2 The Time-Frequency Analysis

What we know is not much. What we do not know is immense.

Pierre-Simon Laplace

Motivated by 'quantum mechanics', in 1946 the physicist Gabor defined elementary time-frequency atoms as waveforms that have a minimal spread in a time-frequency plane. To measure time-frequency 'information' content, he proposed decomposing signals over these elementary atomic waveforms. By showing that such decompositions are closely related to our sensitivity to sounds, and that they exhibit important structures in speech and music recordings, Gabor demonstrated the importance of localized time-frequency signal processing.

Stéphane Mallat

2.1 Introduction

Signals are in general nonstationary. A complete representation of nonstationary signals requires frequency analysis that is local in time, resulting in the time-frequency analysis of signals. The Fourier transform analysis has long been recognized as the great tool for the study of stationary signals and processes where the properties are statistically invariant over time. However, it cannot be used for the frequency analysis that is local in time because it requires all previous as well as future information about the signal to evaluate its spectral density at a single frequency ω. Although time-frequency analysis of signals had its origin almost 60 years ago, there has been major development of the time-frequency distributions approach in the last three decades. The basic idea of the method is to develop a joint function of time and frequency, known as a time-frequency distribution, that can describe the energy density of a signal simultaneously in both time and frequency. In principle, the

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time-frequency distributions characterize phenomena in a two-dimensional time-frequency plane. Basically, there are two kinds of time-frequency representations. One is the quadratic method covering the time-frequency distributions, and the other is the linear approach including the Gabor transform, the Zak transform, the linear canonical transform, and the wavelet transform analysis. So, the time-frequency signal analysis deals with time-frequency representations of signals and with problems related to their definition, estimation, and interpretation, and it has evolved into a widely recognized applied discipline of signal processing. For more detailed information, we refer to Debnath (2001), Grochenig (2001), and Debnath and Shah (2015).

This chapter is devoted to a fairly detailed examination of the joint time-frequency analysis of signals. We start with the time-frequency localization of signals which leads to the windowed Fourier transform. This is followed by the Gabor transform and its basic properties. Included are the Zak transform and its basic properties. Based on the relationship between the Fourier transform and linear canonical transform, a coupled windowed transform, namely, windowed linear canonical transform (WLCT) is introduced.

2.2 The Time-Frequency Localization

To achieve the time-frequency localization of spectral characteristics of a time-varying signal, a window function is introduced into the Fourier transform. A window function $g(t)$ is a function in $L^2(\mathbb{R})$ such that both $g(t)$ and $\hat{g}(\omega)$ have rapid decay, that is, $g(t)$ is well localized in time domain, while $\hat{g}(\omega)$ is well localized in frequency domain. Multiplying a signal $f(t)$ by a window function $g(t)$ before its Fourier transform has the effect of restricting the spectral information of the signal to the domain of influence of the window function. Using the translates of the window function on the time axis to cover the entire time domain, the signal is analyzed for spectral information in localized neighborhoods in time.

Definition 2.2.1 (Window Function). If $g(t) \in L^2(\mathbb{R})$, $\|g\|_2 \neq 0$, and $t \cdot g(t) \in L^2(\mathbb{R})$, then $g(t)$ is called a window function.

It is important to note that $g(t)$ is a window function when its squared magnitude $|g(t)|^2$ has a second-order moment. Therefore, if $g(t)$ is a window function, then $t^{1/2} \cdot g(t)$ also belongs to $L^2(\mathbb{R})$. Writing $g(t) = (1+|t|)^{-1}(1+|t|)g(t)$ and applying the Schwartz inequality, we can obtain

$$\left\|g\right\|_1 \leq \left\|(1+|t|)\right\|_2^{-1} \left\|(1+|t|)\right\|_2 < \infty,$$

which infer that $g(t)$ is integrable on $\mathbb{R}$ and, hence, $\hat{g}(\omega)$ is continuous. Although it follows from the Parseval identity that $\hat{g}(\omega)$ is also in $L^2(\mathbb{R})$, in general, it is not true that $\hat{g} \in L^2(\mathbb{R})$. In other words, it is possible that while g is a window function, $\hat{g}$ is not. An example of such a window function is the Haar function:

$$g(t) = \begin{cases} 1, & 0 \le t < \frac{1}{2} \\ -1, & \frac{1}{2} \le t < 1 \\ 0, & \text{otherwise.} \end{cases} \tag{2.2.1}$$

Example 2.2.1 (Examples of Window Functions).

1. The simplest window function is the rectangular function given by

$$g(t) = \begin{cases} 1, & |t| \le a, \quad a > 0 \\ 0, & |t| > a. \end{cases}$$

   Its Fourier transform is

$$\hat{g}(\omega) = \frac{e^{ia\omega} - e^{-ia\omega}}{i\omega}.$$

   Although $g(t)$ is compactly supported, it gives a bad localization in frequency due to its discontinuous nature. Therefore, usually the more smooth functions are needed.
2. The triangular window or Fejer window is given by

$$g(t) = \begin{cases} 1 + \dfrac{t}{a}, & -a \le t < 0 \\ 1 - \dfrac{t}{a}, & 0 \le t < a \\ 0, & |t| > a. \end{cases}$$

   It can be easily verified that

$$\hat{g}(\omega) = \frac{a \sin^2\left(\dfrac{a\omega}{2}\right)}{\left(\dfrac{a\omega}{2}\right)^2}.$$

   This function provides a good localization in frequency as its spectrum decay at the rate of $\dfrac{1}{\omega^2}$ which is faster than the decay of $\dfrac{1}{\omega}$ exhibited by the Fourier transform of rectangular function.

3. The Hanning window is given by

$$g(t) = \begin{cases} \cos^2\left(\frac{\pi t}{a}\right), & -a/2 \le t \le a/2 \\ 0, & \text{elsewhere.} \end{cases}$$

The corresponding Fourier transform is

$$\hat{g}(\omega) = \frac{a}{4}\sin\left(\frac{a\omega}{2}\right)\left[\frac{1}{\pi - a\omega/2} + \frac{2}{a\omega/2} - \frac{1}{\pi + a\omega/2}\right].$$

4. The Hamming window is given by

$$g(t) = \begin{cases} b + (1-b)\cos^2\left(\frac{\pi t}{a}\right), & -a/2 \le t \le a/2 \\ 0, & \text{elsewhere.} \end{cases}$$

The Fourier transform of this window is

$$\hat{g}(\omega) = \frac{a}{4}\sin\left(\frac{a\omega}{2}\right)\left[\frac{1-b}{\pi - a\omega/2} + \frac{2(1+b)}{a\omega/2} - \frac{1-b}{\pi + a\omega/2}\right].$$

5. The Blackman-Harris window is given by

$$g(t) = \begin{cases} \sum_{k=0}^{3} a_k \cos\left(\frac{2\pi kt}{a}\right), & -a/2 \le t \le a/2 \\ 0, & \text{elsewhere.} \end{cases}$$

The Fourier transform of this function is

$$\hat{g}(\omega) = a\sin\left(\frac{a\omega}{2}\right)\sum_{k=0}^{3} a_k(-1)^k\left[\frac{1}{2\pi k + a\omega} - \frac{1}{2\pi k - a\omega}\right].$$

The most two important parameters for a window function are its center and radius which are defined as below.

Definition 2.2.2 (Center and Radius). If $g(t)$ is a window function, then the center t^* and the root mean square radius σ_t for $g(t)$ are given by

$$t^* = \frac{1}{\|g\|_2^2}\int_{-\infty}^{\infty} t\left|g(t)\right|^2 dt \tag{2.2.2}$$

and

$$\sigma_t = \frac{1}{\|g\|_2}\left\{\int_{-\infty}^{\infty}(t-t^*)^2\big|g(t)\big|^2 dt\right\}^{1/2}, \tag{2.2.3}$$

respectively. The diameter or width of a windowing function $g(t)$ is $2\sigma_g$.

It is immediate from the definition 2.2.2 that the center and standard width of the rectangular window function $g(t) = \chi_{[a,-a]}(t)$ are zero and $2a$, respectively. The function g described above with finite σ_g is called a *time window*. Similarly, we can have a *frequency window* $\hat{g}(\omega)$ with center ω^* and (RMS) radius σ_ω defined analogous to relations (2.2.2) and (2.2.3) as

$$\omega^* = \frac{1}{\|\hat{g}\|_2^2}\int_{-\infty}^{\infty}\omega\big|\hat{g}(\omega)\big|^2 d\omega \tag{2.2.4}$$

$$\sigma_\omega = \frac{1}{\|\hat{g}\|_2}\left\{\int_{-\infty}^{\infty}(\omega-\omega^*)^2\big|\hat{g}(\omega)\big|^2 d\omega\right\}^{1/2}. \tag{2.2.5}$$

For a window function g to be useful in time-frequency analysis, it is necessary that both g and $\hat{g}$ are window functions. Henceforth, we will assume that both g and $\hat{g}$ are window functions with rapid decay in time and frequency, respectively. As we have indicated in the beginning, we could obtain the approximate frequency contents of a signal f in the neighbourhood of some desired location in time, say $t = b$, by first windowing function g to produce the window function $f_b(t) = f(t)\, g(t-b)$ and then taking the Fourier transform of $f_b(t)$. This is called the *windowed Fourier transform* or *short-time Fourier transform* (STFT), or sometimes referred to as *running-windowed Fourier transform.*

Formally, we define the STFT of a function $f \in L^2(\mathbb{R})$ with respect to the window function g evaluated at the location (b,ω) in the time-frequency plane as

$$\mathscr{F}_g f(b,\omega) = \int_{-\infty}^{\infty} f(t)\,\overline{g(t-b)}\,e^{-i\omega t}dt. \tag{2.2.6}$$

Unlike the case of Fourier transform in which the function f must be known for the entire time axis before its spectral component at any single frequency can be computed, STFT needs to know $f(t)$ only in the interval in which $g(t-b)$ is non-zero.

Moreover, equation (2.2.6) gives the localized spectral information of $f(t)$ in the time window

$$\Big[t^* + b - \sigma_t, t^* + b + \sigma_t\Big]. \tag{2.2.7}$$

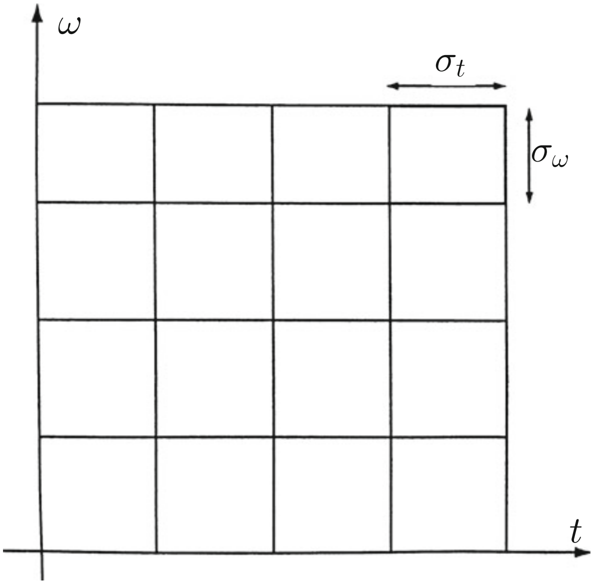

Fig. 2.1 Time-frequency plane for the WFT

To derive the corresponding window in the frequency domain, apply Parseval identity to equation (2.2.6), so that we obtain

$$\begin{aligned}\mathscr{F}_g f(b,\omega) &= \int_{-\infty}^{\infty} f(t)\, g(t-b)\, e^{-i\omega t} dt \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi)\hat{g}(\xi-\omega)\, e^{ib\xi} d\xi \\ &= e^{-i\omega b}\mathscr{F}^{-1}\left[\hat{f}(\xi)\overline{\hat{g}(\xi-\omega)}\right](b) \qquad (2.2.8)\end{aligned}$$

where $\mathscr{F}^{-1}$ is the inverse Fourier transform. If ω^* is the center and σ_ω is the radius of the window function $\hat{g}$, then, (2.2.6) also gives the localized spectral information of f in the frequency window

$$\left[\omega^* + \omega - \sigma_\omega, \omega^* + \omega + \sigma_\omega\right]. \qquad (2.2.9)$$

Thus, we have a time-frequency window

$$\left[t^* + b - \sigma_t, t^* + b + \sigma_t\right] \times \left[\omega^* + \omega - \sigma_\omega, \omega^* + \omega + \sigma_\omega\right] \qquad (2.2.10)$$

centered at $(t^* + b, \omega + \omega^*)$ in the time-frequency plane, with width $2\sigma_t$ and height $2\sigma_\omega$ as shown in Figure 2.1. The width and height of the time-frequency window are constant for all time and frequency values and have a constant area $4\,\sigma_t\sigma_\omega$.

From the above discussion, we conclude that in order to achieve a high degree of localization in time and frequency, we need to choose a window function with sufficiently narrow time and frequency windows. However, the Heisenberg uncertainty principle (Theorem 1.7.1) imposes a theoretical lower bound on the area of the time-frequency window of any window function g and is given by

$$\sigma_t \sigma_\omega \geq \frac{1}{2} \tag{2.2.11}$$

where the equality holds only when g is a Gaussian function.

Example 2.2.2. A sinusoidal wave $f(t) = e^{i\omega_0 t}$ whose Fourier transform is a Dirac delta function given by $\hat{f}(\omega) = 2\pi\,\delta(\omega - \omega_0)$ has a windowed Fourier transform

$$\mathscr{F}_g f(b,\omega) = e^{-ib(\omega-\omega_0)}\hat{g}(\omega - \omega_0)$$

Its energy is spread over the frequency interval

$$\left[\omega_0 - \frac{\sigma_\omega}{2}, \omega_0 + \frac{\sigma_\omega}{2}\right].$$

Example 2.2.3. The windowed Fourier transform of a Dirac delta function defined by $f(t) = f(t - b_0)$ is

$$\mathscr{F}_g f(b,\omega) = e^{-i\omega b_0} g(b_0 - b).$$

Its energy is spread over the time interval

$$\left[b_0 - \frac{\sigma_t}{2}, b_0 + \frac{\sigma_t}{2}\right].$$

2.3 The Gabor Transforms

It has already been stated in previous section that decomposition of a signal into a small number of elementary waveforms that are localized in time and frequency plays a remarkable role in signal processing. Such a decomposition reveals important structures in analyzing nonstationary signals such as speech and music. In order to incorporate both time and frequency localization properties in one single transform function, Gabor (1946) first introduced the windowed Fourier transform (or the Gabor transform) by using a Gaussian distribution function as a window function. His major idea was to use a time localization window function $g_a(t - b)$ for extracting local information from the Fourier transform of a signal, where the parameter a measures the width of the window and the parameter b is used to translate the window in order to cover the whole time domain. The idea is to use this window function in order to localize the Fourier transform, then shift the window to another position, and so on. This remarkable property of the Gabor transform provides the local aspect of the Fourier transform with time resolution equal to the size of the window. In fact, Gabor used

$$g_{t,\omega}(\tau) = e^{i\omega\tau} g(\tau - t) = M_\omega\, T_t\, g(\tau), \tag{2.3.1}$$

as the window function by translating and modulating a function g, where $g(t) = \pi^{-\frac{1}{4}} e^{-t^2/2}$, which is called *canonical coherent states* in quantum physics. The energy associated with the function $g_{t,\omega}$ is localized in the neighborhood of t in an interval of size σ_t measured by the standard deviation of $|g|^2$. Evidently, the Fourier transform of $g_{t,\omega}(\tau)$ with respect to τ is given by

$$\hat{g}_{t,\omega}(\upsilon) = e^{-it(\upsilon-\omega)}\, \hat{g}(\upsilon - \omega). \tag{2.3.2}$$

Obviously, the energy of $\hat{g}_{t,\omega}$ is concentrated near the frequency ω in an interval of size σ_ω which measures the frequency dispersion of $\hat{g}_{t,\omega}$. In a time-frequency (t, ω) plane, the energy spread of the Gabor atom $\hat{g}_{t,\omega}$ can be represented by the rectangle with the center at $\left(t^*, \omega^*\right)$ and sides σ_t (along the time axis) and σ_ω (along the frequency axis). According to the Heisenberg uncertainty principle, the area of the rectangle is at least $\frac{1}{2}$; that is, $\sigma_t\, \sigma_\omega \geq \frac{1}{2}$. This area is minimum when g is a Gaussian function, and the corresponding $g_{t,\omega}$ is called the *Gabor function* or *Gabor wavelet*.

Gabor transform has effectively been applied in many fields of science and engineering, such as image analysis and image compression, object and pattern recognition, computer vision, optics, and filter banks. Since medical signal analysis and medical signal processing play a crucial role in medical diagnostics, the Gabor transform has also been used for the study of brain functions, ECG signals, and other medical signals.

Definition 2.3.1 (The Continuous Gabor Transform). The continuous Gabor transform of a function $f \in L^2(\mathbb{R})$ with respect to a window function $g \in L^2(\mathbb{R})$ is denoted by $\mathscr{G}[f](t, \omega) = \tilde{f}_g(t, \omega)$ and defined by

$$\mathscr{G}[f](t, \omega) = \tilde{f}_g(t, \omega) = \int_{-\infty}^{\infty} f(\tau)\, g(\tau - t)\, e^{-i\omega\tau} d\tau = \left\langle f, \overline{g_{t,\omega}} \right\rangle, \tag{2.3.3}$$

where $g_{t,\omega}(\tau) = \overline{g}(\tau - t) e^{i\omega\tau}$, so $\left\| g_{t,\omega} \right\|_2 = \left\| g \right\|_2$ and, hence, $g_{t,\omega} \in L^2(\mathbb{R})$.

We next discuss the following consequences of the preceding definition:

1. If the window g is real and symmetric with $g(\tau) = g(-\tau)$ and if g is normalized so that $\|g\| = 1$ and $\|g_{t,\omega}\| = \|g(\tau - t)\| = 1$ for any $(t, \omega) \in \mathbb{R}^2$, then the Gabor transform of $f \in L^2(\mathbb{R})$ becomes

$$\mathscr{G}[f](t, \omega) = \langle f, g_{t,\omega} \rangle = \int_{-\infty}^{\infty} f(\tau)\, g(\tau - t)\, e^{-i\omega\tau} d\tau. \tag{2.3.4}$$

This can be interpreted as the *windowed Fourier transform* because the multiplication by $g(\tau - t)$ induces localization of the Fourier integral in the neighborhood of $\tau = t$. Application of the Schwarz inequality to (2.3.4) gives

$$\left| \mathscr{G}[f](t, \omega) \right| = |\langle f, g_{t,\omega} \rangle| \le \|f\| \, \|g_{t,\omega}\| = \|f\| \, \|g\|.$$

This shows that the Gabor transform $\mathscr{G}[f](t, \omega)$ is *bounded*.

2. It follows from definition (2.3.1) with a fixed ω that

$$\mathscr{G}[f](t, \omega) = e^{-i\omega t} \int_{-\infty}^{\infty} f(\tau)\, g(\tau - t)\, e^{i\omega(t-\tau)} d\tau = e^{-i\omega t} \big(f * g_\omega\big)(t), \tag{2.3.5}$$

where $g_\omega(\tau) = e^{i\omega\tau} g(\tau)$ and $g(-\tau) = g(\tau)$. Furthermore, by the Parseval formula, we find

$$\mathscr{G}[f](t, \omega) = \langle f, \overline{g}_{t,\omega} \rangle = \left\langle \hat{f}, \hat{\overline{g}}_{t,\omega} \right\rangle = e^{i\omega t} \int_{-\infty}^{\infty} \hat{f}(\upsilon)\, \hat{g}(\upsilon - \omega)\, e^{-i\upsilon t} d\upsilon. \tag{2.3.6}$$

Note that except for the factor $\exp(i\omega t)$, result (2.3.6) is almost identical with (2.3.3), but the time variable t is replaced by the frequency variable ω and the time window $g(\tau - t)$ is replaced by the frequency window $\hat{g}(\upsilon - \omega)$.

3. For a fixed ω, the Fourier transform of $\mathscr{G}[f](t, \omega)$ with respect to t is given by the following:

$$\mathscr{F}\,\{\mathscr{G}[f](t, \omega)\} = \hat{f}(\upsilon + \omega)\, \hat{g}(\upsilon). \tag{2.3.7}$$

This follows from the Fourier transform of (2.3.5) with respect to t

$$\mathscr{F}\Big\{\mathscr{G}[f](t, \omega)\Big\} = \mathscr{F}\Big\{e^{-i\omega t}\big(f * g_\omega\big)(t)\Big\} = \hat{f}(\upsilon + \omega)\, \hat{g}(\upsilon).$$

4. If $g(t) = e^{-t^2/4}$, then

$$\mathscr{G}[f](t, \omega) = \sqrt{2}\, e^{(i\omega t - \omega^2)} \big(Wf\big)(t + 2i\omega), \tag{2.3.8}$$

where W represents the *Weierstrass transformation* of $f(x)$ defined by

$$W\big[f(x)\big] = \frac{1}{2\sqrt{2}} \int_{-\infty}^{\infty} f(x) e^{-(t-x)^2/4} dx. \tag{2.3.9}$$

5. The time width σ_t around t and the frequency spread σ_ω around ω are independent of t and ω. In fact, we have

$$\sigma_t^2 = \int_{-\infty}^{\infty} (\tau - t)^2 \big|g_{t,\omega}(\tau)\big|^2 d\tau = \int_{-\infty}^{\infty} (\tau - t)^2 \big|g(\tau - t)\big|^2 d\tau = \int_{-\infty}^{\infty} \tau^2 \big|g(\tau)\big|^2 d\tau.$$

Similarly, we obtain, by (2.3.2),

$$\sigma_\omega^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} (\upsilon - \omega)^2 \big|\hat{g}_{t,\omega}(\upsilon)\big|^2 d\upsilon = \frac{1}{2\pi} \int_{-\infty}^{\infty} (\upsilon - \omega)^2 \big|\hat{g}(\upsilon)\big|^2 d\upsilon$$
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \upsilon^2 \big|\hat{g}(\upsilon)\big|^2 d\upsilon.$$

Thus, both σ_t and σ_ω are independent of t and ω. The energy spread of $g_{t,\omega}(\tau)$ can be represented by the Heisenberg rectangle centered at (t, ω) with the area $\sigma_t \sigma_\omega$ which is independent of t and ω. This means that the Gabor transform has the same resolution in the time-frequency plane.

Example 2.3.1. Consider the function f defined by

$$f(\tau) = e^{-a^2\tau^2}, \quad \text{with } g(\tau) = 1.$$

Then, the Gabor transform of f is

$$\mathcal{G}[f](t, \omega) = \int_{-\infty}^{\infty} e^{-\left(a^2\tau^2 + i\omega\tau\right)} d\tau = \frac{\sqrt{\pi}}{a} e^{-\omega^2/4a^2}.$$

Example 2.3.2. Obtain the Gabor transform of function

$$f(\tau) = e^{-i\sigma\tau}.$$

We have

$$\mathcal{G}[f](t, \omega) = \int_{-\infty}^{\infty} e^{-i\tau(\omega+\sigma)} g(\tau - t)\, d\tau = e^{-it(\sigma+\omega)} \hat{g}(\sigma + \omega).$$

Example 2.3.3. Find the Gabor transform of functions

(a) $f(\tau) = 1$, (b) $f(\tau) = \delta(\tau)$, (c) $f(\tau) = \delta(\tau - t_0)$.

Next, we discuss some basic properties of continuous Gabor transform.

Theorem 2.3.1. *Let $f, g, h \in L^2(\mathbb{R})$ and a, b be any two arbitrary constants. Then, the following results hold:*

(a) Linearity: $\mathscr{G}\big[af + bh\big](t, \omega) = a\mathscr{G}\big[f\big](t, \omega) + b\mathscr{G}\big[h\big](t, \omega)$,
(b) Translation: $\mathscr{G}\big[T_a f\big](t, \omega) = e^{-i\omega a}\mathscr{G}\big[f\big](t - a, \omega)$,
(c) Modulation: $\mathscr{G}\big[M_a f\big](t, \omega) = \mathscr{G}\big[f\big](t, \omega - a)$,
(d) Conjugation: $\mathscr{G}\big[\bar{f}\,\big](t, \omega) = \overline{\mathscr{G}}\big[f\big](\tau, -\omega)$.

The proof easily follows from the definition of the Gabor transform and is left as an exercise.

Theorem 2.3.2. *If two signals $f, g \in L^2(\mathbb{R})$, then*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big|\mathscr{G}[f](t, \omega)\Big|^2 dt\, d\omega = \|f\|_2^2\,\|g\|_2^2.$$

Proof. The left-hand side of the above result is equal to

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big|\mathscr{G}[f](t, \omega)\Big|^2 dt\, d\omega \\
&\qquad = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\int_{-\infty}^{\infty} f(\tau)\, g(\tau - t)\, e^{-i\omega\tau} d\tau\right|^2 dt\, d\omega \\
&\qquad = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\int_{-\infty}^{\infty} h_t(\tau)\, e^{-i\omega\tau} d\tau\right|^2 dt\, d\omega, \quad h_t(\tau) = f(\tau)\, g(\tau - t) \\
&\qquad = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty}\Big|\hat{h}_t(\omega)\Big|^2 d\omega \\
&\qquad = \int_{-\infty}^{\infty}\Big\|\hat{h}_t(\omega)\Big\|^2 dt
\end{aligned}$$

$$= \int_{-\infty}^{\infty} \left\| h_t(\tau) \right\|^2 dt, \qquad \text{by Plancherel's theorem}$$

$$= \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} \left| f(\tau) \right|^2 \left| g(\tau - t) \right|^2 d\tau$$

$$= \int_{-\infty}^{\infty} \left| f(\tau) \right|^2 d\tau \int_{-\infty}^{\infty} \left| g(x) \right|^2 dx = \| f \|_2^2 \| g \|_2^2.$$

This completes the proof.

Theorem 2.3.3 (Parseval's Formula). *If the Gabor transforms of the two functions f and h exist with respect to a window function g, then*

$$\left\langle \tilde{f}_g, \tilde{h}_g \right\rangle = \| g \|_2^2 \left\langle f, h \right\rangle. \tag{2.3.10}$$

In particular, if $\| g \|_2 = 1$, then the Gabor transformation is an isometry from $L^2(\mathbb{R})$ into $L^2(\mathbb{R}^2)$.

Proof. We first note that, for a fixed t,

$$\tilde{f}_g(t, \omega) = \mathscr{F}\left\{ f_t(\tau) \right\} = \mathscr{F}\left\{ f(\tau) g_t(\tau) \right\},$$

where $g_t(\tau) = g(\tau - t)$.

Thus, the Parseval formula (1.3.17) for the Fourier transform gives

$$\begin{aligned} \int_{-\infty}^{\infty} \tilde{f}_g(t, \omega)\, \overline{\tilde{h}_g}(t, \omega)\, d\omega &= \left\langle \mathscr{F}\left\{ f g_t \right\}, \mathscr{F}\left\{ h g_t \right\} \right\rangle \\ &= \langle f g_t, h g_t \rangle = \int_{-\infty}^{\infty} f(\tau)\, g(\tau - t)\, \overline{h}(\tau)\, \overline{g(\tau - t)}\, d\tau \\ &= \int_{-\infty}^{\infty} f(\tau)\, \overline{h}(\tau) \left| g(\tau - t) \right|^2 d\tau. \end{aligned}$$

Integrating this result with respect to t from $-\infty$ to ∞ gives

$$\begin{aligned}\langle \tilde{f}_g, \tilde{h}_g \rangle &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{f}_g(t,\omega)\,\overline{\tilde{h}_g}(t,\omega)\,dt\,d\omega \\ &= \int_{-\infty}^{\infty} f(\tau)\,\overline{h}(\tau)\,d\tau \int_{-\infty}^{\infty} \left|g(\tau - t)\right|^2 dt \\ &= \int_{-\infty}^{\infty} f(\tau)\,\overline{h}(\tau)\,d\tau \int_{-\infty}^{\infty} \left|g(x)\right|^2 dx \qquad (\tau - t = x) \\ &= \|g\|_2^2 \langle f, h\rangle.\end{aligned}$$

This proves the result.

If $\|g\|_2 = 1$, then (2.3.10) shows isometry from $L^2(\mathbb{R})$ into $L^2(\mathbb{R}^2)$.

Theorem 2.3.4 (Inversion Formula). *If a function $f \in L^2(\mathbb{R})$, then*

$$f(\tau) = \frac{1}{2\pi}\frac{1}{\|g\|_2^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{f}_g(t,\omega)\,g(\tau - t)\,e^{i\omega\tau}\,d\omega\,dt. \tag{2.3.11}$$

Proof. We apply the inverse Fourier transform of $f(\tau)$ and the Parseval formula to replace $\|g\|_2^2$ by $\frac{1}{2\pi}\|\hat{g}\|_2^2$ so that

$$\begin{aligned}f(\tau)\|g\|_2^2 &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega\tau}\hat{f}(\omega)\,d\omega \cdot \frac{1}{2\pi}\|\hat{g}\|_2^2 \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega\tau}\hat{f}(\omega)\,d\omega \cdot \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\hat{g}(\upsilon)\right|^2 d\upsilon.\end{aligned}$$

Since the integral is true for any arbitrary ω, we replace ω by $\omega + \upsilon$ to obtain

$$\begin{aligned}f(\tau)\|g\|_2^2 &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\tau(\omega+\upsilon)}\hat{f}(\omega+\upsilon)d\omega \cdot \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{g}(\upsilon)\hat{g}(\upsilon)d\upsilon \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega\tau}d\omega \cdot \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\tau\upsilon}\left[\hat{f}(\omega+\upsilon)\,\hat{g}(\upsilon)\right]\hat{g}(\upsilon)\,d\upsilon\end{aligned}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega\tau} d\omega \cdot \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\tau\upsilon}\hat{\tilde{f}}_g(\omega+\upsilon)\,\hat{g}(\upsilon)\,d\upsilon\right], \quad \text{by (2.3.7)}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega\tau} d\omega \cdot \left[\tilde{f}_g(\tau,\omega) * g(\tau)\right], \quad \text{by (1.2.26)}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega\tau} d\omega \int_{-\infty}^{\infty} \tilde{f}_g(t,\omega)\, g(\tau - t)\, dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i\omega\tau}\tilde{f}_g(t,\omega)\, g(\tau - t)\, dt\, d\omega.$$

This proves the inversion theorem.

Theorem 2.3.5 (Conservation of Energy). *If $f \in L^2(\mathbb{R})$, then*

$$\|f\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left|\tilde{f}_g(t,\omega)\right|^2 dt\, d\omega, \tag{2.3.12}$$

where g is a normalized window function $\left(\|g\|_2 = 1\right)$.

Proof. Using (2.3.7) dealing with the Fourier transform of $\tilde{f}_g(t,\omega)$ with respect to t, we apply the Plancherel formula to the right-hand side of (2.3.12) to obtain

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left|\tilde{f}_g(t,\omega)\right|^2 dt\, d\omega$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\mathscr{F}\left\{\tilde{f}_g(t,\omega)\right\}\right|^2 d\upsilon$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\hat{f}(\omega+\upsilon)\right|^2 |\hat{g}(\upsilon)|^2\, d\upsilon$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\hat{f}(\omega)\right|^2 |\hat{g}(\upsilon)|^2\, d\upsilon$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\hat{f}(\omega)\right|^2 d\omega, \quad \text{since } \|\hat{g}\| = \frac{1}{2\pi}\int_{-\infty}^{\infty}|\hat{g}(\upsilon)|^2\, d\upsilon = 1$$

$$= \int_{-\infty}^{\infty}|f(\tau)|^2\, d\tau = \|\hat{f}\|_2^2.$$

This completes the proof.

In many applications to physical and engineering problems, it is more important, at least from a computational viewpoint, to work with discrete transforms rather than continuous ones. In sampling theory, the sample points are defined by $\upsilon = m\omega_0$ and $\tau = nt_0$, where m, n are integers and t_0 and ω_0 are positive quantities. The *discrete Gabor functions* are defined by

$$g_{m,n}(t) = e^{2\pi m\omega_0 t}\, g(t - nt_0) = M_{2\pi m\omega_0}\, T_{nt_0}\, g(t), \tag{2.3.13}$$

where $g \in L^2(\mathbb{R})$ is a fixed function and t_0 and ω_0 are the time shift and the frequency shift parameters, respectively. A typical set of Gabor functions is shown in Figure 2.2.

Definition 2.3.2 (Discrete Gabor Transform). The discrete Gabor transform is defined by

$$\tilde{f}(m,n) = \int_{-\infty}^{\infty} f(t)\,\overline{g}_{m,n}(t)dt = \langle f, g_{m,n}\rangle. \tag{2.3.14}$$

The double series

$$\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}\tilde{f}(m,n)\, g_{m,n}(t) = \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}\langle f, g_{m,n}\rangle\, g_{m,n}(t) \tag{2.3.15}$$

is called the *Gabor series* of f. It is of special interest to find the inverse of the discrete Gabor transform so that $f \in L^2(\mathbb{R})$ can be determined by the formula

$$\tilde{f}(mt_0, n\omega_0) = \int_{-\infty}^{\infty} f(t)\, g_{m,n}(t)\, dt = \langle f, \overline{g}_{m,n}\rangle. \tag{2.3.16}$$

The set of sample points $\{(mt_0, n\omega_0)\}_{m,n=-\infty}^{\infty}$ is called the *Gabor lattice*. The answer to the question of finding the inverse is in the affirmative if the set of functions $\{g_{m,n}(t)\}$ forms an orthonormal basis or, more generally, if the set is a

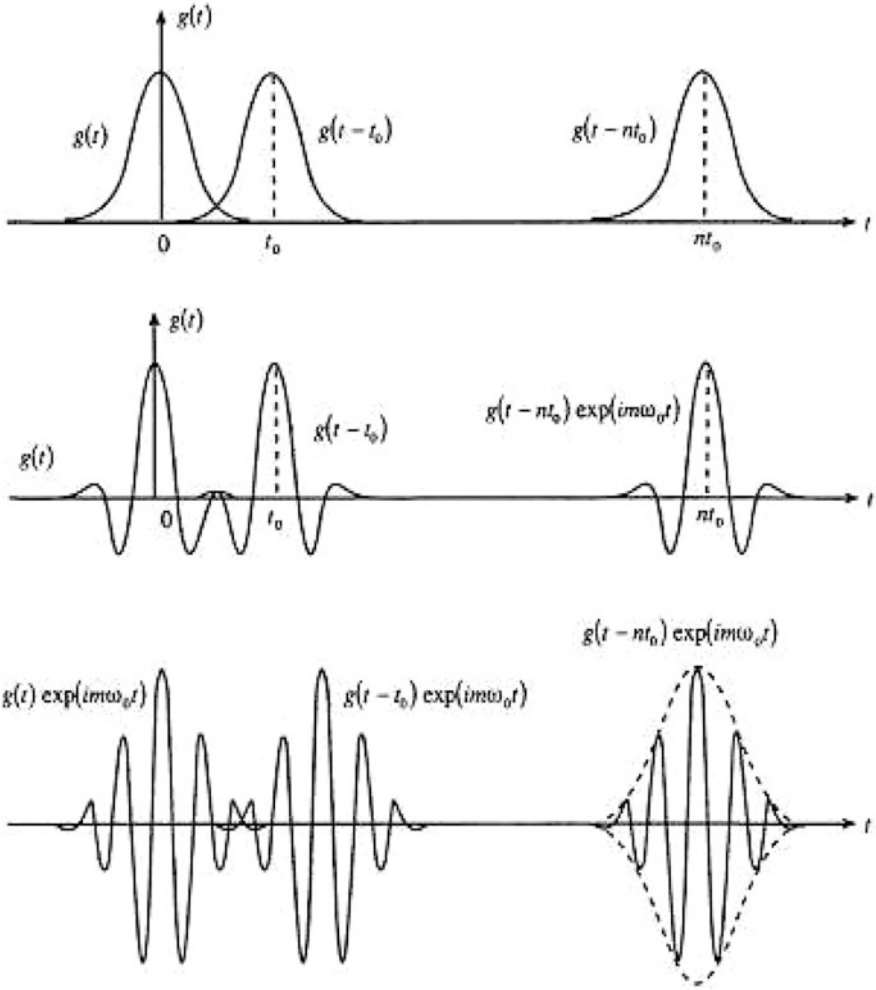

Fig. 2.2 The Gabor elementary functions $g_{m,n}(t)$

frame for $L^2(\mathbb{R})$. A system $\{g_{m,n}(t)\} = \{M_{2\pi m\omega_0} T_{nt_0}\, g(t)\}$ is called a *Gabor frame* or *Weyl-Heisenberg frame* in $L^2(\mathbb{R})$ if there exist two constants $A, B > 0$ such that

$$A\|f\|_2^2 \le \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} \left|\langle f, g_{m,n}\rangle\right|^2 \le B\|f\|_2^2 \tag{2.3.17}$$

holds for all $f \in L^2(\mathbb{R})$. For a Gabor frame $\{g_{m,n}(t)\}$, the *analysis operator* T_g is defined by

$$T_g f = \left\{\left\langle f, g_{m,n}\right\rangle\right\}_{m,n}, \tag{2.3.18}$$

and its *synthesis operator* T_g^* is defined by

$$T_g^* c_{m,n} = \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} c_{m,n}\, g_{m,n}, \tag{2.3.19}$$

where $c_{m,n} \in \ell^2(\mathbb{Z})$. Both T_g and T_g^* are bounded linear operators and in fact are adjoint operators with respect to the inner product $\langle\, ,\rangle$. The *Gabor frame operator* S_g is defined by $S_g = T_g^* T_g$. More explicitly,

$$S_g f = \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} \left\langle f, g_{m,n}\right\rangle g_{m,n}. \tag{2.3.20}$$

If $\left\{g_{m,n} : m, n \in \mathbb{Z}\right\}$ constitutes a Gabor frame for $L^2(\mathbb{R})$, any function $f \in L^2(\mathbb{R})$ can be expressed as

$$f(t) = \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} \left\langle f, g_{m,n}\right\rangle g_{m,n}^* = \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} \left\langle f, g_{m,n}^*\right\rangle g_{m,n}, \tag{2.3.21}$$

where $\left\{g_{m,n}^*\right\}$ is called the *dual frame* given by $g_{m,n}^* = S_g^{-1} g_{m,n}$. Equation (2.3.21) provides an answer for constructing f from its Gabor transform $\left\langle f, g_{m,n}\right\rangle$ for a given window function g.

Finding the conditions on t_0, ω_0, and g under which the Gabor series of f determines f or converges to it, is known as the *Gabor representation problem*. For an appropriate function g, the answer is positive provided that $0 < \omega_0 t_0 < 1$. If $0 < \omega_0 t_0 < 1$, the reconstruction is *stable* and g can have a good time and frequency localization. This is in contrast with the case when $\omega_0 t_0 = 1$, where the construction is *unstable* and g cannot have a good time and frequency localization. For the case when $\omega_0 t_0 > 1$, the reconstruction of f is, in general, impossible no matter how g is selected.

2.4 The Zak Transform

Historically, the Zak transform (ZT), known as the *Weil-Brezin transform* in harmonic analysis, was introduced by Gelfand (1950) in his famous paper on eigenfunction expansions associated with Schrödinger operators with periodic potentials. This transform was also known as the *Gelfand mapping* in the Russian mathematical literature. However, Zak (1967, 1968) independently rediscovered it as the $k - q$ transform in solid state physics to study a quantum-mechanical representation of the motion of electrons in the presence of an electric or magnetic field. Although the Gelfand-Weil-Brezin-Zak transform seems to be a more appropriate name for this transform, there is a general consensus among scientists to name it as the Zak transform since Zak himself first recognized its deep significance and usefulness in a more general setting. In recent years, the Zak transform has been widely used in time-frequency signal analysis, in the coherent states representation in quantum field theory, and also in mathematical analysis of Gabor systems.

Definition 2.4.1 (The Zak Transform). The Zak transform $\big(\mathscr{Z}_a f\big)(t,\omega)$ of a function $f \in L^2(\mathbb{R})$ is defined by the series

$$\big(\mathscr{Z}_a f\big)(t,\omega) = \sqrt{a}\sum_{n\in\mathbb{Z}} f(at+an)\, e^{-2\pi i n\omega}, \tag{2.4.1}$$

where $a > 0$ is a fixed parameter, t and ω are real.

If $f(t)$ represents a signal, then its Zak transform can be treated as the joint time-frequency representation of the signal f. It can also be considered as the discrete Fourier transform of f in which an infinite set of samples in the form $f(at+an)$ is used for $n = 0, \pm 1, \pm 2, \ldots$. Without loss of generality, we set $a = 1$ so that we can write $\big(\mathscr{Z} f\big)(t,\omega)$ in the explicit form

$$\big(\mathscr{Z} f\big)(t,\omega) = F(t,\omega) = \sum_{n\in\mathbb{Z}} f(t+n)\, e^{-2\pi i n\omega}. \tag{2.4.2}$$

This transform satisfies the *periodic relation*

$$\big(\mathscr{Z} f\big)(t,\omega+1) = \big(\mathscr{Z} f\big)(t,\omega), \tag{2.4.3}$$

and the following *quasiperiodic relation*

$$\big(\mathscr{Z} f\big)(t+1,\omega) = e^{2\pi i\omega}\,\big(\mathscr{Z} f\big)(t,\omega), \tag{2.4.4}$$

and therefore the Zak transform $\mathscr{Z} f$ is completely determined by its values on the unit square $S = [0,1]\times[0,1]$.

It is easy to prove that the Zak transform of f can be expressed in terms of the Zak transform of its Fourier transform $\hat{f}(\upsilon) = \mathscr{F}\{f(t)\}$. More precisely,

$$\big(\mathscr{Z} f\big)(t,\omega) = e^{2\pi i\omega t}\big(\mathscr{Z}\hat{f}\,\big)(\omega,-t). \tag{2.4.5}$$

To prove this result, we define a function g for fixed t and ω by

$$g(x) = e^{-2\pi i\omega x} f(x+t).$$

Then, it follows that

$$\begin{aligned}\hat{g}(\upsilon) &= \int_{-\infty}^{\infty} g(x)\, e^{-2\pi i x\upsilon}\, dx \\ &= \int_{-\infty}^{\infty} f(x+t)\, e^{-2\pi i x(\upsilon+\omega)}\, dx \\ &= e^{2\pi i(\upsilon+\omega)t} \int_{-\infty}^{\infty} f(u)\, e^{-2\pi i(\upsilon+\omega)u}\, du \\ &= e^{2\pi i(\upsilon+\omega)t} \hat{f}(\upsilon+\omega).\end{aligned}$$

We next use the Poisson summation formula in the form

$$\sum_{n\in\mathbb{Z}} g(n) = \sum_{n\in\mathbb{Z}} \hat{g}(2\pi n).$$

Or, equivalently,

$$\begin{aligned}\sum_{n\in\mathbb{Z}} f(t+n)\, e^{-2\pi i\omega n} &= e^{2\pi i\omega t} \sum_{n\in\mathbb{Z}} e^{[2\pi i\,(2n\pi)t]} \hat{f}(\omega+2\pi n) \\ &= e^{2\pi i\omega t} \sum_{m\in\mathbb{Z}} \hat{f}(\omega+m) e^{2\pi i m t}.\end{aligned}$$

This gives the desired result (2.4.5).

The following results can be easily verified:

$$\big(\mathscr{Z}\mathscr{F}f\big)(\omega,t) = e^{2\pi i\omega t}\big(\mathscr{Z}f\big)(-t,\omega), \tag{2.4.6}$$

$$\big(\mathscr{Z}\mathscr{F}^{-1}f\big)(\omega,t) = e^{2\pi i\omega t}\big(\mathscr{Z}f\big)(-t,\omega). \tag{2.4.7}$$

If $g_{m,n}(t) = e^{-2\pi i m t} g(t-n)$, then

$$\big(\mathscr{Z}g_{m,n}\big)(\omega,t) = e^{-2\pi i(mt+n\omega)}\big(\mathscr{Z}g(\omega,t)\big).$$

We next observe that $L^2(S)$ is the set of all square-integrable complex-valued functions F on the unit square S, that is,

$$\int_0^1 \int_0^1 \big|F(t,\omega)\big|^2 dt\, d\omega < \infty.$$

It is easy to check that $L^2(S)$ is a Hilbert space with the inner product

$$\langle F, G\rangle = \int_0^1 \int_0^1 F(t,\omega)\, \overline{G}(t,\omega)\, dt\, d\omega \tag{2.4.8}$$

and the norm

$$\|F\|_2 = \left\{ \int_0^1 \int_0^1 \left|F(t,\omega)\right|^2 dt\, d\omega \right\}^{1/2}.$$

The set

$$\left\{ M_{m,n} = M_{2\pi m, 2\pi n}(t,\omega) = e^{2\pi i (mt+n\omega)} : m, n \in \mathbb{Z} \right\} \tag{2.4.9}$$

forms an orthonormal basis of $L^2(S)$.

Example 2.4.1. If

$$\phi_{m,n;a}(t) = \frac{1}{\sqrt{a}}\, T_{na}\, M_{2\pi m/a}\, \chi_{[0,a]}(t),$$

where $a > 0$, then

$$\left(\mathscr{Z}_a \phi_{m,n;a}\right)(t,\omega) = e_m(t)\, e_n(\omega), \quad \text{where } e_k(t) = e^{2\pi i k t}.$$

We have

$$\begin{aligned} \phi_{m,n;a}(t) &= \frac{1}{\sqrt{a}}\, e^{2\pi i m\left(\frac{t-na}{a}\right)}\, \chi_{[0,a]}(t-na) \\ &= \frac{1}{\sqrt{a}}\, e^{\frac{2\pi i m t}{a}}\, \chi_{[na,(n+1)a]}(t). \end{aligned}$$

Thus, we obtain

$$\begin{aligned} \left(\mathscr{Z}_a \phi_{m,n;a}\right)(t,\omega) &= \sum_{k\in\mathbb{Z}} e^{\frac{2\pi i m}{a}\,(at+ak)}\, \chi_{[na,na+a]}(at+ak) \\ &= \sum_{k\in\mathbb{Z}} e_m(t)\, e^{-2\pi i k\omega}\, \chi_{[n-k,n+1-k]}(t) \\ &= e_m(t)\, e_n(\omega). \end{aligned}$$

We shall now discuss the basic properties of continuous Zak transform.

Theorem 2.4.1. *Let $f, g \in L^2(\mathbb{R})$ and a, b be any two arbitrary constants. Then, the following results hold:*

(a) Linearity: $\big[\mathscr{Z}(af + bg)\big](t,\omega) = a\big(\mathscr{Z}f\big)(t,\omega) + b\big(\mathscr{Z}g\big)(t,\omega)$,
(b) Translation: $\big[\mathscr{Z}(T_a f)\big](t,\omega) = \big(\mathscr{Z}f\big)(t-a,\omega)$,
(c) Modulation: $\big[\mathscr{Z}(M_b f)\big](t,\omega) = e^{ibt}\big(\mathscr{Z}f\big)\left(t, \omega - \dfrac{b}{2\pi}\right)$,
(d) Translation and modulation: $\mathscr{Z}\big[M_{2\pi m}T_n f\big](t,\omega) = e^{2\pi i(mt-n\omega)}\big(\mathscr{Z}f\big)(t,\omega)$,
(e) Conjugation: $\big(\mathscr{Z}\bar{f}\big)(t,\omega) = \overline{\big(\mathscr{Z}f\big)}(t,-\omega)$,
(f) Symmetry:

$$\big(\mathscr{Z}f\big)(t,\omega) = \begin{cases} \big(\mathscr{Z}f\big)(-t,-\omega), & \text{if } f \text{ is even} \\ -\big(\mathscr{Z}f\big)(-t,-\omega), & \text{if } f \text{ is odd} \end{cases}$$

(g) Inversion: For $t, \omega \in \mathbb{R}$,

$$f(t) = \int_0^1 \big(\mathscr{Z}f\big)(t,\omega)\,d\omega,$$

$$\hat{f}(\omega) = \int_0^1 \exp(-2\pi i\omega t)\big(\mathscr{Z}f\big)(t,\omega)\,dt,$$

$$f(x) = \int_0^1 \exp(-2\pi ixt)\big(\mathscr{Z}\hat{f}\big)(t,x)\,dt$$

(h) Dilation: $\left(\mathscr{Z}D_{\frac{1}{a}}f\right)(t,\omega) = \big(\mathscr{Z}_a f\big)\left(at, \dfrac{\omega}{a}\right)$,
(i) Product and convolution of Zak transforms.

Results (2.4.3) and (2.4.4) show that the Zak transform is not periodic in the two variables t and ω. The product of two Zak transforms is periodic in t and ω.

Proof. We consider the product

$$F(t,\omega) = \big(\mathscr{Z}f\big)(t,\omega)\overline{\big(\mathscr{Z}g\big)}(t,\omega)$$

and find from (2.4.4) that

$$\overline{\big(\mathscr{Z}g\big)}(t,\omega) = e^{-2\pi i\omega}\big(\mathscr{Z}g\big)(t,\omega).$$

Therefore, it follows that

$$F(t+1,\omega) = \big(\mathscr{Z}f\big)(t,\omega)\overline{\big(\mathscr{Z}g\big)}(t,\omega) = F(t,\omega),$$
$$F(t,\omega+1) = \big(\mathscr{Z}f\big)(t,\omega)\overline{\big(\mathscr{Z}g\big)}(t,\omega) = F(t,\omega).$$

These show that F is periodic in t and ω. Consequently, it can be expanded in a Fourier series on a unit square

$$F(t,\omega) = \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} c_{m,n}\, e^{2\pi imt}\, e^{2\pi in\omega}, \tag{2.4.10}$$

where

$$c_{m,n} = \int_0^1\int_0^1 F(t,\omega)\, e^{-2\pi imt}\, e^{-2\pi in\omega}\, dt\, d\omega.$$

If we assume that the series involved are uniformly convergent, we can interchange the summation and integration to obtain

$$\begin{aligned}
c_{m,n} &= \int_0^1\int_0^1 \left\{\sum_{r\in\mathbb{Z}} f(t+r)\, e^{-2\pi ir\omega}\right\}\left\{\sum_{s\in\mathbb{Z}} \bar{g}(t+s)\, e^{2\pi is\omega}\right\} e^{-2\pi i(mt+n\omega)}\, dt\, d\omega\\
&= \int_0^1 \left\{\sum_{r\in\mathbb{Z}} f(t+r)\right\}\left\{\sum_{s\in\mathbb{Z}} \bar{g}(t+s)\right\} e^{-2\pi imt}\, dt \int_0^1 e^{2\pi i\omega(s-n-r)}\, d\omega\\
&= \int_0^1 \left\{\sum_{r\in\mathbb{Z}} f(t+r)\, \bar{g}(t+n+r)\right\} e^{-2\pi imt}\, dt\\
&= \sum_{r\in\mathbb{Z}} \int_r^{r+1} f(x)\, \bar{g}(x+n) e^{-2\pi im(x-r)}\, dx\\
&= \int_{-\infty}^{\infty} f(x)\, \bar{g}(x+n)\, e^{-2\pi imx}\, dx\\
&= \left\langle f(x), e^{2\pi imx} g(x+n)\right\rangle\\
&= \left\langle f, M_{2\pi m}\, T_{-n} g\right\rangle.
\end{aligned}$$

Consequently, (2.4.10) becomes

$$\big(\mathscr{Z}f\big)(t,\omega)\overline{\big(\mathscr{Z}g\big)}(t,\omega) = \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} \left\langle f, M_{2\pi m}\, T_{-n} g\right\rangle e^{2\pi i(mt+n\omega)}. \tag{2.4.11}$$

This completes the proof.

Theorem 2.4.2. *Suppose H is a function of two real variables t and s satisfying the condition*

$$H\big(t+1, s+1\big) = H\big(t, s\big), \quad \textit{and} \quad h(t) = \int_{-\infty}^{\infty} H(t,s)f(s)\,ds, \quad s, t \in \mathbb{R},$$

where the integral is absolutely and uniformly convergent. Then,

$$\big(\mathscr{Z}f\big)(t,\omega) = \int_0^1 \big(\mathscr{Z}f\big)(s,\omega)\,\Phi(t,s,\omega)\,ds, \tag{2.4.12}$$

where Φ is given by

$$\Phi(t,s,\omega) = \sum_{n\in\mathbb{Z}} H(t+n,s)\,e^{-2\pi in\omega}, \qquad 0 \le t, s, \omega \le l. \tag{2.4.13}$$

Proof. Using the definition of Zak transform, we have

$$\begin{aligned}
\big(\mathscr{Z}h\big)(t,\omega) &= \sum_{k\in\mathbb{Z}} h(t+k)\,e^{-2\pi ik\omega} = \sum_{k\in\mathbb{Z}} e^{-2\pi ik\omega} \int_{-\infty}^{\infty} H(t+k,s)f(s)\,ds \\
&= \sum_{k\in\mathbb{Z}} e^{-2\pi ik\omega} \sum_{m\in\mathbb{Z}} \int_m^{m+1} H(t+k,s)f(s)\,ds \\
&= \sum_{k\in\mathbb{Z}} e^{-2\pi ik\omega} \sum_{m\in\mathbb{Z}} \int_0^1 H(t+k,s+m)f(s+m)\,ds \\
&= \int_0^1 \left\{ \sum_{k\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} H(t+k,s+m)\,f(s+m)\,e^{-2\pi ik\omega} \right\} ds,
\end{aligned}$$

which is, due to (2.4.11),

$$\begin{aligned}
&= \int_0^1 \left\{ \sum_{k\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} H(t+k-m,s)\,f(s+m)\,e^{-2\pi ik\omega} \right\} ds \\
&= \int_0^1 \left\{ \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} H(t+n,s)\,f(s+m)\,e^{-2\pi i(m+n)\omega} \right\} ds \\
&= \int_0^1 \big(\mathscr{Z}f\big)(s,\omega)\,\Phi(t,s,\omega)\,ds.
\end{aligned} \tag{2.4.14}$$

This completes the proof.

In particular, if $H(t,s) = H(t-s)$,

$$\Phi(t,s,\omega) = \sum_{n\in\mathbb{Z}} H(t-s+n)\, e^{-2\pi in\omega} = \big(\mathscr{Z}H\big)(t-s,\omega).$$

Consequently, Theorem 2.4.2 leads to the following convolution theorem.

Theorem 2.4.3 (Convolution Theorem). *If*

$$h(t) = \int_{-\infty}^{\infty} H(t-s)f(s)\, ds = (H*f)(t),$$

then (2.4.12) *reduces to the form*

$$\big(\mathscr{Z}h\big)(t,\omega) = \int_0^1 \big(\mathscr{Z}H\big)(t-s)\,\big(\mathscr{Z}f\big)(s,\omega)\, ds = \mathscr{Z}(H*f)(t,\omega). \tag{2.4.15}$$

Example 2.4.2. If $H(t) = \sum_{k\in\mathbb{Z}} a_k\, \delta(t-k)$, then

$$\mathscr{Z}(H*f)(t,\omega) = A(\omega)\big(\mathscr{Z}f\big)(t,\omega), \tag{2.4.16}$$

where

$$A(\omega) = \sum_{k\in\mathbb{Z}} a_k\, e^{-2\pi ik\omega}.$$

Clearly,

$$\begin{aligned}
\mathscr{Z}(H*f)(t,\omega) &= \mathscr{Z}\left\{\int_{-\infty}^{\infty} H(t-s)f(s)\, ds\right\}(t,\omega) \\
&= \mathscr{Z}\left\{\sum_{k\in\mathbb{Z}} a_k \int_{-\infty}^{\infty} \delta(t-s-k)f(s)\, ds\right\}(t,\omega) \\
&= \mathscr{Z}\left\{\sum_{k\in\mathbb{Z}} a_k f(t-k)\right\}(t,\omega)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k\in\mathbb{Z}} a_k \sum_{n\in\mathbb{Z}} f(t+n-k)\, e^{-2\pi i n\omega} \\
&= \sum_{k\in\mathbb{Z}} a_k \sum_{m\in\mathbb{Z}} f(t+m)\, e^{-2\pi i\omega(m+k)} \\
&= A(\omega)\big(\mathscr{Z}f\big)(t,\omega).
\end{aligned}
$$

Theorem 2.4.4. *The Zak transformation is a unitary mapping from $L^2(\mathbb{R})$ to $L^2(S)$.*

Proof. It follows from the definition of the inner product (2.4.8) in $L^2(S)$ that

$$
\begin{aligned}
\big\langle \mathscr{Z}_a f, \mathscr{Z}_a g\big\rangle &= a\int_0^1\int_0^1 \left\{\sum_{n\in\mathbb{Z}} f(at+an)\, e^{-2\pi i n\omega}\right\}\left\{\sum_{m\in\mathbb{Z}} \overline{g}(at+am)\, e^{2\pi i m\omega}\right\} dt\, d\omega \\
&= a\int_0^1 \left\{\sum_{n\in\mathbb{Z}} f(at+an)\, \overline{g}(at+an)\right\} dt \\
&= \sum_{n\in\mathbb{Z}} \int_{na}^{(n+1)a} f(y)\, \overline{g}(y)\, dy \\
&= \int_{-\infty}^{\infty} f(y)\, \overline{g}(y)\, dy = \langle f, g\rangle. \qquad (2.4.17)
\end{aligned}
$$

In particular, if $f = g$, we obtain from (2.4.17) that

$$
\big\| \mathscr{Z}_a f\big\|_2^2 = \|f\|_2^2. \qquad (2.4.18)
$$

This means that the Zak transform is an isometry from $L^2(\mathbb{R})$ to $L^2(S)$.

Further, Example 2.4.1 shows that $\big\{\phi_{m,n;a}(t) : m, n \in \mathbb{Z}\big\}$ is an orthonormal basis of $L^2(\mathbb{R})$. Hence, the Zak transform is a one-to-one mapping of an orthonormal basis of $L^2(\mathbb{R})$ onto an orthonormal basis of $L^2(S)$. This completes the proof of theorem.

2.5 The Windowed Linear Canonical Transform

In early 1970s, a promising linear integral transform with three free parameters, namely, linear canonical transform (LCT), was proposed by Moshinsky and Quesne (1971a,b) which is considered as one of the most powerful tools for signal and image processing. This transform has also been referred to by different names in the open literature such as quadratic-phase integrals (Bastiaans, 1979), generalized Huygens integrals (Siegman, 1986), the affine Fourier transform (Abe and Sheridan, 1994a,b), ABCD transform (Bernardo, 1996), the generalized Fresnel transform (James and Agarwal, 1996; Palma and Bagini, 1997), the extended fractional Fourier transform (Hua et al., 1997), and Moshinsky-Quesne transform (Healy et al., 2016). Therefore, we can say that the LCT is a generalization of many optical transforms such as the Fourier transform, the fractional Fourier transform (FrFT), the Fresnel transform, the Lorentz transform, and scaling operations. Thus, understanding the LCT may help to gain more insight into its special cases and to carry the knowledge gained from one subject to others.

With more degrees of freedom compared to the Fourier transform and the FrFT, the LCT is more flexible in nature but with similar computation cost as that of conventional Fourier transform (see Healy and Sheridan, 2010). The LCT has found many applications in phase reconstruction, filter design, signal synthesis, pattern recognition, time-frequency analysis, optimal filtering, radar analysis, holographic three-dimensional television, quantum physics, and many others. However, the LCT cannot reveal the local LCT-frequency contents due to its global kernel. On the other hand, the windowed Fourier transform (WFT) with a local window function handles this kind of situation very well, but unfortunately, the WFT often performs unsatisfactorily for its low resolution. Therefore, in order to attain the local contents and high localization properties of a signal, it is desirable to develop a new transform by replacing the Fourier transform kernel with the LCT kernel in the windowed Fourier transform definition. This new transform was first introduced by Bultheel and Martinez-Sulbaran (2007) and is called the *windowed linear canonical transform* (WLCT) which offers a flexible local frequency content, eliminates cross term, and enjoys high resolution of a signal. For more about LCT and their applications to signal and image processing, the reader is to referred to Stankovic et al. (2003), Koc et al. (2008), Tao et al. (2010), Kou and Xu (2012), Shi et al. (2014), Bahri and Ashino (2016), and Healy et al. (2016).

We shall start here with the formal definition of the linear canonical transform (LCT).

Definition 2.5.1. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a unimodular matrix, i.e., $\det(A) = ad - bc = 1$, $a, b, c, d \in \mathbb{R}$ or in $\mathbb{C}$. Then, the continuous linear canonical transform (LCT) with parameter A of any function $f \in L^2(\mathbb{R})$ is defined by

$$\mathscr{L}_A[f](\omega) = \begin{cases} \displaystyle\int_{-\infty}^{\infty} f(t)\, \mathcal{K}_A(t, \omega)\, dt, & b \neq 0 \\ \sqrt{d}\, \exp\left\{ i \dfrac{cd\omega^2}{2} \right\} f(d\omega), & b = 0 \end{cases} \tag{2.5.1}$$

where the kernel $\mathcal{K}_A(t,\omega)$ of LCT is given by

$$\mathcal{K}_A(t,\omega) = \frac{1}{\sqrt{2\pi b}} \exp\left\{\frac{i}{2}\left(\frac{at^2}{b} - \frac{2t\omega}{b} + \frac{d\omega^2}{b} - \frac{\pi}{4}\right)\right\}. \tag{2.5.2}$$

For typographical convenience, we shall often denote the matrix A as $A = (a,b,c,d)$, in the text, but all operations have to be understood in the usual matrix sense. Moreover, we note that when $b = 0$, the LCT becomes a chirp multiplication. Therefore, we only consider the case of $b \neq 0$, and without loss of generality, we assume $b > 0$ throughout this section.

The above definition allows us to make the following comments:

1. Actually, the LCT has three free parameters; if we let $a = \gamma/\beta, b = 1/\beta$, $c = -\beta + \alpha\gamma/\beta, d = \alpha/\beta$, then the LCT of $f(t)$ can be rewritten as

$$\mathscr{L}_A[f](\omega) = \int_{-\infty}^{\infty} f(t)\, \mathcal{K}_A(t,\omega)\, dt, \tag{2.5.3}$$

where

$$\mathcal{K}_A(t,\omega) = \frac{\sqrt{\beta}}{\sqrt{2\pi}} \exp\left\{\frac{i}{2}\left(\gamma t^2 - 2\beta t\omega + \alpha\omega^2 - \frac{\pi}{4}\right)\right\}, \tag{2.5.4}$$

and the parameter matrix is given by

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \gamma/\beta & 1/\beta \\ -\beta + \alpha\gamma/\beta & \alpha/\beta \end{pmatrix} = \begin{pmatrix} \alpha/\beta & -1/\beta \\ \beta - \alpha\gamma/\beta & \gamma/\beta \end{pmatrix}^{-1}. \tag{2.5.5}$$

2. The LCT given by (2.5.1) can be computed via Fourier transform as

$$\mathscr{L}_A[f](\omega) = \frac{1}{\sqrt{2\pi b}} \exp\left\{\frac{i}{2}\left(\frac{d\omega^2}{b} - \frac{\pi}{2}\right)\right\} \mathscr{F}\left\{\exp\left(\frac{iat^2}{2b}\right) f(t)\right\}\left(\frac{\omega}{b}\right). \tag{2.5.6}$$

Note that if we let $h(t) = \dfrac{1}{\sqrt{2\pi b}} \exp\left\{\dfrac{i}{2}\left(\dfrac{at^2}{b} - \dfrac{\pi}{2}\right)\right\} f(t)$, then equation (2.5.6) takes the form

$$\exp\left\{\frac{-id\omega^2}{2b}\right\} \mathscr{L}_A[f](\omega) = \mathscr{F}\{h(t)\}\left(\frac{\omega}{b}\right), \tag{2.5.7}$$

3. As a special case, when $A = (a,b,c,d) = (0,1,-1,0)$, the LCT definition (2.5.1) reduces to the classical Fourier transform.

4. For the parameter matrix $A = (a, b, c, d) = (\cos\alpha, \sin\alpha, -\sin\alpha, \cos\alpha) = \mathscr{R}^{\alpha}$, the LCT multiplied by $e^{i\alpha/2}$ coincides with the FrFT, i.e., $\mathscr{F}^{\alpha} = e^{i\alpha/2}\mathscr{L}_A$ if $A = \mathscr{R}^{\alpha}$.
5. For the matrix $A = (a, b, c, d) = (1, b, 0, 1)$, the LCT reduces to the Fresnel transform.
6. Multiplication by a Gaussian or chirp function is obtained with $A = (a, b, c, d) = (1, 0, c, 1)$.
7. The scaling operator can be viewed as a special case of the LCT with $A = (a, b, c, d) = \left(d^{-1}, 0, 0, d\right)$.

Two interesting and important properties of LCT are the index additivity and reversibility. Index additivity means that the composition of two LCTs with parameter matrices $A_1 = (a_1, b_1, c_1, d_1)$ and $A_2 = (a_2, b_2, c_2, d_2)$, respectively, equals to the LCT with parameter matrix $A_3 = A_2A_1$, that is,

$$\mathscr{L}_{A_1}\left(\mathscr{L}_{A_1}\right)[f] = \mathscr{L}_{A3}[f] = \mathscr{L}_{A_2A_1}[f]. \tag{2.5.8}$$

The inverse of the LCT with parameter matrix $A = (a, b, c, d)$ is the LCT with parameter matrix of $A^{-1} = (d, -b, -c, a)$, that is,

$$\mathscr{L}_{A^{-1}}\left(\mathscr{F}_A\right)(f) = f. \tag{2.5.9}$$

In case the parameter matrices A_1 and A_2 contain complex numbers, then the additivity property (2.5.8) holds if

$$\operatorname{Im}\left(\frac{a_2}{b_2} + \frac{d_1}{b_1}\right) > 0. \tag{2.5.10}$$

However, if $\operatorname{Im}(a_2/b_2 + d_1/b_1) = 0$, then both of b_1 and b_2 must be real. Combining with the inverse property (2.5.9), then b must be real since $A_1 = (a, b, c, d)$ and $A_2 = (d, -b, -c, a) = A_1^{-1}$ by invoking additive property (2.5.8).

Another important property of LCT is the Parseval formula:

$$\langle f, g\rangle = \left\langle \mathscr{L}_A(f), \mathscr{L}_A(g)\right\rangle. \tag{2.5.11}$$

In particular, when $f = g$, we obtain the Plancherel formula for the LCT:

$$\|f\|_2^2 = \left\|\mathscr{L}_A(f)\right\|_2^2. \tag{2.5.12}$$

Following the idea of windowed Fourier transform, we shall try to generalize the LCT to a new transform, namely, windowed linear canonical transform (WLCT).

Before we give the formal definition of WLCT, we first recall that the windowed Fourier transform (2.2.6) of any $f \in L^2(\mathbb{R})$ with respect to the window function $g \in L^2(\mathbb{R})$ is given by

$$\mathscr{G}[f](u,\omega) = \int_{-\infty}^{\infty} f(t)\,\overline{g(t-u)}\,e^{-i\omega t}\,dt = \langle f, g_{u,\omega}\rangle, \tag{2.5.13}$$

where $g_{u,\omega}(t) = e^{i\omega t} g(t-u) = M_\omega T_u g(t)$. Note that the optimal window for time-frequency localization can be achieved only if g is a Gaussian function. Moreover, for fixed $\omega = \omega_0$,

$$g_{u,\omega_0}(t) = e^{i\omega_0 t} g(t-u), \tag{2.5.14}$$

is called a *Gabor filter*. The extension of the Gabor filter to the LCT domain is given by the following definition.

Definition 2.5.2. For a window function $g \in L^2(\mathbb{R})\backslash\{0\}$, its window daughter function associated with LCT is defined by

$$g^A_{u,\omega}(t) = \frac{1}{\sqrt{2\pi b}} \exp\left\{-\frac{i}{2}\left(\frac{at^2}{b} - \frac{2t\omega}{b} + \frac{d\omega^2}{b} - \frac{\pi}{4}\right)\right\} g(t-u). \tag{2.5.15}$$

This function is also called the *linear canonical windowed Fourier kernel.*

We are now in a position to introduce the basic definition of WLCT.

Definition 2.5.3. The windowed linear canonical transform (WLCT) of a function $f \in L^2(\mathbb{R})$ with respect to a window function $g \in L^2(\mathbb{R})$ is denoted by $\mathscr{G}_A[f](u,\omega)$ and defined by

$$\mathscr{G}_A[f](u,\omega) = \int_{-\infty}^{\infty} f(t)\,\overline{g^A_{u,\omega}(t)}\,dt, \tag{2.5.16}$$

where $u, \omega \in \mathbb{R}$, $A = (a,b,c,d)$ with $\det(A) = 1$ and $g^A_{u,\omega}(t)$ is given by (2.5.15).

We next discuss the following consequences of the proceeding definition.

1. It is worth noticing that, when $A = (a,b,c,d) = (0,1,-1,0)$, one recovers the standard definition of WFT (2.5.13). In fact, we have the following relation between WLCT and WFT:

$$\begin{aligned}
&\mathscr{G}_A[f](u,\omega)\\
&\quad= \frac{1}{\sqrt{2\pi b}} \int_{-\infty}^{\infty} f(t)\,\overline{g(t-u)}\exp\left\{\frac{i}{2}\left(\frac{at^2}{b}-\frac{2t\omega}{b}+\frac{d\omega^2}{b}-\frac{\pi}{4}\right)\right\}dt\\
&\quad= \frac{1}{\sqrt{2\pi b}}\exp\left\{\frac{id\omega^2}{2b}\right\}\int_{-\infty}^{\infty}\exp\left\{\frac{i}{2}\left(\frac{at^2}{b}-\frac{\pi}{2}\right)\right\}f(t)\overline{g(t-u)}\,e^{-it\omega/b}dt\\
&\quad= \exp\left\{\frac{id\omega^2}{2b}\right\}\int_{-\infty}^{\infty}h(t)\,\overline{g(t-u)}\,e^{-it\omega/b}dt\\
&\quad= \exp\left\{\frac{id\omega^2}{2b}\right\}\mathscr{G}[h]\left(u,\frac{\omega}{b}\right). \qquad (2.5.17)
\end{aligned}$$

2. If we take the Gaussian function as a window function in (2.5.16), then we get the Gabor linear canonical transform (GLCT).
3. For a fixed u, WLCT (2.5.16) can be interpreted as the LCT of the product of a function f and a conjugate and translated window function g, that is,

$$\mathscr{G}_A[f](u,\omega) = \mathscr{L}_A\left\{f(t)\,\overline{g(t-u)}\right\}(\omega). \qquad (2.5.18)$$

4. Implementing the inverse LCT (2.5.9) to WLCT (2.5.16), we obtain

$$f(t)\,\overline{g(t-u)} = \frac{1}{\sqrt{2\pi b}}\int_{-\infty}^{\infty}\mathscr{G}_A f(u,\omega)\exp\left\{-\frac{i}{2}\left(\frac{at^2}{b}-\frac{2t\omega}{b}+\frac{d\omega^2}{b}-\frac{\pi}{4}\right)\right\}d\omega. \qquad (2.5.19)$$

5. The energy density of the WLCT is defined by

$$\left|\mathscr{G}_A[f](u,\omega)\right|^2 = \left|\frac{1}{\sqrt{2\pi b}}\int_{-\infty}^{\infty}f(t)\,\overline{g(t-u)}\exp\left\{\frac{i}{2}\left(\frac{at^2}{b}-\frac{2t\omega}{b}+\frac{d\omega^2}{b}-\frac{\pi}{4}\right)\right\}dt\right|^2. \qquad (2.5.20)$$

We now investigate some basic properties of WLCT.

Theorem 2.5.1. *Let $f, g, h \in L^2(\mathbb{R})$ and α, β be any two arbitrary constants. Then, the following results hold:*

(a) Linearity: $\mathscr{G}_A[\alpha f+\beta h](u,\omega) = \alpha\,\mathscr{G}_A[f](u,\omega)+\beta\,\mathscr{G}_A[h](u,\omega)$,
(b) Parity: $\mathscr{G}_A[Pf](u,\omega) = \mathscr{G}_A[f](-u,-\omega)$, *where* $Pg(t) = g(-t)$,
(c) Translation: $\mathscr{G}_A[T_{t_0}f](u,\omega) = \exp\left\{it_0\omega c - \dfrac{iat_0^2c}{2}\right\}\mathscr{G}_A[f](u-t_0,\omega-at_0)$,
(d) Modulation: $\mathscr{G}_A[M_{\omega_0}f](u,\omega) = \exp\left\{id\omega\omega_0 - \dfrac{idb\omega_0^2}{2}\right\}\mathscr{G}_A[f](t,\omega-\omega_0 b)$,
(e) Conjugation: $\mathscr{G}_A[\bar{f}](u,\omega) = \overline{\mathscr{G}_A^{-1}[f](u,\omega)}$.

Proof.

(a). The proof of linearity easily follows from the definition of the WLCT.
(b). For every $f \in L^2(\mathbb{R})$, we have

$$
\begin{aligned}
&\mathscr{G}_A[Pf](u,\omega)\\
&= \frac{1}{\sqrt{2\pi b}} \int_{-\infty}^{\infty} f(-t)\, \overline{g\big(-(t-u)\big)} \exp\left\{\frac{i}{2}\left(\frac{at^2}{b} - \frac{2t\omega}{b} + \frac{d\omega^2}{b} - \frac{\pi}{4}\right)\right\} dt\\
&= \frac{1}{\sqrt{2\pi b}} \int_{-\infty}^{\infty} f(-t)\, \overline{g\big(-(t-u)\big)}\\
&\quad \times \exp\left\{\frac{i}{2}\left(\frac{a(-t)^2}{b} - \frac{2(-t)(-\omega)}{b} + \frac{d(-\omega)^2}{b} - \frac{\pi}{4}\right)\right\} dt\\
&= \mathscr{G}_A[f](-u,-\omega).
\end{aligned}
$$

(c). From the definition of WLCT, we have

$$
\begin{aligned}
&\mathscr{G}_A[T_{t_0}f](u,\omega)\\
&= \frac{1}{\sqrt{2\pi b}} \int_{-\infty}^{\infty} f(t-t_0)\, \overline{g(t-u)} \exp\left\{\frac{i}{2}\left(\frac{at^2}{b} - \frac{2t\omega}{b} + \frac{d\omega^2}{b} - \frac{\pi}{4}\right)\right\} dt\\
&= \frac{1}{\sqrt{2\pi b}} \int_{-\infty}^{\infty} f(y)\, \overline{g\big(y-(u-t_0)\big)} \exp\left\{\frac{i}{2}\left(\frac{a(y+t_0)^2}{b} - \frac{2(y+t_0)\omega}{b} + \frac{d\omega^2}{b} - \frac{\pi}{4}\right)\right\} dy\\
&= \frac{1}{\sqrt{2\pi b}} \int_{-\infty}^{\infty} f(y)\, \overline{g\big(y-(u-t_0)\big)}\\
&\qquad \times \exp\left\{\frac{i}{2}\left(\frac{ay^2 + at_0^2 + 2ayt_0}{b} - \frac{2y\omega}{b} - \frac{2t_0\omega}{b} + \frac{d\omega^2}{b} - \frac{\pi}{4}\right)\right\} dy\\
&= \frac{1}{\sqrt{2\pi b}} \int_{-\infty}^{\infty} f(y)\, \overline{g\big(y-(u-t_0)\big)} \exp\left\{\frac{i}{2}\left(\frac{ay^2}{b} - \frac{2y(\omega - t_0 a)}{b} + \frac{d\omega^2}{b} - \frac{\pi}{4}\right)\right\}\\
&\qquad \times \exp\left\{\frac{i}{2}\left(\frac{at_0^2 - 2t_0\omega}{b}\right)\right\} dy\\
&= \frac{1}{\sqrt{2\pi b}} \int_{-\infty}^{\infty} f(y)\, \overline{g\big(y-(u-t_0)\big)} \exp\left\{\frac{i}{2}\left(\frac{ay^2}{b} - \frac{2y(\omega - t_0 a)}{b} + \frac{d(\omega - t_0 a)^2}{b} - \frac{\pi}{4}\right)\right\}
\end{aligned}
$$

$$\times \quad \exp\left\{\frac{i}{2}\left(\frac{2d(\omega - t_0 a)t_0 a + dt_0^2 a^2}{b}\right)\right\} \exp\left\{\frac{i}{2}\left(\frac{at_0^2 - 2t_0\omega}{b}\right)\right\} dy$$

$$= \exp\left\{\frac{i}{2}\left(\frac{2d(\omega - t_0 a)t_0 a + dt_0^2 a^2}{b}\right)\right\} \exp\left\{\frac{i}{2}\left(\frac{at_0^2 - 2t_0\omega}{b}\right)\right\} \mathscr{G}_A f\big(u - t_0, \omega - t_0 a\big)$$

$$= \exp\left\{it_0\omega c - \frac{iat_0^2 c}{2}\right\} \mathscr{G}_A f\big(u - t_0, \omega - t_0 a\big).$$

This completes the proof of part (c).

(d). For every $f \in L^2(\mathbb{R})$, we have

$$\mathscr{G}_A[M_{\omega_0} f](u,\omega)$$

$$= \frac{1}{\sqrt{2\pi b}} \int_{-\infty}^{\infty} e^{i\omega_0 t} f(t)\, \overline{g(t-u)} \exp\left\{\frac{i}{2}\left(\frac{at^2}{b} - \frac{2t\omega}{b} + \frac{d\omega^2}{b} - \frac{\pi}{4}\right)\right\} dt$$

$$= \frac{1}{\sqrt{2\pi b}} \int_{-\infty}^{\infty} f(t)\, \overline{g(t-u)} \exp\left\{\frac{i}{2}\left(\frac{at^2}{b} - \frac{2t\omega}{b} + \frac{d\omega^2}{b} + 2\omega_0 t - \frac{\pi}{4}\right)\right\} dt$$

$$= \frac{1}{\sqrt{2\pi b}} \int_{-\infty}^{\infty} f(t)\, \overline{g(t-u)} \exp\left\{\frac{i}{2}\left(\frac{at^2}{b} - \frac{2t(\omega - \omega_0 b)}{b} + \frac{d\omega^2}{b} - \frac{\pi}{4}\right)\right\} dt$$

$$= \frac{1}{\sqrt{2\pi b}} \int_{-\infty}^{\infty} f(t)\, \overline{g(t-u)} \exp\left\{\frac{i}{2}\left(\frac{at^2}{b} - \frac{2t(\omega - \omega_0 b)}{b} + \frac{d(\omega - \omega_0 b + \omega_0 b)^2}{b} - \frac{\pi}{4}\right)\right\} dt$$

$$= \frac{1}{\sqrt{2\pi b}} \int_{-\infty}^{\infty} f(t)\, \overline{g(t-u)}$$

$$\times \quad \exp\left\{\frac{i}{2}\left(\frac{at^2}{b} - \frac{2t(\omega - \omega_0 b)}{b} + \frac{d(\omega - \omega_0 b)^2}{b} - \frac{\pi}{4} + 2(\omega - \omega_0 b)\omega_0 b + \omega_0^2 b^2\right)\right\} dt$$

$$= \exp\left\{id\omega\omega_0 - \frac{idb\omega_0^2}{2}\right\} \mathscr{G}_A[f](t, \omega - \omega_0 b).$$

(e). For every $f \in L^2(\mathbb{R})$, we have

$$\mathscr{G}_A[\bar{f}\,](u,\omega) = \frac{1}{\sqrt{2\pi b}} \int_{-\infty}^{\infty} \overline{f(t)}\, g(t-u) \exp\left\{\frac{i}{2}\left(\frac{at^2}{b} - \frac{2t\omega}{b} + \frac{d\omega^2}{b} - \frac{\pi}{4}\right)\right\} dt$$

$$= \frac{1}{\sqrt{2\pi b}} \int_{-\infty}^{\infty} \overline{f(t)\, \overline{g(t-u)} \exp\left\{\frac{-i}{2}\left(\frac{at^2}{b} - \frac{2t\omega}{b} + \frac{d\omega^2}{b} - \frac{\pi}{4}\right)\right\} dt}$$

$$= \overline{\mathscr{G}_A^{-1}[f](u,\omega)}.$$

Theorem 2.5.2 (Orthogonality Relation). *If f_1, g_1, f_2 and g_2 belong to $L^2(\mathbb{R})$, then the following formula hold:*

$$\left\langle \mathscr{G}_{A,g_1} f_1, \mathscr{G}_{A,g_2} f_2 \right\rangle = \langle f_1, f_2 \rangle \overline{\langle g_1, g_2 \rangle}. \tag{2.5.21}$$

Proof. Assume that both the window functions $g_1, g_2 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$; then it is obvious that for every $u \in \mathbb{R}$,

$$f_1(t)\,\overline{g_1(t-u)} \in L^2(\mathbb{R}) \quad \text{and} \quad f_2(t)\,\overline{g_2(t-u)} \in L^2(\mathbb{R}).$$

Therefore, it follows from the Parseval formula (2.5.11) for the LCT that
$\left\langle \mathscr{G}_{A,g_1} f_1, \mathscr{G}_{A,g_2} f_2 \right\rangle$

$$\begin{aligned}
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathscr{G}_{A,g_1} f_1(u,\omega)\, \overline{\mathscr{G}_{A,g_2} f_2(u,\omega)}\, du\, d\omega \\
&= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \mathscr{L}_A \left\{ f_1(t)\, \overline{g_1(t-u)} \right\}(\omega)\, \overline{\mathscr{L}_A \left\{ f_2(t)\, g_2(t-u) \right\}(\omega)}\, d\omega \right] du \\
&= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f_1(t)\overline{f_2(t)}\; \overline{g_1(t-u)}\, g_2(t-u)\, \exp\left\{ \frac{-iat^2}{b} \right\} dt \right] du. \qquad (2.5.22)
\end{aligned}$$

By virtue of Fubini theorem, we can interchange the order of integration in (2.5.22) to get

$$\begin{aligned}
\left\langle \mathscr{G}_{A,g_1} f_1, \mathscr{G}_{A,g_2} f_2 \right\rangle &= \int_{-\infty}^{\infty} f_1(t)\overline{f_2(t)}\, \exp\left\{ \frac{-iat^2}{b} \right\} dt \int_{-\infty}^{\infty} \overline{g_1(t-u)}\, g_2(t-u)\, du \\
&= \langle f_1, f_2 \rangle \overline{\langle g_1, g_2 \rangle}
\end{aligned}$$

where the extension to general $g_1, g_2 \in L^2(\mathbb{R})$ has been done by the standard density argument. As it is easy to verify that a fixed $g_1 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, the mapping $g_2 \to \langle \mathscr{G}_{A,g_1} f_1, \mathscr{G}_{A,g_2} f_2 \rangle_{L^2(\mathbb{R}^2)}$ is a linear functional that coincides with $\langle f_1, f_2 \rangle \overline{\langle g_1, g_2 \rangle}$ on a dense subset $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ of $L^2(\mathbb{R})$. It is therefore bounded and extends to all $g_2 \in L^2(\mathbb{R})$. Similarly, for arbitrary f_1, f_2 and $g_2 \in L^2(\mathbb{R})$, the conjugate linear functional $g_1 \to \langle \mathscr{G}_{A,g_1} f_1, \mathscr{G}_{A,g_2} f_2 \rangle_{L^2(\mathbb{R}^2)}$ coincides with $\langle f_1, f_2 \rangle \overline{\langle g_1, g_2 \rangle}$ on $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and extends to all functions of $L^2(\mathbb{R})$.

Corollary 2.5.1 (Conservation of Energy). *If $f \in L^2(\mathbb{R})$, then*

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| \mathscr{G}_{A,g} f(u,\omega) \right|^2 du\, d\omega = \|f\|_2^2. \tag{2.5.23}$$

Proof. Taking $f_1 = f_2 = f$ and $g_1 = g_2 = g$ in (2.5.21), we obtain

$$\left\|\mathcal{G}_{A,g}f\right\|_2^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\mathcal{G}_{A,g}f(u,\omega)\right|^2 du\, d\omega = \|f\|_2^2\,\|g\|_2^2. \tag{2.5.24}$$

The desired result is obtained by taking $\|g\|_2^2 = 1$ in (2.5.24), that is,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\mathcal{G}_{A,g}f(u,\omega)\right|^2 du\, d\omega = \|f\|_2^2, \quad \text{for all } f \in L^2(\mathbb{R}). \tag{2.5.25}$$

Remark.

1. Suppose that $\|f\|_2^2 = 1$. Then, equation (2.5.25) reduces to

$$\left\|\mathcal{G}_{A,g}f\right\|_2^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\mathcal{G}_{A,g}f(u,\omega)\right|^2 du\, d\omega = 1. \tag{2.5.26}$$

   This relation is known as the *radar uncertainty principle* in the WLCT domain.
2. It follows from (2.5.25) that f is completely determined by $\mathcal{G}_{A,g}f$. Furthermore, the condition

$$\mathcal{G}_{A,g}f(u,\omega) = \frac{1}{\sqrt{2\pi b}}\left\langle f, M_\omega T_u g\right\rangle = 0, \quad \forall\, u,\omega \in \mathbb{R},$$

   implies $f = 0$, which means that for each fixed $g \in L^2(\mathbb{R})$, the set $\{M_\omega T_u g : u,\omega \in \mathbb{R}\}$ spans a dense subspace of $L^2(\mathbb{R})$. Therefore, it is interesting to see how f can be recovered from $\mathcal{G}_{A,g}f$. In this regard, we present two proofs for the remarkable inversion formula.

Theorem 2.5.3 (Inversion Theorem). *Suppose that $g_1, g_2 \in L^2(\mathbb{R})$ and $\langle g_1, g_2\rangle \neq 0$. Then for all $f \in L^2(\mathbb{R})$, we have*

$$f(t) = \frac{1}{\langle g_1, g_2\rangle}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathcal{G}_{A,g_1}f(u,\omega)\,\overline{\mathcal{K}_A(t,\omega)}\,g_2(t-u)\,du\,d\omega. \tag{2.5.27}$$

First Proof. Since $\mathcal{G}_{A,g_1}f \in L^2(\mathbb{R})$, it follows by Corollary 2.5.1 that the vector-valued integral

$$\tilde{f}(t) = \frac{1}{\langle g_1, g_2\rangle}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathcal{G}_{A,g_1}f(u,\omega)\,\overline{\mathcal{K}_A(t,\omega)}\,g_2(t-u)\,du\,d\omega$$

is well defined in $L^2(\mathbb{R})$. Using the orthogonality relation (2.5.21), we observe that

$$\begin{aligned}\langle \tilde{f}, h\rangle &= \frac{1}{\sqrt{2\pi b}\,\langle g_1, g_2\rangle}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathcal{G}_{A,g_1}f(u,\omega)\,\overline{\langle h, M_\omega T_u g_2\rangle}\,du\,d\omega\\ &= \frac{1}{\langle g_1, g_2\rangle}\Big\langle \mathcal{G}_{A,g_1}f, \mathcal{G}_{A,g_2}h\Big\rangle\\ &= \langle f, h\rangle.\end{aligned}$$

Thus $f = \tilde{f}$. This proves the inversion theorem.

Second Proof. For every $h \in L^2(\mathbb{R})$, the inverse transform of the WFT (2.5.13) implies that

$$\begin{aligned}h(t) &= \frac{1}{2\pi\,\langle g_1, g_2\rangle}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathcal{G}_{g_1}h(u,\omega)\,e^{i\omega t}\,g_2(t-u)\,du\,d\omega\\ &= \frac{1}{2\pi\,\langle g_1, g_2\rangle}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathcal{G}_{g_1}h\left(u, \frac{\omega}{b}\right)\,e^{i\omega t/b}\,g_2(t-u)\,du\,d\frac{\omega}{b}.\end{aligned} \tag{2.5.28}$$

If we let $h(t) = \dfrac{1}{\sqrt{2\pi b}}\exp\left\{\dfrac{i}{2}\left(\dfrac{at^2}{b}-\dfrac{\pi}{2}\right)\right\}f(t)$ and using the relation (2.5.17), equation (2.5.28) becomes

$$\begin{aligned}&\frac{1}{\sqrt{2\pi b}}\exp\left\{\frac{i}{2}\left(\frac{at^2}{b}-\frac{\pi}{2}\right)\right\}f(t)\\ &\quad= \frac{1}{2\pi\,\langle g_1, g_2\rangle}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left\{\frac{-id\omega^2}{2b}\right\}\,\mathcal{G}_{A,g_1}f(u,\omega)\,e^{i\omega t/b}\,g_2(t-u)\,du\,d\frac{\omega}{b}.\end{aligned}$$

Or, equivalently,

$$\begin{aligned}f(t) &= \frac{1}{\sqrt{2\pi b}\,\langle g_1, g_2\rangle}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathcal{G}_{A,g_1}f(u,\omega)\\ &\qquad\qquad\times\;\exp\left\{\frac{-i}{2}\left(\frac{at^2}{b}+\frac{d\omega^2}{b}-\frac{2t\omega}{b}-\frac{\pi}{2}\right)\right\}g_2(t-u)\,du\,d\omega\\ &= \frac{1}{\langle g_1, g_2\rangle}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathcal{G}_{A,g_1}f(u,\omega)\,\overline{\mathcal{K}_A(t,\omega)}\,g_2(t-u)\,du\,d\omega.\end{aligned}$$

This proves the inversion theorem.

2.6 Exercises

1. For the cosine window

$$g(t) = \begin{cases} \cos\left(\frac{\pi t}{a}\right), & -a/2 \le t \le a/2 \\ 0, & \text{elsewhere} \end{cases}$$

show that its Fourier transform is

$$\hat{g}(\omega) = a\cos\left(\frac{a\omega}{2}\right)\left[\frac{1}{\pi - a\omega} - \frac{1}{\pi + a\omega}\right].$$

2. Consider the hat function g defined by

$$g(t) = \begin{cases} t, & 0 \le t \le 1 \\ 2 - t, & 1 \le t \le 2 \\ 0, & \text{otherwise,} \end{cases}$$

and compute the time-frequency window for g.

3. Use definition 2.2.1 to show that the triplet function

$$g(t) = \begin{cases} e^{-\lambda|t|}\cos^2\left(\frac{\pi t}{a}\right), & -a/2 \le t \le a/2 \\ 0, & \text{elsewhere,} \end{cases}$$

is a window function.

4. Let $f(t) = sin(\pi t)$ and the window function $g(t)$ be the function:

$$g(t) = \begin{cases} 1 + t, & -1 \le t < 0 \\ 1 - t, & 0 \le t < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Determine the windowed Fourier transform for $f(t)$.

5. Let $f(t) = sin(\pi t)$ and the window function $g(t)$ be the symmetrical hat function:

$$g(t) = \begin{cases} 1, & -1 \le t < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Show that the windowed Fourier transform of $f(t)$ is

$$\mathscr{F}_g(t,\omega) = i\left\{\frac{\sin(\pi + \omega)}{\pi + \omega} - \frac{\sin(\pi - \omega)}{\pi - \omega}\right\}.$$

6. For the Gaussian function $g(t) = \dfrac{1}{\sqrt{4\pi a}} e^{-t^2/4a}$, show that
   (a) $\displaystyle\int_{-\infty}^{\infty} \mathscr{G}[f](t,\omega)\,dt = \hat{f}(\omega)$, (b) $\hat{g}(\upsilon) = e^{-a\upsilon^2}$.
7. Find the Gabor transform of the following functions:
   (a) $f(t) = e^{-a^2t^2}$ with $g(t) = 1$, (b) $f(t) = e^{-itb}$.
8. Suppose $g_{t,\omega}(\tau) = g(\tau - t)e^{i\omega\tau}$ where g is a Gaussian window defined in Exercise 6, and show that
   (a) $\hat{g}_{t,\omega}(\upsilon) = e^{-i(\upsilon-\omega)t-a(\upsilon-\omega)^2}$, (b) $\mathscr{G}[f](t,\omega) = \dfrac{1}{2\pi}\langle \hat{f}, \hat{\bar{g}}_{t,\omega}\rangle$.
9. For the Gaussian window defined in Exercise 6, introduce

$$\sigma_t^2 = \frac{1}{\|g\|_2}\left\{\int_{-\infty}^{\infty} \tau^2 g^2(\tau)\,d\tau\right\}^{1/2}.$$

   Show that the radius of the window function is $\sqrt{a}$ and the width of the window is twice the radius.
10. Show that the marginals of the Zak transform are given by

$$\int_0^1 \big(\mathscr{Z}f\big)(t,\omega)\,d\omega = f(t), \quad \text{and} \quad \int_0^1 e^{-2\pi i\omega t}\big(\mathscr{Z}f\big)(t,\omega)\,dt = \hat{f}(\omega).$$

11. If $f(t)$ is time-limited to $-a \le t \le a$ and band-limited to $-b \le \omega \le b$, where $0 \le a, b \le \dfrac{1}{2}$, then the following results hold:
    (a) $\big(\mathscr{Z}f\big)(t,\omega) = f(\tau), \quad |\tau| \le \frac{1}{2},\ \omega \in \mathbb{R}$,
    (b) $\big(\mathscr{Z}f\big)(t,\omega) = e^{2\pi i\omega\tau}\hat{f}(\omega), \quad |\omega| \le \frac{1}{2},\ \tau \in \mathbb{R}$.
    Show that the second of the above results gives the Shannon sampling formula

$$f(t) = \sum_{n\in\mathbb{Z}} \frac{\sin 2\pi b(n-t)}{\pi(n-t)}, \quad t \in \mathbb{R}.$$

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