

Chapter 2

Partial Orders and Pontryagin Duality

Abstract Partial orders, supernatural numbers, and Pontryagin duality, are discussed.

2.1 Partial Orders

Definition 2.1 (i) A partially ordered set is a set S with a relation \prec , and the properties:

- (1) (reflexivity) $a \prec a$, for all $a \in S$;
- (2) (antisymmetry) if $a \prec b$ and $b \prec a$, then $a = b$;
- (3) (transitivity) if $a \prec b$ and $b \prec c$, then $a \prec c$.

(ii) A directed partially ordered set S , is a partially ordered set with the additional property that for $a, b \in S$, there exists $c \in S$ such that $a \prec c$ and $b \prec c$.

a, b are comparable if $a \prec b$ or $b \prec a$. A partially ordered set where any pair of elements is comparable, is a chain (total order).

Definition 2.2 Two partially ordered sets (S, \prec) and (S', \prec') are order isomorphic, if there is a bijective map f from S to S' , and $f(a_1) \prec' f(a_2)$, if and only if $a_1 \prec a_2$.

Example 2.1

- the partial order ‘subgroup’ in a set of groups
- the partial order ‘less or equal’ in the set of natural numbers \mathbb{N} (i.e., $a \prec b$ if $a \leq b$)
- the partial order ‘divisibility’ in the set of natural numbers \mathbb{N} (i.e., $a \prec b$ if $a|b$)

For simplicity we use the same symbol \prec for different partial orders, and its precise meaning is clear from the context.

Definition 2.3 An upper bound of a subset T of the partially ordered set S , is an element $a \in S$ such that $b \prec a$ for all $b \in T$. If the set of all upper bounds of T has a smallest element, it is called the supremum of T .

An element $m \in S$ is called maximal, if there is no element $k \in S$ such that $m < k$. A partially ordered set might have many maximal elements, or it might have no maximal element.

Definition 2.4 A partially ordered set S , is called directed-complete partial order (dcpo) if one of the following two statements, which can be proved to be equivalent to each other [1–3], holds:

- (1) Every directed subset of S has a supremum.
- (2) Every chain in S has a supremum.

A chain which has a supremum, is called a complete chain.

Directed partially ordered sets which are not complete, can sometimes be enlarged into directed-complete partial orders, by adding extra elements.

Example 2.2 The set \mathbb{N} of natural numbers, with divisibility as an order is a directed partially ordered set, but it is not a directed-complete partial order. For example the chain p, p^2, p^3, \dots where $p \in \Pi$, has no supremum. \mathbb{N} has no maximal elements. Below we enlarge this set into the supernatural (Steinitz) numbers, which is a directed-complete partial order.

2.2 The Directed-Complete Partial Order of Supernatural (Steinitz) Numbers

The set \mathbb{N}_S of supernatural (Steinitz) numbers [4, 5] is:

$$\mathbb{N}_S = \left\{ n = \prod p^{e_p} \mid p \in \Pi; \quad e_p \in \mathbb{Z}_0^+ \cup \{\infty\} \right\} \quad (2.1)$$

The index S indicates supernatural or Steinitz. Here:

- The exponents can take the value ∞ .
- The product might contain an infinite number of prime numbers.

In this set only multiplication is well defined, and by definition

$$p^\infty p^e = p^\infty; \quad e \in \mathbb{Z}_0^+ \cup \{\infty\}. \quad (2.2)$$

\mathbb{N} is a subset of \mathbb{N}_S . Indeed, if all $e_p \neq \infty$ and only a finite number of them are different from zero, the $\prod p^{e_p} \in \mathbb{N}$.

Definition 2.5

- Let (e_p) (where $p \in \Pi$ and $e_p \in \mathbb{Z}_0^+ \cup \{\infty\}$) be an infinite sequence of exponents. The $(e_p) < (e'_p)$ indicates that $e_p \leq e'_p$ for all p . By definition all numbers in \mathbb{Z}_0^+ are smaller than ∞ .
- $n = \prod p^{e_p}$ is a divisor of $n' = \prod p^{e'_p}$, if $(e_p) < (e'_p)$. We denote this as $n|n'$ or as $n < n'$.

- \mathcal{E} is the element of \mathbb{N}_S , corresponding to the sequence where all $e_p = 1$:

$$\mathcal{E} = \prod_{p \in \Pi} p \quad (2.3)$$

- Υ is the element of \mathbb{N}_S , corresponding to the sequence where all $e_p = \infty$:

$$\Upsilon = \prod_{p \in \Pi} p^\infty \quad (2.4)$$

Every element of \mathbb{N}_S is a divisor of Υ .

The set \mathbb{N}_S ordered by divisibility (as defined above) is a directed-complete partial order, with Υ as supremum. Examples of complete chains in \mathbb{N}_S , are

$$\begin{aligned} & p, p^2, \dots, p^\infty; \quad p \in \Pi \\ & p_1 < p_1^2 < \dots < p_1^\infty < p_1^\infty p_2 < p_1^\infty p_2^2 < \dots < p_1^\infty p_2^\infty \\ & 2 < 2 \cdot 3 < 2 \cdot 3 \cdot 5 < \dots < \mathcal{E} \\ & 2^\infty < 2^\infty 3^\infty < 2^\infty 3^\infty 5^\infty < \dots < \Upsilon \end{aligned} \quad (2.5)$$

The suprema in these chains are p^∞ , $p_1^\infty p_2^\infty$, \mathcal{E} and Υ , correspondingly. They are examples of the elements that have been added into \mathbb{N} , in order to make it the directed-complete partial order \mathbb{N}_S .

We use the notation $\mathbb{N}_S(p)$ for the complete chain

$$\mathbb{N}_S(p) = \{p, p^2, \dots, p^\infty\}. \quad (2.6)$$

2.3 Pontryagin Duality

Let G be an Abelian group and \tilde{G} its Pontryagin dual group, i.e. the group of its characters (we use the notation χ for characters). For locally compact Abelian groups, the Pontryagin duality theorem states that

$$\tilde{\tilde{G}} \cong G. \quad (2.7)$$

Let \mathfrak{G} be a set of groups, and $\tilde{\mathfrak{G}}$ the set of their Pontryagin dual groups. The partial order subgroup in \mathfrak{G} , endows a partial order in $\tilde{\mathfrak{G}}$, where $\tilde{A} < \tilde{G}$ if $A < G$.

Definition 2.6 Let A be a subgroup of G (we denote this as $A < G$). The annihilator $\text{Ann}_{\tilde{G}}(A)$ of A , is the subgroup of \tilde{G} :

$$\text{Ann}_{\tilde{G}}(A) = \{b \in \tilde{G} \mid \chi_b(a) = 1, \forall a \in A\} \quad (2.8)$$

Table 2.1 The groups G relevant to this monograph, together with their Pontryagin dual groups \tilde{G} , and the corresponding quantum system

G	\tilde{G}	$\Sigma(G, \tilde{G})$
$\mathbb{Z}(d)$	$\mathbb{Z}(d)$	$\Sigma[\mathbb{Z}(d)]$
$GF(p^e)$	$GF(p^e)$	$\Sigma[GF(p^e)]$
\mathbb{Z}_p	$\mathbb{Q}_p/\mathbb{Z}_p$	$\Sigma[\mathbb{Z}_p, (\mathbb{Q}_p/\mathbb{Z}_p)]$
$\hat{\mathbb{Z}}$	\mathbb{Q}/\mathbb{Z}	$\Sigma[\hat{\mathbb{Z}}, (\mathbb{Q}/\mathbb{Z})]$

The following proposition gives the Pontryagin dual group \tilde{A} of a subgroup A of a group G , and we present it without proof (e.g., [6]).

Proposition 2.1 *If $A < G$, then the Pontryagin dual group of A is isomorphic to $\tilde{G}/\text{Ann}_{\tilde{G}}(A)$:*

$$\tilde{A} \cong \tilde{G}/\text{Ann}_{\tilde{G}}(A). \quad (2.9)$$

In quantum mechanics G can be used as the group of ‘positions’, and its Pontryagin dual \tilde{G} as the group of ‘momenta’. We denote such a quantum system as $\Sigma(G, \tilde{G})$. For some groups $G \cong \tilde{G}$, and then we use the simpler notation $\Sigma(G)$ for the corresponding quantum system.

Definition 2.7 $\Sigma(A, \tilde{A})$ is a subsystem of $\Sigma(G, \tilde{G})$ if $A < G$ (in which case the \tilde{A} is related to \tilde{G} as in Eq. (2.9)). We denote this as $\Sigma(A, \tilde{A}) < \Sigma(G, \tilde{G})$.

The groups G relevant to this monograph, together with their Pontryagin dual groups \tilde{G} , and the corresponding quantum system, are shown in Table 2.1.

References

1. Birkhoff, G. (1995). *Lattice theory*. Rhode Island: American Mathematical Society
2. Szasz, G. (1963). *Introduction to lattice theory*. London: Academic.
3. Gratzer, G. A. (2003). *General lattice theory*. Berlin: Springer.
4. Ribes, L., & Zalesskii, P. (2000). *Profinite groups*. Berlin: Springer.
5. Wilson, J. (1998). *Profinite groups*. Oxford: Clarendon.
6. Pontryagin, L. S. (1966). *Topological groups*. New York: Gordon and Breach.

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