Chapter 2
Stein Manifolds

This chapter is a brief survey of the theory of Stein manifolds and Stein spaces, with emphasis on the results that are frequently used in this book. After the initial developments by Karl Weierstrass, Bernhard Riemann, Fritz Hartogs, Eugenio E. Levi, Karl Reinhardt, Hellmuth Kneser, Henri Cartan, Peter Thullen and many others, the main contributions were made in the period 1942–1965 by Kiyoshi Oka, by the French school around Henri Cartan including Pierre Dolbeault, Alexander Grothendieck and Jean-Pierre Serre, and by the Münster school founded by Heinrich Behnke and including Karl Stein, Hans Grauert, Reinhold Remmert and Friedrich Hirzebruch. In 1942, Oka [444, Chap. VI] published the first solution to the Levi problem on two dimensional domains, while the year 1965 marks the publication of Lars Hörmander's fundamental paper [299] in which the $\overline{\partial}$-equation was solved by L^2-methods. Another contemporary work using the L^2-approach on q-convex manifolds is due to Aldo Andreotti and Edoardo Vesentini [28]. Together with the works of Joseph J. Kohn [345, 346], these provide the basis for quantitative methods in complex analysis. Comprehensive accounts of the theory of Stein manifolds and Stein spaces are available in [260, 274, 300], while the article of Schumacher [490] provides a historical survey. An introduction to topics in L^2-theory can be found in Ohsawa's book [441], while his recent book [442] presents an L^2 approach to problems in several complex variables and differential and algebraic geometry.

2.1 Domains of Holomorphy

A basic notion in complex analysis is that of analytic continuation. Karl Weierstrass knew already in 1841 that a holomorphic function in an annulus in the complex plane $\mathbb{C}$ admits a development into what is now called a Laurent series. By estimating the coefficients in this series, Bernhard Riemann showed in his dissertation in 1851 that a function which is analytic in a punctured neighborhood of a point $p \in \mathbb{C}$ and is bounded near p extends to a holomorphic function in a neighborhood of p. It was known early on that on any open relatively compact set $D \Subset \mathbb{C}$ in $\mathbb{C}$ there exist holomorphic functions that do not extend holomorphically across any boundary

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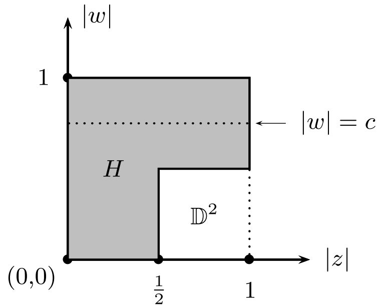
Fig. 2.1 A Hartogs figure in the bidisc



point of D. An explicit example on the disc $\mathbb{D} = \{|z| < 1\}$ is Kronecker's function $f(z) = \sum_{n=1}^{\infty} z^{n^2}$; further examples were given by Weierstrass.

A fundamental discovery was the phenomenon of *simultaneous analytic continuation*. In 1897 Adolph Hurwitz showed in his lecture at the first International Congress of Mathematicians that a holomorphic function of two or more variables does not have isolated singularities. More interesting examples of analytic continuation were found by Friedrich Hartogs in 1906 [280]. The simplest *Hartogs figure* is the domain H in the bidisc $\mathbb{D}^2 \subset \mathbb{C}^2$ defined by

$$H = \left\{(z, w) \in \mathbb{D}^2 : |z| < \frac{1}{2} \text{ or } |w| > \frac{1}{2}\right\}. \tag{2.1}$$

(See Fig. 2.1.) Every function $f \in \mathcal{O}(H)$ extends to a holomorphic function on the bidisc $\mathbb{D}^2$. Indeed, pick a number $\frac{1}{2} < c < 1$ and consider the Cauchy integral

$$F(z, w) = \frac{1}{2\pi \mathrm{i}} \int_{|\zeta|=c} \frac{f(z, \zeta)}{\zeta - w} \, \mathrm{d}\zeta, \quad |z| < 1, \ |w| < c.$$

Then F is a holomorphic function on $D = \mathbb{D} \times c\mathbb{D}$ which agrees with f on $H \cap D$. (Since the disc $\{z\} \times c\mathbb{D}$ is contained in H when $|z| < \frac{1}{2}$, we have $f = F$ there by the Cauchy integral formula; the equality elsewhere follows by the identity principle.) This extends f to a holomorphic function on $H \cup D = \mathbb{D}^2$.

Fifteen years later, Karl Reinhardt [469] studied domains of convergence of power series $\sum_{\alpha \in \mathbb{Z}_+^n} c_\alpha z^\alpha$ in several variables $z = (z_1, \ldots, z_n)$. It is immediate that the domain of convergence is a union of open polydiscs centered at the origin. By introducing the map $\phi \colon \mathbb{C}^n \to (\{-\infty\} \cup \mathbb{R})^n$,

$$\phi(z_1, \ldots, z_n) = \big(\log|z_1|, \ldots, \log|z_n|\big)$$

we see that each union of polydiscs is of the form $\Omega = \phi^{-1}(D)$ where D is a domain in $(\{-\infty\} \cup \mathbb{R})^n$ such that $(x_1, \ldots, x_n) \in D$ and $y_j \le x_j$ for $j = 1, \ldots, n$ implies that $(y_1, \ldots, y_n) \in D$. Reinhardt showed that Ω is the domain of convergence of a power series if and only if the corresponding domain $D \subset (\{-\infty\} \cup \mathbb{R})^n$ is *convex*. This gives analytic continuation of holomorphic functions from a complete Reinhardt domain $\Omega \subset \mathbb{C}^n$ to the smallest logarithmically convex complete Reinhardt domain $\widetilde{\Omega} \subset \mathbb{C}^n$ containing Ω.

In 1932 Hellmuth Kneser reformulated Hartogs' result into a more useful form known as the *Kontinuitätssatz*: Given an embedded family of closed analytic discs $D_t \subset \mathbb{C}^n$ $(t \in [0,1])$ such that D_0 and all the boundaries bD_t belong to a domain $\Omega \subset \mathbb{C}^n$, every holomorphic function on Ω admits an analytic continuation along this family to a neighborhood of the disc D_1.

Hartogs' discovery initiated research on 'natural domains' of holomorphic functions. Analytic continuation in general yields a multi-valued function. Following an idea of Riemann, multi-valued functions are considered as single-valued functions on *Riemann domains* over $\mathbb{C}^n$: a complex manifold X together with a locally biholomorphic map $\pi: X \to \mathbb{C}^n$. The central concept became that of a *domain of holomorphy*—a domain in $\mathbb{C}^n$, or over $\mathbb{C}^n$, with a holomorphic function that does not extend holomorphically to any bigger domain, not even as a multi-valued function. (See Oka [444, Chap. II].) Much of the classical theory developed around the problem of characterizing domains of holomorphy, and of constructing the *envelope of holomorphy* $\widetilde{\Omega}$ of a given domain $\Omega \subset \mathbb{C}^n$—the largest domain such that every holomorphic function on Ω extends to a holomorphic function on $\widetilde{\Omega}$.

Another important discovery was made by Eugenio E. Levi in 1911 [393]. He investigated domains $D \Subset \mathbb{C}^n$ with $\mathcal{C}^2$ boundaries. Let $\rho: \mathbb{C}^n \to \mathbb{R}$ be a $\mathcal{C}^2$ defining function for D, i.e., $D = \{z \in \mathbb{C}^n : \rho(z) < 0\}$ and $d\rho_z \neq 0$ for every $z \in bD = \{\rho = 0\}$. Levi noticed that, if for some boundary point $p \in bD$ and some vector $v \in T^{\mathbb{C}}_p bD$ that is complex tangent to the boundary (i.e., such that $\sum_{j=1}^n \frac{\partial \rho}{\partial z_j}(p) v_j = 0$) the Levi form $\mathcal{L}_{\rho,p}(v) < 0$ is negative, then holomorphic functions on D continue to a neighborhood of p in $\mathbb{C}^n$. The condition $\mathcal{L}_{\rho,p}(v) < 0$ implies that we can holomorphically embed a Hartogs pair $(H, \mathbb{D}^n)$ in $\mathbb{C}^n$ such that H is mapped into D but the image of $\mathbb{D}^n$ contains a neighborhood of p. Levi conjectured that any domain $D \subset \mathbb{C}^n$ as in the following definition is a domain of holomorphy; this became known as the *Levi problem*.

Definition 2.1.1 A domain $D = \{\rho < 0\}$ with a $\mathcal{C}^2$ defining function ρ such that $d\rho \neq 0$ on $bD = \{\rho = 0\}$ is *Levi pseudoconvex* if $\mathcal{L}_{\rho,p}(v) \geq 0$ for every $p \in bD$ and $v \in T^{\mathbb{C}}_z bD$. The domain D is *strongly pseudoconvex* if $\mathcal{L}_{\rho,p}(v) > 0$ for every $p \in bD$ and $0 \neq v \in T^{\mathbb{C}}_p bD$.

It is easily seen that the definition is independent of the choice of a defining function. A strongly pseudoconvex domain is locally at each boundary point biholomorphic to a piece of a strongly convex domain, and is osculated by a ball in suitable coordinates. This is commonly known as *Narasimhan's lemma*, although it was already known to Kneser in 1936 [338].

An important characterization of domains of holomorphy was obtained by Henri Cartan and Peter Thullen in 1932. To a compact set K in a complex space X we associate its $\mathcal{O}(X)$*-convex hull*, also called $\mathcal{O}(X)$*-hull*:

$$\widehat{K}_{\mathcal{O}(X)} = \Big\{ p \in X : |f(p)| \leq \max_{x \in K} |f(x)|, \ \forall f \in \mathcal{O}(X) \Big\}. \tag{2.2}$$

If K is a compact set in $\mathbb{C}^n$ then $\widehat{K} = \widehat{K}_{\mathcal{O}(\mathbb{C}^n)}$ is the *polynomial hull* of K.

Definition 2.1.2 A compact set K in a complex space X is $\mathcal{O}(X)$*-convex* if $K = \widehat{K}_{\mathcal{O}(X)}$. If $X = \mathbb{C}^n$ then such a set K is *polynomially convex*. A complex space X is *holomorphically convex* if for every compact set $K \subset X$ its $\mathcal{O}(X)$-hull $\widehat{K}_{\mathcal{O}(X)}$ is also compact.

Theorem 2.1.3 (Cartan and Thullen [91]) *A Riemann domain over $\mathbb{C}^n$ is a domain of holomorphy if and only if it is holomorphically convex.*

The hull of any compact set in a domain $\Omega \subset \mathbb{C}^n$ is a bounded closed subset of Ω, but it may fail to be compact as is seen in the Hartogs figure (2.1): Since every $f \in \mathcal{O}(H)$ extends to a function in $\mathcal{O}(\mathbb{D}^2)$, the maximum principle shows that the $\mathcal{O}(H)$-hull of the circle $\{(z_0, w) : |w| = \frac{3}{4}\}$ is the intersection of the disc $\{(z_0, w) : |w| \le \frac{3}{4}\}$ with Ω; this set is not compact if $\frac{1}{2} < |z_0| < 1$.

Theorem 2.1.3 is not difficult to prove. On the one hand, the derivatives of a holomorphic function $f \in \mathcal{O}(\Omega)$ satisfy the same bounds on $\widehat{K}_{\mathcal{O}(\Omega)}$ as on K, and hence the Taylor series of f centered around a point $p \in \widehat{K}_{\mathcal{O}(\Omega)}$ has the same domain of convergence as for points in K. If Ω is a domain of holomorphy, it follows that for any compact set $K \subset \Omega$ we have

$$\operatorname{dist}(\widehat{K}_{\mathcal{O}(\Omega)}, b\Omega) = \operatorname{dist}(K, b\Omega), \tag{2.3}$$

so $\widehat{K}_{\mathcal{O}(\Omega)}$ is compact. Conversely, using holomorphic convexity one can easily construct holomorphic functions tending to infinity along a given discrete sequence, so Ω is a domain of holomorphy.

A more challenging problem was to find a geometric characterization of domains of holomorphy. It follows from (2.3) that any closed holomorphic disc D in a domain of holomorphy Ω satisfies $\operatorname{dist}(D, b\Omega) = \operatorname{dist}(bD, b\Omega)$. This condition, which can be formulated in terms of *Hartogs pairs* (biholomorphic images of a standard pair $H \subset \mathbb{D}^n$, where H is a Hartogs figure in the polydisc $\mathbb{D}^n$), is known as *Hartogs pseudoconvexity* of Ω. Essentially it means that an analytic disc in $\overline{\Omega}$ with boundary in Ω must be contained in Ω. Oka showed that in such a case the function $\Omega \ni z \mapsto -\log \operatorname{dist}(z, b\Omega)$ is plurisubharmonic on Ω. Clearly this function blows up at $b\Omega$, so by adding the term $|z|^2$ we get a strongly plurisubharmonic exhaustion function on Ω. Similarly, Levi pseudoconvexity of a domain $\Omega \Subset \mathbb{C}^n$ easily implies that the function $-\log \operatorname{dist}(\cdot\,, b\Omega)$ is plurisubharmonic on Ω.

Could this be a characterization of domains of holomorphy?

This *Levi problem* was solved affirmatively by Oka in 1942 for domains in $\mathbb{C}^2$ [444, Chap. VI]; the higher dimensional case followed ten years later by Oka [444, Chap. IX], Bremermann [67], and Norguet [437]. In summary, we have the following result [300, Theorem 2.6.7].

Theorem 2.1.4 *The following conditions are equivalent for a domain Ω in $\mathbb{C}^n$, or a domain over $\mathbb{C}^n$:*

(a) *Ω is a domain of holomorphy.*
(b) *Ω is Hartogs pseudoconvex.*

(c) *The function* $-\log \operatorname{dist}(\cdot\,, b\Omega)$ *is plurisubharmonic.*
(d) *There exists a (strongly) plurisubharmonic exhaustion function on* Ω.

A domain $\Omega \subset \mathbb{C}^n$ *with* $\mathcal{C}^2$ *boundary is a domain of holomorphy if and only if it is Levi pseudoconvex.*

Every domain in (or over) $\mathbb{C}^n$ admits an envelope of holomorphy which can be constructed by 'pushing analytic discs' countably many times. A construction of the envelope in one step for domains in $\mathbb{C}^2$, and also in any two dimensional Stein manifold, was given by Jöricke in 2009 [314]. For results in this direction see also Merker and Porten [414].

2.2 Stein Manifolds and Stein Spaces

The class of Stein manifolds was introduced by Karl Stein in 1951 [524].

Definition 2.2.1 A complex manifold X is a *Stein manifold* if the following conditions hold:

(a) For every pair of distinct points $x \neq y$ in X there is a holomorphic function $f \in \mathcal{O}(X)$ such that $f(x) \neq f(y)$.
(b) For every point $p \in X$ there exist functions $f_1, \ldots, f_n \in \mathcal{O}(X)$, $n = \dim X$, whose differentials $df_1, \ldots, df_n$ are $\mathbb{C}$-linearly independent at p.
(c) X is holomorphically convex (see Definition 2.1.2).

Property (b) means that global holomorphic functions provide local charts at each point. Property (c) implies that a Stein manifold X admits an exhaustion $K_1 \subset K_2 \subset \cdots \subset \bigcup_{j=1}^{\infty} K_j = X$ by compact $\mathcal{O}(X)$-convex subsets such that $K_j \subset \mathring{K}_{j+1}$ holds for every $j = 1, 2, \ldots$.

Here are some basic properties and examples of Stein manifolds:

- An open set in $\mathbb{C}^n$ is Stein if and only if it is a domain of holomorphy. (This follows from the Cartan-Thullen theorem; see Theorem 2.1.3.)
- A Stein manifold does not contain any compact complex subvariety of positive dimension. (Apply axiom (a) and the maximum principle.)
- The Cartesian product $X \times Y$ of a pair of Stein manifolds is Stein.
- A closed complex submanifold X of $\mathbb{C}^N$ is Stein. (Use coordinate functions restricted to X. For the converse, see Theorem 2.4.1.)
- More generally, a closed complex submanifold of a Stein manifold is Stein.
- Every open Riemann surface is a Stein manifold (Behnke and Stein [50, 51], [260, p. 134]).
- If $X \to Y$ is a holomorphic covering space and Y is Stein, then X is Stein. (This is due to Stein [525].)
- If $X \to Y$ is a finite branched holomorphic covering, then X is Stein if and only if Y is Stein (Gunning [272, p. 151]).

- If $E \to X$ is a holomorphic vector bundle over a Stein base X, then the total space E is also Stein.
- (The Serre problem.) There exist holomorphic fibre bundles over the disc or the plane, with fibre $\mathbb{C}^2$ and with transitions maps given by polynomial automorphisms of $\mathbb{C}^2$, whose total space is not Stein; see Theorems 8.3.12 and 8.3.13.

The notion of a Stein space was introduced by Grauert in 1955 [251]. The standard definition is the following one.

Definition 2.2.2 A second countable complex space X is said to be a *Stein space* if it satisfies properties (a), (c) in Definition 2.2.1 and also

(b′) Every local ring $\mathcal{O}_{X,x}$ is generated by functions in $\mathcal{O}(X)$.

Condition (b′) means that there is a holomorphic map $X \to \mathbb{C}^N$ which embeds a neighborhood of x as a local complex subvariety of $\mathbb{C}^N$. Grauert showed in [251] that one gets an equivalent definition by keeping (c) and replacing (a) and (b) (resp. (b′)) by the following property.

Definition 2.2.3 A complex space X is called *K-complete* if for every point $x \in X$ there is a holomorphic map $f : X \to \mathbb{C}^N$ (with $N = N_x$) such that x is an isolated point of the fibre $f^{-1}(f(x))$.

It is immediate that axiom (a) implies K-completeness. In summary:

Theorem 2.2.4 ([251]) *A complex space X is a Stein space if and only if it is holomorphically convex and it satisfies one of the following two properties*:

(i) *Holomorphic functions separate points on X* (*axiom* (a) *in Definition* 2.2.1).
(ii) *X is K-complete in the sense of Definition* 2.2.3.

For further characterizations of Stein spaces see [260, p. 152].

2.3 Holomorphic Convexity and the Oka-Weil Theorem

The following *Oka-Weil theorem* generalizes Runge's theorem. See Theorem 2.6.8 for an analogous result concerning sections of coherent analytic sheaves.

Theorem 2.3.1 *If X is a Stein space and K is a compact $\mathcal{O}(X)$-convex subset of X, then every holomorphic function in an open neighborhood of K can be approximated uniformly on K by functions in $\mathcal{O}(X)$.*

Theorem 2.3.1 was proved for domains of holomorphy by Oka [444, Chap. I] using his *Oka lemma*; see [300, Lemma 2.7.5]. It is an immediate consequence of the definition that an $\mathcal{O}(X)$-convex set K can be approximated from the outside by *analytic polyhedra*, i.e., by Stein open sets of the form

$$U = \left\{x \in X : \left|h_j(x)\right| < 1,\ j = 1, \dots, m\right\}, \quad h_1, \dots, h_m \in \mathcal{O}(X).$$

By adding more functions, we can ensure that $h = (h_1, \dots, h_m) \colon X \to \mathbb{C}^m$ embeds U properly into the polydisc $\mathbb{D}^m \subset \mathbb{C}^m$. The key point proved by Oka is that for any function $f \in \mathcal{O}(U)$ there is a function $g \in \mathcal{O}(\mathbb{D}^m)$ such that $g \circ h = f$. (This is a special case of the Oka-Cartan extension theorem, see Corollary 2.6.3.) By expanding g in power series and approximating it by Taylor polynomials $P \in \mathbb{C}[z_1, \dots, z_m]$ we get functions $P \circ h \in \mathcal{O}(X)$ approximating f on K.

Another proof of the Oka-Weil theorem can be given by the L^2-methods for solving the nonhomogeneous $\overline{\partial}$-equation. We follows this approach in Sect. 2.8 to prove a stronger parametric version of the Oka-Weil theorem, combined with the Oka-Cartan extension theorem; see Theorem 2.8.4.

Definition 2.3.2 A domain Ω in a complex space X is *Runge in X* if every holomorphic function $f \in \mathcal{O}(\Omega)$ can be approximated uniformly on compacts in Ω by functions in $\mathcal{O}(X)$; equivalently, if the subalgebra $\{f|_\Omega : f \in \mathcal{O}(X)\}$ of $\mathcal{O}(\Omega)$ is dense in $\mathcal{O}(\Omega)$ in the compact-open topology.

Theorem 2.3.3 ([300, p. 91]) *A Stein domain Ω in a Stein space X is Runge in X if and only if for every compact set $K \subset \Omega$ we have $\widehat{K}_{\mathcal{O}(\Omega)} = \widehat{K}_{\mathcal{O}(X)}$.*

There exist several notions of *ambient holomorphic convexity* of a compact set (see [534]); we shall use the following ones.

Definition 2.3.4 Assume that K is a compact set in a complex space X.

(i) K is a *Stein compact* if it admits a basis of Stein neighborhoods in X.
(ii) K is *holomorphically convex* if it admits an open Stein neighborhood Ω in X such that K is $\mathcal{O}(\Omega)$-convex.

2.4 Embedding Stein Manifolds into Euclidean Spaces

An important characterization of Stein manifolds is that they are embeddable as closed complex submanifolds of complex Euclidean spaces. It is an immediate consequence of Definition 2.2.1 that for every relatively compact domain Ω in a Stein manifold X there is a holomorphic map $f \colon X \to \mathbb{C}^N$ for a big enough N such that $f|_\Omega \colon \Omega \to \mathbb{C}^N$ is an injective holomorphic immersion. In 1956, Remmert proved a substantially stronger result that every Stein manifold admits a *proper* holomorphic embedding into some Euclidean space $\mathbb{C}^N$ [470]. In 1960–1961, Bishop and Narasimhan independently proved that Remmert's theorem holds with $N = 2\dim X + 1$. We now summarize these classical results. (For smooth manifolds, part (a) is due to Whitney [571].)

Theorem 2.4.1 ([58], [423, Theorem 5], [424])

(a) *If X is a Stein manifold of dimension n, then the set of proper holomorphic maps $X \to \mathbb{C}^{n+1}$ is dense in $\mathcal{O}(X)^{n+1}$, the set of proper holomorphic immersions $X \to \mathbb{C}^{2n}$ is dense in $\mathcal{O}(X)^{2n}$, and the set of proper holomorphic embeddings $X \to \mathbb{C}^{2n+1}$ is dense in $\mathcal{O}(X)^{2n+1}$.*
(b) *If X is a Stein space of dimension n, then the set of holomorphic maps $X \to \mathbb{C}^{2n+1}$ which are proper, injective, and regular (immersions) on the regular part X_{reg} is dense in $\mathcal{O}(X)^{2n+1}$.*
(c) *If X is a Stein space of dimension n and of finite embedding dimension m, then for $N = \max\{n+m, 2n+1\}$ the set of proper holomorphic embeddings $X \hookrightarrow \mathbb{C}^N$ is dense in $\mathcal{O}(X)^N$.*

More precise embedding theorems for Stein manifolds and Stein spaces are proved in Sects. 9.3–9.5, and for Riemann surfaces in Sects. 9.10–9.11. Unlike Theorem 2.4.1, those results depend on the Oka theory developed in Chaps. 5 and 6.

Since every real analytic manifold admits a Stein complexification [255], we get the following consequence of Theorem 2.4.1 which answers a question of Whitney [571, p. 645].

Corollary 2.4.2 ([255, Theorem 3]) *Every real analytic manifold admits a proper real analytic embedding into a Euclidean space $\mathbb{R}^N$.*

Since Stein manifolds are complex submanifolds of Euclidean spaces, it is not surprising that they can be approximated by affine algebraic manifolds. It was proved by Stout [533] that any relatively compact domain in a Stein manifold is biholomorphically equivalent to a domain in an affine algebraic manifold. (For the real algebraic case, see Nash [428].) More precise algebraic approximation results were obtained by Demailly, Lempert and Schiffman [117, 390] and by Lisca and Matič [398] (see Theorem 10.7.1 on p. 506).

2.5 Characterization by Plurisubharmonic Functions

It is a fundamental fact that Stein manifolds and Stein spaces are characterized by plurisubharmonicity (see Theorem 2.5.2). Quite often, the most efficient way to show that a complex space is Stein is to find a strongly plurisubharmonic exhaustion function on it. This is how Siu proved in 1976 [504] that a Stein subvariety of any complex space has a basis of open Stein neighborhoods (see Theorem 3.1.1 on p. 66). Stein neighborhoods often allow us to transfer a problem on a complex space to a more tractable problem on an ambient Euclidean space; Chap. 3 focuses on such methods.

It follows from holomorphic convexity that every Stein space X is exhausted by an increasing sequence of compacts $K_1 \subset K_2 \subset \cdots \subset \bigcup_{j=1}^{\infty} K_j = X$ such that

$K_j = \widehat{K}_j$ and $K_j \subset \mathring{K}_{j+1}$ for every $j \in \mathbb{N}$. Using such exhaustions and axioms (a), (b′), one can easily find strongly plurisubharmonic exhaustion functions of the form

$$\rho = \sum_{j=1}^{\infty} |f_j|^2 : X \to \mathbb{R}_+, \quad f_j \in \mathcal{O}(X),\ j = 1, 2, \ldots.$$

By a more precise argument one obtains the following result on approximation of compact $\mathcal{O}(X)$-convex subsets of a Stein space by sublevel sets of strongly plurisubharmonic functions (see [300, Theorem 5.1.6, p. 117]).

Proposition 2.5.1 *If K is a compact $\mathcal{O}(X)$-convex set in a Stein space X, then for every open set $U \subset X$ containing K there exists a smooth strongly plurisubharmonic function $\rho: X \to \mathbb{R}$ such that $\rho < 0$ on K and $\rho > 1$ on $X \setminus U$. Furthermore, there exists a plurisubharmonic exhaustion function $\rho: X \to \mathbb{R}_+$ such that $\rho^{-1}(0) = K$ and ρ is strongly plurisubharmonic on $X \setminus K = \{\rho > 0\}$.*

Note that the function $\rho_a: \mathbb{C}^N \to \mathbb{R}_+$ given by $\rho_a = |z-a|^2$ is strongly plurisubharmonic on any complex subvariety $X \subset \mathbb{C}^N$; if X is closed, then this is an exhaustion function on X. Furthermore, if X is smooth, then $\rho_a|_X$ is a Morse function on X for most choices of the point $a \in \mathbb{C}^N$.

These observations show that a Stein space admits plenty of smooth strongly plurisubharmonic exhaustion functions. The following converse is the most useful characterization of Stein manifolds and Stein spaces.

Theorem 2.5.2

(a) [128, 255] *A complex manifold which admits a strongly plurisubharmonic exhaustion function is a Stein manifold.*
(b) [163, 425] *A complex space which admits a strongly plurisubharmonic exhaustion function is a Stein space.*

Furthermore, if $\rho: X \to \mathbb{R}$ is a strongly plurisubharmonic exhaustion function, then each sublevel set $\{x \in X : \rho(x) \le c\}$ is $\mathcal{O}(X)$-convex.

Corollary 2.5.3 *For every compact set K in a Stein space X, the $\mathcal{O}(X)$-hull of K coincides with its plurisubharmonic hull:*

$$\widehat{K}_{\mathcal{O}(X)} = \widehat{K}_{\mathrm{Psh}(X)}.$$

Hence, every holomorphic function in a neighborhood of a compact Psh(X)*-convex set $K = \widehat{K}_{\mathrm{Psh}(X)}$ is a uniform limit on K of functions in $\mathcal{O}(X)$.*

The most efficient proof of Theorem 2.5.2 and Corollary 2.5.3 can be given by the L^2-method for solving nonhomogeneous $\overline{\partial}$-equations with weights of the form $\mathrm{e}^{-\rho}$ with $\rho \in \mathrm{Psh}(X)$ (see e.g. [299, 300, 442]; see also Sect. 2.8 where we prove a parametric version of the Oka-Weil approximation theorem).

Theorem 2.5.2 implies the following solution of the Levi problem.

Corollary 2.5.4 *Let X be a Stein space. If a domain $\Omega \subset X$ admits a plurisubharmonic exhaustion function $\rho\colon \Omega \to \mathbb{R}$, then Ω is Stein. In particular, every Levi (or Hartogs) pseudoconvex domain in a Stein space is Stein.*

The notion of a holomorphically convex set and of a Stein compact was introduced in Definition 2.3.4. Proposition 2.5.1 and Theorem 2.5.2 imply the following characterization of these notions by plurisubharmonicity.

Proposition 2.5.5 *A compact set K in a Stein space X is holomorphically convex if and only if there exists a plurisubharmonic function $\rho\colon U \to \mathbb{R}_+$ in an open neighborhood U of K such that $\rho^{-1}(0) = K$ and ρ is strongly plurisubharmonic on $U \setminus K = \{\rho > 0\}$.*

The sets $\Omega_c = \{x \in U : \rho(x) < c\}$ for small $c > 0$ then form a basis of Stein neighborhoods of K such that K is $\mathcal{O}(\Omega_c)$-convex.

Here is another useful sufficient condition for a set to be Stein compact.

Proposition 2.5.6 *Let K be a compact set in a complex space X. Assume that there exist a neighborhood $U \subset X$ of K, a strongly plurisubharmonic function $\rho\colon U \to \mathbb{R}$, and a weakly plurisubharmonic function $\tau\colon U \to \mathbb{R}_+$ such that $K = \{\tau = 0\}$. Then, K is a Stein compact.*

Proof Fix an open neighborhood $V \Subset U$ of K. It is easy to find a fast growing convex increasing function $\chi\colon \mathbb{R} \to \mathbb{R}$ such that the strongly plurisubharmonic function $\phi = \rho + \chi \circ \tau\colon U \to \mathbb{R}$ satisfies $\phi|_K < 0$ and $K \subset V_c = \{\phi < c\} \Subset V$ for some $c > 0$. The domain V_c is then Stein by Theorem 2.5.2. □

The closure of a smooth weakly pseudoconvex domain $D \Subset \mathbb{C}^n$ need not be a Stein compact; an example is the *worm domain* [121]. For the existence of bounded strongly plurisubharmonic exhaustion functions on weakly Levi pseudoconvex domains see [120].

2.6 Cartan-Serre Theorems A & B

The famous Theorems A and B were proved in Cartan's seminar in 1951–1954; see [87, 90, 260]. It would be impossible to overstate the importance of these results for the development of analytic and algebraic geometry.

Theorem 2.6.1 *Let $\mathcal{F}$ be a coherent analytic sheaf on a Stein space X. Then:*

(A) *The stalk of $\mathcal{F}_x$ of $\mathcal{F}$ at any point $x \in X$ is generated as an $\mathcal{O}_{X,x}$-module by global sections of the sheaf $\mathcal{F}$.*
(B) *$H^p(X; \mathcal{F}) = 0$ for all $p = 1, 2, \ldots$.*

The corresponding results hold for every coherent algebraic sheaf over an affine algebraic variety $X \subset \mathbb{C}^N$ (Serre [497, *p.* 237, *Théorème* 2]).

An analogue of Theorems A and B for coherent analytic sheaves with continuous boundary values on strongly pseudoconvex domains was proved by Heunemann [293] and Leiterer [385].

We recall the relevant notions; a comprehensive account is available in [261]. Let X be a complex space. An *analytic sheaf* (or $\mathcal{O}_X$-sheaf) on X is a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules; that is, a sheaf whose stalk $\mathcal{F}_x$ over any point $x \in X$ is a module over the local ring $\mathcal{O}_{X,x}$. The sheaf $\mathcal{F}$ is *locally finitely generated* if for every point $x_0 \in X$ there exist an open neighborhood $U \subset X$ and finitely many sections $f_1, \ldots, f_k \in \mathcal{F}(U) = \Gamma(U, \mathcal{F})$ whose germs at any point $x \in U$ generate $\mathcal{F}_x$ as an $\mathcal{O}_{X,x}$-module. The simplest example is $\mathcal{O}_X^k$, the direct sum of k copies of the structure sheaf $\mathcal{O}_X$ for any $k \in \mathbb{N}$; this is the sheaf of holomorphic sections of the trivial bundle $X \times \mathbb{C}^k \to X$.

An analytic sheaf is *coherent* if it is locally finitely generated and if for any set of local sections $f_1, \ldots, f_k \in \mathcal{F}(U)$ the corresponding *sheaf of relations* $\mathcal{R} = \mathcal{R}(f_1, \ldots, f_k)$ is also locally finitely generated. The latter sheaf has stalks

$$\mathcal{R}_x = \left\{ (g_{1,x}, \ldots, g_{k,x}) \in \mathcal{O}_{X,x}^k : \sum_{j=1}^k g_{j,x} f_{j,x} = 0 \right\}, \quad x \in U. \tag{2.4}$$

From the above description, we see that an analytic sheaf $\mathcal{F}$ over X is coherent if and only if each point $x \in X$ has an open neighborhood $U \subset X$ and a short exact sequence of analytic sheaf homomorphisms

$$\mathcal{O}_U^m \xrightarrow{\alpha} \mathcal{O}_U^k \xrightarrow{\beta} \mathcal{F}|_U \longrightarrow 0 \tag{2.5}$$

where

$$\beta(g_{1,x}, \ldots, g_{k,x}) = \sum_{j=1}^k g_{j,x} f_{j,x}.$$

Hence, β maps the standard basis sections $e_j = (0, \ldots, 1, \ldots, 0)$ of $\mathcal{O}_U^k$ onto the generators f_j of $\mathcal{F}|_U$ and $\mathcal{R} = \ker \beta = \operatorname{im} \alpha$ is the sheaf of relations (2.4). If X is a Stein space then a resolution (2.5) exists over any relatively compact open subset $U \Subset X$.

Here are the main examples of coherent sheaves on a complex space X:

- The structure sheaf $\mathcal{O}_X$ (Oka [444, Chap. VII]; see also [261, p. 59] and [435]).
- The sheaf of ideals $\mathcal{O}_A$ of a complex subvariety A in X (the Oka-Cartan coherence theorem; [90, p. 631], [261, p. 84], [435]).
- A locally free analytic sheaf, i.e., a sheaf of holomorphic sections of a holomorphic vector bundle. In particular, we have the *tangent sheaf* $\mathcal{T}_X$ and the *cotangent sheaf* $\mathcal{T}_X^*$ on a complex manifold X.
- The Whitney sum $\mathcal{E} \oplus \mathcal{F}$ and the tensor product $\mathcal{E} \otimes \mathcal{F}$ of coherent sheaves.

- The sheaf $\mathcal{H}om(\mathcal{E}, \mathcal{F})$ of $\mathcal{O}_X$-homomorphisms $\mathcal{E} \to \mathcal{F}$ between a pair of coherent analytic sheaves. In particular, the dual $\mathcal{E}^*$ of a coherent analytic sheaf.
- If $\beta: \mathcal{F} \to \mathcal{G}$ is a homomorphism of coherent analytic sheaves, then the kernel $\ker\beta$ and the image $\mathrm{im}\beta$ are coherent analytic sheaves. In summary, given a short exact sequence of homomorphisms of $\mathcal{O}_X$-analytic sheaves

$$0 \longrightarrow \mathcal{E} \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{G} \longrightarrow 0,$$

  if two sheaves are coherent then so is the third one [261, p. 236].
- The direct image of an $\mathcal{O}_X$-coherent sheaf by a proper holomorphic map $X \to Y$ of complex spaces is a coherent $\mathcal{O}_Y$-sheaf (Grauert's coherence theorem; see [261, p. 207]).

Each coherent analytic sheaf $\mathcal{F}$ can be represented as the sheaf of germs of fibrewise linear holomorphic functions on a linear space $\pi: L \to X$ [158]. More precisely, there is a contravariant equivalence between the category of coherent analytic sheaves and the category of linear spaces such that locally free sheaves correspond to vector bundles. The sheaf of germs of holomorphic sections $X \to L$ of any linear space is also coherent [158, p. 53, Corollary].

Given a coherent analytic sheaf $\mathcal{F}$ on a complex space X, the $\mathcal{O}(X)$-module $\mathcal{F}(X) = \Gamma(X, \mathcal{F})$ of all global sections is endowed with a Fréchet space topology (the topology of uniform convergence on compacts in X) such that for every point $x \in X$ the natural restriction map $\mathcal{F}(X) \mapsto \mathcal{F}_x$ is continuous (see [260, Theorem 5, p. 167]). The topology on the stalks $\mathcal{F}_x$ is the *sequence topology* (cf. [259, p. 86ff]). In particular, $\mathcal{F}(X)$ is a Baire space.

We now mention some applications of Theorems A and B; for more on this subject, see [496] and [260, Chap. V].

Corollary 2.6.2 *Let $\beta: \mathcal{F} \to \mathcal{G}$ be an epimorphism of analytic sheaves over a Stein space X. If the kernel $\mathcal{E} = \ker\beta$ is coherent, then the induced map on sections $\mathcal{F}(X) \to \mathcal{G}(X)$, $f \mapsto \beta(f)$ is surjective.*

Proof Since $H^1(X; \mathcal{E}) = 0$ by Theorem 2.6.1, the conclusion follows from the exact cohomology sequence $\mathcal{F}(X) \to \mathcal{G}(X) \to H^1(X; \mathcal{E}) = 0$. □

Applying Corollary 2.6.2 to the exact sequence

$$0 \longrightarrow \mathcal{J}_A \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X/\mathcal{J}_A \longrightarrow 0$$

where A is a closed complex subvariety of X, we obtain

Corollary 2.6.3 (Oka-Cartan extension theorem) *Every holomorphic function on a closed complex subvariety of a Stein space X extends to a holomorphic function on X.*

Corollary 2.6.4 (Cartan's division theorem) *If $\mathcal{F}$ is a coherent analytic sheaf on a Stein space X and if $f_1, \ldots, f_k \in \mathcal{F}(X)$ generate each stalk $\mathcal{F}_x$ ($x \in X$), then every section $f \in \mathcal{F}(X)$ is of the form $f = \sum_{j=1}^k g_j f_j$ for some $g_j \in \mathcal{O}(X)$.*

Proof Consider the exact sequence $0 \to \mathcal{R} \to \mathcal{O}^k \xrightarrow{\beta} \mathcal{F} \to 0$ as in (2.5). Since $\mathcal{R} = \ker \beta$ is coherent, the conclusion follows from Corollary 2.6.2. □

Corollary 2.6.5 *Given a short exact sequence* $0 \to \mathcal{E} \to \mathcal{F} \xrightarrow{\beta} \mathcal{G} \to 0$ *of coherent analytic sheaves on a Stein space such that* $\mathcal{G}$ *is locally free, there exists a sheaf homomorphism* $\phi\colon \mathcal{G} \to \mathcal{F}$ *such that* $\beta \circ \phi = \mathrm{Id}_{\mathcal{G}}$.

Proof Consider the induced exact sequence

$$0 \longrightarrow \mathcal{H}om(\mathcal{G},\mathcal{E}) \longrightarrow \mathcal{H}om(\mathcal{G},\mathcal{F}) \xrightarrow{\beta} \mathcal{H}om(\mathcal{G},\mathcal{G}) \longrightarrow 0.$$

Surjectivity of β is due to $\mathcal{G}$ being locally free. By Theorem B we have $H^1(X;\mathcal{H}om(\mathcal{G},\mathcal{E})) = 0$, and hence β is surjective also on the level of sections. Hence, $\mathrm{Id}_{\mathcal{G}}$ lifts to a homomorphism $\phi\colon \mathcal{G} \to \mathcal{F}$ with $\beta \circ \phi = \mathrm{Id}_{\mathcal{G}}$. □

The following is a special case of Corollary 2.6.5.

Corollary 2.6.6 *If* E' *is a holomorphic vector subbundle of a holomorphic vector bundle* E *over a Stein space* X, *then there exists a holomorphic vector subbundle* E'' *of* E *such that* $E = E' \oplus E''$ *is a holomorphic direct sum.*

Theorem 2.6.7 *On any Stein manifold* X, *the Dolbeault cohomology groups vanish*: $H^{p,q}_{\bar\partial}(X) = 0$ *for all* $p \ge 0,\ q \ge 1$.

Proof The sheaf Ω_p of holomorphic p-forms on X admits a resolution

$$0 \to \Omega_p \hookrightarrow \mathcal{E}_{p,0} \xrightarrow{\bar\partial} \mathcal{E}_{p,1} \xrightarrow{\bar\partial} \mathcal{E}_{p,2} \cdots \xrightarrow{\bar\partial} \mathcal{E}_{p,n} \longrightarrow 0.$$

Since the sheaves $\mathcal{E}_{p,q}$ of smooth (p,q)-forms on X are fine, their cohomology vanishes. Leray's theorem implies that $H^{p,q}_{\bar\partial}(X) \cong H^q(X;\Omega_p)$. Since the sheaf Ω_p is coherent analytic, these groups are zero by Cartan's Theorem B. □

Another proof of Theorem 2.6.7 is obtained by Hörmander's L^2 theory.

Serre proved that each element of a de Rham cohomology group $H^p(X;\mathbb{C})$ ($p = 1,2,\ldots,\dim X$) of a Stein manifold is represented by a closed holomorphic p-form on X (see [496, Theorem 1], [260, p. 155]). The de Rham cohomology of an affine algebraic manifold is represented by algebraic forms (see Grothendieck [270]).

We have the following approximation theorem for sections of coherent analytic sheaves over Stein spaces (see e.g. [260, p. 170]).

Theorem 2.6.8 (Oka-Weil theorem for coherent analytic sheaves) *Let* $\mathcal{F}$ *be a coherent analytic sheaf on a Stein space* X. *If* K *is a compact* $\mathcal{O}(X)$*-convex set in* X, *then any section of* $\mathcal{F}$ *over an open neighborhood of* K *can be approximated uniformly on* K *by sections in* $\mathcal{F}(X)$. *More precisely, if sections* $f_1,\ldots,f_m \in \mathcal{F}(X)$ *generate every stalk* $\mathcal{F}_x$, $x \in K$, *then every section of* $\mathcal{F}$ *over an open neighborhood of* K *can be approximated uniformly on* K *by sections of the form* $\sum_{j=1}^m g_j f_j$ *for some* $g_j \in \mathcal{O}(X)$.

Proof Assume that f is a section of $\mathcal{F}$ over an open neighborhood $\Omega \subset X$ of K. We may assume that Ω is Stein and relatively compact in X. Since a coherent analytic sheaf $\mathcal{F}$ is locally finitely generated, there exist sections $f_1, \ldots f_m \in \mathcal{F}(X)$ which generate every stalk $\mathcal{F}_x$ for $x \in \Omega$. By Corollary 2.6.4 we have $f = \sum_{j=1}^m h_j f_j$ for some functions $h_1, \ldots, h_m \in \mathcal{O}(\Omega)$. By Theorem 2.3.1 we can approximate every h_j uniformly on K by a function $g_j \in \mathcal{O}(X)$. The section $F = \sum_{j=1}^m g_j f_j \in \mathcal{F}(X)$ then approximates f on K. □

2.7 The $\overline{\partial}$-Problem

The $\overline{\partial}$-problem asks for a solution of the equation $\overline{\partial} u = f$ for a given $\overline{\partial}$-closed form f. By Theorem 2.6.7, this problem is always solvable on a Stein manifold. A more direct approach, which also gives L^2 estimates of solutions, is provided by Hörmander's theory [299, 300]; see also Andreotti and Vesentini [28], Kohn [345, 346], and the monographs [95, 442].

We shall frequently use the following result for (0, 1)-forms. Let $\mathrm{d}\lambda$ denote the Lebesgue measure on $\mathbb{C}^n = \mathbb{R}^{2n}$.

Theorem 2.7.1 ([300, Theorem 4.4.2, p. 94]) *Let Ω be a pseudoconvex* (*Stein*) *domain in $\mathbb{C}^n$ and ϕ be a plurisubharmonic function in Ω. For every* (0, 1)*-form $f = \sum f_j \,\mathrm{d}\bar{z}_j$ such that $f_j \in L^2_{loc}(\Omega)$ and $\overline{\partial} f = 0$* (*in the weak sense*) *there exists $u \in L^2_{loc}(\Omega)$ such that*

$$\overline{\partial} u = f \quad \textit{and} \quad \int_\Omega \frac{|u|^2}{(1+|z|^2)^2} \mathrm{e}^{-\phi} \mathrm{d}\lambda \le \int_\Omega \sum_{j=1}^n |f_j|^2 \mathrm{e}^{-\phi} \mathrm{d}\lambda.$$

If f is smooth then so is u.

By taking Ω bounded and $\phi = 0$ we get the estimate

$$\overline{\partial} u = f \quad \text{and} \quad \int_\Omega |u|^2 \mathrm{d}\lambda \le C \int_\Omega \sum_j |f_j|^2 \,\mathrm{d}\lambda, \tag{2.6}$$

where the constant C depends on the radius of Ω and on the dimension n. The analogous results hold on relatively compact domains in Stein manifolds.

To pass from L^2 to $\mathcal{C}^k$ estimates, one needs the following well-known lemma which follows from the Bochner-Martinelli formula [215, Lemma 3.2].

Lemma 2.7.2 (Interior elliptic regularity estimates) *Let $\mathbb{B}^n$ denote the open unit ball in $\mathbb{C}^n$. For each $s \in \mathbb{Z}_+$ there is a constant $c_s > 0$ such that if $f \in \mathcal{C}^{s+1}(r\mathbb{B})$ for some $r > 0$ and $\alpha \in \mathbb{Z}_+^{2n}$ is a multi-index with $|\alpha| = s$ then*

$$c_s \left|\partial^\alpha f(0)\right| \le r^{-n-s} \|f\|_{L^2(r\mathbb{B})} + \sum_{|\beta| \le s} r^{|\beta|+1-s} \left\|\partial^\beta (\overline{\partial} f)\right\|_{L^\infty(r\mathbb{B})}.$$

In particular, we have the sup-norm estimate

$$c_0\,\big|f(0)\big| \le r^{-n}\,\|f\|_{L^2(r\mathbb{B})} + r\,\|\overline{\partial} f\|_{L^\infty(r\mathbb{B})}. \tag{2.7}$$

On bounded strongly pseudoconvex domains in Stein manifolds, the $\overline{\partial}$-equation can also be solved by means of integral formulas with holomorphic kernels. This kernel method gives optimal Hölder estimates. The first results of this type were obtained by Henkin [288] and R. de Arellano (see [289]). We shall use the following result due to Range and Siu [467] and Lieb and Range [395, Theorem 1]; see also [396, 397], [416, Theorem 1′], and [394].

If D is a domain in a complex manifold X, we denote by $\mathcal{C}^l_{p,q}(D)$ the space of (p,q)-forms whose coefficients (in any local chart on X) are of class $\mathcal{C}^l(D)$, i.e., l times continuously differentiable. If D has piecewise $\mathcal{C}^1$ boundary then $\mathcal{C}^l_{p,q}(\bar{D})$ stands for the space of (p,q)-forms on $\bar{D}$ of class $\mathcal{C}^l(\bar{D})$. If $l = k+\alpha$ with $k \in \mathbb{Z}_+$ and $0 < \alpha \le 1$, then $\mathcal{C}^l = \mathcal{C}^{k,\alpha}$ denotes the Hölder space.

Theorem 2.7.3 *Given a relatively compact strongly pseudoconvex domain D in a Stein manifold, there exists a linear operator $T\colon \mathcal{C}^0_{0,1}(D) \to \mathcal{C}^{1/2}(D)$ such that, if $f \in \mathcal{C}^0_{0,1}(\bar{D}) \cap \mathcal{C}^1_{0,1}(D)$ and $\overline{\partial} f = 0$ in D then*

$$\overline{\partial}(Tf) = f \quad \textit{and} \quad \|Tf\|_{\mathcal{C}^{1/2}(\bar{D})} \le c_D\,\|f\|_{\mathcal{C}^0_{0,1}(\bar{D})}.$$

The constant c_D can be chosen uniform for all domains sufficiently $\mathcal{C}^2$-close to D. If D has boundary of class $\mathcal{C}^\ell$ for some $\ell \in \{2,3,\ldots\}$ then there exists a linear operator $T\colon \mathcal{C}^0_{0,1}(D) \to \mathcal{C}^0(D)$ satisfying the following properties:

(i) *If $f \in \mathcal{C}^0_{0,1}(\bar{D}) \cap \mathcal{C}^1_{0,1}(D)$ and $\overline{\partial} f = 0$ then $\overline{\partial}(Tf) = f$.*
(ii) *If $f \in \mathcal{C}^0_{0,1}(\bar{D}) \cap \mathcal{C}^r_{0,1}(D)$ for some $r \in \{1,\ldots,\ell)$ then*

$$\|Tf\|_{\mathcal{C}^{l,1/2}(\bar{D})} \le C_{l,D}\|f\|_{\mathcal{C}^l_{0,1}(\bar{D})}, \quad l = 0,1,\ldots,r. \tag{2.8}$$

Although these results are stated in the original papers for domains with $\mathcal{C}^\infty$ boundaries, one only needs $\mathcal{C}^\ell$ boundary to get estimates up to order ℓ; this is implicitly contained in the paper by Michel and Perotti [416].

2.8 Cartan-Oka-Weil Theorem with Parameters

In this section we prove a parametric version of the classical Oka-Cartan extension theorem combined with the Oka-Weil approximation theorem; see Theorem 2.8.4. We begin with the following simple version on $\mathbb{C}^n$.

Proposition 2.8.1 (The Oka-Weil theorem with parameters on $\mathbb{C}^n$) *Let K be a compact polynomially convex set in $\mathbb{C}^n$ and let $U \subset \mathbb{C}^n$ be an open set containing K. Assume that P is a compact Hausdorff space and $f\colon P \times U \to \mathbb{C}$ is* *a continuous function such that* $f_p = f(p, \cdot)\colon U \to \mathbb{C}$ *is holomorphic for every* $p \in P$. *Given* $\epsilon > 0$ *there exists a continuous function* $F\colon P \times \mathbb{C}^n \to \mathbb{C}$ *such that* $F_p = F(p, \cdot)\colon \mathbb{C}^n \to \mathbb{C}$ *is holomorphic for every* $p \in P$ *and*

$$\sup_{z\in K,\, p\in P} \left|F_p(z) - f_p(z)\right| < \epsilon.$$

Proof Since the set K is polynomially convex, Proposition 2.5.1 gives a smooth strongly plurisubharmonic function $\phi\colon \mathbb{C}^n \to \mathbb{R}$ such that $\phi < 0$ on K and $\phi > 0$ on $\mathbb{C}^n \setminus U$. Let $h\colon \mathbb{R} \to \mathbb{R}_+$ be a smooth convex increasing function such that $h(t) = 0$ for $t \leq 0$ and h is positive and strictly increasing on $t > 0$. Then, the function $\psi = h \circ \phi\colon \mathbb{C}^n \to [0, \infty)$ is smooth plurisubharmonic, it vanishes in an open neighborhood U_0 of K, and is positive on $\mathbb{C}^n \setminus U$. Pick a neighborhood $U_1 \Subset U$ of K such that

$$\psi \geq c > 0 \quad \text{on } \overline{U} \setminus U_1 \tag{2.9}$$

for some positive constant $c > 0$. Choose a smooth function $\chi\colon \mathbb{C}^n \to [0, 1]$ such that $\chi = 1$ on $\overline{U}_1$ and $\operatorname{supp} \chi \subset U$. For every $p \in P$ set

$$\alpha_p = \overline{\partial}(\chi f_p) = f_p\, \overline{\partial}\chi = \sum_{i=1}^{n} \alpha_{i,p} \mathrm{d}\bar{z}_i.$$

Note that α_p is a smooth $(0, 1)$-form on $\mathbb{C}^n$ with compact support contained in $U \setminus U_1$ and depending continuously on $p \in P$. By Theorem 2.7.1 there exists for every $t > 0$ a smooth solution $u_{p,t}\colon \mathbb{C}^n \to \mathbb{C}$ of the equation $\overline{\partial} u_{p,t} = \alpha_p$ satisfying the estimate

$$\int_{\mathbb{C}^n} \frac{|u_{p,t}|^2}{(1 + |z|^2)^2} \mathrm{e}^{-t\psi} \mathrm{d}\lambda \leq \int_U \sum_{i=1}^{n} |\alpha_{i,p}|^2 \mathrm{e}^{-t\psi} \mathrm{d}\lambda.$$

Moreover, since Hörmander's solution to the $\overline{\partial}$-equation is given by a linear solution operator, we can choose solutions depending continuously on the parameters $p \in P$ and $t \in (0, +\infty)$. In view of (2.9), the right hand side of the above estimate approaches 0 when $t \to +\infty$. Since the weight ψ vanishes in U_0, it follows that

$$\int_{U_0} \frac{|u_{p,t}|^2}{(1 + |z|^2)^2} \mathrm{d}\lambda \to 0 \quad \text{as } t \to +\infty$$

and the convergence is uniform in $p \in P$. Note that $u_{p,t}$ is a holomorphic function on U_0 since α_p vanishes there. By the interior elliptic estimate (see Lemma 2.7.2 and in particular the estimate (2.7)) it follows that

$$\lim_{t\to+\infty} \sup_{z\in K,\, p\in P} \left|u_{p,t}(z)\right| = 0.$$

Hence, for a sufficiently big $t > 0$ the function

$$F_p = \chi f_p - u_{p,t} : \mathbb{C}^n \to \mathbb{C}, \quad p \in P$$

satisfies the proposition. □

Before proving the general result, we need some preparations. The first one is Michael's *Convex Selection Theorem* [415] which we now present.

Given topological spaces P and B and a set-valued map $\phi\colon P \to 2^B$ whose values are subsets of B, we say that a map $f\colon P \to B$ is a *selection* of ϕ if $f(p) \in \phi(p)$ for every $p \in P$. Such ϕ is said to be lower semicontinuous if for every open set $V \subset B$ the set $\{p \in P\colon \phi(p) \cap V \neq \varnothing\}$ is open in P. The following is a special case of Michael's theorem; a similar result was proved by Cartan [88, Appendix]. We do not prove it here.

Theorem 2.8.2 ([415]) *Assume that B is a Banach space, P is a paracompact Hausdorff space, and $\phi\colon P \to 2^B$ is a lower semicontinuous set-valued map such that $\phi(p)$ is a nonempty closed convex subset of B for every $p \in P$. For every closed subset P_0 of P and every continuous selection $f\colon P_0 \to B$ of $\phi|_{P_0}$ there exists a continuous selection $F\colon P \to B$ of ϕ extending f.*

Denote by $H^\infty(D)$ the Banach space of all bounded holomorphic functions on a complex space D. We need the following lemma [218, Lemma 3.1] on the existence of a linear bounded extension operator.

Lemma 2.8.3 *Assume that X is a reduced Stein space, X' is a closed complex subvariety of X and $\Omega \Subset \Omega'$ are relatively compact Stein domains in X. There exists a bounded linear extension operator*

$$S\colon H^\infty\big(X' \cap \Omega'\big) \to H^\infty(\Omega)$$

such that $(Sf)(x) = f(x)$ for every $f \in H^\infty(X' \cap \Omega')$ and $x \in X' \cap \Omega$.

Proof We replace X by a relatively compact Stein subdomain containing $\overline{\Omega}'$ and embed it as a closed complex subvariety in a Euclidean space $\mathbb{C}^n$. Since every Stein domain $D \Subset X$ is the intersection $D = X \cap \widetilde{D}$ of X with an open Stein domain $\widetilde{D} \Subset \mathbb{C}^n$, it suffices to prove the lemma for the case $X = \mathbb{C}^n$.

Since Ω' is Stein, the restriction operator $R\colon \mathcal{O}(\Omega') \to \mathcal{O}(X' \cap \Omega')$ is surjective by Cartan's extension theorem (Corollary 2.6.3). Choose a domain $\Omega_1 \subset \mathbb{C}^n$ such that $\Omega \Subset \Omega_1 \Subset \Omega'$. By the open mapping theorem for Fréchet spaces, the image by R of the set $\{f \in \mathcal{O}(\Omega')\colon \|f\|_{L^\infty(\Omega_1)} < 1\}$ contains a neighborhood of the origin in $\mathcal{O}(X' \cap \Omega')$. This means that there are a relatively compact subset $Y \Subset X' \cap \Omega'$ and a constant $M < +\infty$ such that every $h \in \mathcal{O}(X' \cap \Omega')$ extends to a function $h' \in \mathcal{O}(\Omega')$ satisfying the estimate

$$\big\|h'\big\|_{L^\infty(\Omega_1)} \le M\|h\|_{L^\infty(Y)}.$$

We may assume that $\Omega_1 \cap X' \subset Y$. The restriction $h'|_{\Omega_1}$, being bounded, belongs to the Bergman space $H = L^2(\Omega_1) \cap \mathcal{O}(\Omega_1)$. Note that H is a Hilbert space containing the closed subspace $H_0 = \{f \in H\colon f|_{X'} = 0\}$.

Let H_1 be the orthogonal complement of H_0 in H. Projecting h' orthogonally to H_1 gives a function $\tilde{h} \in H_1$ such that $\tilde{h}|_{X' \cap \Omega_1} = h|_{X' \cap \Omega_1}$ and $\tilde{h}$ has the minimal $L^2(\Omega_1)$-norm among all L^2-holomorphic extensions of h to Ω_1. Clearly, such $\tilde{h}$ is unique, and $S: h \to \tilde{h}$ gives a bounded linear operator $S: H^\infty(X' \cap \Omega') \to L^2(\Omega_1)$. By restricting $\tilde{h}$ to $\Omega \Subset \Omega_1$ we get a bounded linear extension operator $S: H^\infty(X' \cap \Omega') \to H^\infty(\Omega)$. □

Theorem 2.8.4 (Cartan-Oka-Weil theorem with parameters) *Let X be a reduced Stein space. Assume that K is an $\mathcal{O}(X)$-convex subset of X, X' is a closed complex subvariety of X, and $P_0 \subset P$ are compact Hausdorff spaces. Let $f: P \times X \to \mathbb{C}$ be a continuous function such that*

(a) *for every $p \in P$, $f(p, \cdot): X \to \mathbb{C}$ is holomorphic on a neighborhood of K (independent of p) and $f(p, \cdot)|_{X'}$ is holomorphic, and*
(b) *$f(p, \cdot)$ is holomorphic on X for every $p \in P_0$.*

Then there exists for every $\epsilon > 0$ a continuous function $F: P \times X \to \mathbb{C}$ satisfying the following conditions:

(i) *$F_p = F(p, \cdot)$ is holomorphic on X for all $p \in P$,*
(ii) *$|F - f| < \epsilon$ on $P \times K$, and*
(iii) *$F = f$ on $(P_0 \times X) \cup (P \times X')$.*

The same result holds for sections of any holomorphic vector bundle over X.

Proof It suffices to show that a function F with the stated properties exists on $P \times D$, where $D \Subset X$ is any given Stein Runge domain in X containing K; the result then follows by an obvious induction over an exhaustion of X. Fix such a domain D and replace X by a relatively compact Stein neighborhood of $\bar{D}$. By Theorem 2.4.1 we can embed this new X as a closed complex subvariety of a Euclidean space $\mathbb{C}^n$. Choose bounded pseudoconvex Runge domains $\Omega \Subset \Omega' \Subset \mathbb{C}^n$ such that $\bar{D} \subset \Omega \cap X$. Lemma 2.8.3 furnishes bounded linear extension operators

$$S: H^\infty\big(X \cap \Omega'\big) \longrightarrow H^\infty(\Omega), \qquad S': H^\infty\big(X' \cap \Omega'\big) \longrightarrow H^\infty(\Omega)$$

such that

$$S(g)|_{X \cap \Omega} = g|_{X \cap \Omega}, \qquad S'(g)|_{X' \cap \Omega} = g|_{X' \cap \Omega}$$

holds for every g is the respective space. With f_p as in the theorem we set

$$h_p = S(f_p|_{X \cap \Omega'}) - S'(f_p|_{X' \cap \Omega'}) \in H^\infty(\Omega), \quad p \in P_0.$$

Then, h_p belongs to the closed subspace of $H^\infty(\Omega)$ defined by

$$H^\infty_{X'}(\Omega) = \big\{h \in H^\infty(\Omega) : h = 0 \text{ on } X' \cap \Omega\big\}.$$

Since these are Banach spaces, Theorem 2.8.2 furnishes a continuous extension of the map $P_0 \to H^\infty_{X'}(\Omega)$, $p \to h_p$, to a map $P \ni p \to \tilde{h}_p \in H^\infty_{X'}(\Omega)$. Set

$$G_p = \tilde{h}_p + S'(f_p|_{X' \cap \Omega'}) \in H^\infty(\Omega), \quad p \in P.$$

We then clearly have

$$G_p|_{X' \cap \Omega} = f_p|_{X' \cap \Omega} \ (\forall p \in P), \qquad G_p|_{X \cap \Omega} = f_p|_{X \cap \Omega} \ (\forall p \in P_0).$$

Thus, the family of holomorphic functions $G_p|_{X\cap\Omega}\colon X\cap\Omega\to\mathbb{C}$ $(p\in P)$ satisfies conditions (i) and (iii) in the theorem, but not necessarily the approximation condition (ii). However, by continuity there is a small open neighborhood $P_0'\subset P$ of P_0 such that condition (ii) does hold for $p\in P_0'$.

To achieve condition (ii) for all $p\in P$, we proceed as follows. Choose functions $\xi_1,\ldots,\xi_m\in\mathcal{O}(\mathbb{C}^n)$ which generate the sheaf of ideals of the subvariety $X'\subset\mathbb{C}^n$ on the subset $\Omega'\Subset\mathbb{C}^n$. By using Lemma 2.8.3 in exactly the same way as above, we can extend the family of holomorphic functions $\{f_p\}_{p\in P}$ from an open neighborhood of the set K in X to an open Stein Runge domain $\Omega_0\subset\mathbb{C}^n$ such that $K\subset\Omega_0\Subset\Omega$. As before, we keep their values on the subvariety $X'\cap\Omega_0$, so we have $G_p=f_p$ on $X'\cap\Omega_0$. Cartan's division theorem (see Corollary 2.6.4) gives

$$G_p=f_p+\sum_{i=1}^m g_{i,p}\xi_i \tag{2.10}$$

where $g_{i,p}\in\mathcal{O}(\Omega_0)$ for $p\in P$ and $i=1,\ldots m$.

We now show that, after shrinking their domain slightly, the families $g_{i,p}$ can be chosen to depend continuously on $p\in P$.

The Oka-Cartan extension theorem (Corollary 2.6.3) shows that the map

$$\begin{aligned}\Phi\colon\mathcal{O}(\Omega_0)^m\to\mathcal{O}_{X'}(\Omega_0)=\bigl\{h\in\mathcal{O}(\Omega_0):h|_{X'}=0\bigr\},\\ \Phi(g_1,\ldots,g_m)=\sum_{i=1}^m g_i\xi_i\end{aligned} \tag{2.11}$$

is surjective. Choose a Stein domain $\Omega_1\subset\mathbb{C}^n$ such that $K\subset\Omega_1\Subset\Omega_0$. Consider the Hilbert spaces

$$H=L^2(\Omega_1)\cap\mathcal{O}(\Omega_1),\qquad H'=\{h\in H:h|_{X'}=0\}.$$

Note that (2.11) defines a linear Hilbert space map $\Phi\colon H\to H'$. Clearly, the functions $(G_p-f_p)|_{\Omega_1}$ $(p\in P)$ belong to H'. Let $g_p=(g_{1,p},\ldots,g_{m,p})\in H$ be the unique preimage of $(G_p-f_p)|_{\Omega_1}$ which is orthogonal to $\ker\Phi$; this family is continuous in $p\in P$. Now, apply Proposition 2.8.1 to approximate g_p by a continuous family of holomorphic maps $\tilde g_p=(\tilde g_{1,p},\ldots,\tilde g_{m,p})\in\mathcal{O}(\Omega)^m$ $(p\in P)$ and set

$$\widetilde G_p=G_p-\sum_{i=1}^m\tilde g_{i,p}\xi_i\in\mathcal{O}(\Omega),\quad p\in P.$$

Comparing with (2.10) we see that, if the approximation of g by $\tilde g$ is close enough, the family $\widetilde G_p$ satisfies conditions (i)–(iii), except that $\widetilde G_p$ need not agree with f_p for $p\in P_0$. This is corrected by choosing a continuous function $\chi\colon P\to[0,1]$ which equals 1 on P_0 and has support contained in P_0', and setting

$$F_p=\chi(p)G_p+\bigl(1-\chi(p)\bigr)\widetilde G_p\in\mathcal{O}(\Omega),\quad p\in P.$$

The family $F_p|_{X\cap\Omega}$ then satisfies Theorem 2.8.4 on the domain $D\subset X$. This completes the proof for functions.

Suppose now that $E \to X$ is a holomorphic vector bundle over X and f_p $(p \in P)$ is a continuous family of sections of E. As before, it suffices to find a family of sections $\{F_p\}_{p\in P}$ satisfying the stated conditions on any given relatively compact open Stein domain $U \Subset X$; the proof is then completed by an induction over an exhaustion of X. Replacing X by such a subset, the bundle $E \to X$ is finitely generated, and hence there exists a surjective holomorphic vector bundle map $\Psi\colon X \times \mathbb{C}^N \to E$ for some $N \in \mathbb{N}$. Let $E' = \ker \Psi$. By Corollary 2.6.6 we can embed E as a holomorphic vector subbundle of $X \times \mathbb{C}^N$ such that $X \times \mathbb{C}^N = E \oplus E'$, and there is a holomorphic vector bundle projection $\phi\colon X \times \mathbb{C}^N \to E$ with $\ker \phi = E'$. Hence, sections of E can be seen as maps $X \to \mathbb{C}^N$. Applying the already proved result for functions componentwise and projecting the resulting map $F\colon P \times X \to \mathbb{C}^N$ back to E by using ϕ completes the proof. $\square$

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