Chapter 2
Laurent Asymptotic Expansions

In Chapter 2, we present operational rules for Laurent asymptotic expansions given in two forms, without and with explicit upper bounds for remainders. Here, we would like to refer to some classic books, for example, Hörmander (1990) and Markushevich (1985), where one can find basic facts about Laurent series, mainly, connected with representation problems for analytic functions. We, however, are interested in simpler objects and problems that are Laurent asymptotic expansions and operational rules for such expansions, with remainders given in the standard form $o(\cdot)$ and with explicit power-type upper bounds for remainders. In the former case, such rules should be possibly considered as generally known, while, in the latter case, asymptotic expansions have some novelty aspects.

2.1 Laurent Asymptotic Expansions with Remainders Given in the Standard Form

In Section 2.1, we present operational rules for Laurent asymptotic expansions with remainders given in the standard form.

2.1.1 Definition of Laurent Asymptotic Expansions with Remainders Given in the Standard Form

Let $A(\varepsilon)$ be a real-valued function defined on an interval $(0, \varepsilon_0]$, for some $0 < \varepsilon_0 \leq 1$, and given on this interval by a Laurent asymptotic expansion,

$$A(\varepsilon) = a_{h_A}\varepsilon^{h_A} + \cdots + a_{k_A}\varepsilon^{k_A} + o_A(\varepsilon^{k_A}), \tag{2.1}$$

where **(a)** $-\infty < h_A \leq k_A < \infty$ are integers, **(b)** coefficients $a_{h_A}, \ldots, a_{k_A}$ are real numbers, **(c)** function $o_A(\varepsilon^{k_A})/\varepsilon^{k_A} \to 0$ as $\varepsilon \to 0$.

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We refer to such Laurent asymptotic expansion as a (h_A,k_A)-expansion with the remainder given in the standard form. This expansion is also a Taylor asymptotic expansion, if $h_A \geq 0$.

We say that (h_A,k_A)-expansion $A(\varepsilon)$ is pivotal if it is known that $a_{h_A} \neq 0$.

Parameter $L_A = k_A - h_A$ is called a length of the asymptotic expansion $A(\varepsilon)$.

Let a function $A(\varepsilon)$ be represented by a standard Taylor $(0,k_A)$-expansion. In applications, $A(\varepsilon)$ may be interpreted as a parameter for some perturbed system or process, and ε as a perturbation parameter. Respectively, $A(0) = a_0$ is interpreted as the value of above parameter for the corresponding unperturbed system or process. The asymptotic representation (2.1) reflects a stability property of parameter $A(\varepsilon)$ with respect to small perturbations, if $k_A = 0$. The asymptotic representation (2.1) corresponds to the model with linear type perturbation, if $k_A = 1$, and to the model with nonlinear type perturbation of order k_A if $k_A > 1$. The particular case of nonlinear type perturbation of order k_A is a polynomial type perturbation, where the corresponding remainder $o(\varepsilon^{k_A}) \equiv 0$. Also, an analytic-type perturbation, where function $A(\varepsilon)$ admits a representation in the form of absolutely convergent power series in interval $(0,\varepsilon_0]$, can be interpreted as a nonlinear type perturbation. Indeed, function $A(\varepsilon)$ can, in this case, be rewritten in the form (2.1), for any $k_A > 1$. In the general case, where $A(\varepsilon)$ is represented by a Laurent (h_A,k_A)-expansion, the perturbation type can be classified via the normalized function $\varepsilon^{-h_A}A(\varepsilon)$, which is a $(0,L_A)$-expansion. In this case, the perturbation type can be characterized as linear or nonlinear if, respectively, $L_A = 1$ or $L_A > 1$.

Let us now explain, why we can restrict consideration by the case, where parameter ε takes only positive values. Let us assume that function $A(\varepsilon)$ is also defined on some interval $[\varepsilon'_0,0)$ and is given on this interval by a Laurent asymptotic expansion $A(\varepsilon) = A'(\varepsilon) = a'_{h'_A}\varepsilon^{h'_A} + \cdots + a'_{k'_A}\varepsilon^{k'_A} + o'_A(\varepsilon^{k'_A})$ analogous to the above one given in relation (2.1). Then $A'(\varepsilon), \varepsilon \in [\varepsilon'_0,0)$ can always be rewritten as a function of positive parameter $-\varepsilon \in (0,-\varepsilon'_0]$ using the formula, $A'(\varepsilon) = A'(-(-\varepsilon)) = (-1)^{h'_A}a'_{h'_A}\cdot(-\varepsilon)^{h'_A} + \cdots + (-1)^{k'_A}a'_{k'_A}\cdot(-\varepsilon)^{k'_A} + o'_A((-1)^{k'_A}(-\varepsilon)^{k'_A})$. Thus, the operational analysis of function $A(\varepsilon)$, in particular computing of coefficients and estimation of remainder for the corresponding asymptotic expansion defined at a two-sided neighborhood of 0 can be reduced to analysis of two functions defined at positive one-sided neighborhoods of 0.

Let us also comment possible variations in representations of Laurent asymptotic expansions. If, for example, it is known that the (h_A,k_A)-expansion (2.1) has the first coefficient $a_{h_A} = 0$, then there is the obvious sense to exclude the first term $a_{h_A}\varepsilon^{h_A}$ from this asymptotic expansion and to rewrite it in the more informative form, $A(\varepsilon) = a_{h_A+1}\varepsilon^{h_A+1} + \cdots + a_{k_A}\varepsilon^{k_A} + o_A(\varepsilon^{k_A})$, i.e., as (h_A+1,k_A)-expansion. Also, if it is known that the remainder of the (h_A,k_A)-expansion (2.1) can be represented in the form $o_A(\varepsilon^{k_A}) \equiv a_{k_A+1}\varepsilon^{k_A+1} + o'_A(\varepsilon^{k_A+1})$, where $o'_A(\varepsilon^{k_A+1})/\varepsilon^{k_A+1} \to 0$ as $\varepsilon \to 0$, then there is the obvious sense to replace the remainder $o_A(\varepsilon^{k_A})$ by the identical function $a_{k_A+1}\varepsilon^{k_A+1} + o'_A(\varepsilon^{k_A+1})$ in the asymptotic expansion (2.1) and rewrite it in the more informative form, $A(\varepsilon) = a_{h_A}\varepsilon^{h_A} + \cdots + a_{k_A}\varepsilon^{k_A} + a_{k_A+1}\varepsilon^{k_A+1} + o'_A(\varepsilon^{k_A+1})$, i.e., as (h_A,k_A+1)-expansion.

Lemma 2.1. *If function* $A(\varepsilon) = a'_{h'_A}\varepsilon^{h'_A} + \cdots + a'_{k'_A}\varepsilon^{k'_A}_A + o'_A(\varepsilon^{k'_A}) = a''_{h''_A}\varepsilon^{h''_A} + \cdots + a''_{k''_A}\varepsilon^{k''_A}_A + o''_A(\varepsilon^{k''_A}), \varepsilon \in (0,\varepsilon_0]$ *can be represented as, respectively,* (h'_A, k'_A)- *and* (h''_A, k''_A)*-expansion, then the asymptotic expansion for function* $A(\varepsilon)$ *can be represented in the following most informative form* $A(\varepsilon) = a_{h_A}\varepsilon^{h_A} + \cdots + a_{k_A}\varepsilon^{k_A} + o_A(\varepsilon^{k_A}), \varepsilon \in (0,\varepsilon_0]$ *of* (h_A,k_A)*-expansion, with parameters* $h_A = h'_A \vee h''_A, k_A = k'_A \vee k''_A$*, and coefficients* $a_{h_A},\ldots,a_{k_A}$*, and remainder* $o_A(\varepsilon^{k_A})$ *given by the following relations*:

(i) $a'_l = 0$*, for* $h'_A \le l < h_A$ *and* $a''_l = 0$*, for* $h''_A \le l < h_A$;

(ii) $a_l = a'_l = a''_l$*, for* $h_A \le l \le \tilde{k}_A = k'_A \wedge k''_A$;

(iii) $a_l = a''_l$*, for* $\tilde{k}_A = k'_A < l \le k_A$ *if* $k'_A < k''_A$;

(iv) $a_l = a'_l$*, for* $\tilde{k}_A = k''_A < l \le k_A$ *if* $k''_A < k'_A$;

(v) $\sum_{\tilde{k}_A < l \le k'_A} a'_l\varepsilon^l + o'_A(\varepsilon^{k'_A}) = \sum_{\tilde{k}_A < l \le k''_A} a''_l\varepsilon^l + o''_A(\varepsilon^{k''_A}), \varepsilon \in (0,\varepsilon_0]$ *and* $o_A(\varepsilon^{k_A})$ *coincides, for* $\varepsilon \in (0,\varepsilon_0]$*, with* $o''_A(\varepsilon^{k''_A})$ *if* $k'_A < k''_A$; $o'_A(\varepsilon^{k'_A}) = o''_A(\varepsilon^{k''_A})$ *if* $k'_A = k''_A$; *or* $o'_A(\varepsilon^{k'_A})$ *if* $k'_A > k''_A$.

The asymptotical expansion $A(\varepsilon)$ *is pivotal if and only if* $a_{h_A} = a'_{h_A} = a''_{h_A} \ne 0$.

Remark 2.1. A constant a can be interpreted as function $A(\varepsilon) \equiv a$. Thus, 0 can be represented, for any integer $-\infty < h \le k < \infty$, as the (h,k)-expansion, $0 = 0\varepsilon^h + \ldots + 0\varepsilon^k + o(\varepsilon^k)$, with remainder $o(\varepsilon^k) \equiv 0$. Also, 1 can be represented, for any integer $0 \le k < \infty$, as the $(0,k)$-expansion, $1 = 1 + 0\varepsilon + \ldots + 0\varepsilon^k + o(\varepsilon^k)$, with remainder $o(\varepsilon^k) \equiv 0$.

2.1.2 Operational Rules for Laurent Asymptotic Expansion with Remainders Given in the Standard Form

Let us consider four Laurent asymptotic expansions, $A(\varepsilon) = a_{h_A}\varepsilon^{h_A} + \cdots + a_{k_A}\varepsilon^{k_A} + o_A(\varepsilon^{k_A})$, $B(\varepsilon) = b_{h_B}\varepsilon^{h_B} + \cdots + b_{k_B}\varepsilon^{k_B} + o_B(\varepsilon^{k_B})$, $C(\varepsilon) = c_{h_C}\varepsilon^{h_C} + \cdots + c_{k_C}\varepsilon^{k_C} + o_C(\varepsilon^{k_C})$, and $D(\varepsilon) = d_{h_D}\varepsilon^{h_D} + \cdots + d_{k_D}\varepsilon^{k_D} + o_D(\varepsilon^{k_D})$ defined for $0 < \varepsilon \le \varepsilon_0$, for some $0 < \varepsilon_0 \le 1$.

Lemma 2.2. *The following operational rules take place for the above Laurent asymptotic expansions*:

(i) *If* $A(\varepsilon), \varepsilon \in (0,\varepsilon_0]$ *is a* (h_A,k_A)*-expansion and* c *is a constant, then* $C(\varepsilon) = cA(\varepsilon), \varepsilon \in (0,\varepsilon_0]$ *is a* (h_C,k_C)*-expansion such that*:

(a) $h_C = h_A, k_C = k_A$;

(b) $c_{h_C+r} = ca_{h_C+r}, r = 0,\ldots,k_C - h_C$;

(c) $o_C(\varepsilon^{k_C}) = co_A(\varepsilon^{k_A})$.

This expansion is pivotal if and only if $c_{h_C} = ca_{h_A} \ne 0$.

(ii) *If* $A(\varepsilon), \varepsilon \in (0,\varepsilon_0]$ *is a* (h_A,k_A)*-expansion and* $B(\varepsilon), \varepsilon \in (0,\varepsilon_0]$ *is a* (h_B,k_B)*-expansion, then* $C(\varepsilon) = A(\varepsilon) + B(\varepsilon), \varepsilon \in (0,\varepsilon_0]$ *is a* (h_C,k_C)*-expansion such that*:

(a) $h_C = h_A \wedge h_B, k_C = k_A \wedge k_B$;

(b) $c_{h_C+r} = a_{h_C+r} + b_{h_C+r}, r = 0,\ldots,k_C - h_C$*, where* $a_{h_C+r} = 0$*, for* $0 \le r < h_A - h_C$ *and* $b_{h_C+r} = 0$*, for* $0 \le r < h_B - h_C$;

(c) $o_C(\varepsilon^{k_C}) = \sum_{k_C<i\le k_A} a_i\varepsilon^i + \sum_{k_C<j\le k_B} b_i\varepsilon^j + o_A(\varepsilon^{k_A}) + o_B(\varepsilon^{k_B})$.

This expansion is pivotal if and only if $c_{h_C} = a_{h_C} + b_{h_C} \neq 0$.

(iii) *If* $A(\varepsilon), \varepsilon \in (0,\varepsilon_0]$ *is a* (h_A,k_A)*-expansion and* $B(\varepsilon), \varepsilon \in (0,\varepsilon_0]$ *is a* (h_B,k_B)*-expansion, then* $C(\varepsilon) = A(\varepsilon)\cdot B(\varepsilon), \varepsilon \in (0,\varepsilon_0]$ *is a* (h_C,k_C)*-expansion such that*:

(a) $h_C = h_A + h_B$, $k_C = (k_A + h_B) \wedge (k_B + h_A)$;

(b) $c_{h_C+r} = \sum_{0\le i\le r} a_{h_A+i} b_{h_B+r-i}$, $r = 0,\ldots,k_C - h_C$;

(c) $o_C(\varepsilon^{k_C}) = \sum_{k_C<i+j, h_A\le i\le k_A, h_B\le j\le k_B} a_i b_j \varepsilon^{i+j} + \sum_{h_A\le i\le k_A} a_i\varepsilon^i o_B(\varepsilon^{k_B})$
$+ \sum_{h_B\le j\le k_B} b_j\varepsilon^j o_A(\varepsilon^{k_A}) + o_A(\varepsilon^{k_A}) o_B(\varepsilon^{k_B})$.

This expansion is pivotal if and only if $c_{h_C} = a_{h_A} b_{h_B} \neq 0$;

(iv) *If* $B(\varepsilon), \varepsilon \in (0,\varepsilon_0]$ *is a pivotal* (h_B,k_B)*-expansion, then there exists* $0 < \varepsilon_0' \le \varepsilon_0$ *such that* $B(\varepsilon) \neq 0, \varepsilon \in (0,\varepsilon_0']$, *and* $C(\varepsilon) = \frac{1}{B(\varepsilon)}, \varepsilon \in (0,\varepsilon_0']$ *is a pivotal* (h_C,k_C)*-expansion such that*:

(a) $h_C = -h_B$, $k_C = k_B - 2h_B$;

(b) $c_{h_C} = b_{h_B}^{-1}$, $c_{h_C+r} = -b_{h_B}^{-1} \sum_{1\le i\le r} b_{h_B+i} c_{h_C+r-i}$, $r = 1,\ldots,k_C - h_C$;

(c) $o_C(\varepsilon^{k_C}) = -\frac{\sum_{k_B-h_B<i+j, h_B\le i\le k_B, h_C\le j\le k_C} b_i c_j \varepsilon^{i+j} + \sum_{h_C\le j\le k_C} c_j\varepsilon^j o_B(\varepsilon^{k_B})}{b_{h_B}\varepsilon^{h_B} + \cdots + b_{k_B}\varepsilon^{k_B} + o_B(\varepsilon^{k_B})}$.

(v) *If* $A(\varepsilon), \varepsilon \in (0,\varepsilon_0]$ *is a* (h_A,k_A)*-expansion, and* $B(\varepsilon), \varepsilon \in (0,\varepsilon_0]$ *is a pivotal* (h_B,k_B)*-expansion, then there exists* $0 < \varepsilon_0' \le \varepsilon_0$ *such that* $B(\varepsilon) \neq 0, \varepsilon \in (0,\varepsilon_0']$, *and* $D(\varepsilon) = \frac{A(\varepsilon)}{B(\varepsilon)}, \varepsilon \in (0,\varepsilon_0']$ *is a* (h_D,k_D)*-expansion such that*:

(a) $h_D = h_A + h_C = h_A - h_B$, $k_D = (k_A + h_C) \wedge (k_C + h_A)$
$= (k_A - h_B) \wedge (k_B - 2h_B + h_A)$;

(b) $d_{h_D+r} = \sum_{0\le i\le r} a_{h_A+i} c_{h_C+r-i}, r = 0,\ldots,k_D - h_D$,

(c) $o_D(\varepsilon^{k_D}) = \sum_{k_D<i+j, h_B\le i\le k_B, h_C\le j\le k_C} b_i c_j \varepsilon^{i+j} + \sum_{h_B\le i\le k_B} b_i\varepsilon^i o_C(\varepsilon^{k_C})$
$+ \sum_{h_C\le j\le k_C} c_j\varepsilon^j o_B(\varepsilon^{k_B}) + o_B(\varepsilon^{k_B}) o_C(\varepsilon^{k_C})$,

where $c_{h_C+j}, j = 0,\ldots,k_C - h_C$ *and* $o_C(\varepsilon^{k_C})$ *are, respectively, the coefficients and the remainder of the* (h_C,k_C)*-expansion* $C(\varepsilon) = \frac{1}{B(\varepsilon)}$ *given in the above proposition* **(iv)**, *or by the following formulas:*

(d) $h_D = h_A - h_B$, $k_D = (k_A - h_B) \wedge (k_B - 2h_B + h_A)$;

(e) $d_{h_D+r} = b_{h_B}^{-1}(a_{h_A+r} - \sum_{1\le i\le r} b_{h_B+i} d_{h_D+r-i})$, $r = 0,\ldots,k_D - h_D$;

(f) $o_D(\varepsilon^{k_D}) = \frac{\sum_{k_A\wedge(k_B+h_A-h_B)<l\le k_A} a_l\varepsilon^l + o_A(\varepsilon^{k_A})}{b_{h_B}e^{h_B} + \cdots + b_{k_B}\varepsilon^{k_B} + o_B(\varepsilon^{k_B})}$
$- \frac{\sum_{k_A\wedge(k_B+h_A-h_B)<i+j, h_B\le i\le k_B, h_D\le j\le k_D} b_i d_j \varepsilon^{i+j} + \sum_{h_D\le j\le k_D} d_j\varepsilon^j o_B(\varepsilon^{k_B})}{b_{h_B}\varepsilon^{h_B} + \cdots + b_{k_B}\varepsilon^{k_B} + o_B(\varepsilon^{k_B})}$.

This expansion is pivotal if and only if $d_{h_D} = a_{h_A} c_{h_C} = a_{h_A}/b_{h_B} \neq 0$.

Remark 2.2. By Lemma 2.1, the Laurent asymptotic expansions for function $D(\varepsilon)$, given by the alternative formulas **(a)**–**(c)** and **(d)**–**(f)** in proposition **(v)** of Lemma 2.2, coincide. Also, these Laurent asymptotic expansions coincide with the expansions given by formulas **(a)**–**(c)** in propositions **(iv)** of Lemma 2.2, if $A(\varepsilon) \equiv 1$. In this

case, 1 should be interpreted as the $(0, k_B - h_B)$-expansion, $1 = 1 + 0\varepsilon + \ldots + 0\varepsilon^{k_B - h_B} + o(\varepsilon^{k_B - h_B})$, with remainder $o(\varepsilon^{k_B - h_B}) \equiv 0$.

Let $A_m(\varepsilon) = a_{h_{A_m},m}\varepsilon^{h_{A_m}} + \cdots + a_{k_{A_m},m}\varepsilon^{k_{A_m}} + o_{A_m}(\varepsilon^{k_{A_m}}), \varepsilon \in (0, \varepsilon_0]$ be a (h_{A_m}, k_{A_m})-expansion, for $m = 1, \ldots, N$.

Lemma 2.3. *The following operational rules for multiple summation and multiplication take place for the above Laurent asymptotic expansions*:

(i) $B_n(\varepsilon) = A_1(\varepsilon) + \cdots + A_n(\varepsilon), \varepsilon \in (0, \varepsilon_0]$ *is, for every* $n = 1, \ldots, N$, *a* (h_{B_n}, k_{B_n})*-expansion, where*:

(a) $h_{B_n} = \min(h_{A_1}, \ldots, h_{A_n}), k_{B_n} = \min(k_{A_1}, \ldots, k_{A_n})$.

(b) $b_{h_{B_n}+l,n} = a_{h_{B_n}+l,1} + \cdots + a_{h_{B_n}+l,n}$, $l = 0, \ldots, k_{B_n} - h_{B_n}$, *where*

$a_{h_{B_n}+l,m} = 0$, *for* $0 \le l < h_{A_m} - h_{B_n}, m = 1, \ldots, n$.

(c) $o_{B_n}(\varepsilon^{k_{B_n}}) = \sum_{1 \le m \le n} \left(\sum_{k_{B_n} < l \le k_{A_m}} a_{l,m}\varepsilon^l + o_{A_m}(\varepsilon^{k_{A_m}}) \right)$.

Expansion $B_n(\varepsilon)$ *is pivotal if and only if* $b_{h_{B_n},n} = a_{h_{A_1},1} + \cdots + a_{h_{A_n},n} \ne 0$.

(ii) $C_n(\varepsilon) = A_1(\varepsilon) \times \cdots \times A_n(\varepsilon), \varepsilon \in (0, \varepsilon_0]$ *is, for every* $n = 1, \ldots, N$, *a* (h_{C_n}, k_{C_n})*-expansion, where*:

(a) $h_{C_n} = h_{A_1} + \cdots + h_{A_n}, k_{C_n} = \min(k_{A_l} + \sum_{1 \le r \le n, r \ne l} h_{A_r}, l = 1, \ldots, n)$.

(b) $c_{h_{C_n}+l,n} = \sum_{l_1 + \cdots + l_n = l, 0 \le l_m \le k_{A_m} - h_{A_m}, 1 \le m \le n} \prod_{1 \le m \le n} a_{h_{A_m}+l_m,m}$,

$l = 0, \ldots, k_{C_n} - h_{C_n}$.

(c) $o_{C_n}(\varepsilon^{k_{C_n}}) = \sum_{k_{C_n} < l_1 + \cdots + l_n, h_{A_m} \le l_m \le k_{A_m}, 1 \le m \le n} \prod_{1 \le m \le n} a_{l_m,m}\varepsilon^{l_1 + \cdots + l_n}$

$+ \sum_{1 \le r \le n} \sum_{1 \le m_1 < \cdots < m_r \le n} \left(\prod_{m \ne m_1, \ldots, m_r, 1 \le m \le n} \left(\sum_{h_{A_m} \le l \le k_{A_m}} a_{l,m}\varepsilon^l \right) \right.$

$\left. \times \prod_{1 \le i \le r} o_{A_{m_i}}(\varepsilon^{k_{A_{m_i}}}) \right)$.

Expansion $C_n(\varepsilon)$ *is pivotal if and only if* $c_{h_{C_n},n} = a_{h_{A_1},1} \times \cdots \times a_{h_{A_n},n} \ne 0$.

In particular, if $A_m(\varepsilon) = A_1(\varepsilon), \varepsilon \in (0, \varepsilon_0]$, *for* $m = 1, \ldots, N$, *then* $C_n(\varepsilon) = A_1(\varepsilon)^n$, $\varepsilon \in (0, \varepsilon_0]$ *is, for every* $n = 1, \ldots, N$, *a* (h_{C_n}, k_{C_n})*-expansion, where*:

(d) $h_{C_n} = nh_{A_1}, k_{C_n} = k_{A_1} + (n-1)h_{A_1}$.

(e) $c_{h_{C_n}+l,n} = \sum_{l_1 + \cdots + l_n = l, 0 \le l_m \le k_{A_1} - h_{A_1}, 1 \le m \le n} \prod_{1 \le m \le n} a_{h_{A_1}+l_m,1}$,

$l = 0, \ldots, k_{C_n} - h_{C_n}$.

(f) $o_{C_n}(\varepsilon^{k_{C_n}}) = \sum_{k_{C_n} < l_1 + \cdots + l_n, h_{A_m} \le l_m \le k_{A_m}, 1 \le m \le n} \prod_{1 \le m \le n} a_{l_m,1}\varepsilon^{l_1 + \cdots + l_n}$

$+ \sum_{1 \le r \le n} \binom{n}{r} \left(\sum_{h_{A_1} \le l \le k_{A_1}} a_{l,1}\varepsilon^l \right)^{n-r} \left(o_{A_1}(\varepsilon^{k_{A_1}}) \right)^r$.

(iii) *Asymptotic expansions for functions* $B_n(\varepsilon) = A_1(\varepsilon) + \cdots + A_n(\varepsilon), n = 1, \ldots, N$ *and* $C_n(\varepsilon) = A_1(\varepsilon) \times \cdots \times A_n(\varepsilon), n = 1, \ldots, N$ *are invariant with respect to any permutation of summation and multiplication order in the above formulas.*

Remark 2.3. An alternative recurrent formulas for parameters, coefficients, and remainders of asymptotic expansions for sums $B_n(\varepsilon)$ and products $C_n(\varepsilon)$ can be obtained using the recurrent identities $B_n(\varepsilon) \equiv B_{n-1}(\varepsilon) + A_n(\varepsilon), n = 1, \ldots, N$, $B_0(\varepsilon) \equiv 0$, and $C_n(\varepsilon) \equiv C_{n-1}(\varepsilon) \cdot A_n(\varepsilon), n = 1, \ldots, N$, $C_0(\varepsilon) \equiv 1$ and applying to them the operational rules for Laurent asymptotic expansions given in Lemma 2.2.

2.1.3 Algebraic Properties of Operational Rules for Laurent Asymptotic Expansions with Remainders Given in the Standard Form

The following lemma summarizes some basic algebraic properties of Laurent asymptotic expansions. It is a corollary of Lemmas 2.1 and 2.2.

Lemma 2.4. *The summation and multiplication operations for Laurent asymptotic expansions defined in Lemma 2.2 possess the following algebraic properties, which should be understood as identities for the corresponding Laurent asymptotic expansions (i.e., identities for the corresponding parameters h,k, coefficients and remainders) of functions represented in two alternative forms in the functional identities given below*:

(i) *The summation and multiplication operations for Laurent asymptotic expansions satisfy the "elimination" identities that are implied by the corresponding functional identities,* $A(\varepsilon)+0\equiv A(\varepsilon)$, $A(\varepsilon)\cdot 1\equiv A(\varepsilon)$, $A(\varepsilon)-A(\varepsilon)\equiv 0$, *and* $A(\varepsilon)\cdot A(\varepsilon)^{-1}\equiv 1$.

(ii) *The summation operation for Laurent asymptotic expansions is commutative and associative that is implied by the corresponding functional identities,* $A(\varepsilon)+B(\varepsilon)\equiv B(\varepsilon)+A(\varepsilon)$ *and* $(A(\varepsilon)+B(\varepsilon))+C(\varepsilon)\equiv A(\varepsilon)+(B(\varepsilon)+C(\varepsilon))$.

(iii) *The multiplication operation for Laurent asymptotic expansions is commutative and associative that is implied by the corresponding functional identities,* $A(\varepsilon)\cdot B(\varepsilon)\equiv B(\varepsilon)\cdot A(\varepsilon)$ *and* $(A(\varepsilon)\cdot B(\varepsilon))\cdot C(\varepsilon)\equiv A(\varepsilon)\cdot(B(\varepsilon)\cdot C(\varepsilon))$.

(iv) *The summation and multiplication operations for Laurent asymptotic expansions possess distributive property that is implied by the corresponding functional identity,* $(A(\varepsilon)+B(\varepsilon))\cdot C(\varepsilon)\equiv A(\varepsilon)\cdot C(\varepsilon)+B(\varepsilon)\cdot C(\varepsilon)$.

Remark 2.4. In proposition **(i)** of Lemma 2.4, 0 should be interpreted as the (h_A,k_A)-expansion, $0=0+0\varepsilon^{h_A}+\ldots+0\varepsilon^{k_A}+o(\varepsilon^{k_A})$, with remainder $o(\varepsilon^{k_A})\equiv 0$, and 1 as $(0,k_A-h_A)$-expansion, $1=1+0\varepsilon+\ldots+0\varepsilon^{k_A-h_A}+o(\varepsilon^{k_A-h_A})$, with remainder $o(\varepsilon^{k_A-h_A})\equiv 0$.

Remark 2.5. The Laurent asymptotic expansion $A(\varepsilon)$ is assumed to be pivotal, in the elimination identity implied by functional identity $A(\varepsilon)\cdot A(\varepsilon)^{-1}\equiv 1$, and to hold, for $0<\varepsilon\le\varepsilon_0'$ such that $A(\varepsilon)\neq 0$, $\varepsilon\in(0,\varepsilon_0']$.

2.2 Laurent Asymptotic Expansions with Explicit Upper Bounds for Remainders

In Section 2.2 we present operational rules for Laurent asymptotic expansions with explicit upper bounds for remainders.

2.2.1 Definition of Laurent Asymptotic Expansions with Explicit Upper Bounds for Remainders

Let $A(\varepsilon)$ be a real-valued function defined on an interval $(0,\varepsilon_0]$, for some $0<\varepsilon_0\le 1$, and given on this interval by a Laurent asymptotic expansion,

$$A(\varepsilon) = a_{h_A}\varepsilon^{h_A} + \cdots + a_{k_A}\varepsilon^{k_A} + o_A(\varepsilon^{k_A}), \tag{2.2}$$

where **(a)** $-\infty < h_A \le k_A < \infty$ are integers, **(b)** coefficients $a_{h_A}, \ldots, a_{k_A}$ are real numbers, **(c)** $|o_A(\varepsilon^{k_A})| \le G_A\varepsilon^{k_A+\delta_A}$, for $0 < \varepsilon \le \varepsilon_A$, where **(d)** $0 < \delta_A \le 1$, $0 \le G_A < \infty$ and $0 < \varepsilon_A \le \varepsilon_0$.

We refer to such Laurent asymptotic expansion as a $(h_A, k_A, \delta_A, G_A, \varepsilon_A)$-expansion.

The $(h_A, k_A, \delta_A, G_A, \varepsilon_A)$-expansion is also a (h_A, k_A)-expansion, according the definition given in Subsection 2.1.1, since, $o_A(\varepsilon^{k_A})/\varepsilon^{k_A} \to 0$ as $\varepsilon \to 0$.

We say that $(h_A, k_A, \delta_A, G_A, \varepsilon_A)$-expansion $A(\varepsilon)$ is pivotal if it is known that $a_{h_A} \ne 0$.

It is useful to note that there is no sense to consider, it seems, a more general case of upper bounds for the remainder $o_A(\varepsilon^{k_A})$, with parameter $\delta_A > 1$. Indeed, let us define $k'_A = k_A + [\delta_A] - \mathrm{I}(\delta_A = [\delta_A])$ and $\delta'_A = \delta_A - [\delta_A] + \mathrm{I}(\delta_A = [\delta_A]) \in (0, 1]$. The $(h_A, k_A, \delta_A, G_A, \varepsilon_A)$-expansion $A(\varepsilon)$ can be rewritten in the equivalent form of the $(h_A, k'_A, \delta'_A, G_A, \varepsilon_A)$-expansion, $A(\varepsilon) = a_{h_A}\varepsilon^{h_A} + \cdots + a_{k_A}\varepsilon^{k_A} + 0\varepsilon^{k_A+1} + \cdots + 0\varepsilon^{k'_A} + o'_A(\varepsilon^{k'_A})$, with the remainder $o'_A(\varepsilon^{k'_A}) = o_A(\varepsilon^{k_A})$, which satisfies inequalities $|o'_A(\varepsilon^{k'_A})| = |o_A(\varepsilon^{k_A})| \le G_A\varepsilon^{k_A+\delta_A} = G_A\varepsilon^{k'_A+\delta'_A}$, for $0 < \varepsilon \le \varepsilon_A$.

Let us also comment the analytic-type case, where function $A(\varepsilon)$ is given by an absolutely convergent power series, $A(\varepsilon) = \sum_{k \ge h_A} a_k\varepsilon^k$, in interval $(0, \varepsilon_0]$. In this case, $A(\varepsilon)$ can be represented in the form of $(h_A, k_A, 1, G_{k_A}, \varepsilon_{k_A})$-expansion $A(\varepsilon) = a_{h_A}\varepsilon^{h_A} + \cdots + a_{k_A}\varepsilon^{k_A} + o_{k_A}(\varepsilon^{k_A})$ with remainder $o_{k_A}(\varepsilon^{k_A}) = \sum_{k \ge k_A+1} a_k\varepsilon^k$, for any $k_A \ge h_A$. Indeed, remainder $o_{k_A}(\varepsilon^{k_A})$ satisfies, for any $\varepsilon_{k_A} \in (0, \varepsilon_0]$, inequalities, $|o_{k_A}(\varepsilon^{k_A})| \le \varepsilon^{k_A+1}G_{k_A}, \varepsilon \in (0, \varepsilon_{k_A}]$, where $G_{k_A} = \sum_{k \ge k_A+1} |a_k| \varepsilon_{k_A}^{k-k_A-1}$.

The above remarks imply that the asymptotic expansion $A(\varepsilon)$ can be represented in different forms. In such cases, we consider forms with larger parameters h_A and k_A as more informative. As far as parameters δ_A, G_A, and ε_A are concerned, we consider as more informative forms, first, with larger values of parameter δ_A, second, with smaller values of parameter G_A and, third, with larger values of parameter ε_A.

Lemma 2.5. *If* $A(\varepsilon) = a'_{h'_A}\varepsilon^{h'_A} + \cdots + a'_{k'_A}\varepsilon^{k'_A} + o'_A(\varepsilon^{k'_A}) = a''_{h''_A}\varepsilon^{h''_A} + \cdots + a''_{k''_A}\varepsilon^{k''_A} + o''_A(\varepsilon^{k''_A}), \varepsilon \in (0, \varepsilon_0]$ *can be represented as, respectively,* $(h'_A, k'_A, \delta'_A, G'_A, \varepsilon'_A)$- *and* $(h''_A, k''_A, \delta''_A, G''_A, \varepsilon''_A)$*-expansion, then the* (h_A, k_A)*-expansion* $A(\varepsilon) = a_{h_A}\varepsilon^{h_A} + \cdots + a_{k_A}\varepsilon^k + o_A(\varepsilon^{k_A}), \varepsilon \in (0, \varepsilon_0]$ *given in Lemma 2.1 is an* $(h_A, k_A, \delta_A, G_A, \varepsilon_A)$*-expansion, with parameters* δ_A, G_A, *and* ε_A *chosen in the following way consistent with the priority order described above*:

$$(\delta_A, G_A, \varepsilon_A) = \begin{cases} (\delta''_A, G''_A, \varepsilon''_A) & \text{if } k'_A < k''_A \text{ or } k'_A = k''_A, \delta'_A < \delta''_A, \\ (\delta'_A = \delta''_A, G'_A \wedge G''_A, \varepsilon'_A \wedge \varepsilon''_A) & \text{if } k'_A = k''_A, \delta'_A = \delta''_A, \\ (\delta'_A, G'_A, \varepsilon'_A) & \text{if } k'_A > k''_A \text{ or } k'_A = k''_A, \delta'_A > \delta''_A. \end{cases} \tag{2.3}$$

Remark 2.6. The following simple upper bounds take place for parameters G_A, δ_A, and ε_A given in Lemma 2.5, $\delta_A \geq \tilde{\delta}_A = \delta'_A \wedge \delta''_A$, $G_A \leq \tilde{G}_A = G'_A \vee G''_A$, and $\varepsilon_A \geq \tilde{\varepsilon}_A = \varepsilon'_A \wedge \varepsilon''_A$, which can be used in the case, where the priority order described above is ignored. Obviously, (h_A, k_A)-expansion $A(\varepsilon)$ is also a $(h_A, k_A, \tilde{\delta}_A, \tilde{G}_A, \tilde{\varepsilon}_A)$-expansion.

Remark 2.7. A constant a can be interpreted as function $A(\varepsilon) \equiv a$. Thus, 0 can be represented, for any integer $-\infty < h \leq k < \infty$, as the $(h, k, \delta_{h,k}, G_{h,k}, \varepsilon_{h,k})$-expansion, $0 = 0\varepsilon^h + \ldots + 0\varepsilon^k + o(\varepsilon^k)$, with remainder $o(\varepsilon^k) \equiv 0$ and, thus, parameters $\delta_{h,k} = 1$, $G_{h,k} = 0$, and $\varepsilon_{h,k} = \varepsilon_0$. Also, 1 can be represented, for any integer $0 \leq k < \infty$, as the $(0, k, \delta_k, G_k, \varepsilon_k)$-expansion, $1 = 1 + 0\varepsilon + \ldots + 0\varepsilon^k + o(\varepsilon^k)$, with remainder $o(\varepsilon^k) \equiv 0$ and, thus, parameters $\delta_k = 1$, $G_k = 0$, and $\varepsilon_k = \varepsilon_0$.

2.2.2 Operational Rules for Laurent Asymptotic Expansion with Explicit Upper Bounds for Remainders

Let us consider four Laurent asymptotic expansions, $A(\varepsilon) = a_{h_A}\varepsilon^{h_A} + \cdots + a_{k_A}\varepsilon^{k_A} + o_A(\varepsilon^{k_A})$, $B(\varepsilon) = b_{h_B}\varepsilon^{h_B} + \cdots + b_{k_B}\varepsilon^{k_B} + o_B(\varepsilon^{k_B})$, $C(\varepsilon) = c_{h_C}\varepsilon^{h_C} + \cdots + c_{k_C}\varepsilon^{k_C} + o_C(\varepsilon^{k_C})$, and $D(\varepsilon) = d_{h_D}\varepsilon^{h_D} + \cdots + d_{k_D}\varepsilon^{k_D} + o_D(\varepsilon^{k_D})$ defined for $0 < \varepsilon \leq \varepsilon_0$, for some $0 < \varepsilon_0 \leq 1$.

Let us denote, $F_A = \max_{h_A \leq i \leq k_A} |a_i|$, $F_B = \max_{h_B \leq i \leq k_B} |b_i|$, $F_C = \max_{h_C \leq i \leq k_C} |c_i|$, and $F_D = \max_{h_D \leq i \leq k_D} |d_i|$.

Lemma 2.6. *The above Laurent asymptotic expansions have the following operational rules for computing remainders*:

(i) *If $A(\varepsilon), \varepsilon \in (0, \varepsilon_0]$ is a $(h_A, k_A, \delta_A, G_A, \varepsilon_A)$-expansion and c is a constant, then $C(\varepsilon) = cA(\varepsilon), \varepsilon \in (0, \varepsilon_0]$ is a $(h_C, k_C, \delta_C, G_C, \varepsilon_C)$-expansion with parameters h_C, k_C and coefficients $c_r, r = h_C, \ldots, k_C$ given in proposition* **(i)** *of Lemma 2.2, and parameters δ_C, G_C, and ε_C given by the formulas*:

(a) $\delta_C = \delta_A = \tilde{\delta}_C$;

(b) $G_C = |c| G_A = \tilde{G}_C$;

(c) $\varepsilon_C = \varepsilon_A = \tilde{\varepsilon}_C = \tilde{\varepsilon}_C$.

(ii) *If $A(\varepsilon), \varepsilon \in (0, \varepsilon_0]$ is a $(h_A, k_A, \delta_A, G_A, \varepsilon_A)$-expansion and $B(\varepsilon), \varepsilon \in (0, \varepsilon_0]$ is a $(h_B, k_B, \delta_B, G_B, \varepsilon_B)$-expansion, then $C(\varepsilon) = A(\varepsilon) + B(\varepsilon)$, $\varepsilon \in (0, \varepsilon_0]$ is a $(h_C, k_C, \delta_C, G_C, \varepsilon_C)$-expansion with parameters h_C, k_C and coefficients $c_r, r = h_C, \ldots, k_C$ given in proposition* **(ii)** *of Lemma 2.2, and parameters δ_C, G_C, and ε_C given by formulas*:

(a) $\delta_C = \delta_A \mathrm{I}(k_A < k_B) + (\delta_A \wedge \delta_B)\mathrm{I}(k_A = k_B) + \delta_B \mathrm{I}(k_B < k_A) \geq \tilde{\delta}_C = \delta_A \wedge \delta_B$;

(b) $G_C = \sum_{k_C < i \leq k_A} |a_i| \varepsilon_C^{i - k_C - \delta_C} + \sum_{k_C < j \leq k_B} |b_j| \varepsilon_C^{j - k_C - \delta_C}$
$+ G_A \varepsilon_C^{k_A + \delta_A - k_C - \delta_C} + G_B \varepsilon_C^{k_B + \delta_B - k_C - \delta_C}$
$\leq \tilde{G}_C = F_A(k_A - k_C) + G_A + F_B(k_B - k_C) + G_B$;

(c) $\varepsilon_C = \varepsilon_A \wedge \varepsilon_B = \tilde{\varepsilon}_C$.

(iii) *If $A(\varepsilon), \varepsilon \in (0, \varepsilon_0]$ is a $(h_A, k_A, \delta_A, G_A, \varepsilon_A)$-expansion and $B(\varepsilon), \varepsilon \in (0, \varepsilon_0]$ is a $(h_B, k_B, \delta_B, G_B, \varepsilon_B)$-expansion, then $C(\varepsilon) = A(\varepsilon) \cdot B(\varepsilon)$, $\varepsilon \in (0, \varepsilon_0]$ is a $(h_C, k_C, \delta_C, G_C, \varepsilon_C)$-expansion with parameters h_C, k_C and coefficients $c_r, r = h_C, \ldots, k_C$ given in proposition* **(iii)** *of Lemma 2.2, and parameters δ_C, G_C, and ε_C given by formulas*:

(a) $\delta_C = \delta_A \mathrm{I}(k_A + h_B < k_B + h_A) \ + \ (\delta_A \wedge \delta_B)\mathrm{I}(k_A + h_B = k_B + h_A)$
$+ \ \delta_B \mathrm{I}(k_A + h_B > k_B + h_A) \geq \tilde{\delta}_C = \delta_A \wedge \delta_B;$

(b) $G_C = \sum_{k_C < i+j, h_A \leq i \leq k_A, h_B \leq j \leq k_B} |a_i||b_j| \varepsilon_C^{i+j-k_C-\delta_C}$
$+ \ G_A \sum_{h_B \leq j \leq k_B} |b_j| \varepsilon_C^{j+k_A+\delta_A-k_C-\delta_C} + \ G_B \sum_{h_A \leq i \leq k_A} |a_i| \varepsilon_C^{i+k_B+\delta_B-k_C-\delta_C}$
$+ \ G_A G_B \varepsilon_C^{k_A+k_B+\delta_A+\delta_B-k_C-\delta_C}$
$\leq \tilde{G}_C = (F_A(k_A - h_A + 1) + G_A)(F_B(k_B - h_B + 1) + G_B);$

(c) $\varepsilon_C = \varepsilon_A \wedge \varepsilon_B = \tilde{\varepsilon}_C.$

(iv) *If* $B(\varepsilon), \varepsilon \in (0, \varepsilon_0]$ *is a pivotal* $(h_B, k_B, \delta_B, G_B, \varepsilon_B)$*-expansion, then there exists* $\varepsilon_C \leq \varepsilon_0' \leq \varepsilon_0$ *such that* $B(\varepsilon) \neq 0, \varepsilon \in (0, \varepsilon_0']$*, and* $C(\varepsilon) = \frac{1}{B(\varepsilon)}, \varepsilon \in (0, \varepsilon_0']$ *is a pivotal* $(h_C, k_C, \delta_C, G_C, \varepsilon_C)$*-expansion with parameters* h_C, k_C *and coefficients* $c_r, r = h_C, \ldots, k_C$ *given in proposition* **(iv)** *of Lemma 2.2, and parameters* $\delta_C, G_C,$ *and* ε_C *given by formulas*:

(a) $\delta_C = \delta_B = \tilde{\delta}_C;$

(b) $G_C = \ (\frac{|b_{h_B}|}{2})^{-1} \big(G_B \sum_{h_C \leq j \leq k_C} |c_j| \varepsilon_C^{j+h_B}$
$+ \sum_{k_B - h_B < i+j, h_B \leq i \leq k_B, h_C \leq j \leq k_C} |b_i||c_j| \varepsilon_C^{i+j-k_B+h_B-\delta_B} \big)$
$\leq \tilde{G}_C = (\frac{|b_{h_B}|}{2})^{-1} \big(G_B F_C(k_C - h_C + 1) + F_B F_C(k_B - h_B + 1)(k_C - h_C + 1) \big)$
$= (\frac{|b_{h_B}|}{2})^{-1} (F_B(k_B - h_B + 1) + G_B) F_C(k_C - h_C + 1);$

(c) $\varepsilon_C = \varepsilon_B \wedge \tilde{\varepsilon}_B' \geq \tilde{\varepsilon}_C = \varepsilon_B \wedge \tilde{\varepsilon}_B$*, where*

$$\tilde{\varepsilon}_B' = \Big(\frac{|b_{h_B}|}{2(\sum_{h_B < i \leq k_B} |b_i| \varepsilon_B^{i-h_B-\delta_B}) + \ G_B \varepsilon_B^{k_B-h_B})} \Big)^{\frac{1}{\delta_B}} \geq \tilde{\varepsilon}_B = \Big(\frac{|b_{h_B}|}{2(F_B(k_B-h_B)+G_B)} \Big)^{\frac{1}{\delta_B}}.$$

(v) *If* $A(\varepsilon), \varepsilon \in (0, \varepsilon_0]$ *is a* $(h_A, k_A, \delta_A, G_A, \varepsilon_A)$*-expansion,* $B(\varepsilon), \varepsilon \in (0, \varepsilon_0]$ *is a pivotal* $(h_B, k_B, \delta_B, G_B, \varepsilon_B)$*-expansion, then there exists* $\varepsilon_D \leq \varepsilon_0' \leq \varepsilon_0$ *such that* $B(\varepsilon) \neq 0, \varepsilon \in (0, \varepsilon_0']$*, and* $D(\varepsilon) = \frac{A(\varepsilon)}{B(\varepsilon)}$ *is a* $(h_D, k_D,\ \delta_D, G_D,\ \varepsilon_D)$*-expansion with parameters* h_D, k_D *and coefficients* $d_r, r = h_D, \ldots, k_D$ *given in proposition* **(v)** *of Lemma 2.2, and parameters* $\delta_D, G_D, \varepsilon_D$ *given by formulas*:

(a) $\delta_D = \delta_A \mathrm{I}(h_C + k_A < h_A + k_C) \ + \ (\delta_A \wedge \delta_C)\mathrm{I}(h_C + k_A = h_A + k_C)$
$+ \ \delta_C \mathrm{I}(h_A + k_C < h_C + k_A) \geq \tilde{\delta}_D = \delta_A \wedge \delta_C = \delta_A \wedge \delta_B;$

(b) $G_D = \sum_{k_D < i+j, h_A \leq i \leq k_A, h_C \leq j \leq k_C} |a_i||c_j| \varepsilon_D^{i+j-k_D-\delta_D}$
$+ \ G_A \sum_{h_C \leq j \leq k_C} |c_j| \varepsilon_D^{j+k_A+\delta_A-k_D-\delta_D} + \ G_C \sum_{h_A \leq i \leq k_A} |a_i| \varepsilon_D^{i+k_C+\delta_C-k_D-\delta_D}$
$+ \ G_A G_C \varepsilon_D^{k_A+k_C+\delta_A+\delta_C-k_D-\delta_D}$
$\leq \tilde{G}_D = (F_A(k_A - h_A + 1) + G_A)(F_C(k_C - h_C + 1) + G_C);$

(c) $\varepsilon_D = \varepsilon_A \wedge \varepsilon_C = \tilde{\varepsilon}_D,$

where coefficients $c_r, r = h_C, \ldots, k_C$ *and parameters* $h_C, k_C, \delta_C, G_C, \varepsilon_C$ *are given for the* $(h_C, k_C, \delta_C, G_C, \varepsilon_C)$*- expansion of function* $C(\varepsilon) = \frac{1}{B(\varepsilon)}$ *in the above proposition* **(iv)***, or by formulas*:

(d) $\delta_D = \delta_A \mathrm{I}(k_A - h_B < k_B - 2h_B + h_A) + (\delta_A \wedge \delta_B)\mathrm{I}(k_A - h_B = k_B - 2h_B + h_A) + \delta_B \mathrm{I}(k_A - h_B > k_B - 2h_B + h_A) \geq \tilde{\delta}_D = \delta_A \wedge \delta_B$;

(e) $G_D = (\frac{|b_{h_B}|}{2})^{-1} \big(\sum_{k_A \wedge (h_A + k_B - h_B) < i \leq k_A} |a_i| \varepsilon_D^{i - h_B - k_D - \delta_D} + G_A \varepsilon_D^{k_A + \delta_A - h_B - k_D - \delta_D}$
$+ \sum_{k_A \wedge (h_A + k_B - h_B) < i + j, h_B \leq i \leq k_B, h_D \leq j \leq k_D} |b_i||d_j| \varepsilon_D^{i + j - k_D - h_B - \delta_D}$
$+ G_B \sum_{h_D \leq j \leq k_D} |d_j| \varepsilon_D^{j + k_B + \delta_B - h_B - k_D - \delta_D} \big)$
$\leq \tilde{G}_D = (\frac{|b_{h_B}|}{2})^{-1} \big(F_A(k_A - k_A \wedge (h_A + k_B - h_B)) + G_A$
$+ F_B F_D (k_B - h_B + 1)(k_D - h_D + 1) + G_B F_D (k_D - h_D + 1) \big)$;

(f) $\varepsilon_D = \varepsilon_A \wedge \varepsilon_B \wedge \tilde{\varepsilon}'_B \geq \tilde{\varepsilon}_D = \varepsilon_A \wedge \varepsilon_B \wedge \tilde{\varepsilon}_B$, *where* $\tilde{\varepsilon}'_B$ *and* $\tilde{\varepsilon}_B$ *are given in relations* **(c)** *of proposition* **(iv)**.

Remark 2.8. Denominators in fractions representing parameters $\tilde{\varepsilon}'_B$ and $\tilde{\varepsilon}_B$ can take value 0. In this case, parameters $\tilde{\varepsilon}'_B$ and $\tilde{\varepsilon}_B$ take value ∞. In particular, this is, according to Remark 2.7, the case, if $B(\varepsilon) \equiv b \neq 0$.

Remark 2.9. Coefficients $\varepsilon_B, \varepsilon_C, \varepsilon_D \in (0,1]$ are taken to nonnegative powers in all terms of the sums, which define parameters G_C, G_D, and $\tilde{\varepsilon}'_B$, in Lemma 2.7. This makes it possible to estimate the corresponding parameters G_C and G_D from above and parameter $\tilde{\varepsilon}'_B$ from below by the corresponding simpler expressions. In particular, the estimates by parameters $\tilde{G}_C$, $\tilde{G}_D$, and $\tilde{\varepsilon}_B$ are obtained by replacing coefficients $\varepsilon_B, \varepsilon_C$, and ε_D by 1.

Remark 2.10. Expansion $C(\varepsilon)$ given in propositions **(i)**–**(iv)** is also a $(h_C, k_C, \tilde{\delta}_C, \tilde{G}_C, \tilde{\varepsilon}_C)$-expansion with parameters $\tilde{\delta}_C, \tilde{G}_C$, and $\tilde{\varepsilon}_C$ given by simpler formulas than parameters δ_C, G_C, and ε_C. Also, expansion $D(\varepsilon)$ given in proposition **(v)** is a $(h_D, k_D, \tilde{\delta}_D, \tilde{G}_D, \tilde{\varepsilon}_D)$-expansion with parameters $\tilde{\delta}_D, \tilde{G}_D$, and $\tilde{\varepsilon}_D$ given by simpler formulas than parameters δ_D, G_D, and ε_D.

Let $A_m(\varepsilon) = a_{h_{A_m},m} \varepsilon^{h_{A_m}} + \cdots + a_{k_{A_m},m} \varepsilon^{k_{A_m}} + o_{A_m}(\varepsilon^{k_{A_m}}), \varepsilon \in (0, \varepsilon_0]$ be a $(h_{A_m}, k_{A_m}, \delta_{A_m}, G_{A_m}, \varepsilon_{A_m})$-expansion, for $m = 1, \ldots, N$.

Also, let us denote $F_{A_m} = \max_{h_{A_m} \leq i \leq k_{A_m}} |a_{i,m}|$, for $m = 1, \ldots, N$.

Lemma 2.7. *The above asymptotic expansions have the following multiple operational rules for computing remainders*:

(i) $B_n(\varepsilon) = A_1(\varepsilon) + \cdots + A_n(\varepsilon), \varepsilon \in (0, \varepsilon_0]$ *is, for every* $n = 1, \ldots, N$, *a* $(h_{B_n}, k_{B_n}, \delta_{B_n}, G_{B_n}, \varepsilon_{B_n})$*-expansion, with parameters* $h_{B_n}, k_{B_n}, n = 1, \ldots, N$ *and coefficients* $b_{h_{B_n}+l,n}, l = 0, \ldots, k_{B_n} - h_{B_n}, n = 1, \ldots, N$ *given in proposition* **(i)** *of Lemma 2.3, and parameters* $G_{B_n}, \delta_{B_n}, \varepsilon_{B_n}, n = 1, \ldots, N$ *given by formulas*:

(a) $\delta_{B_n} = \min_{m \in \mathbb{K}_n} \delta_{A_m} \geq \delta_n^* = \min_{1 \leq m \leq n} \delta_{A_m}$,
where $\mathbb{K}_n = \{m : 1 \leq m \leq n,\ k_{A_m} = \min(k_{A_1}, \ldots, k_{A_n})\}$;

(b) $G_{B_n} = \sum_{1 \leq i \leq n} \big(\sum_{k_{B_n} < j \leq k_{A_i}} |a_{j,i}| \varepsilon_{B_n}^{j - k_{B_n} - \delta_{B_n}} + G_{A_i} \varepsilon_{B_n}^{k_{A_i} + \delta_{A_i} - k_{B_n} - \delta_{B_n}} \big)$
$\leq \sum_{1 \leq i \leq n} \big(F_{A_i}(k_{A_i} - k_{B_n}) + G_{A_i} \big)$;

(c) $\varepsilon_{B_n} = \min(\varepsilon_{A_1}, \ldots, \varepsilon_{A_n})$.

(ii) $C_n(\varepsilon) = A_1(\varepsilon) \times \cdots \times A_n(\varepsilon)$, $\varepsilon \in (0, \varepsilon_0]$ *is, for* $n = 1, \ldots, N$, *a* $(h_{C_n}, k_{C_n}, \delta_{C_n}, G_{C_n}, \varepsilon_{C_N})$*-expansion with parameters* $h_{C_n}, k_{C_n}, n = 1, \ldots, N$ *and coefficients* $c_{h_{C_n}+l,n}$, $l = 0, \ldots, k_{C_n} - h_{C_n}, n = 1, \ldots, N$ *given in proposition* **(ii)** *of Lemma 2.3, and parameters* $G_{C_n}, \delta_{C_n}, \varepsilon_{C_n}, n = 1, \ldots, N$ *given by formulas*:

(a) $\delta_{C_n} = \min_{m \in \mathbb{L}_n} \delta_{A_m} \geq \delta_n^*$, *where* $\mathbb{L}_n = \{m : 1 \leq m \leq n, (k_{A_m} + \sum_{1 \leq r \leq n, r \neq m} h_{A_r}) = \min_{1 \leq l \leq n} (k_{A_l} + \sum_{1 \leq r \leq n, r \neq l} h_{A_r})\}$;

(b) $G_{C_n} = \sum_{k_{C_n} < l_1 + \cdots + l_n, h_{A_m} \leq l_m \leq k_{A_m}, 1 \leq m \leq n} \prod_{1 \leq m \leq n} |a_{l_m,m}| \varepsilon_{C_n}^{l_1 + \cdots + l_n - k_{C_n} - \delta_{C_n}}$
$+ \sum_{1 \leq r \leq n} \sum_{1 \leq m_1 < \cdots < m_r \leq n} \left(\prod_{m \neq m_1, \ldots, m_r, 1 \leq m \leq n} \left(\sum_{h_{A_m} \leq l \leq k_{A_m}} |a_{l,m}| \varepsilon_{C_n}^{l} \right) \right.$
$\left. \times \prod_{1 \leq i \leq r} G_{A_{m_i}} \varepsilon_{C_n}^{k_{A_{m_i}} + \delta_{A_{m_i}} - k_{C_n} - \delta_{C_n}} \right)$
$\leq \prod_{1 \leq m \leq n} (F_{A_m}(k_{A_m} - h_{A_m} + 1) + G_{A_m})$.

(c) $\varepsilon_{C_n} = \min_{1 \leq i \leq n} \varepsilon_{A_i}$.

(iii) *Parameters* $\delta_{B_n}, G_{B_n}, \varepsilon_{B_n}, n = 1, \ldots, N$ *and* $\delta_{C_n}, G_{C_n}, \varepsilon_{C_n}, n = 1, \ldots, N$ *in upper bounds for remainders in the asymptotic expansions, respectively, for functions* $B_n(\varepsilon) = A_1(\varepsilon) + \cdots + A_n(\varepsilon)$, $n = 1, \ldots, N$ *and* $C_n(\varepsilon) = A_1(\varepsilon) \times \cdots \times A_n(\varepsilon), n = 1, \ldots, N$ *are invariant with respect to any permutation, respectively, of summation and multiplication order in the above formulas.*

2.2.3 Algebraic Properties of Operational Rules for Laurent Asymptotic Expansions with Explicit Upper Bounds for Remainders

The summation and multiplication rules for computing of upper bounds for remainders given in propositions **(ii)** and **(iii)** of Lemma 2.6 possess the communicative property, but do not possess the associative and distributional properties.

Lemma 2.6 let us get an effective low bound for parameter δ_A for any $(h_A, k_A, \delta_A, G_A, \varepsilon_A)$-expansion $A(\varepsilon)$ obtained as the result of a finite sequence of operations (described in Lemma 2.6) performed over expansions from some finite set of such expansions.

Lemma 2.8. *The summation and multiplication operations for Laurent asymptotic expansions defined in Lemma 2.6 possess the following algebraic properties, which should be understood as equalities for the corresponding parameters of upper bounds for their remainders*:

(i) *The functional identity,* $C(\varepsilon) \equiv A(\varepsilon) + B(\varepsilon) \equiv B(\varepsilon) + A(\varepsilon)$, *implies that* $\delta_C = \delta_{A+B} = \delta_{B+A}, G_C = G_{A+B} = G_{B+A}$ *and* $\varepsilon_C = \varepsilon_{A+B} = \varepsilon_{B+A}$.

(ii) *The functional identity,* $C(\varepsilon) \equiv A(\varepsilon) \cdot B(\varepsilon) \equiv B(\varepsilon) \cdot A(\varepsilon)$, *implies that* $\delta_C = \delta_{A \cdot B} = \delta_{B \cdot A}, G_C = G_{A \cdot B} = G_{B \cdot A}$ *and* $\varepsilon_C = \varepsilon_{A \cdot B} = \varepsilon_{B \cdot A}$.

(iii) *If* $A(\varepsilon)$ *is* $(h_A, k_A, \delta_A, G_A, \varepsilon_A)$*-expansion obtained as the result of a finite sequence of operations (multiplication by a constant, summation, multiplication, and division) performed over* $(h_{A_i}, k_{A_i}, \delta_{A_i}, G_{A_i}, \varepsilon_{A_i})$*-expansions* $A_i(\varepsilon), i = 1, \ldots, N$, *ac-*

cording to the rules presented in Lemmas 2.2 and 2.6, then $\delta_A \geq \delta_N^* = \min_{1\leq i\leq N}\delta_{A_i}$. *This makes it possible to rewrite* $A(\varepsilon)$ *as the* $(h_A, k_A, \delta_N^*, G_{A,N}^*, \varepsilon_A)$*-expansion, with parameter* $G_{A,N}^* = G_A \varepsilon_A^{\delta_A - \delta_N^*}$.

2.3 Proofs of Lemmas 2.1–2.8

In Section 2.3, we give proofs of Lemmas 2.1–2.8 omitting some known or obvious details. The chosen order of proofs has the sense because of the corresponding pairs of lemmas relate to the same functional identities, as Lemmas 2.1 and 2.5, or the same operational rules, as Lemmas 2.2 and 2.6 or 2.3 and 2.7, or analogous algebraic properties of the corresponding operational rules, as Lemmas 2.4 and 2.8. In each pair, the first lemma relates to Laurent asymptotic expansions with remainders given in the standard form, while the second lemma relates to Laurent asymptotic expansions with explicit upper bounds for remainders.

2.3.1 Lemmas 2.1 and 2.5

Let us, for the moment, use notations $A'(\varepsilon) = a'_{h'_A}\varepsilon^{h'_A} + \cdots + a'_{k'_A}\varepsilon^{k'_A} + o'_A(\varepsilon^{k'_A})$ and $A''(\varepsilon) = a''_{h''_A}\varepsilon^{h''_A} + \cdots + a''_{k''_A}\varepsilon^{k''_A} + o''_A(\varepsilon^{k''_A})$.

In the case $h'_A = h''_A$, proposition **(i)** of Lemma 2.1 is trivial. Let, for example, $h'_A < h''_A = h_A$. In this case, the assumption that proposition **(i)** does not hold, implies that there exists $h'_A \leq l < h_A$ such that $a'_k = 0$, for $h'_A \leq k < l$ and $a'_l \neq 0$. This implies that $e^{-l}A'(\varepsilon) \to a'_l \neq 0$ as $\varepsilon \to 0$, while $e^{-l}A''(\varepsilon) \to 0$ as $\varepsilon \to 0$. This contradicts to the initial identity $A(\varepsilon) \equiv A'(\varepsilon) \equiv A''(\varepsilon)$. Proposition **(i)** let us rewrite $A'(\varepsilon)$ and $A''(\varepsilon)$ in the more informative forms, $A'(\varepsilon) = a'_{h_A}\varepsilon^{h_A} + \cdots + a'_{k'_A}\varepsilon^{k'_A} + o'_A(\varepsilon^{k'_A})$ and $A''(\varepsilon) = a''_{h_A}\varepsilon^{h_A} + \cdots + a''_{k''_A}\varepsilon^{k''_A} + o''_A(\varepsilon^{k''_A})$. These representations imply that $\varepsilon^{-h_A}A'(\varepsilon) \to a'_{h_A}$ as $\varepsilon \to 0$ and $\varepsilon^{-h_A}A''(\varepsilon) \to a''_{h_A}$ as $\varepsilon \to 0$. These relations imply that $a_{h_A} = a'_{h'_A} = a''_{h''_A}$, since $A(\varepsilon) \equiv A'(\varepsilon) \equiv A''(\varepsilon)$. Also, $\varepsilon^{-h_A-1}(A'(\varepsilon) - a_{h_A}\varepsilon^{h_A}) \to a'_{h_A+1}$ as $\varepsilon \to 0$ and $\varepsilon^{-h_A-1}(A''(\varepsilon) - a_{h_A}\varepsilon^{h_A}) \to a''_{h_A+1}$ as $\varepsilon \to 0$. These relations imply that $a_{h_A+1} = a'_{h_A+1} = a''_{h_A+1}$, since $A(\varepsilon) - a_{h_A}\varepsilon^{h_A} \equiv A'(\varepsilon) - a_{h_A}\varepsilon^{h_A} \equiv A''(\varepsilon) - a_{h_A}\varepsilon^{h_A}$. By continuing the above procedure, we can prove that $a_l = a'_l = a''_l$, for $l = h_a \ldots, \tilde{k}_A = k'_A \wedge k''_A$, proposition **(ii)** of the lemma. By canceling the terms, $a_l\varepsilon^l \equiv a'_l\varepsilon^l \equiv a''_l\varepsilon^l, l = h_A, \ldots, \tilde{k}_A$ on the left- and right-hand sides of the initial identity $A(\varepsilon) = A'(\varepsilon) \equiv A''(\varepsilon)$, we get the identity, $\sum_{\tilde{k}_A < l \leq k'_A} a'_l\varepsilon^l + o'_A(\varepsilon^{k'_A}) \equiv \sum_{\tilde{k}_A < l \leq k''_A} a''_l\varepsilon^l + o''_A(\varepsilon^{k''_A})$, given in proposition **(v)**. This identity let us rewrite the Laurent asymptotic expansions $A(\varepsilon)$ in one of the two alternative forms $A(\varepsilon) \equiv A'(\varepsilon) \equiv \sum_{h_A \leq \tilde{k}_A} a_l\varepsilon^l + \sum_{\tilde{k}_A < l \leq k'_A} a'_l\varepsilon^l + o'_A(\varepsilon^{k'_A})$ or $A(\varepsilon) \equiv A''(\varepsilon) \equiv \sum_{h_A \leq \tilde{k}_A} a_l\varepsilon^l + \sum_{\tilde{k}_A < l \leq k''_A} a''_l\varepsilon^l + o''_A(\varepsilon^{k''_A})$. The first or second representation should be chosen as more informative if, respectively, $k'_A > k''_A$ or $k'_A < k''_A$. These representations coincide, if $k'_A = k''_A$. The above remark prove propositions **(iii)–(v)** of Lemma 2.1.

Lemma 2.5, in particular, relation (2.3) directly follows from formulas of remainders of the Laurent asymptotic expansion $A(\varepsilon)$ given in proposition **(v)** of Lemma 2.1 and priority rules for choice of parameters δ_A, G_A, and ε_A for explicit upper bounds for the above remainder, which are used in Lemma 2.5. □

2.3.2 *Lemmas 2.2 and 2.6*

Propositions **(i)** (the multiplication by a constant rule) of Lemmas 2.2 and 2.6 are obvious. Proposition **(ii)** (the summation rules) of Lemmas 2.2 and 2.6 can be obtained by simple accumulation of coefficients for different powers of ε and terms accumulated in the corresponding remainders, and, then, by using obvious upper bounds for absolute values of sums of terms accumulated in the corresponding remainders.

Proposition **(iii)** (the multiplication rule) of Lemma 2.2 can be proved by multiplication of the corresponding asymptotic expansions $A(\varepsilon)$ and $B(\varepsilon)$ and accumulation of coefficients for powers ε^l for $l = h_C, \dots, k_C$ in their product,

$$\begin{aligned} C(\varepsilon) &= A(\varepsilon)B(\varepsilon) \\ &= (a_{h_A}\varepsilon^{h_A} + \cdots + a_{k_A}\varepsilon^{k_A} + o_A(\varepsilon^{k_A}))(b_{h_B}\varepsilon^{h_B} + \cdots + b_{k_B}\varepsilon^{k_B} + o_B(\varepsilon^{k_B})) \\ &= \sum_{h_C \le l \le k_C} \sum_{i+j=l, h_A \le i \le k_A, h_B \le j \le k_B} a_i b_j \varepsilon^l \\ &\quad + \sum_{k_C < i+j, h_A \le i \le k_A, h_B \le j \le k_B} a_i b_j \varepsilon^{i+j} + \sum_{h_B \le j \le k_B} b_j \varepsilon^j o_A(\varepsilon^{k_A}) \\ &\quad + \sum_{h_A \le i \le k_A} a_i \varepsilon^i o_B(\varepsilon^{k_B}) + o_A(\varepsilon^{k_A}) o_B(\varepsilon^{k_B}) = \sum_{h_C \le l \le k_C} c_l \varepsilon^l + o_C(\varepsilon^{k_C}), \end{aligned} \tag{2.4}$$

where

$$\begin{aligned} o_C(\varepsilon^{k_C}) = &\sum_{k_C < i+j, h_A \le i \le k_A, h_B \le j \le k_B} a_i b_j \varepsilon^{i+j} + \sum_{h_B \le j \le k_B} b_j \varepsilon^j o_A(\varepsilon^{k_A}) \\ &+ \sum_{h_A \le i \le k_A} a_i \varepsilon^i o_B(\varepsilon^{k_B}) + o_A(\varepsilon^{k_A}) o_B(\varepsilon^{k_B}). \end{aligned} \tag{2.5}$$

Obviously, $\frac{o_C(\varepsilon^{k_C})}{\varepsilon^{k_C}} \to 0$ as $\varepsilon \to 0$. It should be noted that the accumulation of coefficients for powers ε^l can be made in (2.4) only up to the maximal value $l = k_C = (k_A + h_B) \wedge (k_B + h_A)$, because of the presence in the expression for remainder $o_C(\varepsilon^{k_C})$ terms $b_{h_B}\varepsilon^{h_B} o_A(\varepsilon^{k_A})$ and $a_{h_A}\varepsilon^{h_A} o_B(\varepsilon^{k_B})$.

Also, relation (2.5) and Remark 2.9 readily imply relations **(a)**–**(c)**, which determines parameters $\delta_C, G_C, \varepsilon_C$ and $\tilde{\delta}_C, \tilde{G}_C, \tilde{\varepsilon}_C$ in proposition **(iii)** of Lemma 2.6, in particular, the following inequalities take place, for $\varepsilon \in (0, \varepsilon_C]$,

$$|\varepsilon^{-k_C-\delta_C} o_C(\varepsilon^{k_C})| \le G_C = \sum_{k_C<i+j, h_A\le i\le k_A, h_B\le j\le k_B} |a_i||b_j|\varepsilon_C^{i+j-k_C-\delta_C}$$

$$+ G_A \sum_{h_B\le j\le k_B} |b_j|\varepsilon_C^{j+k_A+\delta_A-k_C-\delta_C} + G_B \sum_{h_A\le i\le k_A} |a_i|\varepsilon_C^{i+k_B+\delta_B-k_C-\delta_C}$$

$$+ G_A G_B \varepsilon_C^{k_A+k_B+\delta_A+\delta_B-k_C-\delta_C}$$

$$\le \tilde{G}_C = F_A F_B (k_A - h_A + 1)(k_B - h_B + 1)$$

$$+ G_A F_B (k_B - h_B + 1) + G_B F_A (k_A - h_A + 1) + G_A G_B$$

$$= (F_A(k_A - h_A + 1) + G_A)(F_B(k_B - h_B + 1) + G_B). \tag{2.6}$$

The assumptions of proposition **(iv)** in Lemma 2.2 imply that $\varepsilon^{-h_B} B(\varepsilon) \to b_{h_B} \neq 0$ as $\varepsilon \to 0$. This relation implies that there exists $0 < \varepsilon_0' \le \varepsilon_0$ such that $B(\varepsilon) \neq 0$ for $\varepsilon \in (0, \varepsilon_0']$, and, thus, function $C(\varepsilon) = \frac{1}{B(\varepsilon)}$ is well defined for $\varepsilon \in (0, \varepsilon_0']$.

Note that $h_B \le k_B$. The assumptions of proposition **(iv)** of Lemma 2.2 imply that, $\varepsilon^{h_B} C(\varepsilon) = (b_{h_B} + \cdots + b_{k_B}\varepsilon^{k_B-h_B} + o_B(\varepsilon^{h_B})\varepsilon^{-h_B})^{-1} \to b_{h_B}^{-1} = c_{h_C}$ as $\varepsilon \to 0$.

This relation means that function $\varepsilon^{h_B} C(\varepsilon)$ can be represented in the form $\varepsilon^{h_B} C(\varepsilon) = c_{h_C} + o(1)$, where $c_{h_C} = b_{h_B}^{-1}$, or, equivalently, that the following representation takes place, $C(\varepsilon) = c_{h_C}\varepsilon^{-h_B} + o_1(\varepsilon^{-h_B}), \varepsilon \in (0, \varepsilon_0']$, where $\frac{o_1(\varepsilon^{-h_B})}{\varepsilon^{-h_B}} \to 0$ as $\varepsilon \to 0$.

The latter two relations prove proposition **(iv)** of Lemma 2.2, for the case $h_B = k_B$. Indeed, these relations mean that function $C(\varepsilon) = \frac{1}{B(\varepsilon)}$ can be represented in the form of (h_C, k_C)-expansion with parameters $h_C = -h_B$, $k_C = k_B - 2h_B = -h_B = h_C$ and coefficient $c_{h_C} = b_{h_B}^{-1}$. Moreover, since $B(\varepsilon) \cdot C(\varepsilon) \equiv 1, 0 < \varepsilon \le \varepsilon_0'$, remainder $c_1(\varepsilon)$ can be found from the following relation, $(b_{h_B}\varepsilon^{h_B} + o(\varepsilon^{h_B}))(c_{h_C}\varepsilon^{-h_B} + o_1(\varepsilon^{-h_B})) \equiv 1$ that yields formula, $o_1(\varepsilon^{-h_B}) = -\frac{c_{h_C}\varepsilon^{-h_B} o_B(\varepsilon^{h_B})}{b_{h_B}\varepsilon^{h_B} + o_B(\varepsilon^{h_B})}$.

This is formula **(c)** from proposition **(iv)** of Lemma 2.2, for the case $h_B = k_B$. Note that, in the case $h_B = k_B$, the above asymptotic expansion for function $C(\varepsilon)$ cannot be extended. Indeed, $\varepsilon^{h_B-1} o_1(\varepsilon^{-h_B}) = \varepsilon^{h_B-1}(C(\varepsilon) - c_{h_C}\varepsilon^{-h_B}) = -\frac{c_{h_C}}{b_{h_B} + o_B(\varepsilon^{h_B})\varepsilon^{-h_B}} \frac{o_B(\varepsilon^{h_B})\varepsilon^{-h_B}}{\varepsilon}$. The term $\frac{o_B(\varepsilon^{h_B})\varepsilon^{-h_B}}{\varepsilon}$ on the right-hand side in the latter relation has an uncertain asymptotic behavior as $\varepsilon \to 0$.

Let us now assume that $h_B + 1 \le k_B$. In this case, the assumptions of proposition **(iv)** of Lemma 2.2 and the above asymptotic relations imply that, $\varepsilon^{h_B-1} o_1(\varepsilon^{-h_B}) = \varepsilon^{h_B-1}(C(\varepsilon) - c_{h_C}\varepsilon^{-h_B}) = (b_{h_B} + \cdots + b_{k_B}\varepsilon^{k_B-h_B} + o_B(\varepsilon^{k_B})\varepsilon^{-h_B})^{-1}(-b_{h_B+1}c_{h_C} - \cdots - b_{k_B}c_{h_C}\varepsilon^{k_B-h_B-1} - o_B(\varepsilon^{k_B})c_{h_C}\varepsilon^{-h_B-1}) \to -b_{h_B}^{-1} b_{h_B+1} c_{h_C} = c_{h_C+1}$ as $\varepsilon \to 0$.

This relation means that function $\varepsilon^{h_B-1} o_1(\varepsilon^{-h_B})$ can be represented in the form $\varepsilon^{h_B-1} o_1(\varepsilon^{-h_B}) = c_{h_C+1} + o(1)$, where $c_{h_C+1} = -b_{h_B}^{-1} b_{h_B+1} c_{h_C}$, or, equivalently, that the following representation takes place, $C(\varepsilon) = c_{h_C}\varepsilon^{-h_B} + c_{h_C+1}\varepsilon^{-h_B+1} + o_2(\varepsilon^{-h_B+1}), \varepsilon \in (0, \varepsilon_0']$, where $\frac{o_2(\varepsilon^{-h_B+1})}{\varepsilon^{-h_B+1}} \to 0$ as $\varepsilon \to 0$.

The latter two relations prove proposition **(iv)** of Lemmas 2.2, for the case $h_B + 1 = k_B$. Indeed, these relations mean that function $C(\varepsilon)$ can be represented in

the form of (h_C,k_C)-expansion with parameters $h_C=-h_B$, $k_C=k_B-2h_B=-h_B+1=h_C+1$ and coefficients $c_{h_C}=b_{h_B}^{-1}$, $c_{h_C+1}=-b_{h_B}^{-1}b_{h_B+1}c_{h_C}$. Moreover, since $B(\varepsilon)\cdot C(\varepsilon)\equiv 1$, the remainder $o_2(\varepsilon^{-h_B+1})$ can be found from the following relation: $(b_{h_B}\varepsilon^{h_B}+b_{h_B+1}\varepsilon^{h_B+1}+o_B(\varepsilon^{h_B+1}))(c_{h_C}\varepsilon^{-h_B}+c_{h_C+1}\varepsilon^{-h_B+1}+o_2(\varepsilon^{-h_B+1}))\equiv 1$. This yields formula, $o_2(\varepsilon^{-h_B+1})=-\frac{b_{h_B+1}c_{h_C+1}\varepsilon^2+(c_{h_C}\varepsilon^{-h_B}+c_{h_C+1}\varepsilon^{-h_B+1})o_B(\varepsilon^{h_B+1})}{b_{h_B}\varepsilon^{h_B}+b_{h_B+1}\varepsilon^{h_B+1}+o_B(\varepsilon^{h_B+1})}$.

This is formula **(c)** from proposition **(iv)** of Lemma 2.2, for the case $h_B+1=k_B$. Note that, in the case $h_B+1=k_B$, the above asymptotic expansion for function $C(\varepsilon)$ cannot be extended. Indeed, $\varepsilon^{h_B-2}o_2(\varepsilon)=\varepsilon^{h_B-2}(C(\varepsilon)-c_{h_C}\varepsilon^{-h_B}-c_{h_C+1}\varepsilon^{-h_B+1})=-\frac{b_{h_B+1}c_{h_C+1}+c_{h_C+1}o_B(\varepsilon^{h_B+1})\varepsilon^{-h_B-1}}{b_{h_B}+b_{h_B+1}\varepsilon+o_B(\varepsilon^{h_B+1})\varepsilon^{-h_B}}-\frac{c_{h_C}}{b_{h_B}+b_{h_B+1}\varepsilon+o_B(\varepsilon^{h_B+1})\varepsilon^{-h_B}}\frac{o_B(\varepsilon^{h_B+1})\varepsilon^{-h_B-1}}{\varepsilon}$. The term $\frac{o_B(\varepsilon^{h_B+1})\varepsilon^{-h_B-1}}{\varepsilon}$ on the right-hand side in the latter relation has an uncertain asymptotic behavior as $\varepsilon\to 0$.

We can repeat the above arguments for the general case $h_B+n=k_B$, for any $n=0,1,\dots$ and to prove that, in the case $h_B+n=k_B$, function $C(\varepsilon)$ can be represented in the form of (h_C,k_C)-expansion with parameters $h_C=-h_B,k_C=k_B-2h_B=-h_B+n=h_C+n$ and coefficients $c_{h_C},\dots,c_{k_C}$ given in proposition **(iv)** of Lemma 2.2. Moreover, identity $B(\varepsilon)\cdot C(\varepsilon)\equiv 1, 0<\varepsilon\le\varepsilon_0'$, let us find the corresponding remainder $o_C(\varepsilon^{k_C})$ from the following relation:

$$(b_{h_B}\varepsilon^{h_B}+\dots+b_{k_B}\varepsilon^{k_B}+o_B(\varepsilon^{k_B}))(c_{h_C}\varepsilon^{h_C}+\dots+c_{h_C}\varepsilon^{k_C}+o_C(\varepsilon^{k_C}))\equiv 1. \quad (2.7)$$

Proposition **(iii)** of Lemma 2.2, applied to the product on the left-hand side in relation (2.7), permits to represent this product in the form of (h,k)-expansion with parameters $h=h_B+h_C=h_B-h_B=0$ and $k=(k_B+h_C)\wedge(h_B+k_C)=(k_B-h_B)\wedge(k_B-2h_B+h_B)=k_B-h_B$. By canceling coefficient for ε^l on the left- and right-hand sides in relation (2.7), for $l=0,\dots,k_B-h_B$, and then, by solving equation (2.7) with respect to the remainder $o_C(\varepsilon^{k_C})$, we get the formula for this remainder given in proposition **(iv)** of Lemma 2.2,

$$\begin{aligned} o_C(\varepsilon^{k_C}) &= -\frac{\sum_{k_B-h_B<i+j,h_B\le i\le k_B,h_C\le j\le k_C} b_ic_j\varepsilon^{i+j}+\sum_{h_C\le j\le k_C}c_j\varepsilon^j o_B(\varepsilon^{k_B})}{b_{h_B}\varepsilon^{h_B}+\dots+b_{k_B}\varepsilon^{k_B}+o_B(\varepsilon^{k_B})} \\ &= -\frac{\sum_{k_B-h_B<i+j,h_B\le i\le k_B,h_C\le j\le k_C} b_ic_j\varepsilon^{i+j-h_B}}{b_{h_B}+\dots+b_{k_B}\varepsilon^{k_B-h_B}+o_B(\varepsilon^{k_B})\varepsilon^{h_B}} \\ &\quad -\frac{\sum_{h_C\le j\le k_C}c_j\varepsilon^{j-h_B}o_B(\varepsilon^{k_B})}{b_{h_B}+\dots+b_{k_B}\varepsilon^{k_B-h_B}+o_B(\varepsilon^{k_B})\varepsilon^{-h_B}}. \end{aligned} \quad (2.8)$$

The assumptions made in proposition **(iv)** of Lemma 2.6 imply that $B(\varepsilon)\ne 0$ and the following inequality holds for $0<\varepsilon\le\varepsilon_C$, where ε_C is given in proposition **(iv)** of Lemma 2.6:

$$\begin{aligned} &|b_{h_B+1}\varepsilon+\dots+b_{k_B}\varepsilon^{k_B-h_B}+o_B(\varepsilon^{k_B})\varepsilon^{-h_B}|=\varepsilon^{\delta_B}(|b_{h_B+1}|\varepsilon^{1-\delta_B}+\dots \\ &\quad +|b_{k_B}|\varepsilon^{k_B-h_B-\delta_B}+G_B\varepsilon^{k_B-h_B})\le\varepsilon^{\delta_B}(F_B(k_B-h_B)+G_B)\le\frac{|b_{h_B}|}{2}, \end{aligned} \quad (2.9)$$

and, thus,

$$|b_{h_B}+b_{h_B+1}\varepsilon+\cdots+b_{k_B}\varepsilon^{k_B-h_B}+o_B(\varepsilon^{k_B})\varepsilon^{-h_B}|\geq|b_{h_B}|$$
$$-(|b_{h_B+1}|\varepsilon+\cdots+|b_{k_B}|\varepsilon^{k_B-h_B}+G_B\varepsilon^{k_B-h_B+\delta_B})\geq\frac{|b_{h_B}|}{2}>0. \quad (2.10)$$

The existence of ε_0' declared in proposition **(iv)** of Lemma 2.6 is obvious. For example, one can choose $\varepsilon_0'=\varepsilon_C$. It is also useful to note that formulas given in proposition **(iv)** of Lemma 2.6 imply that $\varepsilon_C=\varepsilon_B\wedge\tilde{\varepsilon}_B'\in(0,\varepsilon_0]$, since $\varepsilon_B\in(0,\varepsilon_0]$ and $\tilde{\varepsilon}_B'\in(0,\infty]$.

The assumptions made in proposition **(iv)** of Lemma 2.6, inequality (2.10), and Remark 2.8 imply that the following inequality holds, for $0<\varepsilon\leq\varepsilon_C$:

$$\begin{aligned}|\varepsilon^{-k_C-\delta_C}o_C(\varepsilon^{k_C})|&=|\varepsilon^{-k_B+2h_B-\delta_B}o_C(\varepsilon^{k_C})|\\&\leq(\frac{|b_{h_B}|}{2})^{-1}\big(G_B\sum_{h_C\leq j\leq k_C}|c_j|\varepsilon_C^{j+h_B}\\&\quad+\sum_{k_B-h_B<i+j,h_B\leq i\leq k_B,h_C\leq j\leq k_C}|b_i||c_j|\varepsilon_C^{i+j-k_B+h_B-\delta_B}\big)\\&\leq\tilde{G}_C=(\frac{|b_{h_B}|}{2})^{-1}\big(G_BF_C(k_C-h_C+1)\\&\quad+F_BF_C(k_B-h_B+1)(k_C-h_C+1)\big)\\&=(\frac{|b_{h_B}|}{2})^{-1}(G_B+F_B(k_B-h_B+1))F_C(k_C-h_C+1). \quad (2.11)\end{aligned}$$

Inequality (2.11) proofs proposition **(iv)** of Lemma 2.6.

Propositions **(v)** of Lemmas 2.2 and 2.6 and relations **(a)–(c)** given in these propositions can be obtained by direct application, respectively, of propositions **(iii)** and **(iv)** of Lemmas 2.2 and 2.6 to the product $D(\varepsilon)=A(\varepsilon)\cdot\frac{1}{B(\varepsilon)}$.

Now, when it is already known that $D(\varepsilon)=A(\varepsilon)\cdot\frac{1}{B(\varepsilon)}$ is a (h_D,k_D)-expansion, with parameters $h_D=h_A-h_B$ and $k_D=(k_A-h_B)\wedge(k_B-2h_B+h_A)$, multiplication of $D(\varepsilon)$ by $B(\varepsilon)$ yields the following relation holding for $\varepsilon\in(0,\varepsilon_0']$:

$$\begin{aligned}A(\varepsilon)&=D(\varepsilon)B(\varepsilon)=a_{h_A}\varepsilon^{h_A}+\cdots+a_{h_A}\varepsilon^{k_A}+o_A(\varepsilon^{k_A})\\&=(d_{h_D}\varepsilon^{h_D}+\cdots+d_{h_D}\varepsilon^{k_D}+o_D(\varepsilon^{k_D}))(b_{h_B}\varepsilon^{h_B}+\cdots+b_{h_B}\varepsilon^{k_B}+o_B(\varepsilon^{k_B})). \quad (2.12)\end{aligned}$$

By equating coefficients for powers ε^l for $l=h_D,\ldots,k_D$ on the left- and right-hand sides of the third equality in relation (2.12), we get alternative formulas **(e)** for coefficients $d_{h_d},\ldots,d_{k_D}$ given in proposition **(v)** of Lemma 2.2.

Proposition **(iii)** of Lemma 2.2, applied to the product on the right-hand side in (2.12), permits to represent this product in the form of (h,k)-expansion with parameters $h=h_B+h_D=h_B+h_A-h_B=h_A$ and $k=(k_D+h_B)\wedge(k_B+h_D)=((k_A-h_B)\wedge(k_B-2h_B+h_A)+h_B)\wedge(k_B+h_A-h_B)=k_A\wedge(k_B+h_A-h_B)$. By canceling coefficient for ε^l on the left- and right-hand sides in relation (2.12), for

$l = h_A, \ldots, k_A \wedge (k_B + h_A - h_B)$, and then, by solving equation (2.12) with respect to the remainder $o_D(\varepsilon^{k_D})$, we get the formula **(f)** for this remainder given in proposition **(v)** of Lemma 2.2,

$$
\begin{aligned}
o_D(\varepsilon^{k_D}) &= \frac{\sum_{k_A \wedge (k_B+h_A-h_B) < l \le k_A} a_l \varepsilon^l + o_A(\varepsilon^{k_A})}{b_{h_B}\varepsilon^{h_B} + \cdots + b_{k_B}\varepsilon^{k_B} + o_B(\varepsilon^{k_B})} \\
&\quad - \frac{\sum_{k_A \wedge (k_B+h_A-h_B) < i+j, h_B \le i \le k_B, h_D \le j \le k_D} b_i d_j \varepsilon^{i+j}}{b_{h_B}\varepsilon^{h_B} + \cdots + b_{k_B}\varepsilon^{k_B} + o_B(\varepsilon^{k_B})} \\
&\quad - \frac{\sum_{h_D \le j \le k_D} d_j \varepsilon^j o_B(\varepsilon^{k_B})}{b_{h_B}\varepsilon^{h_B} + \cdots + b_{k_B}\varepsilon^{k_B} + o_B(\varepsilon^{k_B})} \\
&= \frac{\sum_{k_A \wedge (k_B+h_A-h_B) < l \le k_A} a_l \varepsilon^{l-h_B} + o_A(\varepsilon^{k_A})\varepsilon^{-h_B}}{b_{h_B} + \cdots + b_{k_B}\varepsilon^{k_B-h_B} + o_B(\varepsilon^{k_B})\varepsilon^{-h_B}} \\
&\quad - \frac{\sum_{k_A \wedge (k_B+h_A-h_B) < i+j, h_B \le i \le k_B, h_D \le j \le k_D} b_i d_j \varepsilon^{i+j-h_B}}{b_{h_B} + \cdots + b_{k_B}\varepsilon^{k_B-h_B} + o_B(\varepsilon^{k_B})\varepsilon^{-h_B}} \\
&\quad - \frac{\sum_{h_D \le j \le k_D} d_j \varepsilon^{j-h_B} o_B(\varepsilon^{k_B})}{b_{h_B} + \cdots + b_{k_B}\varepsilon^{k_B-h_B} + o_B(\varepsilon^{k_B})\varepsilon^{-h_B}}.
\end{aligned}
\tag{2.13}
$$

Inequality (2.10), the assumptions made in proposition **(v)** of Lemma 2.6 and Remark 2.8 finally imply that the following inequality holds, for $0 < \varepsilon \le \varepsilon_D$ given in relation **(f)** of this proposition:

$$
\begin{aligned}
|\varepsilon^{-k_D-\delta_D} o_D(\varepsilon^{k_D})| &\le G_D = \Big(\frac{|b_{h_B}|}{2}\Big)^{-1} \\
&\quad \times \Big(\sum_{k_A \wedge (h_A+k_B-h_B) < i \le k_A} |a_i| \varepsilon_D^{i-h_B-k_D-\delta_D} + G_A \varepsilon_D^{k_A+\delta_A-h_B-k_D-\delta_D} \\
&\quad + \sum_{k_A \wedge (h_A+k_B-h_B) < i+j, h_B \le i \le k_B, h_D \le j \le k_D} |b_i||d_j| \varepsilon_D^{i+j-k_D-h_B-\delta_D} \\
&\quad + G_B \sum_{h_D \le j \le k_D} |d_j| \varepsilon_D^{j+k_B+\delta_B-h_B-k_D-\delta_D} \Big) \\
&\le \tilde{G}_D = \Big(\frac{|b_{h_B}|}{2}\Big)^{-1} \big(F_A(k_A - k_A \wedge (h_A + k_B - h_B)) + G_A \\
&\quad F_B F_D(k_B - h_B + 1)(k_D - h_D + 1) + G_B F_D(k_D - h_D + 1) \big).
\end{aligned}
\tag{2.14}
$$

Inequality (2.14) completes the proof of proposition **(v)** of Lemma 2.6. □

Remark 2.11. Quantity $\frac{|b_{h_B}|}{2}$ can be replaced by quantity $\frac{|b_{h_B}|}{\alpha}$, for any $\alpha > 1$. Respectively, quantity $\frac{|b_{h_B}|}{2}$ can be replaced by quantity $\frac{|b_{h_B}|}{\beta}$, where $\beta = \frac{\alpha}{\alpha - 1}$, on the right-hand side of inequality (2.10) and, in sequel, in inequalities (2.11) and (2.14). This makes it possible to modify propositions **(iv)** and **(v)** of Lemma 1.6 by replacing factor $\frac{|b_{h_B}|}{2}$ by new factor $\frac{|b_{h_B}|}{\alpha}$ in the expression for parameters $\tilde{\varepsilon}'_B, \tilde{\varepsilon}_B$ and by

factor $\frac{|b_{h_B}|}{\beta}$ in the expression for parameters $G_C, \tilde{G}_C, G_D, \tilde{G}_D$. The choice of a smaller value for parameter α diminishes values of parameters $\tilde{\varepsilon}'_B, \tilde{\varepsilon}_B$ and $G_C, \tilde{G}_C, G_D, \tilde{G}_D$.

2.3.3 Lemmas 2.3 and 2.7

Lemma 2.3 is a direct corollary of Lemma 2.2.

Proofs of propositions **(i)** and **(ii)** in Lemma 2.7 are analogous to proofs of propositions **(i)** and **(ii)** in Lemma 2.6. Proposition **(iii)** of Lemma 2.7 is obvious. □

2.3.4 Lemmas 2.4 and 2.8

The first two identities for Laurent asymptotic expansions given in proposition **(i)** of Lemma 2.4 are obvious. The third identity given in this proposition follows in an obvious way from proposition **(i)** of Lemma 2.2. By applying propositions **(iii)** and **(iv)** of Lemma 2.2 to the product $C(\varepsilon) = A(\varepsilon) \cdot A(\varepsilon)^{-1}$, we get parameters $h_C = h_{A \cdot A^{-1}} = h_A - h_A = 0$, $k_C = k_{A \cdot A^{-1}} = (k_A - h_A) \wedge (k_A - 2h_A + h_A) = k_A - h_A$ and coefficients $c_n = \mathrm{I}(n = 0), n = 0, \ldots, k_C$. Also, relations (2.7) and (2.8) imply that the elimination identity $A(\varepsilon) \cdot A(\varepsilon)^{-1} \equiv 1$ holds, since the remainder of Laurent asymptotic expansion for function $A(\varepsilon)^{-1}$ is given by formula **(c)** from proposition **(iv)** of Lemma 2.2. Propositions **(ii)** and **(iii)** of Lemma 2.4 in the parts concerned commutative property of summation and multiplication operations follow from, respectively, propositions **(ii)** and **(iii)** of Lemma 2.2.

Let $D(\varepsilon) = (A(\varepsilon) + B(\varepsilon)) + C(\varepsilon) = A(\varepsilon) + (B(\varepsilon) + C(\varepsilon))$. Using propositions **(ii)** of Lemma 2, we get, $h_D = h_{(A+B)+C} = (h_A \wedge h_B) \wedge h_C = h_A \wedge (h_B \wedge h_C) = h_{A+(B+C)}$ and $k_D = k_{(A+B)+C} = (k_A \wedge k_B) \wedge k_C = k_A \wedge (k_B \wedge k_C) = k_{A+(B+C)}$. These relations and Lemma 2.1 imply equalities for the corresponding coefficients and remainders, for the asymptotic expansions of functions $(A(\varepsilon) + B(\varepsilon)) + C(\varepsilon)$ and $A(\varepsilon) + (B(\varepsilon) + C(\varepsilon))$. The above remarks prove proposition **(ii)** of Lemma 2.4 in the part concerned with the associative property of summation operation for Laurent asymptotic expansions.

Let $D(\varepsilon) = (A(\varepsilon) \cdot B(\varepsilon)) \cdot C(\varepsilon) = A(\varepsilon) \cdot (B(\varepsilon) \cdot C(\varepsilon))$. Using propositions **(iii)** of Lemma 2.2, we get, $h_D = h_{(A \cdot B) \cdot C} = h_{A \cdot B} + h_C = h_A + h_B + h_C = h_A + h_{B \cdot C} = h_{A \cdot (B \cdot C)}$ and $k_D = k_{(A \cdot B) \cdot C} = (k_{A \cdot B} + h_C) \wedge (k_C + h_{A \cdot B}) = ((k_A + h_B) \wedge (k_B + h_A)) + h_C) \wedge (k_C + (h_A + h_B)) = (k_A + h_B + h_C) \wedge (k_B + h_A + h_C) \wedge (k_C + h_A + h_B) = (k_A + (h_B + h_C)) \wedge ((k_B + h_C) \wedge (k_C + h_B)) + h_A) = (k_A + h_{B \cdot C}) \wedge (k_{B \cdot C} + h_A) = k_{A \cdot (B \cdot C)}$. These relations and Lemma 2.1 imply equalities for the corresponding coefficients and remainders, for the asymptotic expansions of functions $(A(\varepsilon) \cdot B(\varepsilon)) \cdot C(\varepsilon)$ and $A(\varepsilon) \cdot (B(\varepsilon) \cdot C(\varepsilon))$. The above remarks prove proposition **(iii)** of Lemma 2.4 in the part concerned with the associative property of multiplication operation for Laurent asymptotic expansions.

Let $D(\varepsilon) = (A(\varepsilon) + B(\varepsilon)) \cdot C(\varepsilon) = A(\varepsilon) \cdot C(\varepsilon) + B(\varepsilon) \cdot C(\varepsilon)$. Using propositions **(ii)** and **(iii)** of Lemma 2, we get, $h_D = h_{(A+B) \cdot C} = h_{A+B} + h_C = h_A \wedge h_B + h_C = (h_A + h_C) \wedge (h_B + h_C) = h_{A \cdot C} \wedge h_{B \cdot C} = h_{A \cdot C + B \cdot C}$ and $k_D = k_{(A+B) \cdot C} = (k_{A+B} + h_C) \wedge (k_C + h_{A+B}) = (k_A \wedge k_B + h_C) \wedge (k_C + h_A \wedge h_B) = (k_A + h_C) \wedge (k_B + h_C) \wedge (k_C + h_A) \wedge (k_C + h_B) = ((k_A + h_C) \wedge (k_C + h_A)) \wedge ((k_B + h_C) \wedge (k_C + h_B)) = k_{A \cdot C} \wedge k_{B \cdot C} = k_{A \cdot C + B \cdot C}$.

These relations and Lemma 2.1 imply equalities for the corresponding coefficients and remainders, for the asymptotic expansions of functions $(A(\varepsilon)+B(\varepsilon))\cdot C(\varepsilon)$ and $A(\varepsilon)\cdot C(\varepsilon)+B(\varepsilon)\cdot C(\varepsilon)$. The above remarks prove proposition **(iv)** of Lemma 2.4 concerned with the distributive property of summation and multiplication operations for Laurent asymptotic expansions.

Propositions **(i)** and **(ii)** of Lemma 2.8 readily follow from, respectively, propositions **(ii)** and **(iii)** of Lemma 2.6. Finally, proposition **(iii)** of Lemma 2.8 follows from relations $\delta_{cA}=\delta_A$, $\delta_{A+B},\delta_{A\cdot B}$, $\delta_{A/B}\geq\delta_A\wedge\delta_B$ and $\delta_{A_1+\cdots+A_N}$, $\delta_{A_1\times\cdots\times A_N}\geq \min_{1\leq m\leq N}\delta_{A_m}$, given, respectively, in Lemmas 2.6 and 2.7. □

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