}_1$ due to the particle of mass m_2 at $\underline{x}_2$. In terms of this, $\underline{\mathrm{F}}^{\mathrm{g}}_{12}$ may furthermore be written as $\underline{\mathrm{F}}^{\mathrm{g}}_{12} = -m_1\underline{\partial}\phi_{12}$. The *gravitational vector field* $\underline{\mathrm{g}} = -\underline{\partial}\phi$ is also useful in the discussion below. Near the surface of the Earth, the magnitude of this is the familiar 'terrestrial gravity' g, whose direction is 'downwards'. Conversely, ϕ is said to be a *scalar potential* for $\underline{\mathrm{g}}$.

Newtonian Gravity is linear, so the Superposition Principle applies as regards building up the gravitational field at each location $\underline{x}$ from each material point source. The total gravitational force due to all of a system's particles is $\underline{\mathrm{F}}^{\mathrm{g}}(\underline{x}) = -m\{\underline{\partial}\,\phi\}(\underline{x})$. In the Field Theoretic formulation, the combination of Newton's Second Law and (2.9) gives $\ddot{\underline{x}} = -\{\underline{\partial}\phi\}(\underline{x})$. Consider the particular case of this for two neighbouring particles at positions $\underline{x}$ and $\underline{x} + \Delta\underline{x}$. By subtraction and the definition of derivative, one arrives at the *tidal equation*

$$\Delta\ddot{\underline{x}} = -\underline{\partial}\{\underline{\partial}\cdot\ \Delta\underline{x}\} \tag{2.10}$$

for the relative acceleration of the two particles. This equation indeed accounts for the tides of the sea in terms of the position of especially the Moon and also the Sun. Moreover, this is but the most familiar of many such effects, and the relative acceleration concept is accorded further theoretical significance in Chap. 7.

A *field equation*—describing how Gravitation is sourced by masses—is also required. In differential form, this gives *Poisson's Law*

$$-\underline{\partial}\cdot\underline{\mathrm{g}} = \triangle\phi = 4\pi G\rho, \tag{2.11}$$

where ρ is the mass density. (2.9) is then recovered as the fundamental solution [220] corresponding to the 3-*d* Laplacian operator.

A small (43 seconds of arc per century) anomalous perihelion precession of Mercury was detected in the late 19th century. At first, this was attributed to a perturbation caused by a 'planet Vulcan' (and then to a cloud of smaller bodies) in close proximity to the Sun. We shall see however in Chap. 7 that Einstein gave an entirely different explanation for this effect, and indeed no 'planet Vulcan' has ever been seen.

2.8 Electrostatics

In this book, we do not take 'Newtonian' to mean 'posited by Newton' but rather 'within Newton's Paradigm for Physics as a whole'. This covered all the Physics that was known for over two centuries after Newton's formulation and remains an excellent approximation for many practical purposes. Within this Paradigm, the next three sections consider further Force Laws that turn out to be based upon fundamental forces.

The phenomenon of static electricity, in the form of rubbing amber with cloth, has been known since the Ancient Greeks. Experimental confirmation of the corresponding force law did not however come until physicist Charles-Augustin Coulomb's work in the 18th century. Coulomb's Law for the force between two charges q_I at positions $\underline{x}_I$ is

$$\underline{\mathrm{F}}_{12} = K\frac{q_1 q_2}{\underline{r}_{12}^2}\underline{\hat{r}}_{12}. \tag{2.12}$$

K is here Coulomb's constant; this has been subsequently interpreted as $1/4\pi\epsilon_0$ for ϵ_0 the permittivity of space (the value of this quantity is given and explained in Chap. 3.5).

The development of Field Theory was particularly significant for the study of Electricity and Magnetism and their eventual unification. Vector Calculus subsequently provided an efficient language for this. For now, Coulomb's Law can be recast as a particular case of *Gauss' Law*, in terms of a vector electric field $\underline{\mathrm{E}}$ or a scalar potential Φ such that $\underline{\mathrm{E}} = -\underline{\partial}\,\Phi$. The differential form of Gauss's Law is now

$$-\triangle\Phi = \underline{\partial}\cdot\underline{\mathrm{E}} = \rho_{\mathrm{e}}/\epsilon_0 \tag{2.13}$$

for ρ_{e} the charge density. The passage from the 3-*d* Laplacian to the inverse-square fundamental solution is just a mathematical reworking, by which Coulomb's Law is recovered analogously to how Newton's Universal Law of Gravitation is retrieved from Poisson's Law in the previous section. Working in terms of Φ superposition is again immediate, so Gauss's Law readily covers a wider range of configurations of charges.

2.9 Gravitation and Electrostatics Compared

Let us comment further here on the extent of the similarity between Coulomb's Law and Newton's Universal Law of Gravitation. Both are inverse square laws between 'charges' that feel the force in question; here these are electric charges, whereas for Gravitation they are masses. We shall see in Chaps. 3 and 7, however, that the above similarity turns out to be a coincidence of simplified regimes rather than some deep inter-relation. Moreover, electric charges come with two possible signs: positive and negative, whereas Gravitation has only one sign of 'charge': positive mass. Nor need all macroscopic bodies possess any electric charge, whereas they do all possess mass. Finally note that it is unclear at this stage that Gravitation will turn out to be very significant for further theoretical reasons beyond Newton's unification of Terrestrial and Celestial Mechanics.

Also N.B. that

$$\text{Gravitation is } 10^{40} \text{ times weaker than electrostatic attraction.} \tag{2.14}$$

This is a rough order of magnitude estimate, which holds for the range of constituent elementary particles of ordinary matter.

Moreover, mass already featured in a different manner in the conceptualization of Newtonian Mechanics. This might be interpreted as mass being a two-use concept, or as there actually being two different concepts of mass [520] that are not a priori to be assumed to be the same. I.e. inertial mass in Newton's Second Law and gravitational mass in Newton's Law of Gravitation. The latter can furthermore be split into active and passive subcases [520].

Let us next consider Newton's Second Law in a rotating rather than inertial frame. Dividing by the inertial mass m_{i}, this is [814]

$$\underline{\ddot{x}} = \underline{\mathrm{a}} + \frac{m_{\mathrm{g}}}{m_{\mathrm{i}}}\underline{\mathrm{g}} + \frac{1}{m_{\mathrm{i}}}\underline{F} - \left\{2\underline{\Omega} \times \underline{\dot{x}} + \underline{\dot{\Omega}} \times \underline{x} + \underline{\Omega} \times \{\underline{\Omega} \times \underline{x}\}\right\}, \tag{2.15}$$

since nothing can shield gravity, and where $\underline{\mathrm{a}} = \underline{\mathrm{a}}(\underline{x}, \underline{\dot{x}})$ is an acceleration field. It has additionally been noted experimentally (in e.g. 'Eötvös-type' experiments [910], named after Baron Roland von Eötvös) that $\frac{m_{\mathrm{g}}}{m_{\mathrm{i}}}$ cannot be measured, i.e. that this happens to be independent of material composition. Furthermore, elevating this from an experimental summary to a physical principle constitutes a type of *Equivalence Principle*:[4] a significant matter to which we return in Chap. 7. On the other hand, distinct electric charge-to-mass ratios are readily observed.

We end this discussion of mass with how the unit of mass—the kilogram—has been defined since 1889 as the mass of some carefully preserved lump of metal. It is presently a surface area minimizing cylinder of the Pt–10Ir alloy.

2.10 Magnetostatics

For now, take magnets to concern a further force known since antiquity to be exhibited by a few minerals such as lodestone, to be transferable unto some metals, and to be pervasive as some kind of weak background. E.g. the compass was invented in Ancient China as a tool of navigation; the background it picks up is now known to be sourced by the interior of the Earth. Magnetism was found to be sourced by electric currents; the steady-current regime case of this began to be understood as a

[4] This contradicts the Aristotelian doctrine that heavy bodies fall faster than lighter ones. On the other hand, another Ancient Greek philosopher—Epicurus—did entertain such a concept [520].

Force Law between wires due to Biot and Savart in the early 19th century. E.g. for wire elements $\mathrm{d}\underline{l}_I$ carrying steady currents I_I at positions r_I,

$$\underline{\mathrm{F}}_{12} = \frac{\mu_0}{4\pi}\frac{I_1\mathrm{d}\underline{l}_1 \cdot I_2\mathrm{d}\underline{l}_2}{r_{12}^2}\underline{\hat{r}}_{12}. \tag{2.16}$$

Here μ_0 is the permeability of space, which is defined to take the exact value $4\pi \times 10^{-7}$ ampère metres. Formulating the above in Field Theoretic terms gives *Ampère's Law*

$$\underline{\partial} \times \underline{\mathrm{B}} = \mu_0 \underline{\mathrm{j}}. \tag{2.17}$$

Equation (2.16) is now readily recovered as a particular solution of this.

Magnets have two kinds of poles—termed North and South. No pole of one kind has ever been observed in the absence of an opposite pole, e.g. bar magnets are observed to have a North pole at one end and an equal-strength South pole at the other. If a bar magnet is split into shorter bars, *each piece* has one of each kind of pole, so one cannot consider a region of space containing one kind of pole but not the other. This '*non-observation of magnetic monopoles*' is encoded as a further Law,

$$\underline{\partial} \cdot \underline{\mathrm{B}} = 0. \tag{2.18}$$

Compare this with Gauss's Law (2.13), which in general has a non-zero source charge term on its right hand side. Magnetism has no such thing as a source pole term!

2.11 Light Flashes

For now, let us consider these just in Newtonian terms so as to enable their use as thought-experiment probe devices so as to compare [831] Newtonian Mechanics and SR in Chap. 4.

2.12 Cartesian and Curvilinear Tensors Within the Newtonian Paradigm

$\mathbb{R}^p$ can be viewed[5] as an inner product space or normed space (Appendix A.3) associated with a matrix $\mathbb{I}$ whose components are δ_{ij} (the Kronecker delta symbol). This is an efficient way to encode length of a vector $\|\underline{v}\|$, distance between points

[5]Most usually this is $\mathbb{R}^3$, but some issues are sufficiently well-illustrated by $\mathbb{R}$ or $\mathbb{R}^2$, and $\mathbb{R}^p$ for $p > 3$ also enter at the level of configuration space. Moreover, there is no extra complication in treating this in arbitrary dimension.

with position vectors $\underline{q}_1$ and $\underline{q}_2$: $\|\underline{q}_1 - \underline{q}_2\|$, ratios of lengths $\|\underline{v}\|/\|\underline{u}\|$, and angles between vectors, $\arccos(\frac{\underline{v}\cdot\underline{u}}{\|\underline{v}\|\|\underline{u}\|})$. One can now treat (the p-dimensional version of) Euclidean Geometry in these more modern terms.

The transformations preserving (,), $\|\ \|$ or $\mathbb{I}$ are translations $Tr(p)$, rotations $Rot(p)$ and reflections Ref. These form a group of Euclidean transformations; we focus on the case without reflections, for which we denote the group by $Eucl(p)$.[6]

Cartesian coordinate systems on $\mathbb{R}^p$ are interrelated by $\bar{x}_i = R_{ij}x_j + T_i$. For rotations R_{ij} and translations T_i, considering vectors on $\mathbb{R}^p$ is natural because inner product spaces are vector spaces; vectors are furthermore well-known to model many physical quantities. Under rotation of frames, vectors transform as

$$\bar{v}_i = R_{ij}v_j. \tag{2.19}$$

'Vector proportionality laws', such as *Ohm's Law* $\underline{\mathrm{j}} = \sigma\underline{\mathrm{E}}$ for electrical conductivity tensor σ, end up taking forms such as $\mathrm{j}_i = \sigma_{ij}\mathrm{E}_j$: laws involving 'matrix valued' quantities. This leads to one asking how such a σ_{ij} itself transforms under the group under which the vectors were already held to transform. One can keep on repeating this process for objects with increasing numbers of indices. The most well-known of these is probably $\mathrm{s}_{ij} = \mathrm{C}_{ijkl}\mathrm{e}_{ij}$ from elasticity theory, for **s** the stress, **e** the strain and **C** the elasticity, i.e. a tensorial rendition of *Hooke's Law*. Additionally, the tensor transformation law for the 2-index object is

$$\bar{\mathrm{T}}_{ij} = R_{il}\mathrm{T}_{lm}R^{\mathrm{T}}_{mj} = R_{il}R_{jm}\mathrm{T}_{lm}. \tag{2.20}$$

Tensors are of wide importance in Physics. Within this framework, the *Quotient Theorem* furthermore gives back that '(tensor) = (unknown) · (tensor)' implies that the unknown entity also transform as a tensor, justifying the above means of envisaging the need for tensors as well as just vectors.

δ_{ij} itself is a special such tensor, blessed with *isotropy*. This means that the entity is the same in all directions, in the case in which $\mathbb{R}^n$ is being interpreted as space, this property is a significant postulated attribute. This is in contrast with all other 2-tensors, which exhibit preferred directions as per the previous paragraph's argument. Having preferred directions corresponds to exhibiting *anisotropy*. A notable example of this is the electrical conductivity of graphite, which is large in a plane of directions and small in the perpendicular direction. Since observing this, it has been found out to result from a parallel-layer structure on the slightly larger than atomic scale. It is later also further insightful to identify δ_{ij} as a *metric* (indeed, it is a metric tensor and the associated distance is the basis of a metric space). This is specifically the *Euclidean metric*, and the rigid transformations $Tr(p)$, $Rot(p)$ and

[6] See Appendix A.2 for an outline of Group Theory. $Tr(p)$ and $Rot(p)$ are continuous transformations. Ref are discrete. Let us call $Eucl(p)$ as defined here the *proper* alias *special Euclidean group*, whereas including Ref would involve a *full* Euclidean group) 'Proper' and 'special' are more widely used of cases excluding discrete reflection transformations. See also Appendix B as regards other types of Flat Geometry corresponding to the preservation of other combinations of the previous paragraph's quantities.

Ref are now the corresponding *isometries*: metric-preserving transformations (see Appendix B). The word ‘metric’ indeed means a measurer of the basic geometrical entities: the above expressions for lengths, distances, ratios of distances and angles. The corresponding line element is

$$\mathrm{d}s^2 = \|\mathrm{d}\underline{x}\|^2 = \mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2, \tag{2.21}$$

for $\mathrm{d}\Omega^2$ the $\{d-1\}$-sphere line element; in particular $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\phi^2$ for a 2-sphere in 3-d.

The second form of this is in curvilinear coordinates (in this case spherical polar coordinates). Some problems (or models) match some curvilinear coordinate system well in symmetry, and some workings are solvable in particular coordinates. In general distinction between upstairs (contravariant) and downstairs (covariant) indices is required (see Appendix D.2 for more). The coordinate transformation now involves

$$\mathrm{J}^i{}_j := \frac{\partial \bar{x}^i}{\partial x^j} \tag{2.22}$$

—the *Jacobian (transformation) matrix*—in place of R_{ij}. See Appendix D.2 for a more general treatment of coordinate transformations and consequently of tensors. R_{ij} and $\mathrm{J}^i{}_j$ are but the first two cases of this encountered in this book. Each case has a concept of tensor transformation law as associated with the corresponding group of transformations. So e.g. a *Cartesian tensor* is really a $[Rot(p) = SO(p)]$-*tensor* for $SO(p)$ the d-dimensional special orthogonal group: see Appendix E). The group corresponding to $\mathrm{J}^i{}_j$ itself will be introduced in Chap. 7.

Moreover, R_{ij} maps simple cases to simple cases within the Newtonian Paradigm, but that $\mathrm{J}^i{}_j$ seldom preserves simplicity. This is useful in those few problems which admit judicious choices of simplifying coordinates. Also note the difference between ‘laws are simple in these restricted frames’ and ‘this specific example’s mathematics is simple’.

We finally point to curvilinear transformations are in general not valid over the whole of $\mathbb{R}^n$. For instance, polar coordinates are not valid at the origin since ϕ is undefined there. This is accompanied by a breakdown in the $\mathrm{J}^i{}_j$ in moving from coordinates valid in some region to coordinates invalid there. The absolute value of the determinant of $\mathrm{J}^i{}_j$—the *Jacobian* J itself—furthermore features as a factor in integrands. The familiar Vector Calculus is the most common and simple case of Tensor Calculus, and is used in both the Cartesian and curvilinear contexts.

2.13 Principles of Dynamics (PoD) formulations of Mechanics

Other than in the case of a single particle, instantaneous configuration of a system is a distinct notion from space. In the former, a system’s configuration is represented by a single point in *configuration space* $\mathfrak{q}$, and the evolution of a system by a single curve therein. E.g. for N particles in d-dimensional space $\mathbb{R}^d$ the configuration space [598] is $\mathbb{R}^{dN}$ [598].

Theoreticians can moreover choose to describe the position of a particle in (absolute) Euclidean 3-space $\mathbb{R}^3$ by 3 curvilinear coordinates. It is convenience, rather than any physical reality, which underlies which particular choice is made.

Configuration spaces are a starting point for the Principles of Dynamics (below), and are also central to Presentism and Fully Timeless Approaches. Mechanical systems are usually taken to be second order, so that the initial position of the particle does not suffice to determine the motion. One requires also such as the initial velocity or the initial momentum. For N particles in $\mathbb{R}^d$, naïvely one requires the prescription of dN coordinates to describe their positions. However, the particles may not be free to move in all possible ways, e.g. some of them could be attached by means of strings, springs or rods. Such constitute *constrained* mechanical systems, which can be described in terms of less than $3n$ independent coordinates, Q^{A}.[7] Whereas one may attempt to study such particle systems directly using Newton's Laws, this may be cumbersome and requires knowledge of all the forces acting at each point in the system.

A method based on energy considerations, which is often of computational value and extends to Field Theory, was formalized by Euler and Lagrange in the 18th century: the Principles of Dynamics (Appendix J). Firstly, one considers a system's potential energy $V = V(\boldsymbol{Q})$ and kinetic energy T. A typical form for the latter is $M_{\mathsf{AB}}\dot{Q}^{\mathsf{A}}\dot{Q}^{\mathsf{B}}/2 := \|\dot{\boldsymbol{Q}}\|_{\boldsymbol{M}}^2/2$. M_{AB} is here the configuration space metric, the most common case of which is the kinetic mass metric $m_I\delta^{ij}\delta^{IJ}$ for $I = 1$ to N particles in d-dimensional space. This indicates that configuration spaces are geometrical entities: a theme developed in Appendix G. $\|\;\|_{\boldsymbol{M}}$ is a usefully concise notation here, as per Appendix A.3. One next forms the *Lagrangian* $L := T - V$: a single function, knowledge of which permits one to write down a set of equations of motion equivalent to Newton's. Consult Appendices J.1-4 as regards subsequent significant developments.

In the present case, the Principles of Dynamics' generalized momentum produces the vectorial approach's usual notion of momentum (2.1), and the Euler–Lagrange equations amount to a recovery of the vectorial approach's Newton's Second Law. Some significant Mechanics examples of Poisson brackets evaluations are the fundamental bracket

$$\left\{q^i, p_j\right\} = \delta^i{}_j, \tag{2.23}$$

and the angular momentum bracket

$$\{L_i, L_j\} = \epsilon_{ij}{}^k L_k, \qquad \left\{q^i, L_j\right\} = \epsilon^i{}_{jk} q^k, \qquad \{p_i, L_j\} = \epsilon_{ij}{}^k p_k. \tag{2.24}$$

The first Poisson bracket in (2.24) signifies that angular momentum corresponds to the $SO(3)$ group of rotations (Appendix E), and the second and third that q^i and p_i are good objects—vectors—under $SO(3)$ transformations.

[7] My capital sans-serif indices are general indices, many of which are reserved for particular uses in this book. This book also uses bold font for index-free presentations, so e.g. the Q^{A} are denoted more succinctly by $\boldsymbol{Q}$.

In conclusion, the Principles of Dynamics readily extends to formulation in curvilinear coordinates: these last two sections on ‘useful tools’ additionally combine well. These lie at the root of much efficient problem-solving within the Newtonian Paradigm. Additionally, this is not their only purpose, for they are built out of concepts that extend much further across Physics, Via a large family of Tensor Calculi and of metric geometries, and by Principles of Dynamics approaches applying to all branches of Classical Physics—involving such as fields as well as particles—such tools apply in *whichever* Paradigm of Classical Physics rather than just Newton’s.

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