

Chapter 2

Introduction to Fuzzy Systems

Robert Czabanski, Michal Jezewski and Jacek Leski

Abstract The following chapter describes the basic concepts of fuzzy systems and approximate reasoning. The study focuses mainly on fuzzy models based on Zadeh's compositional rule of inference. The presentation begins with an introduction of fundamental ideas of fuzzy conditional (if-then) rules. A collection of fuzzy if-then rules formulates the so-called knowledge base, which formally represents the knowledge to be processed during approximate reasoning. The subsequent sections present formal definitions related to the compositional rule of inference and approximate reasoning using a knowledge base. Theoretical considerations are supplemented with practical examples of fuzzy systems as the foundation of many modern structures. The description includes fuzzy systems proposed by Mamdani and Assilan, Takagi, Sugeno and Kang, and Tsukamoto.

2.1 Introduction

The main inspiration behind the introduction of fuzzy sets theory was the necessity for modeling real-world phenomena, which are inherently vague and ambiguous. Human knowledge about complex problems can be successfully represented using the imprecise terms of natural language. The theories of fuzzy sets and fuzzy logic provide formal tools for mathematical representation and efficient processing of such information.

The term “system” is usually understood as a set of interacting components with well-defined structure and organized as an intricate whole that can be distinguished from the “external” environment. A system communicates with the environment

R. Czabanski (✉) · M. Jezewski · J. Leski
Institute of Electronics, Silesian University of Technology, Akademicka Str. 16,
44-100 Gliwice, Poland
e-mail: robert.czabanski@polsl.pl

M. Jezewski
e-mail: michal.jezewski@polsl.pl

J. Leski
e-mail: jacek.leski@polsl.pl

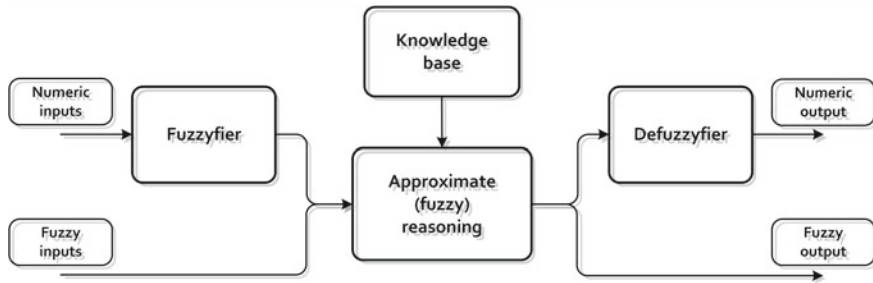


Fig. 2.1 The typical structure of a fuzzy system

through so-called inputs and outputs. Fuzzy systems are structures based on fuzzy techniques oriented towards information processing, where the usage of classical sets theory and binary logic is impossible or difficult. In the literature, terms such as fuzzy system, fuzzy model, system based on fuzzy rules, fuzzy controller, or fuzzy associative memory are used interchangeably depending on the application type [16]. Their main characteristic involves symbolic knowledge representation in a form of fuzzy conditional (if-then) rules.

The typical structure of a fuzzy system (Fig. 2.1) consists of four functional blocks: the fuzzifier, the fuzzy inference engine, the knowledge base, and the defuzzifier. Both linguistic values (defined by fuzzy sets) and crisp (numerical) data can be used as inputs for a fuzzy system. If crisp data are applied, then the inference process is preceded by fuzzification, which assigns the appropriate fuzzy set to the nonfuzzy input. The values of input variables are mapped into linguistic values of the output variable by means of the appropriate method of approximate reasoning (inference engine) using expert knowledge, which is represented as a collection of fuzzy conditional rules (knowledge base). In addition to the linguistic values, the numerical data may be required as the fuzzy system output. In such cases defuzzification methods are used, which assign the representative crisp data to the resultant output fuzzy set.

Practical applications of fuzzy systems include problems for which the complete mathematical description is unavailable, or where the usage of the precise (non-fuzzy) model is uneconomical or highly inconvenient. The ability to process inaccurate information makes a fuzzy system an excellent tool, for example, for control processes [12, 19], system identification [11, 20], decision support [24, 33], and signal and image processing [4, 23].

In the following sections only static fuzzy systems (i.e., systems where the outputs are determined only on the basis of the current input values) are considered. Included are concepts of knowledge representation in the form of fuzzy conditional rules, the idea of approximate reasoning, and the description of basic structures of fuzzy systems.

2.2 Fuzzy Conditional Rules

One of the fundamental concepts of fuzzy sets theory is a linguistic variable [34]. Its values are the statements of natural language (terms), which are the labels (descriptions) of fuzzy sets defined on a given universe (space) of discourse. Formally, a linguistic variable is defined as a quintuple [35]:

$$X = (\mathcal{N}, \mathcal{L}(\mathbf{G}), \mathbb{X}, \mathbf{G}, \mathbf{S}), \quad (2.1)$$

where \mathcal{N} is a name of the linguistic variable, $\mathcal{L}(\mathbf{G})$ denotes the family of values of the linguistic variable being a collection of labels of the fuzzy sets defined on the universe \mathbb{X} , \mathbf{G} is the set of syntactic rules defined by a grammar determining all terms in $\mathcal{L}(\mathbf{G})$, and \mathbf{S} represents the semantics of the variable X , that defines the meaning of all labels.

As an example we can use a linguistic variable describing the fetal heart rate (FHR). The name of the variable can be defined as $\mathcal{N} = \text{"mean FHR"}$. According to FIGO guidelines [21], the set of possible linguistic values is a collection of three labels describing the fetal state as: $\mathcal{L} = \{\text{"normal," "suspicious," "pathological"}\}$. To each of the labels we can assign a fuzzy set $A_i : i = 1, 2, \dots, 5$, defined on $\mathbb{X} = [0, 250]$ bpm, which represents the range of possible number of heart beats per min [3]. The examples of membership functions $\mu_{A_i}(x)$ of the fuzzy sets A_i are shown in Fig. 2.2.

An elementary statement for the linguistic variable X is the fuzzy expression:

$$X \text{ is } L_A, \quad (2.2)$$

where L_A is a label from the collection $\mathcal{L}(\mathbf{G})$, defined by a fuzzy set A on the universe \mathbb{X} . The logical value of the expression is determined on the basis of membership function $\mu_A(x)$ of the fuzzy set A . In the preceding example, an elementary statement is:

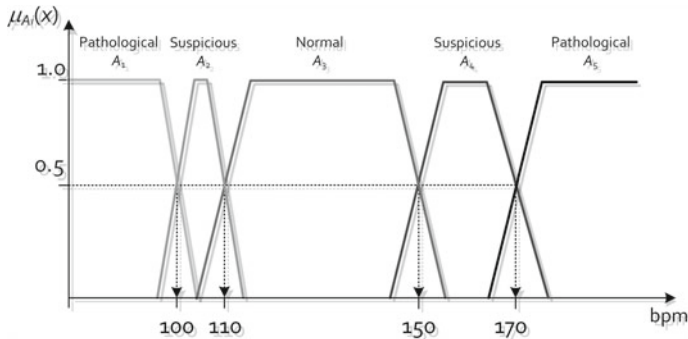


Fig. 2.2 Examples of membership functions of fuzzy sets defining the values of the linguistic variable $X = \text{"mean FHR"}$

“mean FHR” is “normal”,

which value for the measurement 110 bpm is equal to $\mu_{A_3}(x) = 0.5$ (see Fig. 2.2).

A more complex fuzzy expression can be obtained by combining two or more elementary expressions. It can be presented in the conjunctive:

$$(X_1 \text{ is } L_{A_1}) \text{ and } (X_2 \text{ is } L_{A_2}), \quad (2.3)$$

or the disjunctive form:

$$(X_1 \text{ is } L_{A_1}) \text{ or } (X_2 \text{ is } L_{A_2}), \quad (2.4)$$

where X_1, X_2 are linguistic variables with labels L_{A_1}, L_{A_2} defined by the fuzzy sets A_1 and A_2 , respectively, on the universes \mathbb{X}_1 and \mathbb{X}_2 .

The value of a complex fuzzy expression for $x_1 \in \mathbb{X}_1$ and $x_2 \in \mathbb{X}_2$ is determined on the basis of the membership functions of fuzzy sets A_1 and A_2 [16]:

$$\mu_{A_1}(x_1) \star_T \mu_{A_2}(x_2), \quad (2.5)$$

for the conjunctive form, and

$$\mu_{A_1}(x_1) \star_S \mu_{A_2}(x_2), \quad (2.6)$$

for the disjunctive form, where \star_T denotes a t -norm, and \star_S an s -norm.

An elementary fuzzy statement can also be expressed in the form of an implication forming a fuzzy if-then rule (fuzzy conditional statement):

$$\text{if } (X \text{ is } L_A), \text{ then } (Y \text{ is } L_B), \quad (2.7)$$

defining a relationship between linguistic variables. The statement “ $X \text{ is } L_A$ ” is called the antecedent (premise), and the statement “ $Y \text{ is } L_B$ ” is called the consequent (conclusion).

A generalized form of the fuzzy conditional statement can be defined as an implication of complex fuzzy expressions. For the conjunctive form it can be written as:

$$\begin{aligned} &\text{if } (X_1 \text{ is } L_{A_1}) \text{ and } (X_2 \text{ is } L_{A_2}) \text{ and } \dots \text{ and } (X_N \text{ is } L_{A_N}), \\ &\text{then } (Y_1 \text{ is } L_{B_1}), (Y_2 \text{ is } L_{B_2}), \dots, (Y_M \text{ is } L_{B_M}), \end{aligned} \quad (2.8)$$

and for the disjunctive form as:

$$\begin{aligned} &\text{if } (X_1 \text{ is } L_{A_1}) \text{ or } (X_2 \text{ is } L_{A_2}) \text{ or } \dots \text{ or } (X_N \text{ is } L_{A_N}), \\ &\text{then } (Y_1 \text{ is } L_{B_1}), (Y_2 \text{ is } L_{B_2}), \dots, (Y_M \text{ is } L_{B_M}), \end{aligned} \quad (2.9)$$

where X_1, X_2, \dots, X_N are the input linguistic variables; Y_1, Y_2, \dots, Y_M are the output linguistic variables; $L_{A_1}, L_{A_2}, \dots, L_{A_N}$, and $L_{B_1}, L_{B_2}, \dots, L_{B_M}$ are their linguistic values, defined with fuzzy sets A_1, A_2, \dots, A_N and B_1, B_2, \dots, B_M on universes $\mathbb{X}_1, \mathbb{X}_2, \dots, \mathbb{X}_N$, and $\mathbb{Y}_1, \mathbb{Y}_2, \dots, \mathbb{Y}_M$, respectively.

Both implications are the fuzzy if-then rules with multiple inputs and multiple outputs (MIMO). The MIMO fuzzy rule can be decomposed into the corresponding set of canonical fuzzy if-then rules [16], which are the MISO (multiple inputs and single output) type of fuzzy conditional statements with conjunctive antecedent:

$$\text{if and } \left(\bigwedge_{n=1}^N (X_n \text{ is } L_{A_n}) \right), \text{ then } Y \text{ is } L_B. \quad (2.10)$$

Canonical fuzzy conditional statements are the basics for representing expert knowledge in a fuzzy system. Using pseudo-vector notation, the canonical fuzzy if-then rule can be written as

$$\text{if } (\mathbf{X} \text{ is } \mathbf{L}_A), \text{ then } (Y \text{ is } L_B), \quad (2.11)$$

which is an $N + 1$ -nary fuzzy relation [4]:

$$R = ((A_1 \times A_2 \times \dots \times A_N) \implies B) = (\mathbf{A} \implies B), \quad (2.12)$$

defined on $\mathbb{X}_1 \times \mathbb{X}_2 \times \dots \times \mathbb{X}_N \times \mathbb{Y}$, with the membership function:

$$\mu_R(x_1, \dots, x_N, y) = \Phi(\mu_A(\mathbf{x}), \mu_B(y)), \quad (2.13)$$

where $\mathbf{x} = [x_1, \dots, x_N]^T \in \mathbb{X}_1 \times \mathbb{X}_2 \times \dots \times \mathbb{X}_N$, $y \in \mathbb{Y}$, and depending on the interpretation of the fuzzy if-then rule, $\Phi(\cdot, \cdot)$ denotes a t -norm (a conjunctive interpretation) [8, 16] or fuzzy implication (logical interpretation) [8, 9, 16].

If the conjunction “and” in the antecedents of the fuzzy if-then rules is represented by a t -norm T , then:

$$\mu_A(\mathbf{x}) = \mu_{A_1}(x_1) \star_T \mu_{A_2}(x_2) \star_T \dots \star_T \mu_{A_N}(x_N), \quad (2.14)$$

where A_1, A_2, \dots, A_N are fuzzy sets representing the values of linguistic variables in the antecedent of the canonical fuzzy rule.

Hence, for the conjunctive interpretation we get:

$$\begin{aligned} \mu_R(\mathbf{x}, y) &= \mu_R(x_1, \dots, x_N, y) = \mu_A(\mathbf{x}) \star_{T_r} \mu_B(y) = \\ &= \mu_{A_1}(x_1) \star_T \mu_{A_2}(x_2) \star_T \dots \star_T \mu_{A_N}(x_N) \star_{T_r} \mu_B(y), \end{aligned} \quad (2.15)$$

where \star_{T_r} is a t -norm representing the fuzzy if-then rule, whereas for logical interpretation:

$$\mu_R(\mathbf{x}, y) = \mu_R(x_1, \dots, x_N, y) = \Psi(\mu_A(\mathbf{x}), \mu_B(y)) = \Psi(\mu_{A_1}(x_1) \star_T \mu_{A_2}(x_2) \star_T \dots \star_T \mu_{A_N}(x_N), \mu_B(y)), \quad (2.16)$$

where $\Psi(\cdot, \cdot)$ denotes a fuzzy implication.

Fuzzy implication is usually introduced using an axiomatic approach [9], where it is defined as a continuous function $\Psi : [0, 1] \times [0, 1] \rightarrow [0, 1]$, which for each $a, b, c \in [0, 1]$ fulfills five necessary (general) conditions:

- P1: if $a \leq c$, then $\Psi(a, b) \geq \Psi(c, b)$,
- P2: if $b \leq c$, then $\Psi(a, b) \leq \Psi(a, c)$,
- P3: $\Psi(0, b) = 1$,
- P4: $\Psi(a, 1) = 1$,
- P5: $\Psi(1, 0) = 0$,

and eight recommended (specific) conditions [4]. Properties P3, P4, and P5 are called falsity, neutrality, and Booleanity, respectively [4, 22]. As examples we can use Lukasiewicz:

$$\Psi(a, b) = \min(1 - a + b, 1), \quad (2.17)$$

Reichenbach:

$$\Psi(a, b) = 1 - a + ab, \quad (2.18)$$

and Zadeh fuzzy implication:

$$\Psi(a, b) = \max(1 - a, \min(a, b)). \quad (2.19)$$

A single fuzzy rule describes a local relationship between the input and output variables of the fuzzy system within the limits defined by the domain of fuzzy sets in the rule antecedent. The complete input–output mapping is represented by the whole collection of fuzzy if-then rules from the knowledge (rule) base. For further considerations we assume a base consisting of I rules in the form:

$$\mathcal{R} = \{R^{(i)}\}_{i=1}^I = \left\{ \text{if } \bigwedge_{n=1}^N (X_n \text{ is } L_{A_n}^{(i)}), \text{ then } Y \text{ is } L_B^{(i)} \right\}_{i=1}^I. \quad (2.20)$$

A well-defined fuzzy rule base should be complete, consistent, and continuous [31]. The completeness means that for each value from the input space at least one rule is activated, that is $\exists_{i=1,2,\dots,I} \mu_{A^{(i)}}(\mathbf{x}) \neq 0$. The knowledge base is consistent if there are no rules with the same antecedent but different consequents. And finally, the knowledge base is continuous if there are no neighboring rules, for which the result of intersection of fuzzy sets in their consequents is an empty set.

The knowledge base is constructed first by acquiring knowledge about the modeled phenomenon, and next by representing it in a form of fuzzy conditional rules. In practice, there are three basic methods to create a fuzzy rule base [16]:

- by using knowledge of a human expert or based on the physical laws describing the phenomenon (white box modeling),
- by automatically extracting the rules based on numerical data representing the relationship between inputs and outputs of the phenomenon (black box modeling),
- mixed, where part of the knowledge is derived from a human expert and part from automated extraction (grey box modeling).

The possible applications of a fuzzy system depend, however, not only on the properly defined knowledge base, but also on the appropriate design of an inference engine.

2.3 Approximate Reasoning

Inference methods originating from classical logic are based on so-called rules of inference. A rule of inference is a pattern of reasoning that explains how a conclusion may be logically derived from a given premise previously assumed to be true. One of the most commonly used rules of inference is the rule of detachment, often referred to as modus ponendo ponens (“the way that affirms by affirming”). Modus ponendo ponens (MPP) is based on two premises. The first is the conditional statement $p \implies q$, namely that “ p implies q ”. The second assumes that the antecedent p of the conditional statement is true. From these two premises it can be concluded that the consequent q is true. The MPP rule can be written as [4]:

$$\begin{array}{l} \text{Premise I (fact): } p \\ \text{Premise II (rule): } p \implies q \\ \hline \text{Conclusion: } q \end{array}$$

or symbolically:

$$(p \wedge (p \implies q)) \implies q. \quad (2.21)$$

Binary logic assumes only two possibilities: total compliance or total noncompliance of the fact with the implication antecedent. In contrast, fuzzy inference engines use an approximate reasoning based on the generalized rules of inference. The generalized modus ponendo ponens (GMPP) may be written as [34]:

$$\begin{array}{l} \text{Premise I (fact): } p' \\ \text{Premise II (rule): } p \implies q \\ \hline \text{Conclusion: } q' \end{array}$$

or:

$$[p' \wedge (p \implies q)] \implies q', \quad (2.22)$$

where statements p' and q' are similar, respectively, to p and q .

A conditional fuzzy rule can be defined as a fuzzy relation, and hence, the statements in antecedents and consequents as fuzzy sets. The statement X **is** $L_{A'}$ is a fact, where $L_{A'}$ denotes the label of a linguistic variable X defined by a fuzzy set A' on the universe \mathbb{X} . The knowledge is represented by the fuzzy conditional rule “**if** X **is** L_A , **then** Y **is** L_B ,” where L_A and L_B are the linguistic values of linguistic variables X and Y , defined by fuzzy sets A and B , on the universes \mathbb{X} and \mathbb{Y} , respectively. Consequently, the inference scheme of GMPP takes the form:

$$\begin{array}{l} \text{Premise I (fact): } X \text{ is } L_{A'} \\ \text{Premise II (rule): if } X \text{ is } L_A, \text{ then } Y \text{ is } L_B \\ \hline \text{Conclusion: } Y \text{ is } L_{B'} \end{array}$$

or:

$$[(X \text{ is } L_{A'}) \wedge (X \text{ is } L_A \implies Y \text{ is } L_B)] \implies Y \text{ is } L_{B'}. \quad (2.23)$$

The fuzzy set B' is determined using Zadeh’s compositional rule of inference [34].

2.3.1 Compositional Rule of Inference

The compositional rule of inference (CRI), also known as supremum-star composition [34], is a generalization of an operation for determining the function value. The first stage of CRI is to construct a cylindrical extension of a fuzzy set $A'(x)$ from the universe \mathbb{X} to $\mathbb{X} \times \mathbb{Y}$:

$$\forall_{(x,y) \in \mathbb{X} \times \mathbb{Y}} \quad \mu_{\text{Ce}(A')}(x, y) = \mu_{A'}(x). \quad (2.24)$$

Secondly, an intersection (logical product) of cylindrical extension $\text{Ce}(A')$ and fuzzy relation R is constructed using t -norm T :

$$\begin{aligned} \forall_{(x,y) \in \mathbb{X} \times \mathbb{Y}} \quad \mu_{\text{Ce}(A') \cap R}(x, y) &= \mu_{\text{Ce}(A')}(x, y) \star_T \mu_R(x, y) \\ &= \mu_{A'}(x) \star_T \mu_R(x, y). \end{aligned} \quad (2.25)$$

The final CRI outcome is a result of the $\text{Ce}(A') \cap R$ projection on \mathbb{Y} :

$$\forall_{y \in \mathbb{Y}} \quad \mu_{B'}(y) = \sup_{x \in \mathbb{X}} [\mu_{A'}(x) \star_T \mu_R(x, y)]. \quad (2.26)$$

The fuzzy set B' can also be presented as a composition of a fuzzy set A' , which is an unary fuzzy relation, with conditional fuzzy rule R being a binary fuzzy relation:

$$B' = A' \circ R, \quad (2.27)$$

where \circ is the operator of the supremum- t -norm composition.

The GMPP for the i th canonical fuzzy if-then rule (2.20) can be written as [16]:

$$B'^{(i)} = A' \circ R^{(i)} = A' \circ (A^{(i)} \implies B^{(i)}), \quad (2.28)$$

where $A' = A'_1 \times A'_2 \times \cdots \times A'_N$ is a multidimensional fuzzy set that defines the value of the multidimensional input linguistic variable on the space $\underline{X} = \underline{X}_1 \times \underline{X}_2 \times \cdots \times \underline{X}_N$.

The membership function of the conclusion $B'^{(i)}$ is calculated as follows.

$$\begin{aligned} \mu_{B'^{(i)}}(y) &= \sup_{\mathbf{x} \in \underline{X}} [\mu_{A'}(\mathbf{x}) \star_{T_s} \mu_{R^{(i)}}(\mathbf{x}, y)] = \\ &= \sup_{\mathbf{x} \in \underline{X}} [\mu_{A'_1}(x_1) \star_T \mu_{A'_2}(x_2) \star_T \cdots \star_T \mu_{A'_N}(x_N) \star_{T_s} \mu_{R^{(i)}}(x_1, \dots, x_N, y)], \end{aligned} \quad (2.29)$$

where T_s is a t -norm of the supremum- t -norm composition. In the case of the conjunctive interpretation (2.15) we can write:

$$\begin{aligned} \mu_{B'^{(i)}}(y) &= \sup_{\mathbf{x} \in \underline{X}} [\mu_{A'}(\mathbf{x}) \star_{T_s} \mu_{A^{(i)}}(\mathbf{x}) \star_{T_r} \mu_{B^{(i)}}(y)] = \\ &= \sup_{\mathbf{x} \in \underline{X}} [(\mu_{A'_1}(x_1) \star_T \mu_{A'_2}(x_2) \star_T \cdots \star_T \mu_{A'_N}(x_N)) \star_{T_s} \\ &\quad (\mu_{A_1^{(i)}}(x_1) \star_T \mu_{A_2^{(i)}}(x_2) \star_T \cdots \star_T \mu_{A_N^{(i)}}(x_N)) \star_{T_r} \mu_{B^{(i)}}(y)]. \end{aligned} \quad (2.30)$$

And for logical interpretation (2.16) we get:

$$\begin{aligned} \mu_{B'^{(i)}}(y) &= \sup_{\mathbf{x} \in \underline{X}} [\mu_{A'}(\mathbf{x}) \star_{T_s} \Psi(\mu_{A^{(i)}}(\mathbf{x}), \mu_{B^{(i)}}(y))] = \\ &= \sup_{\mathbf{x} \in \underline{X}} [(\mu_{A'_1}(x_1) \star_T \mu_{A'_2}(x_2) \star_T \cdots \star_T \mu_{A'_N}(x_N)) \star_{T_s} \\ &\quad \Psi(\mu_{A_1^{(i)}}(x_1) \star_T \mu_{A_2^{(i)}}(x_2) \star_T \cdots \star_T \mu_{A_N^{(i)}}(x_N), \mu_{B^{(i)}}(y))]. \end{aligned} \quad (2.31)$$

Under certain conditions [5], logical and conjunctive interpretation of fuzzy conditional rules leads to equivalent inference results.

Equations (2.30) and (2.31) define the membership function of a fuzzy set representing the resulting conclusion of an inference using only one fuzzy if-then rule. For a knowledge base consisting of many fuzzy conditional statements it is necessary to combine conclusions from all individual rules.

2.3.2 Approximate Reasoning with Knowledge Base

Generally, there are two methods of approximate reasoning that can be applied to determine the outcome fuzzy set B' on the basis of a collection of fuzzy if-then rules [4]:

- composition-based inference (first aggregate then infer: FATI), where first a combination of all rules from the knowledge base is constructed, and then inference using the supremum-star composition is conducted,
- individual rule-based inference (first infer then aggregate: FITA), in which the first step involves inference using the supremum-star composition for each of the rules individually and then, a combination of inference results is performed.

The FATI process of combining the rules, as well as the stage in the FITA schema of determining the resulting conclusion, is called aggregation [10]. The aggregation can be defined by introduction of the concept of the aggregation operator [16], which for I values $x_1, x_2, \dots, x_I \in [0, 1]$ represents a mapping $\oplus : [0, 1]^I \Rightarrow [0, 1]$:

$$x = \bigoplus_{i=1}^I x_i = \oplus (x_1, x_2, \dots, x_I). \quad (2.32)$$

There are various definitions of aggregation operator including logical sum, represented by an s -norm (Mamdani combination [19]), logical product, represented by a t -norm (Gödel combination [16]), as well as nonmonotonic fuzzy operations that allow conducting the inference even if part of the knowledge is missing [32]. Most of them can be defined as special cases of the generalized average operator [4]:

$$\uplus_{(\alpha)} (x_1, \dots, x_I) = \biguplus_{(\alpha)} x_i = \left[\frac{1}{I} \sum_{i=1}^I (x_i)^\alpha \right]^{\frac{1}{\alpha}}, \quad (2.33)$$

for $\alpha \in \mathbb{R} \setminus \{0\}$.

Consequently, the first stage of the FATI method can be defined as:

$$\mathcal{R} = \bigoplus_{i=1}^I R^{(i)}, \quad (2.34)$$

where $R^{(i)}$ is the i th fuzzy relation.

Next, the outcome fuzzy set B'_{FATI} is determined for an input fuzzy set \mathbf{A}' using the GMPP:

$$B'_{FATI} = \mathbf{A}' \circ \mathcal{R} = \mathbf{A}' \circ \left[\bigoplus_{i=1}^I R^{(i)} \right], \quad (2.35)$$

the membership function of which is defined as

$$\begin{aligned}\mu_{B'_{FATI}}(y) &= \sup_{\mathbf{x} \in \underline{X}} [\mu_{A'}(\mathbf{x}) \star_{T_s} \mu_{\mathcal{R}}(\mathbf{x}, y)] \\ &= \sup_{\mathbf{x} \in \underline{X}} \left\{ \mu_{A'}(\mathbf{x}) \star_{T_s} \left[\bigoplus_{i=1}^I \mu_{R^{(i)}}(\mathbf{x}, y) \right] \right\}.\end{aligned}\quad (2.36)$$

In the case of the FITA method, first the conclusion of each fuzzy if-then rule is determined:

$$\forall_{i=1,2,\dots,I} \quad B'^{(i)} = A' \circ R^{(i)}, \quad (2.37)$$

the membership function of which is written as:

$$\mu_{B'^{(i)}}(y) = \sup_{\mathbf{x} \in \underline{X}} [\mu_{A'}(\mathbf{x}) \star_{T_s} \mu_{R^{(i)}}(\mathbf{x}, y)]. \quad (2.38)$$

During the next stage, these partial results of the inference are aggregated forming the outcome fuzzy set:

$$B'_{FITA} = \bigoplus_{i=1}^I (A' \circ R^{(i)}), \quad (2.39)$$

defined by the membership function:

$$\mu_{B'_{FITA}}(y) = \bigoplus_{i=1}^I \sup_{\mathbf{x} \in \underline{X}} [\mu_{A'}(\mathbf{x}) \star_{T_s} \mu_{R^{(i)}}(\mathbf{x}, y)]. \quad (2.40)$$

It can be proven [7], that the results of the FATI method are a subset of those obtained using the FITA procedure:

$$B'_{FATI} \subseteq B'_{FITA}, \quad (2.41)$$

that is:

$$\forall_{y \in \mathbb{Y}} \quad \mu_{B'_{FATI}}(y) \leq \mu_{B'_{FITA}}(y). \quad (2.42)$$

Usually, for simplicity of calculations, the B'_{FATI} is used instead of B'_{FITA} , under the assumption that the difference is insignificant [4].

2.3.3 Fuzzification and Defuzzification

In many applications inputs of the fuzzy systems are defined as crisp numerical data. However, approximate reasoning requires inputs to be represented as fuzzy sets. The process of mapping real values $\mathbf{x}_0 = [x_{01}, x_{02}, \dots, x_{0N}]^T \in \underline{\mathbb{X}} \subset \mathbb{R}^N$ to an N -dimensional fuzzy set \mathbf{A}' defined on $\underline{\mathbb{X}}$ is called fuzzification. The fuzzification can be symbolically expressed as a transformation of N -dimensional space into a multitude of fuzzy sets [16]:

$$\underline{\mathbb{X}} \Rightarrow \mathcal{F}(\underline{\mathbb{X}}). \quad (2.43)$$

Using membership functions we can write:

$$\underline{\mathbb{X}} \Rightarrow \{ \mu_{\mathbf{A}'}(\mathbf{x}) \mid \mathbf{x} \in \underline{\mathbb{X}}, \mu_{\mathbf{A}'}(\mathbf{x}) \in [0, 1] \}. \quad (2.44)$$

Among many definitions of a fuzzification operator, the singleton fuzzifier can be distinguished:

$$\mu_{\mathbf{A}'}(\mathbf{x}) = \delta_{\mathbf{x}, \mathbf{x}_0} = \begin{cases} 1, & \mathbf{x} = \mathbf{x}_0, \\ 0, & \mathbf{x} \neq \mathbf{x}_0, \end{cases} \quad (2.45)$$

for which both methods of approximate reasoning (FATI and FITA) provide equivalent inference results [5].

The result of approximate reasoning is a fuzzy set $B'(y)$, which can be associated with a specific linguistic label. However, there are applications that require a crisp numerical inference outcome. The process of calculating a representative numerical output $y_0 \in \mathbb{Y}$ from the outcome fuzzy set $B'(y)$ on \mathbb{Y} is called defuzzification. Defuzzification is a mapping of a multitude of fuzzy sets defined on the space \mathbb{Y} to a single numerical value from \mathbb{Y} [16]:

$$\mathcal{F}(\mathbb{Y}) \rightarrow \mathbb{Y}. \quad (2.46)$$

Using membership functions we get:

$$\{ \mu_{B'}(y) \mid y \in \mathbb{Y}, \mu_{B'}(y) \in [0, 1] \} \rightarrow \mathbb{Y}. \quad (2.47)$$

Due to the different criteria for determining which element y_0 of the fuzzy set $B'(y)$ should be regarded as the most representative one, there are many definitions of the defuzzification procedure [6, 14, 31]. One of the most popular is a center of gravity method (COG), which specifies the result as a center of the area under the membership function $\mu_{B'}(y)$:

$$y_0 = \frac{\int_{\mathbb{Y}} y \mu_{B'}(y) dy}{\int_{\mathbb{Y}} \mu_{B'}(y) dy}. \quad (2.48)$$

2.4 Basic Types of Fuzzy Systems

Due to a wide range of possible applications there are many different types of fuzzy systems that have been proposed in the literature thus far [4, 16, 22, 23, 31]. But new solutions characterized by decreased computation complexity, improved modeling quality, or greater ease of the linguistic interpretation of the inference results are still the topic of research. The model proposed by E.H. Mamdani and S. Assilan [19] is generally regarded as the first fuzzy system presented in the literature. Currently, it can be considered as the foundation of the fuzzy models family based on if-then rules with fuzzy sets in antecedents as well as consequents.

2.4.1 Mamdani–Assilan Fuzzy Model

The Mamdani–Assilan fuzzy system (MAFS) uses a set of conditional fuzzy rules in the canonical form (2.20), which can be determined by a human expert. The MAFS is based on the conjunctive interpretation of fuzzy rules, where the conjunctive “and” of a rule antecedent is defined with the t -norm minimum (\wedge). The inference results from individual rules are aggregated by applying the s -norm maximum (\vee). The numerical inputs $\mathbf{x}_0 = [x_{01}, x_{02}, \dots, x_{0N}]^T$ are mapped into fuzzy sets with the singleton fuzzifier, and the numerical outcome is calculated using the COG method. The approximate reasoning schema is realized on the basis of Eq. (2.40), which takes the form:

$$\mu_{B'}(y) = \bigvee_{i=1}^I [\mu_{A^{(i)}}(\mathbf{x}_0) \wedge \mu_{B^{(i)}}(y)], \quad (2.49)$$

where

$$\mu_{A^{(i)}}(\mathbf{x}_0) = \mu_{A_1^{(i)}}(x_{01}) \wedge \mu_{A_2^{(i)}}(x_{02}) \wedge \dots \wedge \mu_{A_N^{(i)}}(x_{0N}). \quad (2.50)$$

The above equation defines the so-called firing strength of the i th rule, denoted as $F^{(i)}(\mathbf{x}_0)$. Hence, the formula (2.49) can also be written as

$$\mu_{B'}(y) = \bigvee_{i=1}^I [F^{(i)}(\mathbf{x}_0) \wedge \mu_{B^{(i)}}(y)]. \quad (2.51)$$

Using the COG defuzzification we get:

$$y_0 = \frac{\int_{\mathbb{Y}} y \mu_{B'}(y) dy}{\int_{\mathbb{Y}} \mu_{B'}(y) dy}. \quad (2.52)$$

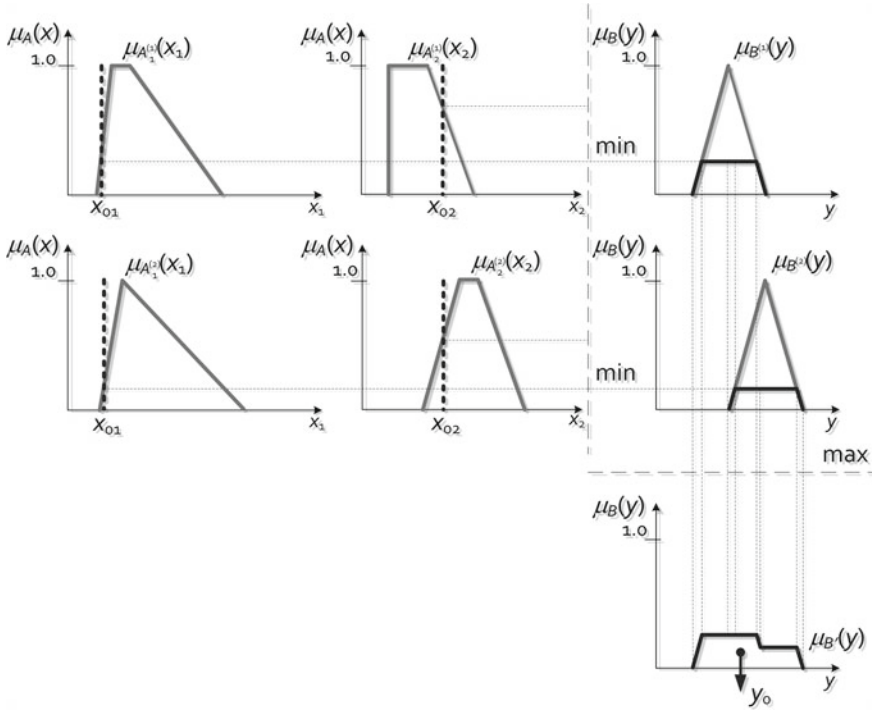


Fig. 2.3 Example of fuzzy inference using the Mamdani–Assilian fuzzy system with two inputs and the knowledge base consisting of two conditional fuzzy rules

Figure 2.3 shows an example of fuzzy inference using MAFS with two inputs and the knowledge base consisting of two conditional fuzzy rules.

The defuzzification requires high computational complexity, however, some simplifications can be applied. Using the algebraic product t -norm and the arithmetic mean as the aggregation operator we obtain a Larsen fuzzy system, which is defined as [16]:

$$\mu_{B'}(y) = \frac{1}{I} \sum_{i=1}^I F^{(i)}(\mathbf{x}_0) \mu_{B^{(i)}}(y). \quad (2.53)$$

By substitution of (2.53) into (2.52) we get:

$$y_0 = \frac{\sum_{i=1}^I F^{(i)}(\mathbf{x}_0) \int_{\mathbb{Y}} y \mu_{B^{(i)}}(y) dy}{\sum_{j=1}^I F^{(j)}(\mathbf{x}_0) \int_{\mathbb{Y}} \mu_{B^{(j)}}(y) dy}. \quad (2.54)$$

Denoting the area under a membership function of the fuzzy set $B^{(i)}(y)$ as

$$\mathcal{A}(\mu_{B^{(i)}}(y)) = \int_{\mathbb{Y}} \mu_{B^{(i)}}(y) dy, \quad (2.55)$$

and its center of gravity as $y^{(i)}$, we can write:

$$y_0 = \frac{\sum_{i=1}^I F^{(i)}(\mathbf{x}_0) \mathcal{A}(\mu_{B^{(i)}}(y)) y^{(i)}}{\sum_{j=1}^I F^{(j)}(\mathbf{x}_0) \mathcal{A}(\mu_{B^{(j)}}(y))}. \quad (2.56)$$

The above solution requires only a single calculation of the areas under the membership functions and centers of gravity locations for all fuzzy rules. By assuming additionally that $\mathcal{A}(\mu_{B^{(i)}}(y))$ are the same for all I consequents, we get the Sugeno–Yasukawa fuzzy model [26].

Approximate reasoning without the defuzzification necessity was presented in papers by Takagi and Sugeno [27] and Sugeno and Kang [25]. The proposed model, called the Takagi–Sugeno–Kang fuzzy system (TSKFS), is described in the following subsection.

2.4.2 Takagi–Sugeno–Kang Fuzzy System

The knowledge base of the TSKFS consists of conditional fuzzy rules with the consequents in the form of classical functions, the arguments of which are the input numerical data:

$$\mathcal{R} = \{R^{(i)}\}_{i=1}^I = \left\{ \text{if } \bigwedge_{n=1}^N \left(x_{0n} \text{ is } L_{A_n}^{(i)} \right), \text{ then } y = y^{(i)}(\mathbf{x}_0) \right\}_{i=1}^I, \quad (2.57)$$

where x_{0n} is an input singleton, $\mathbf{x}_0 = [x_{01}, x_{02}, \dots, x_{0N}]^\top$, and $y^{(i)}(\mathbf{x})$ is the function in the i th consequent.

The output of each fuzzy rule is a crisp numerical datum $y = y^{(i)}(\mathbf{x}_0)$, and the TSKFS outcome is calculated as a weighted average of individual outputs:

$$y_0 = \frac{\sum_{i=1}^I F^{(i)}(\mathbf{x}_0) y^{(i)}(\mathbf{x}_0)}{\sum_{j=1}^I F^{(j)}(\mathbf{x}_0)}, \quad (2.58)$$

where

$$F^{(i)}(\mathbf{x}_0) = \mu_{A_1^{(i)}}(x_{01}) \star_T \mu_{A_2^{(i)}}(x_{02}) \star_T \cdots \star_T \mu_{A_N^{(i)}}(x_{0N}), \quad (2.59)$$

is the firing strength and \star_T is a t -norm (usually a minimum or algebraic product).

Equation (2.58) can be interpreted as a mixture of experts, each modeled by a single fuzzy rule. Each rule defines the relationship between outputs and inputs of the system in the relevant input range. The weighted average of statements from all local experts (rules) determines the reasoning result. The weight, represented by the firing strength of the rule, specifies the influence level of a single expert on the final inference outcome.

The consequent of the i th TSKFS fuzzy rule can also be understood as a singleton [4], the location of which is determined by the function $y^{(i)}(\mathbf{x})$:

$$\mu_{B^{(i)}}(y) = \delta_{y, y^{(i)}} = \begin{cases} 1, & y = y^{(i)}(\mathbf{x}_0), \\ 0, & y \neq y^{(i)}(\mathbf{x}_0). \end{cases} \quad (2.60)$$

Hence, the TSKFS is usually referred to as the fuzzy system with “moving” singletons. The term “moving” relates to the relationship between a singleton location and the input numerical data. The amplitude (height) of the singleton after the approximate reasoning is defined by the firing strength of a rule.

The TSKFS consequents are frequently defined as linear functions (first-order polynomials):

$$y^{(i)}(\mathbf{x}_0) = p_0^{(i)} + p_1^{(i)}x_{01} + p_2^{(i)}x_{02} + \cdots + p_N^{(i)}x_{0N} = \mathbf{p}^{(i)\top} \mathbf{x}'_0, \quad (2.61)$$

where $\mathbf{p}^{(i)}$ is the $(N + 1)$ -dimensional vector of parameters of the function $y^{(i)}(\mathbf{x})$, and \mathbf{x}'_0 denotes the extended input vector:

$$\mathbf{x}'_0 = [1 \ x_0]^\top. \quad (2.62)$$

A collection of simple linear functions $y^{(i)}(\mathbf{x})$ allows for modeling the most complex input–output relationships. Overlapping areas of antecedents in neighboring rules ensure smooth switching between the local models.

An example of TSKFS inference with two inputs and two conditional fuzzy rules is shown in Fig. 2.4. The main advantage of the TSKFS is the low computational effort required to determine the numerical output of the system as the inference process does not involve defuzzification. However, it does not allow for the application of different interpretations of the fuzzy rules and different types of aggregation operators. This is due to the application of singletons in the rules consequents. The artificial neural network based fuzzy inference system (ANNBFIS) [17] is devoid of such disadvantages. The ANNBFIS combines the benefits of the usage of a fuzzy set in the rule consequent (as in the MAFS) together with the dependency of the consequent location on system inputs (as in the TSKFS) [4, 15, 16]. Another extension of the

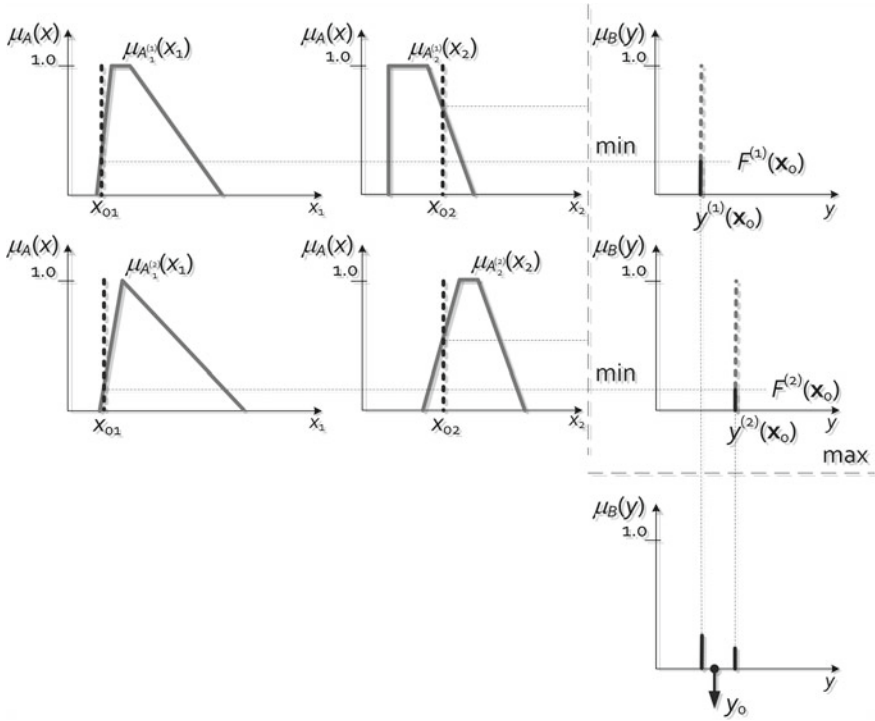


Fig. 2.4 An example of approximate reasoning with Takagi–Sugeno–Kang fuzzy system with two inputs and two fuzzy if-then rules

TSKFS is the Tsukamoto fuzzy system (TFS) [28]. The main difference between TSKFS and TFS is the method of determining the singleton location in the consequent of the fuzzy rule. In TFS it is defined using a monotonic function as well as a firing strength of the rule.

2.4.3 Tsukamoto Fuzzy System

The knowledge base of TFS is a collection of fuzzy conditional statements in the form:

$$R^{(i)} = \text{if } \bigwedge_{n=1}^N (x_{0n} \text{ is } L_{A_n}^{(i)}) , \text{ then } y = f_i^{-1} (F^{(i)}(\mathbf{x}_0)) , \quad (2.63)$$

where $f_i(y)$ is a monotonic function in the i th consequent.

For the firing strength equal to $F^{(i)}(\mathbf{x}_0)$ the consequent is a singleton with the amplitude $F^{(i)}(\mathbf{x}_0)$ and the location $y^{(i)}$ such that $F^{(i)}(\mathbf{x}_0) = f_i(y^{(i)})$:

$$\mu_{B^{(i)}}(y) = F^{(i)}(\mathbf{x}_0) \delta_{y, y^{(i)}} = \begin{cases} F^{(i)}(\mathbf{x}_0), & y = y^{(i)}, \\ 0, & y \neq y^{(i)}, \end{cases} \quad (2.64)$$

where $y^{(i)} = f_i^{-1}(F^{(i)}(\mathbf{x}_0))$.

The inference outcome of the TFS is calculated as a weighted average of singleton locations from all rules, with weights defined as the rules firing strengths:

$$y_0 = \frac{\sum_{i=1}^I F^{(i)}(\mathbf{x}_0) y^{(i)}}{\sum_{j=1}^I F^{(j)}(\mathbf{x}_0)} = \frac{\sum_{i=1}^I F^{(i)}(\mathbf{x}_0) f_i^{-1}(F^{(i)}(\mathbf{x}_0))}{\sum_{j=1}^I F^{(j)}(\mathbf{x}_0)}. \quad (2.65)$$

An example of the Tsukamoto approximate reasoning with two inputs and two fuzzy if-then rules is shown in Fig. 2.5.

The TFS is rarely used due to the difficulty in obtaining the conditional fuzzy rules from a human expert in the form (2.63). For the same reasons the Baldwin fuzzy system (BFS) [1, 2] is difficult to apply in practice. The BFS represents a

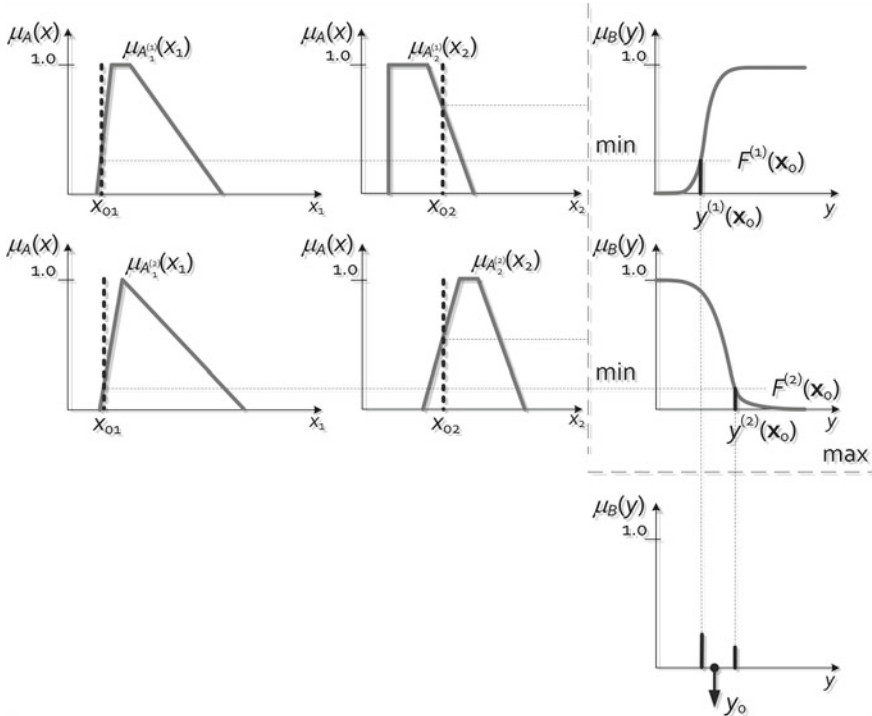


Fig. 2.5 Example of the Tsukamoto approximate reasoning with two inputs and two fuzzy if-then rules

different approach to fuzzy modeling, which is not based on Zadeh's compositional rule of inference but on reasoning using fuzzy truth value restrictions. The literature describes many other interesting proposals of fuzzy models, including those based on interval-valued fuzzy sets and type-2 fuzzy sets. A detailed overview can be found, for example, in [13, 18, 29, 30].

2.5 Summary

In this chapter we discussed basic problems related to the idea of fuzzy systems based on the Zadeh compositional rule of inference. The presentation started with explaining the concepts of the linguistic variable and fuzzy conditional statement. Next, different types of the fuzzy if-then rules and various methods of their mathematical representation were presented. Also, an overview of the compositional rule of inference proposed by Zadeh was introduced. General theoretical considerations on approximate reasoning were supplemented with examples of elementary fuzzy models. We described the basic solutions being the foundation of many modern constructions including fuzzy systems of Mamdani–Assilan, Takagi–Sugeno–Kang, and Tsukamoto.

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Theory and Applications of Ordered Fuzzy Numbers

A Tribute to Professor Witold Kosiński

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