Implications and Dependencies Between Attributes

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Abstract. This work is a translation of "Implikationen und Abhängigkeiten zwischen Merkmalen" by Bernhard Ganter and Rudolf Wille, Technische Hochschule Darmstadt, Preprint-Number 1017, 1986. The manuscript has originally been published in "Die Klassifikation und ihr Umfeld", edited by P. O. Degens, H. J. Hermes, and O. Opitz, Indeks-Verlag, Frankfurt, 1986 (rights now with Ergon-Verlag).

This translation has been authorized by Bernhard Ganter.

1 Introduction

Implications and dependencies between attributes require primal interest when one wants to study the connection between objects and attributes in application domains. In the following, it shall be shown how implications and dependencies between attributes can be investigated within the framework of Formal Concept Analysis. It is assumed that the reader has knowledge about the basic notions of Formal Concept Analysis as described in [12,13]. Note that attributes of a context (G, M, I) shall also be denoted as single-valued attributes, in order to characterize them as a special case of many-valued attributes [12]. In the following, many aspects will only be addressed briefly and further references will be provided. It should be mentioned that there are many computer programs available for Formal Concept Analysis that can also be used to investigate implications and dependencies between attributes [6].

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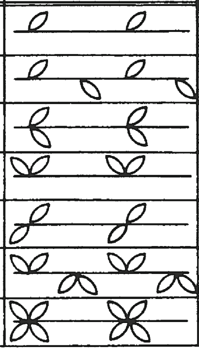

Typ	Beispiel	a	b	c	d	e
1		×				
2		×	×			
3		×		×		
4		×			×	
5		×				×
6		×	×		×	×
7		×		×	×	×

The attributes mean:

a: translation
b: glide reflection
c: horizontal reflection
d: vertical reflection
e: rotation by 180 degrees

Fig. 1. Context of symmetry types of one-sided frieze patterns ("Beispiel" is "Example")

2 Implications Between Single-Valued Attributes

Let us first illustrate by means of an example what implications between attributes of a context are. For that purpose we choose the context from Fig. 1, containing the symmetry transformations of (one-sided) frieze patterns [9]. Objects of this context are the different types of symmetries, each comprising the frieze patterns with similar symmetry transformations; each type is represented by a corresponding pattern. Attributes of the context are possible transformations of frieze patterns. A cross in the table indicates which transformations are permitted by the corresponding symmetry type.

In this context the attributes "glide reflection" and "horizontal reflection" together imply the attribute "rotation by 180°", because each symmetry type with the first two attributes also has the attribute "rotation by 180°". Analogously, the attributes "vertical reflection" and "rotation by 180°" together imply the attribute "horizontal reflection". The attribute "translation" even is implied by every other attribute separately. The attribute "glide reflection" is not implied by the attributes "vertical reflection" and "horizontal reflection", because Symmetry Type 7 has the latter two attributes, but not "glide reflection" (remark: in the given context, the attribute "glide reflection" only denotes glide reflections with a translation by half the period of the corresponding frieze pattern).

In general an implication between sets A and B of attributes of a context (G, M, I) is defined as follows: A implies B (in symbols: $A \to B$), if $B \subseteq A''$ holds; in particular, an attribute m of the context is implied by A (in symbols: $A \to m$), if $m \in A''$ holds [12]. The implications can be read from the concept lattice $\underline{\mathfrak{B}}(G, M, I)$: A implies m if and only if

$$\mu m := (\{m\}', \{m\}'') \geq \bigwedge \{\mu n \mid n \in A\},$$

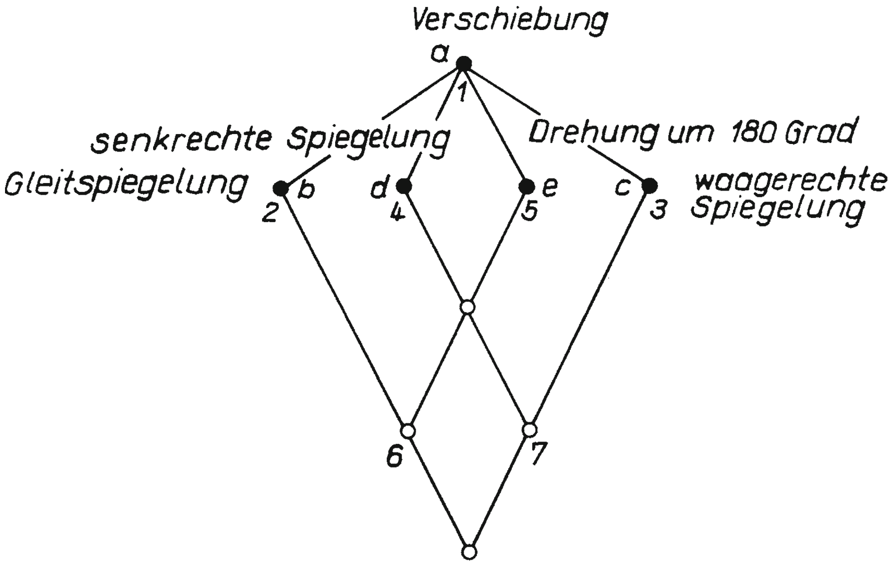

Fig. 2. Concept lattice of the context from Fig. 1 ("Gleitspiegelung" is "glide reflection", "senkrechte Spiegelung" is "vertical reflection", "Verschiebung" is "translation", "Drehung um 180 Grad" is "rotation by 180°", "waagerechte Spiegelung" is "horizontal reflection")

i.e., if in the line diagram of the concept lattice, the concept labeled with m is above or equal to the infimum of all those concepts labeled with some n from A. The implications mentioned above as well as others can thus be seen directly in the line diagram of the corresponding concept lattice as shown in Fig. 2.

How to determine and describe all attribute implications of a context (G, M, I)? Trivially, $A \to B$ holds if B is contained in A, or if $B \subseteq C$ and $A \to C$ is true, which is why it is not necessary to list such implications separately. Furthermore, $A_t \to B_t$ $(t \in T)$ directly entails $(\bigcup\{A_t \mid t \in T\}) \to (\bigcup\{B_t \mid t \in T\})$, and thus such derivable implications also do not need to be listed separately. This motivates the following definition: a set of attributes A of a context (G, M, I) is called a proper premise if

$$A \neq A'' \neq \bigcup\{(A \setminus \{n\})'' \mid n \in A\};$$

in particular, $\emptyset$ is a proper premise if $\emptyset'' \neq \emptyset$. $A \to B$ shall be called a proper implication if A is a proper premise and

$$B = A'' \setminus \bigcup\{(A \setminus \{n\})'' \mid n \in A\}.$$

To obtain a complete overview over all attribute implications of (G, M, I), it is sufficient to list all proper implications of (G, M, I), because from these implications the rest can be derived directly by the above mentioned rules. For the context from Fig. 1 one can read the following implications from Fig. 2: $\emptyset \to a$, $\{b,d\} \to e$, $\{b,e\} \to d$, $\{c,d\} \to e$, $\{c,e\} \to d$, $\{b,c\} \to \{d,e\}$. Implications like the last one in this list, where no objects satisfy the premise, will also be omitted in cases where such implications do not contribute to the understanding of the domain in question.

For contexts with fixed numbers of objects and attributes, the following rule of thumb roughly holds true: the more concepts, the fewer implications. Hence it is usually advisable to switch from the original context (G, M, I) to its complementary context $(G, M, (G \times M) \setminus I)$. For $m \in M$ and $A \subseteq M$ the following equivalences are true: $m \in A'' \iff \{m\} \subseteq A'' \iff A' \subseteq \{m\}' \iff \bigcap\{\{n\}' \mid n \in A\} \subseteq \{m\}' \iff G \setminus \{m\}' \subseteq \bigcup\{G \setminus \{n\}' \mid n \in A\}$. Therefore $A \to m$ is true in (G, M, I) if and only if in the complementary context every object having attribute m also has at least one attribute n from A. Figure 4 illustrates that this condition can easily be read from the line diagram of the concept lattice of the complementary context; the original context in Fig. 3 has 54 concepts, whereas the complementary context only has 17.

	Urlaubsort	a	b	c	d	e	f	g	h	i	j	k	a	b	c	d	e	f	g	h	i	j	k
1	Kassel	X	X	X	X	X	X	X	X	X	X												X
2	Bad Karlshafen		X	X	X	X	X	X	X		X	X	X								X		
3	Naumburg	X	X			X	X	X	X		X	X			X	X					X		
4	Emstal	X	X	X		X	X		X		X	X				X			X		X		
5	Reinhardshagen		X	X		X		X	X		X		X			X		X			X		X
6	Arolsen	X	X	X	X	X	X	X	X		X	X									X		
7	Diemelsee	X		X	X	X	X	X	X		X	X		X							X		
8	Willingen		X	X	X	X	X	X	X		X	X	X								X		
9	Bad Wildungen		X	X	X	X	X	X	X	X	X	X	X										
10	Waldeck	X		X	X	X	X	X	X		X	X		X							X		
11	Battenberg		X	X	X	X	X	X	X		X	X	X								X		
12	Vöhl	X		X	X	X	X	X	X		X	X		X							X		
13	Frankenau		X	X		X	X	X	X		X	X	X			X					X		
14	Bad Hersfeld		X	X	X	X	X	X	X		X	X	X								X		
15	Kirchheim	X		X	X	X	X	X	X		X	X		X							X		
16	Ronshausen		X	X	X	X	X	X	X		X		X								X		X
17	Rotenburg	X	X	X	X	X	X	X	X		X	X									X		
18	Knüllwald	X	X	X	X	X	X	X	X		X	X									X		
19	Melsungen		X	X		X	X	X	X		X		X			X					X		X
20	Neukirchen	X		X	X	X	X	X	X		X	X		X							X		
21	Zwesten			X	X	X	X	X	X		X	X									X		
22	Bad Sooden-Allendorf		X	X	X	X	X	X	X		X	X	X								X		
23	Witzenhausen	X	X	X	X	X	X	X	X		X	X									X		
24	Wanfried	X				X		X	X		X	X		X	X	X		X			X		
25	Ringgau		X	X	X	X	X	X	X		X		X								X		X

Fig. 3. Context of holiday resorts in the Hessian Highlands [10] and its complement. The attributes have the following meaning: a: open-air bath, b: heated open-air bath, c: public indoor pool, d: hotel indoor pool, e: bowling alley, f: riding, g: tennis, h: minigolf, i: golf, j: fishing, k: farm holidays; ("Urlaubsort" is "holiday resort")

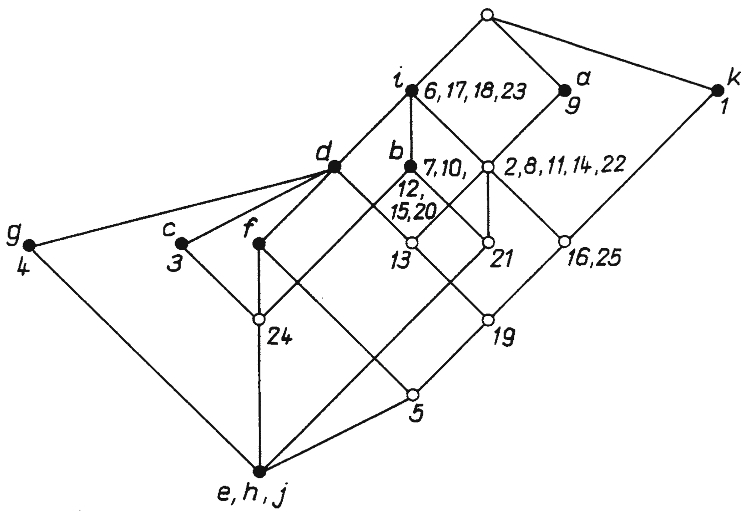

Fig. 4. Concept lattice of the completementary context in Fig. 3

For example, Fig. 4 shows that only the attributes d and f are implied by proper premises with more than one attribute, because these are the only attributes labeling concepts that are not labeled with objects.

3 Concept Lattices as Structures of Attribute Implications

As explained in the first section, attribute implications of a context (G, M, I) can be read from its concept lattice. Conversely, the concept lattice is uniquely determined by the attribute implications of (G, M, I): if $\mathcal{L}$ denotes the list of all proper implications of (G, M, I) and $A \subseteq M$, then

$$A'' = A \cup \bigcup \{Y \mid (X \to Y) \in \mathcal{L} \text{ where } X \subseteq A\}.$$

Hence the concept lattice of (G, M, I) can be considered as the structure of the attribute implications of (G, M, I) [14].

The question arises whether it is possible for each set $\mathcal{L}$ of pairs $(X, Y), X, Y \subseteq M$ to find a context (G, M, I), such that $X \to Y$ are attribute implications for all $(X, Y) \in \mathcal{L}$ and the concept lattice of (G, M, I) is uniquely determined by these attribute implications. Since concept intents are closed under attribute implications, it is not far-fetched to call a subset A of M to be $\mathcal{L}$-closed, if $(X, Y) \in \mathcal{L}$ and $X \subseteq A$ implies $Y \subseteq A$. The $\mathcal{L}$-closed subsets of M form a closure system $\mathcal{H}(\mathcal{L})$, consisting exactly of the concept intents of the context $(\mathcal{H}(\mathcal{L}), M, \ni)$ [4]. This yields an affirmative answer to the original question. The intents of $(\mathcal{H}(\mathcal{L}), M, \ni)$ can also be generated iteratively. For this we define for $A \subseteq M$ the set

$$A^{\mathcal{L}} := A \cup \bigcup \{Y \mid (X, Y) \in \mathcal{L} \text{ where } X \subseteq A\}.$$

Obviously, $A^{\mathcal{L}}$ is contained in the concept intent A'', but does not have to be equal to it. Forming the sets $A^{\mathcal{L}}$, $A^{\mathcal{L}\mathcal{L}}$, $A^{\mathcal{L}\mathcal{L}\mathcal{L}}$, ... iteratively, one eventually arrives at a set $\mathcal{L}(A)$ satisfying $\mathcal{L}(A) = \mathcal{L}(A)^{\mathcal{L}}$, i.e., $\mathcal{L}(A) = A''$ in the context determined by $\mathcal{L}$ (if M is infinite, one may have to repeat this procedure transfinitely).

Given a context (G, M, I), how to find suitable lists of attribute implications, by which the concept lattice of the context is uniquely determined? As explained before, the proper implications of (G, M, I) constitute such a list. By means of a simple procedure, this list can be generated gradually: one successively tests all subsets of M in such a manner that a set is always considered later than all of its proper subsets. Consequently, one starts with the empty subset of M; one sets $\mathcal{L} := \emptyset$. For a subset A of M it is tested whether $A^{\mathcal{L}} = A''$ for

```
 1. {b} => {a,c,f,q,s,t,v}
 2. {c} => {f}
 3. {d} => {a,b,c,f,g,h,k,l,m,q,r,s,t,u,v,w}
 4. {e} => {n}
 5. {f} => {c}
 6. {a,c,f} => {b,q,s,t,v}
 7. {g} => {l}
 8. {h} => {g,k,l,m,q,r,v,w}
 9. {i} => {e,g,h,j,k,l,m,n,o,p,q,r,v,w}
10. {k} => {g,h,l,m,q,r,v,w}
11. {l} => {g}
12. {m} => {g,h,k,l,q,r,v,w}
13. {n} => {e}
14. {c,e,f,j,n} => {a,b,q,s,t,v}
15. {e,g,l,n} => {p}
16. {o} => {e,g,h,i,j,k,l,m,n,p,q,r,v,w}
17. {g,l,p} => {e,n}
18. {e,n,p} => {g,l}
19. {e,g,j,l,n,p} => {h,i,k,m,o,q,r,v,w}
20. {c,f,q} => {a,b,s,t,v}
21. {g,l,q} => {h,k,m,r,v,w}
22. {e,j,n,q} => {v}
23. {j,p,q} => {e,g,h,i,k,l,m,n,o,r,v,w}
24. {r} => {g,h,k,l,m,q,v,w}
25. {a,s} => {b,c,f,q,t,v}
26. {c,f,s} => {a,b,q,t,v}
27. {e,j,n,s} => {a,b,c,f,q,t,v}
28. {q,s} => {a,b,c,f,t,v}
29. {t} => {a,b,c,f,q,s,v}
30. {u} => {a,b,c,d,f,g,h,k,l,m,q,r,s,t,v,w}
31. {v} => {q}
32. {a,q,v} => {b,c,f,s,t}
33. {e,n,q,v} => {j}
34. {w} => {g,h,k,l,m,q,r,v}
35. {g,h,j,k,l,m,q,r,v,w} => {e,i,n,o,p}
36. {a,b,c,f,g,h,k,l,m,q,r,s,t,v,w} => {d,u}
```

Fig. 5. Minimal list of implications for Fig. 6

Viereck	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w
1					X		X	X	X	X	X	X	X	X	X	X	X	X				X	X
2	X	X	X	X		X	X	X			X	X	X				X	X	X	X	X	X	X
3	X	X	X		X	X				X				X			X		X	X		X	
4			X		X	X	X					X		X		X							
5	X									X							X						
6							X			X		X							X				
7	X				X		X					X		X		X							
8			X			X	X			X		X											
9			X			X				X						X							
10	X						X			X		X											
11	X									X						X							
12	X	X	X			X										X	X		X	X		X	
13	X	X	X			X				X							X		X	X		X	
14					X		X					X		X		X			X				
15	X				X					X				X									
16	X				X									X			X						
17	X															X	X						
18										X						X			X				

Fig. 6. Context of properties of planar quadrangles, as given in [8] (A. Jung has participated in the creation of this context). The attributes mean the following: a: two neighboring sides have same length, b: sides are divided into pairs of neighboring sides, each of which have same length, c: sum of length of opposite sides is equal, d: sides have same length, e: sides are chords of a circle, f: sides touch a circle, g: two sides are parallel, h: opposite sides are parallel, i: opposite sides are perpendicular, j: one interior angle is perpendicular, k: neighboring angles add up to a straight angle, l: interior angles form pairs of neighboring angles, which add up to a straight angle, m: opposite angles are equal, n: opposite angles add up to a straight angle, o: interior angles are equal, p: diagonals have same length, q: one diagonal bisects the other, r: diagonals bisect each other, s: diagonals are perpendicular to each other, t: one diagonal bisects interior angles, u: each diagonal bisects interior angles, v: one diagonal partitions into congruent triangles, w: each diagonal partitions into congruent triangles. The quadrangles in the context have the following vertices:

$1:\ (0,0),(2,0),(2,1),(0,1),\quad 2:\ (0,0),(2,1),(3,3),(1,2),$
$3:\ (0,0),(3,9),(0,10),(-3,9),\quad 4:\ (-4,0),(4,0),(1,4),(-1,4),$
$5:\ (0,0),(5,0),(8,16),(0,10),\quad 6:\ (0,0),(4,0),(1,2),(0,2),$
$7:\ (0,0),(11,0),(8,4),(3,4),\quad 8:\ (0,0),(6,0),(3,4),(0,4),$
$9:\ (-5,0),(3,-4),(5,0),(3.7309\ldots,5.9733\ldots)$
$10:(0,0),(8,0),(5,4),(0,4)\quad 11:(0,0),(5,0),(7,1),(0,5),$
$12:(2,0),(0,1),(-2,0),(0,-3),\ 13:(0,0),(1,0),(2,2),(0,1),$
$14:(0,0),(2,2),(0,3),(-1,2)\quad 15:(0,0),(5,5),(4,8),(-5,5),$
$16:(0,0),(1,-3),(2,4),(1,3),\quad 17:(0,0),(11,4),(6,8),(1,4),$
$18:(0,0),(12,0),(5,12),(0,5).$

(“Viereck” is “quadrangle”)

the currently known $\mathcal{L}$; in case of inequality, $\mathcal{L}$ is extended by the pair (A, B), where $B := A'' \setminus \bigcup\{Y \mid (X, Y) \in \mathcal{L} \text{ where } X \subseteq A\}$. One obtains a minimal list of attribute implications that determine the concept lattice of (G, M, I), if in the given procedure one tests $\mathcal{L}(A) = A''$ instead of $A^{\mathcal{L}} = A''$ [2]. In this procedure considerable amounts of computation time can be saved if one exploits the fact that among all subsets A with equal closure $\mathcal{L}(A)$ there is always a lexicographically first one [4,5].

Minimal lists of attribute implications are interesting when one wants to verify the correctness of a context by means of its implications. It is claimed for the context from Fig. 6 that it has exactly those implications that are valid between the listed attributes of planar quadrangles. This can be verified by using the above mentioned procedure to generate a minimal list of attribute implications that determine the concept lattice, and then proving these implications within the realm of planar Euclidean geometry. After this it is certain that the concept lattice in Fig. 7 reflects the implications between geometric properties. A variant of the procedure allows for incrementally generating the context from a fixed set of attributes in the domain of objects, such that the resulting context admits exactly the valid attribute implications in this domain [4,5]. We have implemented this as an interactive computer program: the computer asks for each newly found implication whether it should be included in the list, or whether instead the context could be extended by a new object that rejects the proposed implication. In this way, both a minimal list of implications and a context of objects, which can be considered as counterexamples for non-valid implications, are generated.

4 Dependencies Between Many-Values Attributes

In a many-valued context (G, M, W, I), a set of objects G, a set of attributes M, and a set of attribute manifestations or attribute values W, are connected by a ternary relation $I \subseteq G \times M \times W$ in such a way that $(g, m, v) \in I$ and $(g, m, w) \in I$ always implies $v = w$ [7,12]. For each object an attribute can thus have at most one value, for different objects, however, multiple values are possible, which is why we call such an attribute many-valued. If $(g, m, w) \in I$, then this is also denoted by writing $m(g) = w$, where m is considered as a partial function from G to W; the domain of m is then $\mathrm{Def}(m) := \{g \in G \mid (g, m, w) \in I \text{ for some } w \in W\}$.

The model of relational databases from database theory can be considered as a many-valued context, suggesting to define functional dependencies between many-valued attributes as in database theory [11]. In a many-valued context (G, M, W, I) an attribute m is called nominally dependent from a set of attribues A (in symbols: $A \xrightarrow{n} m$), if $\mathrm{Def}(m) \subseteq \mathrm{Def}(n)$ for all $n \in A$ and if for all $g, h \in \mathrm{Def}(m)$ with $m(g) \neq m(h)$ there exists at least one $n \in A$ where $n(g) \neq n(h)$. Thus one has $A \xrightarrow{n} m$, if and only if there exists a function f from the set of all tuples $(n(g) \mid n \in A)$ with $g \in \mathrm{Def}(m)$, such that $f(n(g) \mid n \in A) = m(g)$ for all

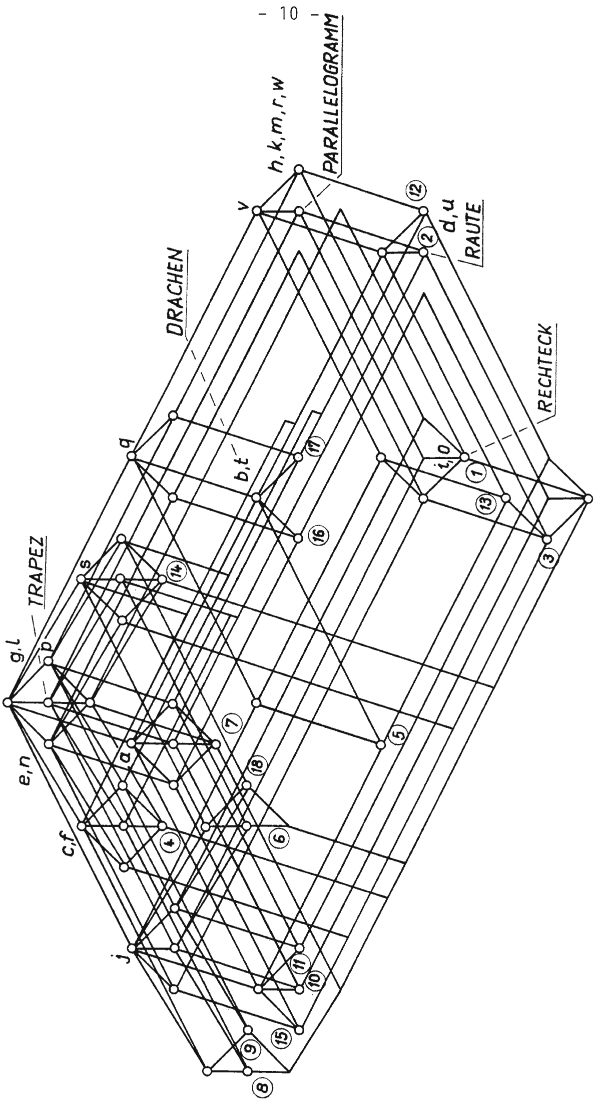

Fig. 7. Concept lattice of context in Fig. 6 ("Trapez" is "trapezoid", "Drachen" means "kite", "Parallelogramm" is "parallelogram", "Raute" is "rhombus", "Rechteck" is "rectangle")

$g \in \text{Def}(m)$; from this it is apparent why nominal dependencies are also called functional dependencies.

To determine all nominal dependencies of a many-valued context (G, M, W, I) it is beneficial to assign to each $m \in M$ the relation $E(m) := \{(g, h) \in \text{Def}(m) \times \text{Def}(m) \mid m(g) = m(h)\}$ on G. Obviously, for $m \in M$ and $A \subseteq M$ one has $A \xrightarrow{n} m$ if and only if $\bigcap\{E(n) \mid n \in A\} \cap \mathrm{Def}(m) \times \mathrm{Def}(m) \subseteq E(m)$. This characterization suggests to compute nominal dependencies as attribute implications of a single-valued contexts. We therefore define for the many-valued context $\mathbb{K} := (G, M, W, I)$ the (single-valued) context $\mathbb{K}_n := (\mathfrak{P}_2(G), M \dot\cup \hat{M}, I_n)$, where $\mathfrak{P}_2(G)$ denotes the set of all two-elemental subsets of G, $\hat{M} := \{\hat{m} \mid m \in M \text{ where } \mathrm{Def}(m) \neq G\}$, and $\{g,h\} I_n m \iff (g,h) \in E(m)$ and $\{g,h\} I_n \hat{m} \iff (g,h) \in \mathrm{Def}(m) \times \mathrm{Def}(m)$. In $\mathbb{K}$ an attribute m is nominally dependent on A, if and only if m is implied by A or $A \cup \{\hat{m}\}$ (in case $\mathrm{Def}(m) \neq G$) in $\mathbb{K}_n$. If $\mathrm{Def}(m) = G$ for all attributes m of $\mathbb{K}$, then the nominal dependencies of $\mathbb{K}$ correspond exactly to the attribute implications of $\mathbb{K}_n$. The procedure thus outlined to determine all nominal dependencies shall be illustrated at the context of Chinese pots from [3] (also see [13], p. 41). This two-valued context $\mathbb{K}$ is given in Fig. 8, and the derived context $\mathbb{K}_n$ (in reduced form) in Fig. 9.

a	b	c	d	e	f	g	h
+	+	+	+	+	+	+	+
+	+	+	−	+	+	+	+
+	−	+	+	+	+	+	+
+	−	−	+	+	+	+	+
+	+	+	+	−	+	+	+
+	−	+	+	−	+	+	+
+	−	−	+	−	+	+	+
+	−	−	−	−	+	+	+
+	−	−	−	−	−	+	+
+	−	−	−	+	+	+	+
+	−	−	−	+	−	+	+
+	−	−	−	+	−	+	−
+	−	−	−	+	+	−	−
+	−	−	−	+	−	−	−
+	−	−	−	−	−	−	−
−	−	−	+	−	+	−	−
−	−	−	+	−	−	−	−

Fig. 8. Two-valued context $\mathbb{K}$ from [3]

	a	b	c	d	e	f	g	h
{1,2}	X	X	X		X	X	X	X
{1,3}	X		X	X	X	X	X	X
{3,4}	X	X		X	X	X	X	X
{1,5}	X	X	X	X		X	X	X
{8,9}	X	X	X	X	X		X	X
{11,12}	X	X	X	X	X	X	X	
{12,14}	X	X	X	X	X	X		X
{7,16}		X	X	X	X	X		
{15,17}		X	X		X	X	X	X

Fig. 9. Single-valued context $\mathbb{K}_n$ for $\mathbb{K}$ from Fig. 8

To visualize the attribute implications of $\mathbb{K}$, it is advisable to depict the concept lattice of the context complementary to $\mathbb{K}_n$, as explained in the first section, which has been done in Fig. 10. In Fig. 10 one can see directly that $\{d,g\} \to a$ and $\{d,h\} \to a$ are the only proper implications; this means that $\{d,g\} \xrightarrow{n} a$ and $\{d,h\} \xrightarrow{n} a$ are the only "proper" nominal dependencies of $\mathbb{K}$.

In some many-valued context not the nominal dependencies are of interest, but rather a form of attribute dependency that additionally takes into account the structure on the attribute values. This shall be discussed on the basis of the school grade context from Fig. 11. In this context, the attribute "English" depends nominally on {"Greek", "Mathematics", "Chemistry"}; however, as one can already see at the first eight students, in spite of better grades in Greek,

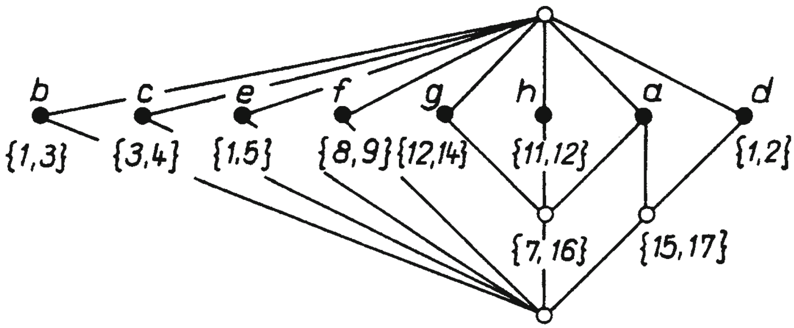

Fig. 10. Concept lattice of the complementary context for the context in Fig. 9

		Betragen	Fleiß	Aufmerksamk.	Ordnung		Deutsch	Geschichte	Sozialkunde	Latein	Griechisch	Englisch	Mathematik	Chemie	Biologie	Kunst	Leibeserz.
1	Anna	3	4	3	4		4	4	4	5	4	5	3	4	3	3	3
2	Berend	3	4	3	4		4	4	4	3	4	4	5	2	3	3	3
3	Christa	2	2	2	2		4	3	3	4	3	4	3	2	3	2	2
4	Dieter	1	1	1	1		1	1	1	2	2	2	1	1	1	3	3
5	Ernst	2	2	2	2		3	3	3	3	2	3	3	3	3	3	2
6	Fritz	2	1	2	2		2	2	2	2	2	2	2	2	3	3	1
7	Gerda	2	2	2	3		3	3	3	3	2	2	2	2	3	2	2
8	Horst	2	2	2	3		4	3	4	4	2	4	3	2	3	3	2
9	Ingolf	2	3	3	2		3	4	2	3	3	4	2	2	3	4	2
10	Jürgen	2	2	3	2		2	3	2	4	2	4	4	2	2	2	1
11	Karl	2	3	2	2		3	2	2	3	3	3	2	1	3	3	2
12	Linda	2	1	2	2		4	3	3	3	2	3	4	3	4	2	3
13	Manfred	2	2	2	2		2	3	4	1	1	2	3	2	2	2	3
14	Norbert	3	3	2	3		3	4	3	3	3	3	4	2	2	3	2
15	Olga	1	1	2	2		2	3	3	1	1	2	2	2	3	2	2
16	Paul	1	1	1	1		2	1	2	1	1	2	2	1	1	2	1
17	Quax	2	2	2	2		3	2	2	3	2	3	2	1	2	3	3
18	Rudolf	3	4	3	3		4	4	4	4	4	5	5	4	4	4	2
19	Stefan	1	1	1	1		1	1	1	1	1	2	2	1	1	1	2
20	Till	1	1	1	1		2	3	2	1	1	2	1	1	2	1	2
21	Uta	1	1	2	2		2	2	2	1	1	2	3	2	3	2	2
22	Volker	2	2	3	2		4	3	2	4	4	4	4	1	2	3	3
23	Walter	3	4	3	4		4	5	4	4	4	2	4	2	1	4	3
24	Xaver	1	1	1	1		2	2	1	2	1	2	2	2	3	3	2
25	Zora	2	2	2	2		2	4	4	3	3	3	3	4	4	2	2

Fig. 11. Context of school grades of a secondary school (1981) ("Betragen" is "Conduct", "Fleiß" is "Diligence", "Aufmerksamk(eit)" is "Attention","Ordnung" is "Tidiness", "Deutsch" is "German", "Geschichte" is "History", "Sozialkunde" is "Social Studies", "Latein" is "Latin", "Griechisch" is "Greek", "Englisch" is "English", "Mathematik" is "Mathematics", "Chemie" is "Chemistry", "Biologie" is "Biology", "Kunst" is "Art", "Leibeserz(iehung)" is "Physical Education")

Mathematics, and Chemistry, the grade in English can turn out to be worse. Clearly, those dependencies between school grades which avoid this are particularly interesting. Hence, we define for a many-valued context (G, M, W, I) together with an order relation $\leq$ on W that an attribute m is ordinally dependent from an attribute set A (in symbols: $A \xrightarrow{o} m$), if $\mathrm{Def}(m) \subseteq \mathrm{Def}(n)$ for all $n \in A$ and $m(g) \leq m(h)$ for $g, h \in \mathrm{Def}(m)$ if $n(g) \leq n(h)$ for all $n \in A$. The anti-symmetry of the order relation implies that every ordinal dependency is also a nominal dependency, but the converse is not true. In the school grade context roughly every fifth nominal dependency is also an ordinal one.

In order to determine all ordinal dependencies of a many-valued context $\mathbb{K} := (G, M, W, I)$ in which W is ordered by $\leq$, we define for each $m \in M$ a relation on G by $D(m) := \{(g, h) \in \mathrm{Def}(m) \times \mathrm{Def}(m) \mid m(g) \leq m(h)\}$. For $m \in M$ and $A \subseteq M$ one obviously has $A \xrightarrow{o} m$, if and only if $\bigcap\{D(n) \mid n \in A\} \bigcap \mathrm{Def}(m) \times \mathrm{Def}(m) \subseteq D(m)$. Now let $\mathbb{K}_o := (G \times G, M \dot{\cup} \hat{M}, I_o)$, where $(g, h)\ I_o\ m \iff (g, h) \in D(m)$ and $(g, h)\ I_o\ \hat{m} \iff (g, h) \in \mathrm{Def}(m) \times \mathrm{Def}(m)$. In $\mathbb{K}$ the attribute m is ordinally dependent on A if and only if m is implied in $\mathbb{K}_o$ by A or by $A \cup \{\hat{m}\}$ (in case $\mathrm{Def}(m) \neq G$). One has thus obtained a method to determine all ordinal dependencies. For the context of school grades, this yields that only the attributes "Conduct", "Diligence", "Attention", "German", and "English" ordinally depend on other attributes.

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