2

Basics of Moving Averages

In the preceding chapter we considered the simplest type of a moving average where equal weights are given to each price observation in the window of data. This chapter introduces the general weighted moving average and discusses how to quantitatively assess the two important characteristics of a moving average: the average lag time and the smoothness.

2.1 General Weighted Moving Average

Moving averages are computed using the averaging window of size n. Specifically, a moving average at time t is computed using the last closing price P_t and $n-1$ lagged prices P_{t-i}, $i \in [1, n-1]$. Generally, each price observation in the rolling window of data has its own weight in the computation of a moving average. More formally, a general weighted moving average of price series P at time t is computed as

$$MA_t(n, P) = \frac{w_0 P_t + w_1 P_{t-1} + w_2 P_{t-2} + \cdots + w_{n-1} P_{t-n+1}}{w_0 + w_1 + w_2 + \cdots + w_{n-1}} = \frac{\sum_{i=0}^{n-1} w_i P_{t-i}}{\sum_{i=0}^{n-1} w_i}, \tag{2.1}$$

where w_i is the weight of price P_{t-i} in the computation of the weighted moving average. It is worth observing that, in order to compute a moving average, one has to use at least two prices; this means that one should have $n \geq 2$. Note that when the number of price observations used to compute a moving average equals one, a moving average becomes the last closing price, that is, $MA_t(1, P) = P_t$.

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The formula for a weighted moving average can alternatively be written as

$$MA_t(n, P) = \sum_{i=0}^{n-1} \psi_i P_{t-i}, \tag{2.2}$$

where

$$\psi_i = \frac{w_i}{\sum_{j=0}^{n-1} w_j}.$$

Observe that weights ψ_i are normalized. Specifically, whereas the sum of weights w_i is not equal to one, it is easy to check that the sum of weights ψ_i equals one

$$\sum_{i=0}^{n-1} \psi_i = 1.$$

The set of weights given by either $\{w_0, w_1, \ldots, w_{n-1}\}$ or $\{\psi_0, \psi_1, \ldots, \psi_{n-1}\}$ is usually called a (price) "weighting function". Each type of a moving average has its own distinct weighting function. The most common shapes of a weighting function are: equal-weighting of prices, over-weighting the most recent prices, and hump-shaped form with under-weighting both the most recent and most distant prices.

The moving average is a linear operator. Specifically, if X and Y are two time series and a, b, and c are three arbitrary constants, then it is easy to prove the following property:

$$MA_t(n, aX + bY + c) = a \times MA_t(n, X) + b \times MA_t(n, Y) + c. \tag{2.3}$$

In the subsequent exposition, as a rule a moving average is computed using the series of prices P. Therefore, to shorten the notation, we will often drop the variable P in the notation of a moving average; that is, we will write $MA_t(n)$ instead of $MA_t(n, P)$.

2.2 Average Lag Time of a Moving Average

The weighting function of a moving average fully characterizes its properties and allows us to estimate the average lag time of the moving average. The idea behind the computation of the average lag time is to calculate the average "age" of the data included in the moving average.[1] In particular, the price observation at time $t - i$ has weight w_i in the calculation of a moving average and lags behind the most recent observation at time t by i periods. Consequently, the incremental delay from observation at $t - i$ amounts to $w_i \times i$. The average lag time is the lag time at which all the weights can be considered to be "concentrated". This idea yields the following identity:

$$\underbrace{(w_0 + w_1 + w_2 + \cdots + w_{n-1})}_{\text{Sum of all weights}} \times \text{Lag time}$$
$$= \underbrace{w_0 \times 0 + w_1 \times 1 + w_2 \times 2 + \cdots + w_{n-1} \times (n-1)}_{\text{Weighted sum of delays of individual observations}}.$$

Therefore the average lag time of a weighted moving average can be computed using the following formula

$$\text{Lag time}(MA) = \frac{\sum_{i=1}^{n-1} w_i \times i}{\sum_{i=0}^{n-1} w_i} = \sum_{i=1}^{n-1} \psi_i \times i. \tag{2.4}$$

Notice that since the most recent observation has the lag time 0, the weight w_0 disappears from the computation of the weighted sum of delays of individual observations.

The formula for the average lag time can be rewritten as follows. First, we write $\sum_{i=1}^{n-1} w_i \times i$ as a double sum (we just replace i with $\sum_{j=1}^{i} 1$)

$$\sum_{i=1}^{n-1} w_i \times i = \sum_{i=1}^{n-1} w_i \sum_{j=1}^{i} 1.$$

Second, interchanging the order of summation in the double sum above yields

$$\sum_{i=1}^{n-1} w_i \sum_{j=1}^{i} 1 = \sum_{j=1}^{n-1} \sum_{i=j}^{n-1} w_i.$$

[1] A similar idea is used in physics to compute the center of mass and in finance to compute the bond duration (Macaulay duration).

Finally, we rewrite the formula for the average lag time as

$$\text{Lag time}(MA) = \frac{\sum_{j=1}^{n-1}\sum_{i=j}^{n-1} w_i}{\sum_{i=0}^{n-1} w_i} = \sum_{j=1}^{n-1} \phi_j, \tag{2.5}$$

where the weight ϕ_j is given by

$$\phi_j = \frac{\sum_{i=j}^{n-1} w_i}{\sum_{i=0}^{n-1} w_i} = \sum_{i=j}^{n-1} \psi_j. \tag{2.6}$$

The usefulness of Eq. (2.5) will become clear shortly.

2.3 Alternative Representation of a Moving Average

The alternative representation of a moving average is motivated by the fact that a series of stock prices can be considered as a dynamic process in time. We introduce the notation

$$\Delta P_{t-i} = P_{t-i+1} - P_{t-i}$$

which is the change in the stock price over the time interval from $t-i$ to $t-i+1$. Using this notation, we can write

$$P_{t-i} = P_t - \Delta P_{t-1} - \Delta P_{t-2} - \cdots - \Delta P_{t-i} = P_t - \sum_{j=1}^{i} \Delta P_{t-j}, \qquad i \geq 1.$$

The formula for the weighted moving average (given by Eq. (2.1)) can be rewritten as

$$\begin{aligned} MA_t(n) &= \frac{w_0 P_t + \sum_{i=1}^{n-1} w_i \left(P_t - \sum_{j=1}^{i} \Delta P_{t-j}\right)}{\sum_{i=0}^{n-1} w_i} \\ &= P_t - \frac{\sum_{i=1}^{n-1} w_i \sum_{j=1}^{i} \Delta P_{t-j}}{\sum_{i=0}^{n-1} w_i}. \end{aligned}$$

Interchanging the order of summation in the double sum above yields

$$MA_t(n) = P_t - \frac{\sum_{j=1}^{n-1}\left(\sum_{i=j}^{n-1} w_i\right)\Delta P_{t-j}}{\sum_{i=0}^{n-1} w_i} = P_t - \sum_{j=1}^{n-1} \phi_j \Delta P_{t-j}, \quad (2.7)$$

where ϕ_j is given by Eq. (2.6). Therefore, all right-aligned moving averages can be represented as the last closing price minus the weighted sum of the previous price changes. Note that in the ordinary moving averages (to be considered in the next chapter) the weights are positive, $w_i > 0$ for all i. As a result, in this case the sequence of weights ϕ_j is decreasing with increasing j

$$\phi_1 > \phi_2 > \cdots > \phi_{n-1}.$$

Consequently, regardless of the shape of the weighting function for prices w_i, the weighting function ϕ_j always over-weights the most recent price changes.

In the subsequent exposition, we will call the weighting function ψ_i $(i \geq 0)$ the (normalized) "price weighting function" and the weighting function ϕ_j $(j \geq 1)$ the "price-change weighting function".

The alternative representation of a moving average provides very insightful information on the relationship between the stock price P_t, the value of the moving average $MA_t(n)$, and the average lag time. Therefore, let us elaborate more on this.

Equation (2.7) can be rewritten as

$$P_t - MA_t(n) = \sum_{j=1}^{n-1} \phi_j \Delta P_{t-j}.$$

This equation implies that the value of the moving average generally is not equal to the last closing price unless $\sum_{j=1}^{n-1} \phi_j \Delta P_{t-j} = 0$. For example, this happens when the price remains on the same level (the prices move sideways) in the averaging window. In this case $\Delta P_{t-j} = 0$ for all j and, as a result, the value of the moving average equals the last closing price.

If the prices move upward (downward) such that $\Delta P_{t-j} > 0$ ($\Delta P_{t-j} < 0$) for all j, then $P_t - MA_t(n) > 0$ ($P_t - MA_t(n) < 0$). *Therefore, when the prices are in uptrend, the moving average tends to be below the last closing price. In contrast, when the prices move downward, the moving average tends to be above the last closing price.* The stronger the trend, the larger the discrepancy between the last closing price and the value of a moving average.

Suppose that the change in the stock price follows a Random Walk process with a drift

$$\Delta P_{t-j} = E[\Delta P] + \sigma \varepsilon_j, \tag{2.8}$$

where $E[\Delta P]$ is the expected price change, σ is the standard deviation of the price change, and ε_j is a sequence of independent and identically distributed random variables with mean zero and unit variance ($E[\varepsilon_j] = 0$, $Var[\varepsilon_j] = 1$). In this case the expected difference between the last closing price and the value of the moving average equals

$$\begin{aligned} E\left[P_t - MA_t(n)\right] &= E\left[\sum_{j=1}^{n-1} \phi_j \Delta P_{t-j}\right] = \sum_{j=1}^{n-1} \phi_j E[\Delta P_{t-j}] \\ &= \text{Lag time}(MA) \times E[\Delta P], \end{aligned} \tag{2.9}$$

where the last equality follows from Eq. (2.5). In words, the expected difference between the last closing price and the value of the moving average equals the average lag time times the average price change. Equation (2.9) is very insightful and implies that, in periods where variation in ΔP_{t-j} is rather small (for example, when prices are steadily increasing or decreasing), all moving averages with the same lag time move largely together *regardless of the shapes of their weighting functions and the sizes of their averaging windows.*[2] This property will be illustrated a number of times in the subsequent chapter.

It is instructive to illustrate graphically the relationship between the time series of stock prices, the moving average of prices, and the average lag time. For the sake of simplicity of illustration, we assume that the stock price steadily increases between times 0 and t. Specifically, we suppose that the stock price dynamic is given by

$$P_t = P_0 + \Delta P \times t,$$

where $\Delta P > 0$ is some arbitrary constant. The value of the moving average at time t is given by

$$MA_t(n) = P_t - \sum_{j=1}^{n-1} \phi_j \Delta P = P_0 + \Delta P \left(t - \sum_{j=1}^{n-1} \phi_j\right). \tag{2.10}$$

[2] Note that the average lag time is computed using the sequence of the weights ψ_i, $1 \le i \le n-1$. Many alternative sequences of weights can produce exactly the same value of the average lag time.

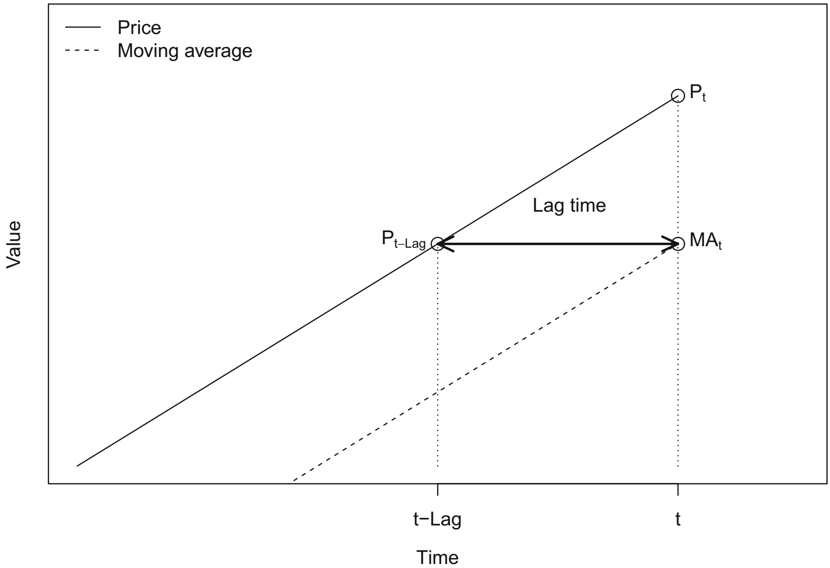

Fig. 2.1 Illustration of the lag time between the time series of stock prices and the moving average of prices

In this illustration, the lag time "Lag" between the time series of prices and the moving average of prices can be defined by the following relationship[3]

$$MA_t(n) = P_{t-\text{Lag}}.$$

This gives us the following equality

$$P_0 + \Delta P\left(t - \sum_{j=1}^{n-1} \phi_j\right) = P_0 + \Delta P(t - \text{Lag}).$$

The result is

$$\text{Lag} = \sum_{j=1}^{n-1} \phi_j,$$

which can be considered as an alternative derivation of the formula for the average lag time of a moving average. Graphically, the relationship between the stock price, the value of the moving average, and the average lag time is depicted in Fig. 2.1. It is important to emphasize that this relationship again implies

[3]In words, "Lag" is the required number of backshift operations applied to the time series of $\{MA_t(n)\}$ that makes it coincide with the time series of prices $\{P_t\}$.

that, when prices increase (or decrease) steadily, then all moving averages, that have exactly the same average lag time, move together regardless of the shapes of their weighting functions and the sizes of their averaging windows.

It is worth observing an additional interesting relationship between the dynamic of the price and the dynamic of a moving average of prices when prices increase or decrease steadily. Equation (2.10) implies that the change in the value of a moving average between times t and $t+1$ is given by

$$\Delta MA_t(n) = MA_{t+1}(n) - MA_t(n) = \Delta P.$$

This is a very insightful result. In words, this result means that, when prices increase or decrease steadily (meaning that ΔP is virtually constant), the change in the value of a moving average equals the price change *regardless of the size of the averaging window and the shape of the weighting function.* That is, in this case both the price and all moving averages (with different average lag times) move parallel in a graph.

It is important to emphasize that the notion of the "average lag time" should be understood literally. That is, at each given moment the lag time depends on the weighting function of the moving average and the price changes in the averaging window. However, if we average over all specific lag times, then the average lag time will be given by Eq. (2.4) or alternatively by (2.5). Only in cases where the prices are steadily increasing or decreasing, the "average lag time" provides a correct numerical characterisation of the time lag between the price and the value of the moving average.

2.4 Smoothness of a Moving Average

Besides the average lag time, the other important characteristic of a moving average is its smoothness. The smoothness of a time series is often evaluated by analysing the properties of the first difference of the time series. In our context, to evaluate the smoothness of a moving average $MA_t(n)$, we start with the computation of the first difference

$$\Delta MA_t(n) = MA_{t+1}(n) - MA_t(n).$$

The idea is that the smoother the time series $MA_t(n)$ is, the lesser the variation in its first difference $\Delta MA_t(n)$. Using Eq. (2.2), the formula above can be rewritten as

$$\Delta MA_t(n) = \sum_{i=0}^{n-1} \psi_i P_{t+1-i} - \sum_{i=0}^{n-1} \psi_i P_{t-i} = \sum_{i=0}^{n-1} \psi_i \Delta P_{t-i}. \tag{2.11}$$

One possible estimate of the smoothness of a moving average is the variance of $\Delta MA_t(n)$. In this case, small values of variance correspond to smoother series. If we assume that the change in the stock price follows a Random Walk process with a drift given by (2.8), then the variance of $\Delta MA_t(n)$ is equal to

$$\mathrm{Var}(\Delta MA_t(n)) = \sigma^2 \sum_{i=0}^{n-1} \psi_i^2 = \sigma^2 \times HI(MA), \tag{2.12}$$

where

$$HI(MA) = \sum_{i=0}^{n-1} \psi_i^2$$

is the well-known Herfindahl index (a.k.a. Herfindahl-Hirschman Index, or HHI). This index is a commonly accepted measure of market concentration and competition among market participants. This index is also used to measure the investment portfolio concentration (see, for example, Ivkovic et al. 2008). Therefore Eq. (2.12) says that the variance of $\Delta MA_t(n)$ is directly proportional to the measure of concentration of weights in the price weighting function of a moving average and the variance of the price changes.[4]

The reciprocal of the Herfindahl index, $HI^{-1}(MA)$, computed using the (normalized) price weighting function of a moving average, represents a very convenient way to measure the smoothness of a moving average. The reasons for this are as follows. First, the properties of this index are well known. Second, to evaluate the smoothness, in this case one needs only to know the weighting function of a moving average; there is no need to estimate the smoothness empirically using some particular price series data. Third, in many cases it is possible to derive a closed-form solution for the smoothness of a specific moving average.

Using the properties of the Herfindahl index, the lowest smoothness of a moving average is attained when some $\psi_i = 1$ and all other weights are zero; in this case $HI = 1$. For some fixed n, the highest smoothness is attained when all weights are equal; in this case $HI = \frac{1}{n}$. That is, equal weighting of

[4]There is a large strand of econometric literature that demonstrates that volatility of financial assets is not constant over time. Specifically, there are alternating calm and turbulent periods in financial markets. Therefore, in real markets the smoothness of a moving average is not constant over time. In particular, the smoothness improves in calm periods and worsens in turbulent periods.

prices in a moving average produces the smoothest moving average for a given size n of the averaging window. As expected, when prices are equally weighted, increasing the size of the averaging window decreases the Herfindahl index and therefore increases the smoothness of a moving average.

2.5 Chapter Summary

Each specific moving average is uniquely characterized by its price weighting function. This price weighting function allows us to compute the two central characteristics of a moving average: the average lag time and smoothness. We demonstrated that the smoothing properties of a moving average can be evaluated by the inverse of the Herfindahl index. It turns out that both the average lag time and the Herfindahl index of a moving average are related to the concentration of weights in the price weighting function. Whereas the Herfindahl index directly measures the concentration of weights in the weighting function (the higher the concentration, the worse the smoothness), the average lag time provides the exact location of the weight concentration.

At each current time, the value of the moving average of prices generally deviates from the last closing price. Our analysis shows explicitly that when stock prices are steadily trending upward, the moving average lies below the price. In contrast, when stock prices are steadily trending downward, the moving average lies above the price.[5] On average, the discrepancy between the value of the moving average and the last closing price equals the average lag time times the average price change. Only when the prices are trending sideways (that is, they stay on about the same level) the value of the moving average is close to the last closing price.

The analysis provided in this chapter reveals two important properties of moving averages when prices trend steadily. The first property says that in this case all moving averages with the same average lag time move largely together (as a single moving average) regardless of the shapes of their weighting functions and the sizes of their averaging windows. As an immediate corollary to this property, the behavior of the moving averages with the same average lag time differs due to their different reactions to the changes in the stock price trend. The second property says that, when prices trend steadily, both the price and all moving averages (with different average lag times) move parallel in a graph regardless of the sizes of their averaging windows and the shapes of their weighting functions. As an immediate corollary to this property, a change in

[5] It is worth emphasizing that this relationship holds only when stock prices trend steadily in one direction. This relationship does not hold when the direction of the trend changes frequently.

the direction of the price trend causes moving averages with various average lag times to move in different directions in a graph.

Reference

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