Approximation by C^1 Splines on Piecewise Conic Domains

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Abstract We develop a Hermite interpolation scheme and prove error bounds for C^1 bivariate piecewise polynomial spaces of Argyris type vanishing on the boundary of curved domains enclosed by piecewise conics.

Keywords Curved elements · C^1 elements · Bivariate splines

1 Introduction

Spaces of piecewise polynomials defined on domains bounded by piecewise algebraic curves and vanishing on parts of the boundary can be used in the finite element method as an alternative to the classical mapped curved elements [10, 12]. Since implicit algebraic curves and surfaces provide a well-known modeling tool in CAGD [1], these methods are inherently isogeometric in the sense of [14]. Moreover, this approach does not suffer from the usual difficulties of building a globally C^1 or smoother space of functions on curved domains (see [4, Sect. 4.7]) shared by the classical curved finite elements and the B-spline-based isogeometric analysis.

In particular, a space of C^1 piecewise polynomials on domains enclosed by piecewise conic sections has been studied in [10] and applied to the numerical solution of fully nonlinear elliptic equations. These piecewise polynomials are quintic on the interior triangles of a triangulation of the domain and sextics on the boundary triangles (pie-shaped triangles with one side represented by a conic section as well as those triangles that share with them an interior edge with one endpoint on the boundary) and generalize the well-known Argyris finite element. Although local bases for

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these spaces have been constructed in [10] and numerical examples demonstrated the convergence orders expected from a piecewise quintic finite element, no error bounds have been provided.

In this paper, we study the approximation properties of the spaces introduced in [10]. We define a Hermite-type interpolation operator and prove an error bound that shows the convergence order $\mathscr{O}(h^6)$ of the residual in L_2-norm and order $\mathscr{O}(h^{6-k})$ in Sobolev spaces $H^k(\Omega)$. This extends the techniques used in [12] for C^0 splines to Hermite interpolation.

The paper is organized as follows. We introduce in Sect. 2 the spaces $\mathbb{S}^{1,2}_{d,0}(\triangle)$ of C^1 piecewise polynomials on domains bounded by a number of conic sections, with homogeneous boundary conditions, define in Sect. 3 our interpolation operator in the case $d = 5$, and investigate in Sect. 4 its approximation error for functions in Sobolev spaces $H^m(\Omega)$, $m = 5, 6$, vanishing on the boundary.

2 C^1 Piecewise Polynomials on Piecewise Conic Domains

We make the same assumptions on the domain and its triangulation as in [10, 12], as outlined below.

Let $\Omega \subset \mathbb{R}^2$ be a bounded curvilinear polygonal domain with $\Gamma = \partial\Omega = \bigcup_{j=1}^{n} \overline{\Gamma}_j$, where each Γ_j is an open arc of an algebraic curve of at most second order (i.e., either a straight line or a conic). For simplicity, we assume that Ω is simply connected, so that its boundary Γ is a closed curve without self-intersections. Let $Z = \{z_1, \ldots, z_n\}$ be the set of the endpoints of all arcs numbered counterclockwise such that z_j, z_{j+1} are the endpoints of Γ_j, $j = 1, \ldots, n$, with $z_{j+n} = z_j$. Furthermore, for each j, we denote by ω_j the internal angle between the tangents τ_j^+ and τ_j^- to Γ_j and Γ_{j-1}, respectively, at z_j. We assume that $\omega_j \in (0, 2\pi)$ for all j. Hence, Ω is a Lipschitz domain.

Let $\triangle$ be a *triangulation* of Ω, i.e., a subdivision of Ω into triangles, where each triangle $T \in \triangle$ has at most one edge replaced with a curved segment of the boundary $\partial\Omega$, and the intersection of any pair of the triangles is either a common vertex or a common (straight) edge if it is non-empty. The triangles with a curved edge are said to be *pie-shaped*. Any triangle $T \in \triangle$ that shares at least one edge with a pie-shaped triangle is called a *buffer* triangle, and the remaining triangles are *ordinary*. We denote by $\triangle_0$, $\triangle_B$, and $\triangle_P$ the sets of all ordinary, buffer, and pie-shaped triangles of $\triangle$, respectively, such that $\triangle = \triangle_0 \cup \triangle_B \cup \triangle_P$ is a disjoint union, see Fig. 1. Let $V, E, V_I, E_I, V_\partial, E_\partial$ denote the set of all vertices, all edges, interior vertices, interior edges, boundary vertices, and boundary edges, respectively.

For each $j = 1, \ldots, n$, let $q_j \in \mathbb{P}_2$ be a polynomial such that $\Gamma_j \subset \{x \in \mathbb{R}^2 : q_j(x) = 0\}$, where $\mathbb{P}_d$ denotes the space of all bivariate polynomials of total degree at most d. By changing the sign of q_j if needed, we ensure that $q_j(x)$ is positive for points in Ω near the boundary segment Γ_j. For simplicity, we assume in this paper that all boundary segments Γ_j are curved. Hence, each q_j is an irreducible quadratic polynomial and

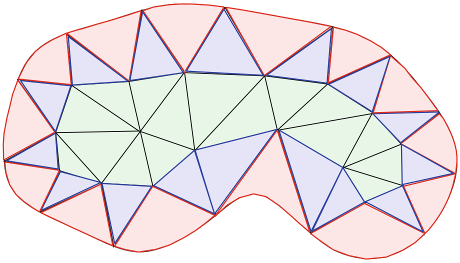

Fig. 1 A triangulation of a curved domain with ordinary triangles (*green*), pie-shaped triangles (*pink*), and buffer triangles (*blue*)

$$\nabla q_j(x) \neq 0 \quad \text{if} \quad x \in \Gamma_j. \tag{1}$$

We assume that $\triangle$ satisfies the following conditions:

(A) $Z = \{z_1, \dots, z_n\} \subset V_\partial$.
(B) No interior edge has both endpoints on the boundary.
(C) No pair of pie-shaped triangles shares an edge.
(D) Every $T \in \triangle_P$ is star-shaped with respect to its interior vertex v.
(E) For any $T \in \triangle_P$ with its curved side on Γ_j, $q_j(z) > 0$ for all $z \in T \setminus \Gamma_j$.
(F) No pair of buffer triangles shares an edge.

It can be easily seen that (B) and (C) are achievable by a slight modification of a given triangulation, while (D) and (E) hold for sufficiently fine triangulations. The assumption (F) is made for the sake of simplicity of the analysis. Note that the triangulation shown in Fig. 1 does not satisfy (F).

For any $T \in \triangle$, let h_T denote the diameter of T, and let ρ_T be the radius of the disk B_T inscribed in T if $T \in \triangle_0 \cup \triangle_B$ or in $T \cap T^*$ if $T \in \triangle_P$, where T^* denotes the triangle obtained by joining the boundary vertices of T by a straight line, see Fig. 2. Note that every triangle $T \in \triangle$ is star-shaped with respect to B_T. In particular, for $T \in \triangle_P$, this follows from Condition (D) and the fact that the conics do not possess inflection points.

We define the *shape regularity constant* of $\triangle$ by

$$R = \max_{T \in \triangle} \frac{h_T}{\rho_T}. \tag{2}$$

For any $d \geq 1$, we set

$$\begin{aligned}
\mathbb{S}^1_d(\triangle) &:= \{s \in C^1(\Omega) : s|_T \in \mathbb{P}_d,\ T \in \triangle_0, \ \text{and}\ s|_T \in \mathbb{P}_{d+1},\ T \in \triangle_P \cup \triangle_B\},\\
\mathbb{S}^{1,2}_{d,I}(\triangle) &:= \{s \in \mathbb{S}^1_d(\triangle) : s \text{ is twice differentiable at any } v \in V_I\},\\
\mathbb{S}^{1,2}_{d,0}(\triangle) &:= \{s \in \mathbb{S}^{1,2}_{d,I}(\triangle) : s|_\Gamma = 0\}.
\end{aligned}$$

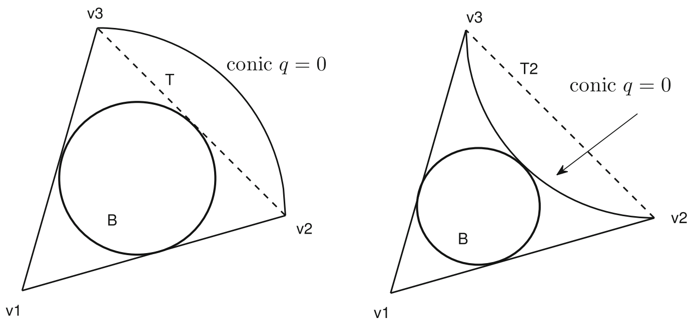

Fig. 2 A pie-shaped triangle with a curved edge and the associated triangle T^* with straight sides and vertices v_1, v_2, v_3. The curved edge can be either outside (*left*) or inside T^* (*right*)

We refer to [10] for the construction of a local basis for the space $\mathbb{S}^{1,2}_{5,0}(\triangle)$ and its applications in the finite element method.

Our goal is to obtain an error bound for the approximation of functions vanishing on the boundary by splines in $\mathbb{S}^{1,2}_{5,0}(\triangle)$. This is done through the construction of an interpolation operator of Hermite type. Note that a method of *stable splitting* was employed in [6–8] to estimate the approximation power of C^1 splines vanishing on the boundary of a polygonal domain. C^1 finite element spaces with a stable splitting are also required in Böhmer's proofs of the error bounds for his method of numerical solution of fully nonlinear elliptic equations [2]. A stable splitting of the space $\mathbb{S}^{1,2}_{5,I}(\triangle)$ will be obtained if a stable local basis for a stable complement of $\mathbb{S}^{1,2}_{5,0}(\triangle)$ in $\mathbb{S}^{1,2}_{5,I}(\triangle)$ is constructed, which we leave to a future work.

3 Interpolation Operator

We denote by $\partial^\alpha f$, $\alpha \in \mathbb{Z}^2_+$, the partial derivatives of f and consider the usual Sobolev spaces $H^m(\Omega)$ with the seminorm and norm defined by

$$|f|^2_{H^m(\Omega)} = \sum_{|\alpha|=m} \|\partial^\alpha f\|^2_{L^2(\Omega)}, \quad \|f\|^2_{H^m(\Omega)} = \sum_{k=0}^{m} |f|^2_{H^k(\Omega)} \quad (H^0(\Omega) = L^2(\Omega)),$$

where $|\alpha| := \alpha_1 + \alpha_2$. We set $H^1_0(\Omega) = \{f \in H^1(\Omega) : f|_{\partial\Omega} = 0\}$.

In this section, we construct an interpolation operator $I_\triangle : H^5(\Omega) \cap H^1_0(\Omega) \to \mathbb{S}^{1,2}_{5,0}(\triangle)$ and estimate its error for the functions in $H^m(\Omega) \cap H^1_0(\Omega)$, $m = 5, 6$, in the next section.

As in [12], we choose domains $\Omega_j \subset \Omega$, $j = 1, \ldots, n$, with Lipschitz boundary such that

(a) $\quad \partial\Omega_j \cap \partial\Omega = \Gamma_j$,

(b) $\partial\Omega_j \setminus \partial\Omega$ is composed of a finite number of straight line segments,
(c) $q_j(x) > 0$ for all $x \in \overline{\Omega}_j \setminus \Gamma_j$, and
(d) $\Omega_j \cap \Omega_k = \emptyset$ for all $j \neq k$.

In addition, we assume that the triangulation $\triangle$ is such that

(e) $\overline{\Omega}_j$ contains every triangle $T \in \triangle_P$ whose curved edge is part of Γ_j

and that q_j satisfy (without loss of generality)

(f) $\max_{x\in\overline{\Omega}_j} \|\nabla q_j(x)\|_2 \leq 1$ and $\|\nabla^2 q_j\|_2 \leq 1$, for all $j = 1, \ldots, n$,

where $\nabla^2 q_j$ denotes the (constant) Hessian matrix of q_j.

Note that (e) will hold with the same set $\{\Omega_j : j = 1, \ldots, n\}$ for any triangulations obtained by subdividing the triangles of $\triangle$.

The following lemma can be shown following the lines of the proof of [13, Theorem 6.1], see also [12, Theorem 3.1].

Lemma 1 *There is a constant K depending only on Ω, the choice of Ω_j, $j = 1, \ldots, n$, and $m \geq 1$, such that for all j and $u \in H^m(\Omega) \cap H^1_0(\Omega)$,*

$$|u/q_j|_{H^{m-1}(\Omega_j)} \leq K\|u\|_{H^m(\Omega_j)}. \tag{3}$$

Given a unit vector $\tau = (\tau_x, \tau_y)$ in the plane, we denote by D_τ the directional derivative operator in the direction of τ in the plane, so that

$$D_\tau f := \tau_x D_x f + \tau_y D_y f, \quad D_x f := \partial f/\partial x, \quad D_y f := \partial f/\partial y.$$

Given $f \in C^{\alpha+\beta}(\triangle)$, $\alpha, \beta \geq 0$, any number

$$\eta f = D^\alpha_{\tau_1} D^\beta_{\tau_2} (f|_T)(z),$$

where $T \in \triangle$, $z \in T$, and τ_1, τ_2 are some unit vectors in the plane, is said to be a *nodal value* of f, and the linear functional $\eta : C^{\alpha+\beta}(\triangle) \to \mathbb{R}$ is a *nodal functional*, with $d(\eta) := \alpha + \beta$ being the *degree* of η.

For some special choices of z, τ_1, τ_2, we use the following notation:

- If v is a vertex of $\triangle$ and e is an edge attached to v, we set

$$D^\alpha_e f(v) := D^\alpha_\tau (f|_T)(v), \quad \alpha \geq 1,$$

  where τ is the unit vector in the direction of e away from v, and $T \in \triangle$ is one of the triangles with edge e.
- If v is a vertex of $\triangle$ and e_1, e_2 are two consecutive edges attached to v, we set

$$D^\alpha_{e_1} D^\beta_{e_2} f(v) := D^\alpha_{\tau_1} D^\beta_{\tau_2} (f|_T)(v), \quad \alpha, \beta \geq 1,$$

where $T \in \triangle$ is the triangle with vertex v and edges e_1, e_2, and τ_i is the unit vector in the e_i direction away from v.

- For every edge e of the triangulation $\triangle$, we choose a unit vector $\tau^\perp$ (one of two possible) orthogonal to e and set

$$D^\alpha_{e^\perp} f(z) := D^\alpha_{\tau^\perp} f(z), \quad z \in e, \quad \alpha \geq 1,$$

provided $f \in C^\alpha(z)$.

On every edge e of $\triangle$, with vertices v' and v'', we define three points on e by

$$z_e^j := v' + \frac{j}{4}(v'' - v'), \quad j = 1, 2, 3.$$

For every triangle $T \in \triangle_0$ with vertices v_1, v_2, v_3 and edges e_1, e_2, e_3, we define $\mathcal{N}_T^0$ to be the set of nodal functionals corresponding to the nodal values

$$D_x^\alpha D_y^\beta f(v_i), \quad 0 \leq \alpha + \beta \leq 2, \quad i = 1, 2, 3,$$

$$D_{e_i^\perp} f(z_{e_i}^2), \quad i = 1, 2, 3,$$

see Fig. 3 (left), where the nodal functionals are depicted in the usual way by dots, segments, and circles as, for example, in [5].

Let $T \in \triangle_P$. We define $\mathcal{N}_T^P$ to be the set of nodal functionals corresponding to the nodal values

$$D_x^\alpha D_y^\beta f(v_1), \quad 0 \leq \alpha + \beta \leq 2,$$

$$D_x^\alpha D_y^\beta f(v_i), \quad 0 \leq \alpha + \beta \leq 1, \quad i = 2, 3,$$

$$D_x^\alpha D_y^\beta f(c_T), \quad 0 \leq \alpha + \beta \leq 1,$$

where v_1 is the interior vertex of T, v_2, v_3 are boundary vertices, and c_T is the center of the disk B_T, see Fig. 4.

Let $T \in \triangle_B$ with vertices v_1, v_2, v_3. We define $\mathcal{N}_T^{B,1}$ to be the set of nodal functionals corresponding to the nodal value

$$f(c_T), \quad c_T := (v_1 + v_2 + v_3)/3.$$

Also, we define $\mathcal{N}_T^{B,2}$ to be the set of nodal functionals corresponding to the nodal values

$$f(z_{e_i}^2), \quad i = 1, 2, 3,$$

$$D_x^\alpha D_y^\beta f(v_i), \quad 0 \leq \alpha + \beta \leq 2, \quad i = 1, 2, 3,$$

$$D_{e_i^\perp} f(z_{e_i}^j), \quad j = 1, 3, \quad i = 1, 2, 3,$$

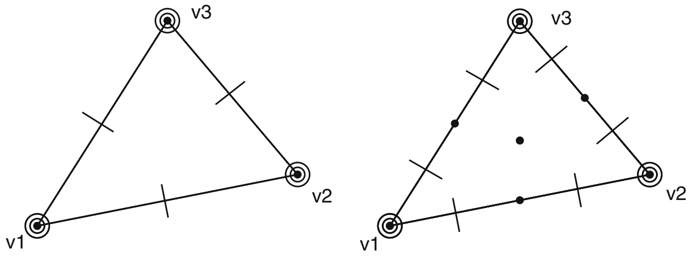

Fig. 3 Nodal functionals corresponding to $\mathcal{N}_T^0$ (*left*) and $\mathcal{N}_T^B$ (*right*)

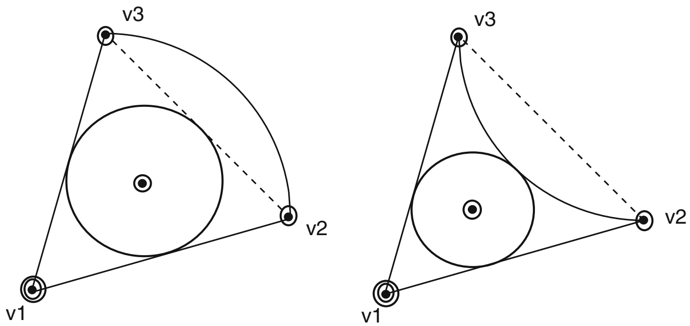

Fig. 4 Nodal functionals corresponding to $\mathcal{N}_T^P$

where v_1 is the boundary vertex, and v_2, v_3 are the interior vertices of T. We set

$$\mathcal{N}_T^B := \mathcal{N}_T^{B,1} \cup \mathcal{N}_T^{B,2},$$

see Fig. 3 (right).

We define an operator $I_\triangle : H^5(\Omega) \cap H_0^1(\Omega) \to \mathbb{S}_{5,0}^{1,2}(\triangle)$ of interpolatory type. Let $u \in H^5(\Omega) \cap H_0^1(\Omega)$. By Sobolev embedding, we assume without loss of generality that $u \in C^3(\overline{\Omega})$. For all $T \in \triangle_0 \cup \triangle_P$, we set $I_\triangle u|_T = I_T(u|_T)$, with the local operators I_T defined as follows.

If $T \in \triangle_0$, then $p := I_T u$ is the polynomial of degree 5 that satisfies the following interpolation conditions:

$$\eta p = \eta u, \quad \text{for all } \eta \in \mathcal{N}_T^0.$$

This is a well-known Argyris interpolation scheme, see, e.g., [15, Sect. 6.1], which guarantees the existence and uniqueness of the polynomial p.

Let $T \in \triangle_P$ with the curved edge on Γ_j. Then, $I_T u := pq_j$, where $p \in \mathbb{P}_4$ satisfies the following interpolation condition:

$$\eta p = \eta(u/q_j), \quad \text{for all } \eta \in \mathcal{N}_T^P. \tag{4}$$

The nodal functionals in $\mathcal{N}_T^P$ are well defined for u/q_j even though the vertices v_2, v_3 of T lie on the boundary Γ_j because $u/q_j \in H^4(\Omega_j)$ by Lemma 1, and hence, u/q_j may be identified with a function $\tilde{u} \in C^2(\overline{\Omega}_j)$ by Sobolev embedding. The interpolation scheme (4) defines a unique polynomial $p \in \mathbb{P}_4$, which will be justified in the proof of Lemma 3. In addition, we will need the following statement.

Lemma 2 *The polynomial p defined by* (4) *satisfies*

$$D_x^\alpha D_y^\beta (pq_j)(v) = D_x^\alpha D_y^\beta u(v), \quad 0 \le \alpha + \beta \le 2,$$

where v is any vertex of the pie-shaped triangle T.

Proof By (4), $p(v)q_j(v) = \tilde{u}(v)q_j(v) = u(v)$, where $\tilde{u} \in C^2(\overline{\Omega}_j)$ is the above function satisfying $u = \tilde{u}q_j$. Moreover,

$$\begin{aligned}\nabla(pq_j)(v) &= \nabla p(v)q_j(v) + p(v)\nabla q_j(v)\\ &= \nabla\tilde{u}(v)q_j(v) + \tilde{u}(v)\nabla q_j(v)\\ &= \nabla(\tilde{u}q_j)(v) = \nabla u(v).\end{aligned}$$

Similarly, if v is the interior vertex of T, then

$$\begin{aligned}\nabla^2(pq_j)(v) &= \nabla^2 p(v)q_j(v) + \nabla p(v)(\nabla q_j(v))^T + \nabla q_j(v)(\nabla p(v))^T + p(v)\nabla^2 q_j(v)\\ &= \nabla^2\tilde{u}(v)q_j(v) + \nabla\tilde{u}(v)(\nabla q_j(v))^T + \nabla q_j(v)(\nabla\tilde{u}(v))^T + \tilde{u}(v)\nabla^2 q_j(v)\\ &= \nabla^2 u(v).\end{aligned}$$

If v is one of the boundary vertices, then $q_j(v) = 0$, and hence,

$$\begin{aligned}\nabla^2(pq_j)(v) &= \nabla p(v)(\nabla q_j(v))^T + \nabla q_j(v)(\nabla p(v))^T + p(v)\nabla^2 q_j(v)\\ &= \nabla\tilde{u}(v)(\nabla q_j(v))^T + \nabla q_j(v)(\nabla\tilde{u}(v))^T + \tilde{u}(v)\nabla^2 q_j(v)\\ &= \nabla^2 u(v).\end{aligned}$$

□

It is easy to deduce from Lemma 2 that the interpolation conditions for p at the boundary vertices v_2, v_3 of T can be equivalently formulated as follows: For $i = 2, 3$,

$$\begin{gathered} p(v_i) = \frac{\partial u}{\partial n_i}(v_i) \Big/ \frac{\partial q_j}{\partial n_i}(v_i), \\ \frac{\partial p}{\partial n_i}(v_i) = \frac{1}{2}\frac{\partial^2 u}{\partial n_i^2}(v_i) \Big/ \frac{\partial q_j}{\partial n_i}(v_i), \qquad \frac{\partial p}{\partial \tau_i}(v_i) = \frac{\partial^2 u}{\partial n_i \partial \tau}(v_i) \Big/ \frac{\partial q_j}{\partial n_i}(v_i), \end{gathered} \tag{5}$$

where n_i and τ_i are the normal and the tangent unit vectors to the curve $q_j(x) = 0$ at v_i.

Finally, assume that $T \in \triangle_B$ with vertices v_1, v_2, v_3 where v_1 is a boundary vertex. Then, $I_T u = p \in \mathbb{P}_6$ satisfies the following interpolation conditions:

$$\eta p = \eta u, \quad \text{for all } \eta \in \mathcal{N}_T^{B,1},$$

and

$$\eta p = \eta I_{T_i} u, \quad \text{for all } \eta \in \mathcal{N}_i \subset \mathcal{N}_T^{B,2}, \quad i = 1, 2, 3,$$

where T_1 is a triangle in $\triangle_0$ sharing an edge $e_1 = \langle v_2, v_3 \rangle$ with T, and $\mathcal{N}_1$ corresponds to the nodal values

$$f(z_{e_1}^2), \quad D_{e_1^\perp} f(z_{e_1}^i), \quad i = 1, 3,$$

$$D_x^\alpha D_y^\beta f(v_i), \quad 0 \le \alpha + \beta \le 2, \quad i = 2, 3;$$

T_2 is a triangle in $\triangle_P$ sharing an edge $e_2 = \langle v_1, v_2 \rangle$ with T, and $\mathcal{N}_2$ corresponds to the nodal values

$$f(z_{e_2}^2), \quad D_{e_2^\perp} f(z_{e_2}^i), \quad i = 1, 3,$$

$$D_x^\alpha D_y^\beta f(v_1), \quad 0 \le \alpha + \beta \le 2;$$

and T_3 is a triangle in $\triangle_P$ sharing an edge $e_3 = \langle v_1, v_3 \rangle$ with T, and $\mathcal{N}_3$ corresponds to the nodal values

$$f(z_{e_3}^2), \quad D_{e_3^\perp} f(z_{e_3}^i), \quad i = 1, 3.$$

Since $\mathcal{N}_T^{B,2} = \mathcal{N}_1 \cup \mathcal{N}_2 \cup \mathcal{N}_3$ and $\mathcal{N}_T^B = \mathcal{N}_T^{B,1} \cup \mathcal{N}_T^{B,2}$ are a well posed interpolation scheme [16] for polynomials of degree 6, it follows that p is uniquely defined by the above conditions.

Theorem 1 *Let $u \in H^5(\Omega) \cap H_0^1(\Omega)$. Then $I_\triangle u \in \mathbb{S}_{5,0}^{1,2}(\triangle)$.*

Proof By the above construction, $I_\triangle u$ is a piecewise polynomial of degree 5 on all triangles in $\triangle_0$ and degree 6 on the triangles in $\triangle_P \cup \triangle_B$. Moreover, $I_\triangle u$ vanishes on the boundary of Ω.

To see that $I_\triangle u \in \mathbb{S}_{5,0}^{1,2}(\triangle)$ we thus need to show the C^1 continuity of $I_\triangle u$ across all interior edges of $\triangle$. If e is a common edge of two triangles $T', T'' \in \triangle_0$, then the C^1 continuity follows from the standard argument for C^1 Argyris finite element, see [4, Chap. 3] and [15, Sect. 6.1].

Next, we will show the C^1 continuity of $I_\triangle u$ across edges shared by buffer triangles with either ordinary or pie-shaped triangles. Let $T \in \triangle_B$ and $T' \in \triangle_0 \cup \triangle_P$ with common edge $e' = \langle v', v'' \rangle$, and let $p = I_T u$ and $s = I_{T'} u$. Consider the univariate polynomials $p|_{e'}$ and $s|_{e'}$, and let $q = p|_{e'} - s|_{e'}$. Assuming that the edge e' is parameterized by $t \in [0, 1]$, then q is a univariate polynomial of degree 6 with respect to the parameterization $v' + t(v'' - v')$, $t \in [0, 1]$. Similarly, we consider the orthogonal/normal derivatives $D_{e'^\perp} p|_{e'}$ and $D_{e'^\perp} s|_{e'}$, and let $r = D_{e'^\perp} p|_{e'} - D_{e'^\perp} s|_{e'}$; then, r is a univariate polynomial of degree 5 with respect to the same parameter t. The C^1 continuity will follow if we show that both q and r are zero functions.

If $T' = T_1 \in \triangle_0$, then using the interpolation conditions corresponding to $\mathscr{N}_1 \subset \mathscr{N}_T^{B,2}$, we have

$$q(0) = q'(0) = q''(0) = q(1/2) = q(1) = q'(1) = q''(1) = 0,$$
$$r(0) = r'(0) = r(1/4) = r(3/4) = r(1) = r'(1) = 0,$$

which implies $q \equiv 0$ and $r \equiv 0$.

If $T' = T_2 \in \triangle_P$, then the interpolation conditions corresponding to $\mathscr{N}_2 \subset \mathscr{N}_T^{B,2}$ imply

$$q(0) = q'(0) = q''(0) = q(1/2) = 0,$$
$$r(0) = r'(0) = r(1/4) = r(3/4) = 0.$$

In view of Lemma 2, we have

$$D_x^\alpha D_y^\beta s(v_2) = D_x^\alpha D_y^\beta u(v_2) = D_x^\alpha D_y^\beta p(v_2), \quad 0 \le \alpha + \beta \le 2,$$

which implies

$$q(1) = q'(1) = q''(1) = 0, \quad r(1) = r'(1) = 0,$$

and hence, $q \equiv 0$ and $r \equiv 0$.

If $T' = T_3 \in \triangle_P$, then the interpolation conditions corresponding to $\mathscr{N}_3 \subset \mathscr{N}_T^{B,2}$ imply

$$q(1/2) = 0, \quad r(1/4) = r(3/4) = 0,$$

whereas Lemma 2 gives

$$q(0) = q'(0) = q''(0) = 0, \quad r(0) = r'(0) = 0,$$
$$q(1) = q'(1) = q''(1) = 0, \quad r(1) = r'(1) = 0,$$

which completes the proof. □

It follows from Lemma 2 that $I_\triangle u$ is twice differentiable at the boundary vertices, and thus,

$$I_\triangle u \in \{s \in \mathbb{S}_5^1(\triangle) \;:\; s \text{ is twice differentiable at any vertex and } s|_\Gamma = 0\}.$$

Moreover, $I_\triangle u$ satisfies the following interpolation conditions:

$$D_x^\alpha D_y^\beta I_\triangle u(v) = D_x^\alpha D_y^\beta u(v), \quad 0 \le \alpha + \beta \le 2, \quad \text{for all } v \in V,$$

$$D_{e^\perp} I_\triangle u(z_e^2) = D_{e^\perp} u(z_e^2), \quad \text{for all edges } e \text{ of } \triangle_0,$$

$$D_x^\alpha D_y^\beta I_\triangle u(c_T) = D_x^\alpha D_y^\beta u(c_T), \quad 0 \le \alpha+\beta \le 1, \quad \text{for all } T \in \triangle_P,$$

$$I_\triangle u(c_T) = u(c_T), \quad \text{for all} T \in \triangle_,$$

where c_T denotes the center of the disk B_T inscribed into T^* if T is a pie-shaped triangle and the barycenter of T if T is a buffer triangle. In view of (5), $I_\triangle u \in \mathbb{S}_{5,0}^{1,2}(\triangle)$ is uniquely defined by these conditions for any $u \in C^2(\overline{\Omega})$.

4 Error Bounds

In this section, we estimate the error $\|u - I_\triangle u\|_{H^k(\Omega)}$ for functions $u \in H^m(\Omega) \cap H_0^1(\Omega)$, $m = 5, 6$. Similar to [12, Sect. 3], we follow the standard finite element techniques involving the Bramble–Hilbert Lemma (see [4, Chap. 4]) on the ordinary triangles and make use of the estimate (3) on the pie-shaped triangles. Since the spline $I_\triangle u$ on the buffer triangles is constructed in part by interpolation and in part by the smoothness conditions, the estimate of the error on such triangles relies in particular on the estimates of the interpolation error on the neighboring ordinary and buffer triangles.

Lemma 3 *If $p \in \mathbb{P}_4$ and $T \in \triangle_P$, then*

$$\|p|_{T^*}\|_{L^\infty(T^*)} \le \max_{\eta \in \mathcal{N}_T^P} h_{T^*}^{d(\eta)} |\eta p|, \tag{6}$$

where T^* is the triangle obtained by replacing the curved edge of T by the straight line segment, and h_{T^*} is the diameter of T^*. Similarly, if $p \in \mathbb{P}_6$ and $T \in \triangle_B$, then

$$\|p|_T\|_{L^\infty(T)} \le \max_{\eta \in \mathcal{N}_T^B} h_T^{d(\eta)} |\eta p|, \tag{7}$$

where h_T is the diameter of T.

Proof To show the estimate (6) for T^*, we follow the proof of [11, Lemma 3.9]. We note that we only need to show that the interpolation scheme for pie-shaped triangles is a valid scheme, that is, we need to show that $\mathcal{N}_T^P$ is $\mathbb{P}_4$-unisolvent, and the rest of the proof can be done similar to that of [11, Lemma 3.9]. Recall that a set of functionals $\mathcal{N}$ are said to be $\mathbb{P}_d$-*unisolvent* if the only polynomial $p \in \mathbb{P}_d$ satisfying $\eta p = 0$ for $\eta \in \mathcal{N}$ is the zero function.

Let $T^* = \langle v_1, v_2, v_3 \rangle$, where v_1 is the interior vertex. Set $e_1 := \langle v_1, v_2 \rangle$, $e_2 := \langle v_2, v_3 \rangle$, $e_3 := \langle v_3, v_1 \rangle$, see Fig. 4. The interpolation conditions along e_1, e_3 imply that s vanishes on these edges. After splitting out the linear polynomial factors which vanish along the edges e_1, e_3, we obtain a valid interpolation scheme for quadratic polynomials with function values at the three vertices, and function and gradient values at the the barycenter c of $B_T \subset T^*$. The validity of this scheme can be seen by looking at a straight line ℓ through c and any one of the vertices of T^*. Along the line ℓ, a function value is given at the vertex and a function value together with the first derivative is given at the point c, and this set of data are unisolvent for the univariate quadratic polynomials, which means s must vanish along ℓ. After factoring out the respective linear polynomial, we are left with function values at three non-collinear points, which defines a valid interpolation scheme for the remaining linear polynomial factor of s.

To show the estimate (7) for $T \in \triangle_B$, the proof is similar. We need to show the set of functionals $\mathscr{N}_T^B$ is $\mathbb{P}_6$-unisolvent but this follows from the standard scheme of [16] for polynomials of degree six.

We note that the argument of the proof of [11, Lemma 3.9] applies to affine invariant interpolation schemes, that is the schemes that use the edge derivatives. As our scheme relies on the standard derivatives in the direction of the x, y axes, we need to express the edge derivatives as linear combinations of the x, y derivatives as follows. Assume that e_1, e_2 are two edges that emanate from a vertex v. Let $\tau_i = (\tau_{i1}, \tau_{i2})$ be the unit vector in the direction of e_i away from v, $i = 1, 2$. Then, we can easily obtain the following identities

$$D_{e_i} f(v) = \tau_{i1} D_x f(v) + \tau_{i2} D_y f(v),$$

$$D_{e_i}^2 f(v) = \tau_{i1}^2 D_x^2 f(v) + 2\tau_{i1}\tau_{i2} D_x D_y f(v) + \tau_{i2}^2 D_y^2 f(v),$$

$$D_{e_1} D_{e_2} f(v) = \tau_{11}\tau_{21} D_x^2 f(v) + (\tau_{11}\tau_{22} + \tau_{12}\tau_{21}) D_x D_y f(v) + \tau_{12}\tau_{22} D_y^2 f(v).$$

□

Lemma 4 *Let $T \in \triangle_P$ and its curved edge $e \subset \Gamma_j$. Then*

$$\|I_T u\|_{L^\infty(T)} \le C_1 \max_{0 \le \ell \le 2} h_T^{\ell+1} |u/q_j|_{W_\infty^\ell(T)} \quad \textit{if } u \in H^5(\Omega) \cap H_0^1(\Omega), \tag{8}$$

where C_1 depends only on h_T/ρ_T. Moreover, if $5 \le m \le 6$, then for any $u \in H^m(\Omega) \cap H_0^1(\Omega)$,

$$\|u - I_T u\|_{H^k(T)} \le C_2 h_T^{m-k} |u/q_j|_{H^{m-1}(T)}, \quad k = 0, \dots, m-1, \tag{9}$$

$$|u - I_T u|_{W_\infty^k(T)} \le C_3 h_T^{m-k-1} |u/q_j|_{H^{m-1}(T)}, \quad k = 0, \dots, m-2, \tag{10}$$

where C_2, C_3 depend only on h_T/ρ_T.

Proof We will denote by $\tilde{C}$ constants which may depend only on h_T/ρ_T and on Ω. Assume that $u \in H^5(\Omega) \cap H_0^1(\Omega)$ and recall that by definition $I_T u = pq_j$, where $p \in \mathbb{P}_4$ satisfies the interpolation conditions (4). Since $u \in H^5(\Omega_j) \cap H_0^1(\Omega_j)$, it follows that $u/q_j \in H^4(\Omega_j)$ by Lemma 1, and hence, $u/q_j \in C^2(\overline{\Omega}_j)$ by Sobolev embedding. From Lemma 3, we have

$$\|p\|_{L^\infty(T^*)} \leq \max_{\eta\in\mathcal{N}_T^P} h_{T^*}^{d(\eta)}|\eta p|, \tag{11}$$

and hence

$$\|p\|_{L^\infty(T^*)} \leq \max_{\eta\in\mathcal{N}_T^P} h_{T^*}^{d(\eta)}|\eta(u/q_j)| \leq \tilde{C} \max_{0\leq\ell\leq 2} h_T^\ell |u/q_j|_{W_\infty^\ell(T)}.$$

As in the proof of [12, Theorem 3.2], we can show that for any polynomial of degree at most 6,

$$\|s\|_{L^\infty(T)} \leq \tilde{C}\|s\|_{L^\infty(T^*)} \quad \text{and} \quad \|s\|_{L^\infty(T^*)} \leq \tilde{C}\|s\|_{L^\infty(T)}. \tag{12}$$

By using (f), it is easy to show that $\|q_j\|_{L^\infty(T)} \leq h_T$, and hence,

$$\|I_T u\|_{L^\infty(T)} = \|pq_j\|_{L^\infty(T)} \leq h_T\|p\|_{L^\infty(T)},$$

which completes the proof of (8).

Moreover, since the area of T is less than or equal to $\frac{\pi}{4}h_T^2$ and $\partial^\alpha(I_T u) \in \mathbb{P}_{6-k}$ if $|\alpha| = k$, it follows that

$$\|\partial^\alpha(I_T u)\|_{L^2(T)} \leq \frac{\sqrt{\pi}}{2} h_T\|\partial^\alpha(I_T u)\|_{L^\infty(T)} \leq \tilde{C}h_T\|\partial^\alpha(I_T u)\|_{L^\infty(T^*)}.$$

By Markov inequality (see, e.g., [15, Theorem 1.2]), we get furthermore

$$\|\partial^\alpha(I_T u)\|_{L^\infty(T^*)} \leq \tilde{C}\rho_T^{-k}\|I_T u\|_{L^\infty(T^*)},$$

and hence in view of (12)

$$|I_T u|_{H^k(T)} \leq \tilde{C}h_T^{1-k}\|I_T u\|_{L^\infty(T)}.$$

In view of (8), we arrive at

$$|I_T u|_{H^k(T)} \leq \tilde{C} \max_{0\leq\ell\leq 2} h_T^{\ell+2-k}|u/q_j|_{W_\infty^\ell(T)}, \quad \text{if } u \in H^5(\Omega)\cap H_0^1(\Omega). \tag{13}$$

Let $m \in \{5, 6\}$, and let $u \in H^m(\Omega)\cap H_0^1(\Omega)$. It follows from Lemma 1 that $u/q_j \in H^{m-1}(T)$. By the results in [4, Chap. 4], there exists a polynomial $\tilde{p} \in \mathbb{P}_{m-2}$ such that

$$\begin{aligned} \|u/q_j - \tilde{p}\|_{H^k(T)} &\leq \tilde{C}h_T^{m-k-1}|u/q_j|_{H^{m-1}(T)}, \quad k = 0, \ldots, m-1, \\ |u/q_j - \tilde{p}|_{W_\infty^k(T)} &\leq \tilde{C}h_T^{m-k-2}|u/q_j|_{H^{m-1}(T)}, \quad k = 0, \ldots, m-2. \end{aligned} \tag{14}$$

Indeed, a suitable $\tilde{p}$ is given by the *averaged Taylor polynomial* [4, Definition 4.1.3] with respect to the disk B_T, and the inequalities in (14) follow from [4, Lemma 4.3.8]

(Bramble–Hilbert Lemma) and an obvious generalization of [4, Proposition 4.3.2], respectively. It is easy to check by inspecting the proofs in [4] that the quotient h_T/ρ_T can be used in the estimates instead of the chunkiness parameter used there.

Since

$$u - I_T u = (u/q_j - \tilde{p})q_j - I_T(u - \tilde{p}q_j),$$

we have for any norm $\|\cdot\|$,

$$\|u - I_T u\| \le \|(u/q_j - \tilde{p})q_j\| + \|I_T(u - \tilde{p}q_j)\|.$$

In view of (f) and (14), for any $k = 0, \dots, m-2$,

$$\begin{aligned} |(u/q_j - \tilde{p})q_j|_{W^k_\infty(T)} &\le h_T |u/q_j - \tilde{p}|_{W^k_\infty(T)} + \|u/q_j - \tilde{p}\|_{W^{k-1}_\infty(T)} \\ &\le \tilde{C} h_T^{m-k-1} |u/q_j|_{H^{m-1}(T)}, \end{aligned}$$

and for any $k = 0, \dots, m-1$,

$$\begin{aligned} \|(u/q_j - \tilde{p})q_j\|_{H^k(T)} &\le \tilde{C} h_T \|u/q_j - \tilde{p}\|_{H^k(T)} + \tilde{C}\|u/q_j - \tilde{p}\|_{H^{k-1}(T)} \\ &\le \tilde{C} h_T^{m-k} |u/q_j|_{H^{m-1}(T)}. \end{aligned}$$

Furthermore, by the Markov inequality, (8), (13), and (14),

$$|I_T(u - \tilde{p}q_j)|_{W^k_\infty(T)} \le \tilde{C} \max_{0\le \ell \le 2} h_T^{\ell+1-k} |u/q_j - \tilde{p}|_{W^\ell_\infty(T)} \le \tilde{C} h_T^{m-k-1} |u/q_j|_{H^{m-1}(T)},$$

$$\|I_T(u - \tilde{p}q_j)\|_{H^k(T)} \le \tilde{C} \max_{0\le \ell \le 2} h_T^{\ell+2-k} |u/q_j - \tilde{p}|_{W^\ell_\infty(T)} \le \tilde{C} h_T^{m-k} |u/q_j|_{H^{m-1}(T)}.$$

By combining the inequalities in the five last displays, we deduce (9) and (10). □

We are ready to formulate and prove our main result.

Theorem 2 *Let $5 \le m \le 6$. For any $u \in H^m(\Omega) \cap H^1_0(\Omega)$,*

$$\Big(\sum_{T\in\triangle} \|u - I_\triangle u\|^2_{H^k(T)}\Big)^{1/2} \le C h^{m-k} \|u\|_{H^m(\Omega)}, \quad k = 0, \dots, m-1, \tag{15}$$

where h is the maximum diameter of the triangles in $\triangle$, and C is a constant depending only on Ω, the choice of Ω_j, and the shape regularity constant R of $\triangle$.

Proof We estimate the norms of $u - I_T u$ on all triangles $T \in \triangle$. The letter C stands below for various constants depending only on the parameters mentioned in the formulation of the theorem.

If $T \in \triangle_0$, then $s|_T$ is a macroelement as defined in [15, Chap. 6]. Furthermore, by [15, Theorem 6.3], the set of linear functionals $\mathcal{N}_T^0$ give rise to a stable local nodal basis, which is in particular uniformly bounded. Hence, by [9, Theorem 2], we obtain a Jackson estimate in the form

$$\|u - I_T u\|_{H^k(T)} \le C h_T^{m-k} |u|_{H^m(T)}, \quad k = 0, \ldots, m, \tag{16}$$

where C depends only on h_T/ρ_T. If $T \in \triangle_P$, with the curved edge $e \subset \Gamma_j$, then the Jackson estimate (9) holds by Lemma 4.

Let $T \in \triangle_B$, $p := I_\triangle u|_T$, and let $\tilde{p} \in \mathbb{P}_6$ be the interpolation polynomial that satisfies $\eta \tilde{p} = \eta u$ for all $\eta \in \mathcal{N}_T^B$. Then

$$\eta(\tilde{p} - p) = \begin{cases} 0 & \text{if } \eta \in \mathcal{N}_T^{B,1}, \\ \eta(u - I_{T'} u) & \text{if } \eta \in \mathcal{N}_T^{B,2}, \end{cases}$$

where $T' = T'_\eta \in \triangle_0 \cup \triangle_P$. Hence, by Markov inequality and (7) of Lemma 3, we conclude that for $k = 0, \ldots, m$,

$$\|\tilde{p} - p\|_{H^k(T)} \le C h_T^{1-k} \|\tilde{p} - p\|_{L^\infty(T)},$$

with

$$\|\tilde{p} - p\|_{L^\infty(T)} \le C \max\{h_T^\ell |u - I_{T'} u|_{W_\infty^\ell(T')} : 0 \le \ell \le 2,\ T' \in \triangle_0 \cup \triangle_P,\ T' \cap T \ne \emptyset\},$$

whereas by the same arguments leading to (16) we have

$$\|u - \tilde{p}\|_{H^k(T)} \le C h_T^{m-k} |u|_{H^m(T)},$$

with the constants depending only on h_T/ρ_T. If $T' \in \triangle_0 \cup \triangle_P$, then by (10) and the analogous estimate for $T' \in \triangle_0$, comparing [4, Corollary 4.4.7], we have for $\ell = 0, 1, 2$,

$$|u - I_{T'} u|_{W_\infty^\ell(T')} \le C h_{T'}^{m-\ell-1} \begin{cases} |u|_{H^m(T')} & \text{if } T' \in \triangle_0, \\ |u/q_j|_{H^{m-1}(T')} & \text{if } T' \in \triangle_P, \end{cases}$$

where C depends only on $h_{T'}/\rho_{T'}$. By combining these inequalities, we obtain an estimate of $\|u - I_T u\|_{H^k(T)}$ by $C\tilde{h}^{m-k}$ times the maximum of $|u|_{H^m(T)}$, $|u|_{H^m(T')}$ for $T' \in \triangle_0$ sharing edges with T, and $|u/q_j|_{H^{m-1}(T')}$ for $T' \in \triangle_P$ sharing edges with T. Here, C depends only on the maximum of h_T/ρ_T and $h_{T'}/\rho_{T'}$, and $\tilde{h}$ is the maximum of h_T and all $h_{T'}$ for $T' \in \triangle_0 \cup \triangle_P$ sharing edges with T.

By using (16) on $T \in \triangle_0$, (9) on $T \in \triangle_P$ and the estimate of the last paragraph on $T \in \triangle_B$, we get

$$\sum_{T \in \triangle} \|u - I_\triangle u\|^2_{H^k(T)} \le C h^{2(m-k)} \Big(\sum_{T \in \triangle_0 \cup \triangle_B} |u|^2_{H^m(T)} + \sum_{T \in \triangle_P} |u/q_{j(T)}|^2_{H^{m-1}(T)} \Big),$$

where $j(T)$ is the index of Γ_j containing the curved edge of $T \in \triangle_P$. Clearly,

$$\sum_{T\in\triangle_0\cup\triangle_B} |u|^2_{H^m(T)} \le |u|^2_{H^m(\Omega)} \le \|u\|^2_{H^m(\Omega)},$$

whereas by Lemma 1,

$$\sum_{T\in\triangle_P} |u/q_{j(T)}|^2_{H^{m-1}(T)} \le \sum_{j=1}^{n} |u/q_j|^2_{H^{m-1}(\Omega_j)} \le K\|u\|^2_{H^m(\Omega)},$$

where K is the constant of (3) depending only on Ω and the choice of Ω_j. □

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References

1. J. Bloomenthal et al., *Introduction to Implicit Surfaces* (Morgan-Kaufmann Publishers Inc., San Francisco, 1997)
2. K. Böhmer, On finite element methods for fully nonlinear elliptic equations of second order. SIAM J. Numer. Anal. **46**, 1212–1249 (2008)
3. K. Böhmer, *Numerical Methods for Nonlinear Elliptic Differential Equations: A Synopsis* (Oxford University Press, Oxford, 2010)
4. S.C. Brenner, L.R. Scott, *The Mathematical Theory of Finite Element Methods* (Springer, New York, 1994)
5. P.G. Ciarlet, *The Finite Element Method for Elliptic Problems* (North-Holland, Amsterdam, 1978)
6. O. Davydov, Smooth finite elements and stable splitting, Berichte “Reihe Mathematik” der Philipps-Universität Marburg, 2007-4 (2007). An adapted version has appeared as [3, Sect. 4.2.6]
7. O. Davydov, A. Saeed, Stable splitting of bivariate spline spaces by Bernstein-Bézier methods, in *Curves and Surfaces - 7th International Conference, Avignon, France, June 24–30, 2010*, eds. by J.-D. Boissonnat et al. LNCS, vol. 6920 (Springer, Berlin, 2012), pp. 220–235
8. O. Davydov, A. Saeed, Numerical solution of fully nonlinear elliptic equations by Böhmer’s method. J. Comput. Appl. Math. **254**, 43–54 (2013)
9. O. Davydov, W.P. Yeo, Macro-element hierarchical Riesz bases, in *Mathematical Methods for Curves and Surfaces: 8th International Conference, Oslo, 2012*, eds. by M. Floater et al. LNCS, vol. 8177 (Springer, Berlin, 2014), pp. 112–134
10. O. Davydov, A. Saeed, C^1 quintic splines on domains enclosed by piecewise conics and numerical solution of fully nonlinear elliptic equations. Appl. Numer. Math. **116**, 172–183 (2017)
11. O. Davydov, G. Nürnberger, F. Zeilfelder, Bivariate spline interpolation with optimal approximation order. Constr. Approx. **17**, 181–208 (2001)
12. O. Davydov, G. Kostin, A. Saeed, Polynomial finite element method for domains enclosed by piecewise conics. CAGD **45**, 48–72 (2016)
13. K. Höllig, U. Reif, J. Wipper, Weighted extended B-spline approximation of Dirichlet problem. SIAM J. Numer. Anal. **39**(2), 442–462 (2001)

14. T.J.R. Hughes, J.A. Cottrel, Y. Bazilevs, Isogeometric analysis: CAD, finite elements, NURBS, exact geometry and mesh refinement. Comput. Methods Appl. Mech. Eng. **194**, 4135–4195 (2005)
15. M.J. Lai, L.L. Schumaker, *Spline Functions on Triangulations* (Cambridge University Press, Cambridge, 2007)
16. L.L. Schumaker, On super splines and finite elements. SIAM J. Numer. Anal. **26**(4), 997–1005 (1989)

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