Riemann on Geometry, Physics, and Philosophy—Some Remarks

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Abstract Riemann's paper 'On the hypotheses that lie at the foundations of geometry' is one of the fundamental papers in the creation of modern geometry. We analyse its content, look at the influence the work of Gauss and Herbart exercised on Riemann, and discuss other of Riemann's papers that shed light on his ideas, in particular on his appreciation of the concept of curvature.

1 Introduction

Riemann's paper 'On the hypotheses that lie at the foundations of geometry' (henceforth, *Hypotheses*) [24] is generally regarded as one of the most important papers ever written in mathematics. As such, it was read by generations of mathematicians, most notably in the Göttingen tradition that reached from him to Hermann Weyl, and its ideas continue to influence mathematics today. Without it, Einstein's general theory of relativity would have been unthinkable.

Unsurprisingly, therefore, it has been worked over by historians of mathematics, historically-minded mathematicians, and philosophers of mathematics looking for its key ideas and a historical and intellectual context into which to put them. The results are intriguingly meagre. The *Hypotheses* is not the last step in a complicated chain of arguments involving Riemann with numerous predecessors, nor is it the response to a perceived crisis. Rather, it is, as it is presented, the next step after the work of Gauss [12] and, partly, as a response to shifting philosophical ideas about the nature of geometry that may have also caught Riemann's attention because of their implications for physics.

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This paper will first look at the *Hypotheses* in detail, and then consider its relation to Gauss's ideas about differential geometry. Then it will turn to the connections to physics and philosophy, and conclude by looking at the other relevant paper Riemann wrote, the *Commentatio* [27] and the the discussions that it provoked.

2 The *Hypotheses*

Riemann's essay [24] was published posthumously in 1867, and is based on the lecture he gave in 1854 as part of the process of obtaining his Habilitation, the necessary and sufficient qualification for obtaining a teaching position at a German university. As such it was given to the Philosophy Faculty at Göttingen, of which Mathematics was a Department, with Gauss as one of the examiners. These circumstances explain the unfortunate absence of formulae that would otherwise have assisted subsequent readers.

Riemann began by remarking that geometry "takes for granted the notion of space" as well as the first principles of constructions in space. The basic concepts have only nominal definitions and the crucial properties are determined from axioms, but this leaves the relationships between the axioms obscure; in fact, said Riemann, it is not even clear a priori if the relationships are possible.

This opening paragraph makes two points. First, that it is not clear what the axioms or postulates of Euclidean geometry are about; second, that it is not even clear that they are consistent.

But Riemann did not take his analysis in the direction of a more refined study of the axioms. It is likely that he saw the work of Legendre, whose name he mentions, as indicative of the poverty of such work, not only because in all the editions of his *Géométrie* [16] Legendre had failed to prove the parallel postulate, but because Riemann thought that the whole axiomatic attempt to give a geometrical account of physical space was misguided. In unpublished notes from the early 1850 s he called such enquiries "extremely unfruitful" (see Scholz ([30] p. 218)), which, as Scholz points out, makes it very unlikely that Riemann had seen any of Lobachevskii's work.

Instead, Riemann began his paper [24] by remarking that the general notion of quantity was multi-dimensional, and "it emerges from this study that a multi-dimensional object is capable of being measured in different ways and that space is only a particular example of a three-dimensional quantity." Moreover, the properties that characterise space among all three-dimensional quantities can only be determined experimentally. From this perspective the axioms of Euclid are only hypotheses, although highly probable outside "the realms of both the immeasurably great and the immeasurably small."

This opening page is one of the first places where a characteristically modern mathematical approach is taken to what many had seen as a straight-forward question. Riemann did not say that Euclidean geometry needs fixing and offer a proposal. He stepped back and asked himself: what are we studying when we study geometry? His answer was quantity, and for him that was a multi-dimensional object—the sort of thing that is described, as we shall see, with coordinates. In Riemann's opinion a geometer, even a practical one, should not concentrate on the passage from space to a mathematical theory of space (still less take one for granted) but first build up a theory of multi-dimensional quantities, and this he turned to do.

He began Part I of the paper by regretting that apart from some ideas of Gauss [14] published in his second memoir on biquadratic residues and some philosophical remarks of Herbart [15] there was little to guide him. He indicated that there are continuous and discrete manifolds depending on how the elements are determined, and here a manifold is a vague term, little more than a collection of elements. Discrete manifolds in this sense are frequently encountered, said Riemann, but continuous ones less so, and he gave examples of the location of material objects and their colours. To make sense of continuous manifolds we rely on measurement, which presupposes a measuring unit that can be freely transported. When this is not available we have only the general concept of a manifold, and this difficulty may be why the work of Lagrange, Pfaff, and Jacobi on many-valued analytic functions has been so unfruitful so far.

Riemann had already rewritten the theory of functions of a complex variable in his doctoral theses [23], published in 1851. It is likely that his ideas about surfaces were what he was alluding to here, although the great work on Abelian functions would not be published until 1857.

How then to determine position in a manifold? Riemann explained that if the manifold has dimension one then position is determined by moving forwards and backwards using some unspecified concept of length. If this one-dimensional manifold is itself then moved forwards and backwards in a different dimension then a two-dimensional manifold is obtained, and so on. The converse also holds: one can break an n-dimensional manifold down into smaller ones along which some function has a constant value, and exceptional cases aside these sets where the function takes a fixed value are $n-1$-dimensional submanifolds.

In Part II of his paper [24] Riemann explained how to introduce metrical relations in a manifold on the assumption that lines have a length independent of their position and every line can be measured by every other line. Here he was happy to acknowledge the work of Gauss [12] on curved surfaces (which we shall look at below).

Riemann now supposed that the position of a point in perhaps some region of an n-manifold is determined by its n coordinates $(x_1, x_2, \ldots, x_n)$. He restricted his attention to continuous systems in which the coordinates can vary by amounts dx and sought an expression for the line element ds in terms of the $dx_1, dx_2, \ldots, dx_n$. He further assumed that the length of a line element is unaltered if all its points undergo the same infinitesimal displacement. If moreover distance increases as points move away from the origin and the first and second derivatives are finite then the first derivative must vanish and the second cannot be negative, so Riemann took it to be positive. He deduced that the line element ds could be "the square root of an everywhere positive quadratic form in the variables dx", as for example we take to be the case for space when we write

$$ds = \sqrt{\sum (dx)^2}.$$

Riemann noted that there are other possibilities. For example, ds could be the fourth root of a fourth power expression, but he did not see many possibilities for geometry there and he set this aside.

The quadratic form, however, did interest him. It contains $n(n+1)/2$ coefficients, of which n can be altered by a change of variables, so it depends essentially on $n(n-1)/2$ coefficients that are determined by the manifold. The example of $\sqrt{\sum (dx)^2}$ is therefore special, and Riemann proposed to call such manifolds flat.

To proceed further, Riemann considered the infinitesimal triangle with one vertex at the origin, one on a geodesic out of the origin to the point $(x_1, x_2, \ldots, x_n)$, and one on a geodesic out of the origin to the point $(dx_1, dx_2, \ldots, dx_n)$. The quotient of $\sqrt{\sum (dx)^2}$ by the area of this triangle measures the departure of this infinitesimal region from flatness, and divided by $-3/4$ is in fact the Gaussian curvature of the surface. So the curvature of an n-manifold can be understood by knowing $n(n-1)/2$ surface curvatures (the sectional curvatures, as we would say).

This led Riemann to explain the difference between intrinsic and extrinsic properties of a surface. He explained that for a sphere, which has an intrinsic geometry different from a plane, the Gaussian curvature multiplied by the area of an infinitesimal geodesic triangle is half the excess of the sum of its angles over π. This allowed him to express the belief that the geometry of an n-dimensional manifold could be understood by understanding its sectional curvatures.

Flat manifolds, he observed, have every sectional curvature zero. They are therefore a special case of the manifolds of constant curvature (that is, having the same sectional curvatures everywhere) and in these manifolds geometric figures can be freely moved around without stretching. To give examples of such manifolds he wrote down the metric

$$ds = \frac{1}{1 + \frac{\alpha}{4} \sum x^2} \sqrt{\sum (dx)^2},$$

where α is the curvature.

It was evident that the surfaces of constant positive curvature are spheres; the sphere of radius r has curvature r^{-2}. Riemann gave a complicated description of how to fit all the surfaces of constant curvature into one family. On this description, the cylinder is the example of a surface of zero curvature, and surfaces of constant negative curvature are locally like the saddle-shaped part of a torus.

In the third and final part of his paper [24] Riemann discussed how his ideas might apply to space. If all the sectional curvatures are zero, the space is Euclidean. But if we assume only that there is free mobility of bodies then space is described as a three-dimensional manifold of constant curvature, which can be determined from the knowledge of the sum of the angles in any triangle. Or, one could assume that length is independent of position but not direction.

As for empirical confirmation, the topological structures available to describe three-dimensional manifolds form a discrete set, so exact statements can be made about them even if one can never be certain of their truth. As for the metrical relations, however, these are necessarily inexact because every measurement is imprecise. This has implications for the immeasurably large and the immeasurably small.

The immeasurably large divided into spaces that are infinite and spaces that are merely unbounded, as for example a sphere. That said, Riemann regarded questions about the immeasurably large as irrelevant to the elucidation of natural phenomena, if only because existing astronomical measurements show that any non-zero sectional curvature of space can only be detected in regions vastly greater than the range of our telescopes. This seems to belong to a Göttingen tradition going back to Gauss and extending at least as far as Schwarzschild, who reported on the implications of measurements of the parallel of stars for the curvature of space in 1899 (see (Epple [7])).

Not so questions about the immeasurably small. Here "the concept of a rigid body, and the concept of a light ray, cease to be meaningful". But, Riemann concluded his lecture, these questions take us into physics "which the nature of today's occasion does not allow us to explore".

3 Influences

3.1 Gauss

It is possible to read Riemann's paper [24] in various ways. A modern mathematician can supply the missing details, or, at the other extreme, regard it as almost incoherent. It is possible for a mathematician to offer a comparably deep vision of new mathematics today, but it would be couched in a language of possible definitions, possible methods, and likely theorems that, conjecturally, resolve outstanding problems. Riemann's paper is more philosophical–in the good sense of challenging one to be clear about what is involved in an enquiry–and more speculative.

As Riemann made clear, among the few antecedents he could acknowledge were two papers by Gauss [11, 12]. The one on differential geometry is easy to appreciate. In the 1810s and 1820s Gauss had re-defined the subject in two memoirs. In his *Disquisitiones generales circa superficies curvas* of 1828 he introduced the concept of the intrinsic curvature of a surface. Gauss began his exposition by taking his readers through three definitions of a surface: in the first a surface is given by an expression of the form $z = f(x, y)$; in the second by an expression of the form $f(x, y, z) = 0$; and in the third in the parameterised form $(x(u, v), y(u, v), z(u, v))$. For each of these approaches he showed what the implications were calculating the curvatures of the principal curves at each point, which Euler [8] had showed are a good way to understand how curved a surface is at each point.

Gauss then introduced the map later known as the 'Gauss map'. At each point P of the surface he supposed there was a vector of unit length and normal to the surface, PP', and he considered the unit vector OQ parallel to PP' that has its base point at the centre of a fixed sphere of unit radius. The image of P' on the surface under the Gauss map is the point Q on the unit sphere.

Gauss then proved that the 'Gauss map' has a simple effect on areas: it multiplies the infinitesimal area around a point by an amount equal to the product of the principal curvatures. This product he proposed to call the curvature of the surface, and he showed that it depends only on E, F, and G and their derivatives with respect to u and v, but not on $x(u, v)$, $y(u, v)$, and $z(u, v)$. It is therefore intrinsic to the surface—a result that surprised him so much he called it the Theorema egregium or exceptional theorem.

One reason Gauss regarded the third form of presenting a surface as not only the most general but the most important, was because it allows u and v to be used as coordinates, and because it allows for a study of maps between one surface and another. In particular, given two surfaces defined by

$$\mathbf{r} = (x(u, v), y(u, v), z(u, v)) \text{ and } \mathbf{r}' = (x'(u', v'), y'(u', v'), z'(u', v'))$$

and a map between them, one can compare the line elements

$$ds^2 = E(u, v)du^2 + 2F(u, v)dudv + G(u, v)dv^2$$

and

$$ds'^2 = E'(u', v')du'^2 + 2F'(u', v')du'dv' + G'(u', v')dv'^2,$$

where $E(u, v) = \mathbf{r}_u.\mathbf{r}_u, \ldots, G'(u', v') = \mathbf{r}'_{v'}.\mathbf{r}'_{v'}$.

For example, the map is conformal or angle preserving if

$$ds^2 = \Phi(u', v')ds'^2,$$

for some function Φ, so in particular a map between a plane and a surface with $ds^2 = E(u, v)du^2 + 2F(u, v)dudv + G(u, v)dv^2$ is conformal if and only if $E = G$ and $F = 0$. Gauss [12] had made a detailed study of maps in connection with the survey of Hannover and South Denmark on which he worked in the 1820s, and had explicitly remarked that a map between planes is conformal if and only if it is given by a complex analytic function; this was only one of several occasions when he hinted at a theory of such functions that he was never to pull together and publish. It is very likely, however, that Riemann knew some of these ideas, but it is often impossible to say if he learned of them in discussions with Gauss or only by reading Gauss's papers [11, 12] after Gauss died. However, he wrote explicitly in his doctoral paper (1851) that the conformal nature of a complex analytic map was something that he learned from Gauss's paper [12] on conformal maps, and he stressed the importance of this geometrical aspect of the maps.

So Riemann took two ideas from Gauss's work on geometry. The conformal nature of a complex analytic map (away from any branch points) surely suggested to Riemann that there was significant geometrical features of a surface as early as 1851. But the idea of the intrinsic curvature of a surface was one Riemann took far beyond what Gauss had done with it.

Gauss [12] had identified the intrinsic feature of the geometry of a surface in $\mathbb{R}^3$, but he continued to think of surfaces as lying in $\mathbb{R}^3$. The idea that a region—a surface—with two coordinates u and v and a metric

$$ds^2 = E(u, v)du^2 + 2F(u, v)dudv + G(u, v)dv^2$$

is a fit subject for geometry already, whether or not there an embedding of it in space given by functions $x(u, v)$, $y(u, v)$, and $z(u, v)$ is due entirely to Riemann. It is almost certainly what Gauss had in mind when he said to his friend Wilhelm Weber after listening to Riemann's lecture that the profundity of the ideas that Riemann had put forward had greatly astonished him (see (Dedekind [6] 581)).

But, as Riemann's paltry citations indicate, there was very little done to extend Gauss's ideas of the intrinsic geometry of surfaces in the three decades that separate Gauss's memoir [12] from Riemann's lecture. One of the few papers written on the subject was by H.F. Minding [21], who investigated surfaces of constant negative curvature in his (1839). Bonnet, and Liouville in (Monge [22]), brought Gauss's theory to France, but Riemann went only once to Paris, in 1860, and it is not clear what he knew of French work before.

Riemann had also mentioned Gauss's second memoir on biquadratic residues (Gauss [14]). This is the work in which Gauss introduced what are today called the Gaussian integers, and explained at some length what he had been less overt in his *Disquisitiones Arithmeticae*, that the complex numbers can be thought of as points in a plane. Here (see §16) he stressed the highly intuitive character of this representation, and also–which is what Riemann surely picked up on–that this illuminated the true metaphysics of imaginary quantities.[1]

Gauss went on (§20) to stress that one goes beyond the positive numbers only when what is counted has an opposite (a negative) and what is then counted is not a substance (an object thinkable in itself) but a relation between two objects. More generally one creates new objects when one has a relation that admits a concept of opposite. Then (§22) "The mathematician abstracts totally from the nature of the objects and the content of their relations; he is concerned solely with the counting and the comparison of the relations among themselves." Nonetheless, intuitive representations are helpful and once an intuitive meaning for $\sqrt{-1}$ is completely established "one needs nothing further to admit this quantity into the domain of objects of arithmetic."

By 1850 Riemann did not need to be told that complex numbers were admitted into mathematics. Like Gauss, and Cauchy, he knew that the problem was not with complex numbers but with how to define complex functions. But he may well have

[1] See also the English translation, Ewald ([9], I, 312–313).

appreciated the metaphysics, the abstract character of mathematical objects and their relations, and the connection to intuition that, as we shall now see, was also a theme of Herbart's philosophy [15] and Riemann's physics.[2]

3.2 Herbart

The emphasis Riemann placed on Herbart's ideas came from Riemann's interest in philosophy. Herbart had been a philosopher at Göttingen from 1805 until his death in 1841, and his main book, the *Psychology as science newly founded on experience, metaphysics and mathematics* [15], appealed strongly to Riemann. But Riemann was also critical: he wrote in the philosophical passages in his collected works (1990, 539) that he could agree with almost all of Herbart's earliest research, but could not agree with his later speculations at certain essential points to do with his Naturphilosophie and psychology. He also identified himself as a Herbartian in psychology and epistemology, but not in ontology and synechology (a discipline concerned with space, time, and motion, and in particular with intelligible space, regarded as a mental construct that makes the explanation of matter possible).

In Riemann's view, natural science is the attempt to comprehend nature by precise concepts, and if concepts yield inaccurate predictions then the concepts must be modified. As a result, the more of nature we understand the more it sinks below the surface of phenomena. Riemann approved of Herbart's anti-Kantian epistemology, because Herbart [15] had argued that all our concepts arise by modifying earlier ones, and the most primitive concepts originate from attempts to understand what our senses tell us, which is why we have the possibility of forming concepts adequate for natural science. In particular they need not be a priori, as the Kantian ones are.

Herbart was a powerful source for the idea of varying quantities–ultimately, manifolds–although Herbart remained fixed on the idea that geometry was necessarily three-dimensional. But Riemann aimed at constructing coherent systems of concepts that could then be matched against the coherence of the natural world. He did not agree with Herbart's account of how our ideas of space are generated from experience, and went directly to systems of mathematical concepts. The elucidation of fundamental concepts was characteristic of Riemann's work, and it was an approach he shared with Herbart even when he did not use the same concepts himself.

3.3 Physics: Newton and Euler [8]

In a note Riemann [28] made on his work (*Werke*, 539) he wrote

> My main work consists in a new formulation of the known natural laws – expressing them in terms of other fundamental ideas – so as to make possible the use of experimental data

[2]This account draws on Bottazzini and Tazzioli (1995 [2]) and Scholz ([31] 1982b).

> on the exchanges between heat, light, magnetism, and electricity. In researching their interrelationship, I have been guided principally by the study of the work of Newton, Euler and – on the other side – Herbart.

This is a striking assessment; Riemann belongs to a list of brilliant mathematicians whose lasting contributions are more in mathematics than physics, contrary to their hope.

Riemann had no sympathy for action at a distance, and Dedekind [6] in his *Life of Riemann* (Riemann *Werke* [28]) tells us that Riemann was very pleased to discover from Brewster's biography of Newton that Newton too disliked the idea. Instead, Riemann imagined space filled with an ether, whose properties were responsible for the transmission of force and other physical quantities from place to place, and he hoped to unify in this way the theories of gravitation, electromagnetism, heat, and light.

Riemann imagined a substance that flowed between and through atoms, being created in some and vanishing in others. A point-particle is surrounded in this model by something like an elastic medium or ether that is described by a system of curvilinear coordinates centred at the point and varying in time. Deformations in the medium are captured by the equivalent of the strain tensor in elasticity theory, and variations in the metric reduce a force that is propagated through space because the point-particle opposes the deformation.

By 1853 he had brought these very vague ideas to the point where they provided a framework in which to speculate about how heat, light, and gravitation propagate. The mechanism was to be entirely through the action of neighbouring points, and this would involve the point-particle resisting a change in volume and the physical line element associated with the coordinate frame opposing a change in length (see [28], p. 564; [29], p. 511).

> Both classes of phenomena may be explained, if we suppose that the whole of infinite space is filled with a uniform substance, and each particle of substance acts only on its immediate neighbourhood.
>
> The mathematical law according to which this occurs can be considered as divided into
>
> 1) the resistance of a particle of substance to alteration in volume;
>
> 2) the resistance of a physical line element to alteration in length.
>
> Gravitation and electrostatic attraction and repulsion are founded on the first part; propagation of light and heat, and electrodynamic or magnetic attraction and repulsion on the second.

He then investigated "the laws of motion of a substance in empty space". He regarded the motion as the sum $u + v$ of a term u associated with the propagation of gravity and of light respectively. The usual separate equations for each process in a system of equations that Riemann believed gave an account of how the motion of a particle depends only on the particles around it.

As Speiser [32] (1927) was the first to point out, some of these ideas go back to Euler [8], who had attempted to formulate a theory of gravitation, light, electricity and magnetism in terms of an infinite, flowing ether. He had set out this view in his *Letters to a German Princess* in the early 1760s, and succeeded in using it to discuss the propagation of light. Speiser reported that Euler's [8] views were well regarded in his day but have since been largely forgotten.

In 1858 Riemann pushed his ideas further, and came up with a flawed theory of electrodynamics that is nonetheless interesting. The derivation of the equations rested at one point on a faulty exchange of the order of integration of two integrals, which may be why Riemann withdrew it from publication, and his theory involved a retarded potential. In this theory electromagnetism travelled at a speed α, which Riemann related it to the velocity of light, c, by the equation $\alpha^2 = \frac{1}{2}c^2$. In subsequent lectures, although not in the paper itself, he tried to ground his theory in the propagation of light between neighbouring particles.

All this gives weight to the observation that Pearson raised when editing (Clifford 1885, 203) and that Bottazzini and Tazzioli usefully repeat (1995, 32): "whether physicists might not find it simpler to assume that space is capable of varying curvature, and of a resistance to that variation, than to suppose the existence of a subtle medium pervading an invariable homaloidal [Euclidean] space."However, there is no evidence that Riemann took that step.

4 Heat Diffusion and the *Commentatio*

The *Hypotheses* paper [24] was far from being helpful to mathematicians, who might well have preferred more formulae to help them work out Riemann's visionary ideas. They had, in fact, one other paper to refer to, known as the *Commentatio* [27] or Paris memoir, recently and ably discussed in (Cogliati [4] 2014) and (Darrigol [5] 2015). This was an essay, written in Latin, that Riemann submitted, unsuccessfully, for a prize competition on the diffusion of heat in 1861, and which was published with several other of his unpublished papers in the first edition of his collected works [28].[3]

The question asked for conditions on the distribution of heat in an infinite, homogeneous, solid body so that a system of isothermal curves would remain a system of isothermal curves for an indefinite period of time, and moreover the temperature will become a function of time and two other variables.

Riemann viewed the question as concerning a positive definite quadratic form at each point that governed the flow of heat, and because the body is assumed to be homogeneous the coefficients entering the quadratic form are constants. He then looked for the conditions under which a quadratic form with variable coefficients $b_{i,j}$ can be diagonalised. He wrote the quadratic form as a differential form, so the question became one of finding

> conditions under which the expression $\sum_{i,j} b_{i,j} ds_i ds_j$ can be transformed into the form $\sum_{i,j} a_{i,j} dx_i dx_j$, with constant coefficients $a_{i,j}$, by taking the quantities s to be suitable functions of x.

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[3] See also Spivak ([33] 1970–1975, Chap. 6, Add. 2).

This turned the question into one of reducing the first form to $\sum_i dx_i^2$, because any positive definite quadratic form with constant coefficients can be so reduced.

A sketchy analysis that was difficult to follow but surely rested on some good unstated reasons led Riemann to claim that the reduction can be carried out provided a very complicated expression in the derivatives of the coefficients $b_{i,j}$, that Riemann abbreviated to (i, i', i'', i''') vanishes, so the question became: what is the meaning of this quantity?

At this point Riemann wrote "In order to understand the structure of these equations better, we form the expression [X]". Here, following Darrigol [5] (2015) we have introduced the symbol X for a complicated three-term expression that will be defined below.

A variational argument now gave Riemann a coordinate-free expression for X that involves (i, i', i'', i'''), and at this point he produced a geometrical analogy. He wrote that the expression $\sqrt{\sum_{i,j} b_{i,j} ds_i ds_j}$ can be interpreted as the line element in a general n-dimensional space, and the invariant just obtains appears in this setting as the curvature of the surface at a point. In the case at hand there are three variables, and so six equations that the $b_{i,j}$ must satisfy, of which only three are independent. In short, the reduction of the quadratic form in the heat diffusion problem to a sum of squares with constant coefficients is possible under exactly the same conditions as the reduction of a metric to the Euclidean case: it depends on the curvature vanishing.

When the paper appeared in Riemann's *Werke* [28], Heinrich Weber, one of the editors, supplied a lengthy commentary based on some remarks by Dedekind [6], the other editor. In the second edition he replaced these remarks with some new ones, in which he noticed that several authors had also looked at Riemann's essay: Christoffel, Lipschitz, and Beez among them. In fact, Christoffel [3] and Lipschitz [19] had set themselves the task independently, in attempts to understand the effect of coordinate transformations on quadratic forms in the wake of the publication of Riemann's *Hypotheses* [24] in 1867. Lipschitz returned to the subject in 1876, and Richard Beez's contribution [1] was an attempt write the matter up fully. Thereafter several mathematicians were drawn to *Commentatio* [27], notably Levi-Civita in his paper [17] on parallel displacement (1917).

Much of Weber's commentary consists in very helpfully going through Riemann's calculations more slowly and in more detail, first for a geodesic normal system of coordinates and then by indicating the changes that must be made to deal with a general coordinate system. It was a sensible strategy, but even so Weber made mistakes, and admitted that he had not been able to clear up the paper entirely. And indeed, Riemann had also made a mistake, and attempts to clarify it occupy a fair number of pages in the subsequent literature. Thus it seems that Levi-Civita [17] was led astray in his explanation of Riemann's reasoning, but that Lipschitz [18–20], and Beez [1] before him had understood it better.

Darrigol (2014) gives a thorough account of the developments from Riemann to Levi-Civita, and is particularly interested in on how Riemann came to his final results. It is only too clear that in Riemann's paper [27] we meet for the first time the sheer complexity that we handle today with Christoffel symbols, tensor analysis, Bianchi identities and the like, and Darrigol [5] investigates whether Riemann took a largely algebraic path or one guided by some identifiable geometric intuitions. On the basis of some previously unpublished notes in the Riemann *Nachlaß* [29] he concludes that a geometric insight into how curvature varies suggested some algebraic methods to Riemann.

It is when Riemann turned to the geometric analogy that we have to examine the symbol X. It is defined as

$$X = \delta^2 \sum b_{i,j} ds_i ds_j - 2d\delta \sum b_{i,j} ds_i \delta s_j + d^2 \sum b_{i,j} \delta s_i \delta s_j,$$

and Riemann immediately wrote it as

$$\sum (ij, kl)(ds_i \delta s_j - ds_j \delta s_i)(ds_k \delta s_l - ds_l \delta s_k).$$

The problem here is, as Beez [1] was the first to point out, the deduction is seemingly invalid, but it becomes valid if the term

$$2d\delta \sum b_{i,j} ds_i \delta s_j$$

is replaced by

$$d\delta \sum b_{i,j} ds_i \delta s_j + \delta d \sum b_{i,j} ds_i \delta s_j.$$

Both Darrigol [5] and Cogliati [4] point out that in fact this disparity disappears because the second-order terms are contracted with $d_i ds_j \delta s_k \delta s_l$, but Cogliati adds that Riemann's expression is a natural one to find if Riemann had worked with a normal coordinate system and then appealed to the invariance.

From a historian's point of view, one important point out which Cogliati [4] and Darrigol [5] agree, against some recent historical interpretations, is that Riemann and all his mathematical successors interpreted the expression (i, i', i'', i''') as a curvature and appreciated the use of geometrical reasoning in a problem on heat conduction. The alternative view, that much of this work was a species of tensor calculus without geometrical significance, seems to be an untenable distinction in the period.

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