Stein Manifolds and Holomorphic Mappings: Errata

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Abstract This note contains errata for my book *Stein Manifolds and Holomorphic Mappings (The Homotopy Principle in Complex Analysis)*, Second Edition, Springer, Cham, 2017.

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page 28: in the penultimate item, ρ must be replaced by u.

page 71: The third paragraph of *Proof*: replace $K_0 = K \cap X_0$ by $K_0 = K \cap Y$.

page 74: In Proposition 3.3.2, the space Z must be Stein, but X need not be Stein.

page 75: Theorem 3.3.5 is stated incorrectly. A counterexample is the projection $\pi : Z = \mathbb{C}^2 \to \mathbb{C} = X$, $\pi(x, y) = x$, and $S = \{(x, y) : x = y^2\}$. The theorem holds (with the proof given in [202, Proposition 3.2]) under the following additional condition:

For every point $z_0 \in S$ there are a neighbourhood $U \subset Z$ and a holomorphic map $\psi = (\psi_1, \ldots, \psi_d) : U \to \mathbb{C}^d$, where d is the codimension of the fibres S_x in Z_x $(x \in X)$, such that $S \cap U = \{\psi = 0\}$ and the vertical differential $VD(\psi)$ has maximal rank d.

This condition implies that S admits a normal bundle in Z which is a holomorphic subbundle of the vertical tangent bundle $VT(Z)|_S$ restricted to S. This obviously fails in the above example: for any holomorphic function f on $\mathbb{C}^2$ near $(0, 0)$ vanishing on S, the function $f(0, \cdot\,)$ has a zero of order at least 2 at the origin.

Applications of Theorem 3.3.5 in the book are consistent with this additional condition.

page 116, the sentence below (4.26): Sternberg considered the smooth case in $\mathbb{R}^n$.

page 153, Lemma 4.11.5: On line 3, replace $[v] \neq \omega(A)$ by $[v] \notin \omega(A)$. On line 3 of the proof, the statement "Since $[z] \notin \omega(A)$" must be replaced by "Since $[v] \notin \omega(A)$".

page 156, Theorem 4.11.13: the correct attribution is [322, Theorem 6].

page 225: a part of the argument in the second paragraph of the proof of Theorem 5.6.5 is incomplete since we do not have a Stein domain in E which is needed to apply Theorem 3.3.5. The problem is resolved by passing to the Stein graphs of the respective maps f, h and using the following result (see [194, Lemma 3.4]).

Lemma 1. *Let $\pi : \Omega \to Y$ be a holomorphic submersion of a Stein manifold Ω onto a complex manifold Y. Then there are an open Stein domain $W \subset Y \times \Omega$ containing the submanifold $S := \{(y, z) \in Y \times \Omega : \pi(z) = y\}$ and a holomorphic retraction $\widetilde{\rho} : W \to S$ of the form $\widetilde{\rho}(y, z) = (y, \rho(y, z))$ for every $(y, z) \in W$.*

It follows that for every $y \in Y$ the map $\rho_y = \rho(y, \cdot\,)$ is a holomorphic retraction of an open neighbourhood of the fibre $\Omega_y = \pi^{-1}(y)$ in Ω onto Ω_y, depending holomorphically on y. Note that Lemma 1 follows from Theorem 3.3.5 applied to the submersion $Y \times \Omega \to Y$, $(y, z) \mapsto y$, and the submanifold S defined in the lemma.

The proof of Theorem 5.6.5 can now be completed as follows. Consider the manifolds $\widetilde{X} = \mathbb{C}^n \times X$, $\widetilde{E} = \mathbb{C}^n \times E$ and the projection $\tilde{\pi} : \widetilde{E} \to \widetilde{X}$, $\tilde{\pi}(z, e) = (z, \pi(e))$. Recall that $U \subset \mathbb{C}^n$ is an open convex neighbourhood of the compact convex set $K \subset \mathbb{C}^n$ and $h : U \to E$ and $f = \pi \circ h : U \to X$ are holomorphic maps. Their graphs

$$\Gamma_f = \{(z, f(z)) : z \in U\} \subset \widetilde{X}, \qquad \Gamma_h = \{(z, h(z)) : z \in U\} \subset \widetilde{E}$$

are locally closed Stein submanifolds of $\widetilde X$ and $\widetilde E$, respectively, which therefore admits open Stein neighborhoods $Y \subset \widetilde X$ and $\Omega \subset \widetilde E$. These can be chosen such that $\tilde\pi|_\Omega : \Omega \to Y$ is a surjective holomorphic submersion. Let ρ_y for $y \in Y$ be a holomorphic retraction onto the fibre Ω_y (depending holomorphically on y), furnished by Lemma 1. If the approximating map $f_1 : V \to X$ from an open set $V \supset K$ in $\mathbb{C}^n$ (see the book) is sufficiently uniformly close to f on K, then the point $(z, h(z)) \in \Omega$ lies in the domain of the retraction $\rho_{(z,f_1(z))}$ for all z in some neighbourhood $U_1 \subset U$ of K. For every $z \in U_1$ let $h_1(z) \in E$ denote the projection of the point $\rho_{(z,f_1(z))}(z, h(z)) \in \Omega_{(z,f_1(z))} \subset \widetilde E$ to E. Then, the map $h_1 : U_1 \to E$ is holomorphic, uniformly close to h on K, and it satisfies $\pi \circ h_1(z) = f_1(z)$ for $z \in U_1$. The proof can now be completed as in the book.

page 233: the first statement in Proposition 5.6.23 is false; a counterexample is any compact hyperbolic manifold. Here is the correct statement supported by the proof.

Proposition 5.6.23 *A complex manifold Y with the density property whose tangent bundle is pointwise generated by holomorphic vector fields on Y is flexible in the sense of Arzhantsev et al., and hence an Oka manifold. In particular, a Stein manifold with the density property is elliptic.*

page 235, line 8: condition $A(r) \Subset D'_1$ should read $A(r) \Subset D'_0$.

page 252: Proposition 5.13.1 is immediate if the subspace P_0 of the parameter space P is a deformation retract of a neighbourhood of P_0 in P, since the retraction allows a suitable reparametrization of the given family of sections. This suffices for most applications.

page 285: The condition in the last line should read $f_{(p,0)}(\overline{V}) \subset D_j$ (replace K by $\overline{V}$).

page 287: In Definition 6.6.5, Condition HAP is stated incorrectly; the same condition must hold for any local holomorphic spray of sections with parameter in a ball $\mathbb{B} \subset \mathbb{C}^n$. Equivalently, the stated condition must apply to the trivial extensions $Z \times \mathbb{B} \to X \times \mathbb{B}$. This holds for any subelliptic submersions $h : Z \to X$ in view of Theorem 6.6.2.

page 295: Proposition 6.10.1 gives a homotopy connecting a global continuous section to a complex of holomorphic sections of a holomorphic submersion. It has been pointed out that the parametric case, which is the first step in the proof of Theorem 6.2.2, is not explained. The proof is a simple application of partitions of unity on the parameter space. See Remark 3 in the paper *Developments in Oka theory since 2017*.

page 317, line 11: the statement $H^2(X,\mathbb{Z}) = H^2(M,\mathbb{Z})/(D)$ is not correct: we only have $H^2(X,\mathbb{Z}) \supset H^2(M,\mathbb{Z})/(D)$. This proper inclusion is enough for the argument.

page 370: In the paragraph below Definition 8.5.2 there is the following statement:

"Conversely, the tubular neighborhood theorem for Stein manifolds (see Theorem 3.3.3 on p. 74) implies that any Stein submanifold Y in an arbitrary complex manifold X is an ideal theoretic complete intersection in an open neighborhood $U \subset X$ of Y."

This holds under the assumption that the normal bundle of Y in X is trivial.

page 379, line 3: Then a generic holomorphic *maps*... (replace by *map*)

page 488: formula (10.15) should read $g(C) = (d-1)(d-2)/2$.

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