Chapter 2
Profinite Graphs

Unless otherwise specified, in this chapter $\mathcal{C}$ is a pseudovariety of finite groups, i.e., a nonempty class of finite groups closed under subgroups, quotients and finite direct products.

2.1 First Notions and Examples

A *profinite graph* is a profinite space Γ with a distinguished nonempty subset $V(\Gamma)$, the *vertex set* of the graph Γ, and two continuous maps

$$d_0, d_1 : \Gamma \to V(\Gamma)$$

whose restrictions to $V(\Gamma)$ are the identity map $\mathrm{id}_{V(\Gamma)}$ (to simplify the notation, we sometimes write $d_i m$, rather than $d_i(m)$ $(m \in \Gamma, i = 0, 1)$). This implies that the distinguished subset $V(\Gamma)$ is necessarily closed. The elements of $V(\Gamma)$ are called the *vertices* of Γ, the elements of $E(\Gamma) = \Gamma - V(\Gamma)$ are the *edges* of Γ, and $d_0(e)$ and $d_1(e)$ are the *initial* and *terminal* vertices of an edge e, respectively (also called the *origin* and *terminus* of e). An edge e with $d_0(e) = d_1(e) = v$ is called a *loop* or a loop based at v. We refer to d_0 and d_1 as the *incidence maps* of the graph Γ.

Observe that a profinite graph is also a graph in the usual sense, or, more precisely, an oriented graph (see Appendix A), if we dispense with the topology. The set of edges $E(\Gamma)$ of a profinite graph Γ need not be a closed subset of Γ. If $E(\Gamma)$ is closed (and therefore compact), it is enough to check the continuity of d_0 and d_1 on $V(\Gamma)$ and $E(\Gamma)$ separately, since then $V(\Gamma)$ and $E(\Gamma)$ are disjoint and clopen.

Associated with each edge e of Γ we introduce symbols e^1 and e^{-1}. We identify e^1 with e. Define incidence maps for these symbols as follows: $d_0(e^{-1}) = d_1(e)$ and $d_1(e^{-1}) = d_0(e)$. Given vertices v and w of Γ, a *path* p_{vw} from v to w is a finite sequence $e_1^{\varepsilon_1}, \dots, e_m^{\varepsilon_m}$, where $m \geq 0$, $e_i \in E(\Gamma)$, $\varepsilon_i = \pm 1$ $(i = 1, \dots, m)$ such that $d_0(e_1^{\varepsilon_1}) = v$, $d_1(e_m^{\varepsilon_m}) = w$ and $d_1(e_i^{\varepsilon_i}) = d_0(e_{i+1}^{\varepsilon_{i+1}})$ for $i = 1, \dots, m-1$. Such a path is said to have *length* m. Observe that a path is always meant to be finite. The *underlying graph* of the path p_{vw} consists of the edges $e_1, \dots, e_m$ and their

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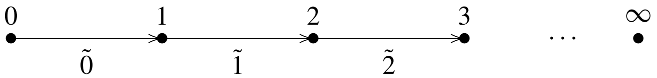
vertices $d_i(e_j)$ $(i = 0, 1;\ j = 1, \ldots, m)$. The path p_{vw} is called *reduced* if whenever $e_i = e_{i+1}$, then $\varepsilon_i = \varepsilon_{i+1}$, for all $i = 1, \ldots, m-1$.

Example 2.1.1 (a) A finite abstract graph Γ (see Appendix A) with the discrete topology is a profinite graph.

(b) Let $\mathbf{N} = \{0, 1, 2, \ldots\}$ and $\tilde{\mathbf{N}} = \{\tilde{n} \mid n \in \mathbf{N}\}$ be copies of the set of natural numbers (with the discrete topology). Define

$$I = \mathbf{N} \,\dot\cup\, \tilde{\mathbf{N}} \,\dot\cup\, \{\infty\}$$

to be the one-point compactification of the space $\mathbf{N} \,\dot\cup\, \tilde{\mathbf{N}}$. Recall that then in the topology of I each set $\{n\}$ and $\{\tilde{n}\}$ is open ($n \in \mathbf{N}$), and the basic open neighbourhoods of ∞ are the complements of finite subsets of $\mathbf{N} \,\dot\cup\, \tilde{\mathbf{N}}$. Clearly I is a profinite space. We make I into a profinite graph by setting $V(I) = \mathbf{N} \,\dot\cup\, \{\infty\}$, $E(I) = \tilde{\mathbf{N}}$, $d_0(\tilde{n}) = n$, $d_1(\tilde{n}) = n+1$, for $\tilde{n} \in E(I)$, and $d_i(n) = n$, for $n \in V(I)$ $(i = 1, 2)$.



Observe that in this case the subset of edges $E(I)$ is open, but not closed in I.

(c) Let p be a prime number and let $\mathbf{Z}_p$ be the additive group of the ring of p-adic integers. Define a graph

$$\Gamma = \Gamma\big(\mathbf{Z}_p, \{1\}\big)$$

with set of vertices $V = V(\Gamma) = \mathbf{Z}_p$ and whose set of edges is $E = E(\Gamma) = \{(\alpha, 1) \mid \alpha \in \mathbf{Z}_p\}$. Then $V(\Gamma)$ and $E(\Gamma)$ are profinite spaces. We define the topology of

$$\Gamma = V(\Gamma) \,\dot\cup\, E(\Gamma)$$

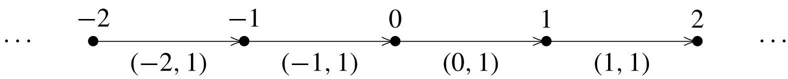
to be the disjoint topology: a subset A of Γ is open if and only if $A \cap V$ is open in V and $A \cap E$ is open in E. One easily sees that Γ is a profinite space. Observe that the subset of edges $E = E(\Gamma)$ of Γ is both open and closed (clopen) in the topology of Γ. The incidence maps are the continuous maps

$$d_i : \Gamma \longrightarrow V \quad (i = 0, 1)$$

defined as $d_0(\alpha) = \alpha$, $d_0(\alpha, 1) = \alpha$ and $d_1(\alpha) = \alpha$, $d_1(\alpha, 1) = \alpha + 1$ $(\alpha \in \mathbf{Z}_p)$. With these definitions Γ becomes a profinite graph. [This is an instance of profinite graphs obtained from profinite groups in a standard manner, the so-called Cayley graphs: see Example 2.1.12.] The subgroup of integers $\mathbf{Z} = \langle 1 \rangle$ is dense in $\mathbf{Z}_p$ and the topology of $\mathbf{Z}$ induced by the topology of $\mathbf{Z}_p$ is the discrete topology. Let

$$\Gamma\big(\mathbf{Z}, \{1\}\big) = \big\{\alpha \in V(\Gamma) \bigm| \alpha \in \mathbf{Z}\big\} \,\dot\cup\, \big\{(\alpha, 1) \bigm| \alpha \in \mathbf{Z}\big\}.$$

Then $\Gamma(\mathbf{Z}, \{1\})$ is an abstract discrete graph



which is dense in the profinite graph $\Gamma = \Gamma(\mathbf{Z}_p, \{1\})$.

More generally, let β be a fixed element of $\mathbf{Z}_p$, and define

$$\Gamma\big(\mathbf{Z}+\beta,\{1\}\big)=\big\{\alpha\in V(\Gamma)\;\big|\;\alpha\in\mathbf{Z}+\beta\big\}\,\dot\cup\,\big\{(\alpha,1)\in E(\Gamma)\;\big|\;\alpha\in\mathbf{Z}+\beta\big\}.$$

Then $\Gamma(\mathbf{Z}+\beta,\{1\})$ is an abstract discrete graph

$$\cdots\quad \beta-2 \xrightarrow[(\beta-2,1)]{} \beta-1 \xrightarrow[(\beta-1,1)]{} \beta \xrightarrow[(\beta,1)]{} \beta+1 \xrightarrow[(\beta+1,1)]{} \beta+2 \quad\cdots$$

which is also dense in the profinite graph $\Gamma=\Gamma(\mathbf{Z}_p,\{1\})$. Note that $\Gamma(\mathbf{Z}_p,\{1\})$ is a disjoint union of uncountably many abstract discrete graphs of the form $\Gamma(\mathbf{Z}+\beta,\{1\})$:

$$\Gamma\big(\mathbf{Z}_p,\{1\}\big)=\dot{\bigcup_{\lambda\in\Lambda}}\,\Gamma\big(\mathbf{Z}+\beta_\lambda,\{1\}\big),$$

where $\{\beta_\lambda \mid \lambda\in\Lambda\}$ is a complete set of representatives of the cosets of the subgroup $\mathbf{Z}$ in the group $\mathbf{Z}_p$.

Let Γ and Δ be profinite graphs. A *qmorphism or a quasi-morphism of profinite graphs* or a *map of graphs*

$$\alpha:\Gamma\to\Delta$$

is a continuous map such that $d_j(\alpha(m))=\alpha(d_j(m))$, for all $m\in\Gamma$ and $j=0,1$. If in addition $\alpha(e)\in E(\Delta)$ for every $e\in E(\Gamma)$, we say that α is a *morphism*.

The composition of qmorphisms of profinite graphs is again a qmorphism, so that profinite graphs and their qmorphisms form a category. Similarly profinite graphs and their morphisms form a category. If α is a surjective (respectively, injective, bijective) qmorphism, we say that α is an *epimorphism* (respectively, *monomorphism*, *isomorphism*). An isomorphism $\alpha:\Gamma\to\Gamma$ of the graph Γ to itself is called an *automorphism*. Note that a monomorphism of graphs sends edges to edges, and hence it is always a morphism. A nonempty closed subset Γ of a profinite graph Δ is called a *profinite subgraph* of Δ if whenever $m\in\Gamma$, then $d_j(m)\in\Gamma$ $(j=0,1)$.

The equality $d_j(\alpha(m))=\alpha(d_j(m))$ $(j=0,1;\,m\in\Gamma)$ implies that a qmorphism of profinite graphs maps vertices to vertices. However, the next example shows that a qmorphism can map an edge to a vertex.

Example 2.1.2 (Subgraph collapsing) Let Δ be a profinite subgraph of a profinite graph Γ. Consider the natural continuous map $\alpha:\Gamma\to\Gamma/\Delta$ to the quotient space Γ/Δ with the quotient topology [the points of Γ/Δ are the equivalence classes of the relation $\sim$ on Γ defined as follows: if $m,m'\in\Gamma$, then $m\sim m'$ if and only if either $m=m'$ or $m,m'\in\Delta$; if $m\in\Gamma$, then $\alpha(m)$ is the equivalence class of m; a subset U of Γ/Δ is open if $\alpha^{-1}(U)$ is open in Γ]. Define a structure of profinite graph on the space Γ/Δ as follows: $V(\Gamma/\Delta)=\alpha(V(\Gamma))$, $d_0(\alpha(m))=\alpha(d_0(m)), d_1(\alpha(m))=\alpha(d_1(m))$, for all $m\in\Gamma$. Then clearly α is a qmorphism of graphs and Γ/Δ becomes a quotient graph of Γ. We shall say that Γ/Δ is obtained from Γ by *collapsing* Δ to a point. Observe that α maps any edge of Γ which is in Δ to a vertex of Γ/Δ.

We note that if $\alpha : \Gamma \to \Delta$ is an epimorphism of profinite graphs, then Δ has the quotient topology (i.e., for $A \subseteq \Delta$, one has that A is open in Δ if and only if $\alpha^{-1}(A)$ is open in Γ), since Γ and Δ are compact Hausdorff spaces. We then say that Δ is a *quotient graph* of Γ and α is a *quotient qmorphism of graphs*.

If Γ is a profinite graph and $\varphi : \Gamma \to Y$ is a continuous surjection onto a profinite space Y, there is no assurance that there exists a profinite graph structure on Y so that φ is a qmorphism of graphs. The following construction provides necessary and sufficient conditions for this to happen.

Construction 2.1.3 *Let Γ be a profinite graph and let $\varphi : \Gamma \to Y$ be a continuous surjection onto a profinite space Y. Then we construct a quotient qmorphism of graphs*

$$\tilde{\varphi} : \Gamma \to \Gamma_\varphi$$

with the following properties.

(a) *There is a continuous surjection of topological spaces $\psi_\varphi : \Gamma_\varphi \to Y$ such that the diagram*

$$\begin{tikzcd} \Gamma \arrow[r, "\varphi"] \arrow[d, "\tilde{\varphi}"'] & Y \\ \Gamma_\varphi \arrow[ur, "\psi_\varphi"'] & \end{tikzcd}$$

commutes.

(b) *If Y admits a profinite graph structure so that φ is a qmorphism, then ψ_φ is an isomorphism of profinite graphs.*

(c) *Consequently, there exists a profinite graph structure on Y such that φ is a qmorphism of graphs if and only if whenever $m, m' \in \Gamma$ with $\varphi(m) = \varphi(m')$, then $\varphi d_0(m) = \varphi d_0(m')$ and $\varphi d_1(m) = \varphi d_1(m')$. If this is the case, then that structure is unique (isomorphic to Γ_φ) and the incidence maps of Y are defined by $d_i \varphi(m) = \varphi d_i(m)$ $(m \in \Gamma, i = 0, 1)$.*

(d) *If $E(\Gamma)$ is a closed subset of Γ and $\varphi(E(\Gamma)) \cap \varphi(V(\Gamma)) = \emptyset$, then $\tilde{\varphi}$ is a morphism of profinite graphs and $\psi_\varphi(E(\Gamma_\varphi)) \cap \psi_\varphi(V(\Gamma_\varphi)) = \emptyset$.*

To construct Γ_φ, define a map

$$\tilde{\varphi} : \Gamma \longrightarrow Y \times Y \times Y$$

by

$$\tilde{\varphi}(m) = \big(\varphi(m), \varphi d_0(m), \varphi d_1(m)\big) \quad (m \in \Gamma).$$

Let $\Gamma_\varphi = \tilde{\varphi}(\Gamma)$. Then Γ_φ admits a unique graph structure such that $\tilde{\varphi} : \Gamma \to \Gamma_\varphi$ is a qmorphism of graphs, namely one is forced to define the incidence maps $\tilde{d}_0$ and $\tilde{d}_1$ of Γ_φ by

$$\tilde{d}_0\big(\varphi(m), \varphi d_0(m), \varphi d_1(m)\big) = \big(\varphi d_0(m), \varphi d_0(m), \varphi d_0(m)\big) \quad (m \in \Gamma)$$

and

$$\tilde{d}_1\big(\varphi(m), \varphi d_0(m), \varphi d_1(m)\big) = \big(\varphi d_1(m), \varphi d_1(m), \varphi d_1(m)\big) \quad (m \in \Gamma)$$

(one easily checks that these are well defined, and that $\tilde{\varphi}$ is indeed a qmorphism of profinite graphs). Next note that there exists a unique map $\psi_\varphi : \Gamma_\varphi \to Y$ such that $\psi_\varphi \tilde{\varphi} = \varphi$, namely, $\psi_\varphi(\varphi(m), \varphi d_0(m), \varphi d_1(m)) = \varphi(m)$.

If Y is a profinite graph and φ is a qmorphism of profinite graphs, then ψ_φ is an isomorphism of graphs because in this case the map $\rho : Y \to \Gamma_\varphi$ given by $\rho\varphi(m) = (\varphi(m), \varphi d_0(m), \varphi d_1(m))$ is a well-defined qmorphism of graphs and it is inverse to ψ_φ. This proves properties (a) and (b). Property (c) is clear. Property (d) is easily verified. $\square$

Before stating the following proposition we recall briefly the concept of an inverse limit in the category of graphs (see Sect. 1.1). Let $(I, \preceq)$ be a directed partially ordered set (a directed poset). An inverse system of profinite graphs $\{\Gamma_i, \varphi_{ij}, I\}$ over the directed poset I consists of a collection of profinite graphs Γ_i indexed by I and qmorphisms of profinite graphs $\varphi_{ij} : \Gamma_i \to \Gamma_j$, whenever $i \succeq j$, in such a way that $\varphi_{ii} = \mathrm{Id}_i$, for all $i \in I$, and $\varphi_{jk}\varphi_{ij} = \varphi_{ik}$, whenever $i \succeq j \succeq k$. The inverse limit (or projective limit) of such a system

$$\Gamma = \varprojlim_{i \in I} \Gamma_i$$

is the subset of $\prod_{i \in I} \Gamma_i$ consisting of those tuples (m_i) with $\varphi_{ij}(m_i) = m_j$, whenever $i \succeq j$. Such an inverse limit is in a natural way a profinite graph whose space of vertices is

$$V(\Gamma) = \varprojlim_{i \in I} V(\Gamma_i).$$

Observe that the natural projections $\varphi_i : \Gamma \to \Gamma_i$ are qmorphisms of profinite graphs. Note that if each φ_{ij} is a morphism, then so are the canonical projections φ_i.

Let Γ be a profinite graph and consider the set $\mathcal{R}$ of all open equivalence relations R on the set Γ (i.e., the equivalence classes xR are open for all $x \in \Gamma$). For $R \in \mathcal{R}$, denote by $\varphi_R : \Gamma \to \Gamma/R$ the corresponding quotient map as topological spaces. One defines a partial ordering $\preceq$ on $\mathcal{R}$ as follows: for $R_1, R_2 \in \mathcal{R}$, we say that $R_1 \succeq R_2$ if there exists a map $\varphi_{R_1,R_2} : \Gamma/R_1 \to \Gamma/R_2$ such that the diagram

$$\begin{tikzcd} & \Gamma/R_1 \arrow[dd, "\varphi_{R_1,R_2}"] \\ \Gamma \arrow[ur, "\varphi_{R_1}"] \arrow[dr, "\varphi_{R_2}"'] & \\ & \Gamma/R_2 \end{tikzcd}$$

commutes. Then (cf. RZ, Theorem 1.1.2) $(\mathcal{R}, \preceq)$ is in fact a directed poset, $\{\Gamma/R, \varphi_{R_1,R_2}\}$ is an inverse system over $\mathcal{R}$, and, as topological spaces, the collection of quotient maps $\{\varphi_R \mid R \in \mathcal{R}\}$ induces a homeomorphism from Γ to $\varprojlim_{R\in\mathcal{R}} \Gamma/R$; in fact we identify these two spaces by means of this homeomorphism and write

$$\Gamma = \varprojlim_{R\in\mathcal{R}} \Gamma/R. \tag{2.1}$$

Consider now the subset $\mathcal{R}'$ of $\mathcal{R}$ consisting of those $R \in \mathcal{R}$ such that Γ/R admits a graph structure (which is unique according to part (c) of Construction 2.1.3) so that $\varphi_R : \Gamma \to \Gamma/R$ is a qmorphism of profinite graphs. We check next that the poset $(\mathcal{R}', \preceq)$ is directed. Indeed, let $R_1, R_2 \in \mathcal{R}'$. Since $\mathcal{R}$ is directed, there exists an $R \in \mathcal{R}$ such that $R \succeq R_1, R_2$. Let $\varphi_R : \Gamma \to \Gamma/R$ be the corresponding quotient map. Let Γ_{φ_R} and $\widetilde{\varphi_R} : \Gamma \to \Gamma_{\varphi_R}$ be as in Construction 2.1.3. Then $\Gamma_{\varphi_R} = \Gamma/\tilde{R}$, where $\tilde{R}$ is the equivalence relation on Γ whose equivalence classes are $\{\widetilde{\varphi_R}^{-1}(x) \mid x \in \Gamma_{\varphi_R}\}$. Clearly $\tilde{R} \in \mathcal{R}'$ and $\tilde{R} \succeq R$; hence $\tilde{R} \succeq R_1, R_2$, as needed.

Observe that if $R_1, R_2 \in \mathcal{R}'$ and $R_1 \succeq R_2$, then the map $\varphi_{R_1,R_2} : \Gamma/R_1 \to \Gamma/R_2$ is in fact a qmorphism of finite graphs. Therefore the collection $\{\Gamma/R, \varphi_{R_1,R_2}\}$ of all finite quotient graphs of Γ is an inverse system of finite graphs and qmorphisms over the directed poset $\mathcal{R}'$.

Proposition 2.1.4 *Let Γ be a profinite graph.*

(a) *Γ is the inverse limit of all its finite quotient graphs*:

$$\Gamma = \varprojlim_{R\in\mathcal{R}'} \Gamma/R.$$

Consequently

$$V(\Gamma) = \varprojlim_{R\in\mathcal{R}'} V(\Gamma/R).$$

(b) *If the subset $E(\Gamma)$ of edges of Γ is closed, then a directed subposet $\mathcal{R}''$ of $\mathcal{R}'$ can be chosen so that whenever $R_1, R_2 \in \mathcal{R}''$ with $R_1 \succeq R_2$, then φ_{R_1,R_2} : $\Gamma/R_1 \to \Gamma/R_2$ is a morphism of graphs and*

$$\Gamma = \varprojlim_{R\in\mathcal{R}''} \Gamma/R.$$

Consequently,

$$V(\Gamma) = \varprojlim_{R\in\mathcal{R}''} V(\Gamma/R) \quad \textit{and} \quad E(\Gamma) = \varprojlim_{R\in\mathcal{R}''} E(\Gamma/R).$$

Proof (a) In view of (2.1) one simply has to show that $\mathcal{R}'$ is cofinal in $\mathcal{R}$, i.e., one has to show that whenever $R \in \mathcal{R}$, there exists an $R' \in \mathcal{R}'$ with $R' \succeq R$. But this is clear from property (a) of Construction 2.1.3.

(b) Suppose that $E(\Gamma)$ is closed. Then $\Gamma = V(\Gamma) \mathbin{\dot\cup} E(\Gamma)$ and $V(\Gamma)$ and $E(\Gamma)$ are clopen subsets of Γ. Let $\tilde{\mathcal{R}}$ be the subset of $\mathcal{R}$ consisting of those equivalence relations $R \in \mathcal{R}$ whose equivalence classes xR are contained in either $E(\Gamma)$

or $V(\Gamma)$; this implies that if $\varphi_R : \Gamma \to \Gamma/R$ is the canonical projection, then $\varphi_R(V(\Gamma)) \cap \varphi_R(E(\Gamma)) = \emptyset$. Then one shows that $\tilde{\mathcal{R}}$ is cofinal in $\mathcal{R}$, so that

$$\Gamma = \varprojlim_{R\in\tilde{\mathcal{R}}} \Gamma/R.$$

One can argue now as in part (a); we just indicate the main points: let $\mathcal{R}''$ be the subset of $\tilde{\mathcal{R}}$ consisting of those equivalence relations R'' such that Γ/R'' has the structure of a graph in such a way that $\varphi_{R''} : \Gamma \to \Gamma/R''$ is a morphism of profinite graphs; note that $\mathcal{R}''$ is also a subset of $\mathcal{R}'$; using property (d) of Construction 2.1.3 one shows that $\mathcal{R}''$ is cofinal in $\tilde{\mathcal{R}}$, and hence the result easily follows as above. □

Lemma 2.1.5 *Let $\{\Gamma_i, \varphi_{ij}, I\}$ be an inverse system of profinite graphs and qmorphisms over a directed poset I, and set*

$$\Gamma = \varprojlim_{i\in I} \Gamma_i. \tag{2.2}$$

Let $\rho : \Gamma \to \Delta$ be a qmorphism into a finite graph Δ. Then there exists a $k \in I$ such that ρ factors through Γ_k, i.e., there exists a qmorphism $\rho' : \Gamma_k \to \Delta$ such that $\rho = \rho'\varphi_k$, where $\varphi_k : \Gamma \to \Gamma_k$ is the projection.

Proof For $i \in I$ denote by $\mathcal{R}_i$ the set of all equivalence relations R of Γ_i such that the quotient Γ_i/R is a finite discrete graph and the natural projection $\Gamma_i \to \Gamma_i/R$ is a qmorphism. Define an ordering on the set of pairs

$$\mathcal{A} = \{(i, R) \mid i \in I, R \in \mathcal{R}_i\}$$

by setting $(i, R_i) \succeq (j, R_j)$, if $i \succeq j$ and $(\varphi_{ij} \times \varphi_{ij})(R_i) \subseteq R_j$. Let us prove that $(\mathcal{A}, \preceq)$ is a directed poset. Fix $i, j \in I$ and $R_i \in \mathcal{R}_i$, $R_j \in \mathcal{R}_j$. Since I is a directed poset, there exists some $k \in I$ with $k \succeq i, j$. By Proposition 2.1.4, Γ_k is the inverse limit of all its finite quotient graphs; therefore there exists an $R_k \in \mathcal{R}_k$ with $(\varphi_{ki} \times \varphi_{ki})(R_k) \subseteq R_i$ and $(\varphi_{kj} \times \varphi_{kj})(R_k) \subseteq R_j$, so that $(k, R_k) \succeq (i, R_i), (j, R_j)$, as needed.

Now it is easy to see that

$$\Gamma = \varprojlim_{(i,R)\in\mathcal{A}} \Gamma_i/R.$$

Thus from now on we may assume that each Γ_i in the decomposition (2.2) is finite.

Assume first that each projection $\varphi_i : \Gamma \to \Gamma_i$ is surjective. Let S be the equivalence relation on Γ whose equivalence classes are the clopen sets $\rho^{-1}(m), m \in \Delta$; then $\Gamma/S = \Delta$ and ρ is the natural projection $\Gamma \to \Gamma/S$. Similarly, for $i \in I$, let S_i be the equivalence relation on Γ whose equivalence classes are the clopen sets $\varphi_i^{-1}(m), m \in \Gamma_i$, so that φ_i is the natural projection $\Gamma \to \Gamma/S_i$. Since $\Gamma = \varprojlim_{i\in I}\Gamma_i$, we have that $\bigcap_{i\in I} S_i$ is the trivial equivalence relation, i.e., $\bigcap_{i\in I} S_i = D$, where D is the diagonal subset of $\Gamma \times \Gamma$. Note that S and S_i $(i \in I)$ are clopen subsets of $\Gamma \times \Gamma$. Hence, it follows from the compactness of $\Gamma \times \Gamma$ that there exists a finite subset F of I such that $\bigcap_{j\in F} S_j \subseteq S$. Since the poset I is directed, there exists a $k \in I$ with $S_k \subseteq \bigcap_{j\in F} S_j \subseteq S$. This means that there exists a qmorphism of graphs $\rho_k : \Gamma_k = \Gamma/S_k \to \Delta = \Gamma/S$ such that $\rho = \rho_k\varphi_k$.

Consider now a general φ_i. By the above, there exists some $k' \in I$ and a qmorphism of graphs $\rho_{k'} : \varphi_{k'}(\Gamma) \to \Delta$ such that $\rho = \rho_{k'}\varphi_{k'}$. Since $\Gamma_{k'}$ is finite, there exists a $k \succeq k'$ such that $\varphi_{kk'}(\Gamma_k) \subseteq \varphi_{k'}(\Gamma)$. Then $\rho' = \rho_{k'}\varphi_{kk'}$ is the required qmorphism. □

An alternative proof of Lemma 2.1.5 above can be obtained along the lines of the proof of Lemma 1.1.16 in RZ.

A profinite graph Γ is said to be *connected* if whenever $\varphi : \Gamma \to A$ is a qmorphism of profinite graphs onto a finite graph, then A is connected as an abstract graph (see Sect. A.1 in Appendix A).

Proposition 2.1.6

(a) *Every quotient graph of a connected profinite graph is connected.*

(b) *If*

$$\Gamma = \varprojlim_{i\in I} \Gamma_i$$

and each Γ_i is a connected profinite graph, then Γ is a connected profinite graph.

(c) *Let Γ be a connected profinite graph. If $|\Gamma| > 1$, then Γ has at least one edge. Furthermore, if the set of edges $E(\Gamma)$ of Γ is closed in Γ, then for any vertex $v \in V(\Gamma)$, there exists an edge $e \in E(\Gamma)$ such that either $v = d_0(e)$ or $v = d_1(e)$.*

(d) *Let Γ be a profinite graph, and let Δ be a connected profinite subgraph of Γ. Consider the quotient graph Γ/Δ obtained by collapsing Δ to a point and let $\alpha : \Gamma \to \Gamma/\Delta$ be the natural projection. Then the inverse image $\tilde{\Lambda} = \alpha^{-1}(\Lambda)$ in Γ of a connected profinite subgraph Λ of Γ/Δ is a connected profinite subgraph.*

Proof Part (a) is obvious. Let A be a finite quotient graph of Γ. Then (see Lemma 2.1.5) there exists an $i \in I$ such that A is also a quotient graph of Γ_i. It follows that A is connected, proving (b).

To check the first assertion in (c) observe that by Proposition 2.1.4, Γ has a finite quotient graph with at least two elements; since such a finite quotient graph is connected, it has at least one edge, and hence so does Γ. To check the second assertion in (c), write Γ as an inverse limit $\Gamma = \varprojlim_{i\in I} \Gamma_i$ of finite quotient graphs Γ_i in such a way that

$$E(\Gamma) = \varprojlim_{i\in I} E(\Gamma_i)$$

(see Proposition 2.1.4(b)). For $i \in I$, let $\varphi_i : \Gamma \to \Gamma_i$ denote the canonical projection, and if $i, j \in I$ with $i \succeq j$, let $\varphi_{ij} : \Gamma_i \to \Gamma_j$ denote the canonical morphism. Put $v_i = \varphi_i(v)$ $(i \in I)$. Since Γ_i is a connected finite graph, the set $S_i =$

$d_0^{-1}(v_i) \cup d_1^{-1}(v_i)$ of edges of Γ_i starting or ending at v_i is nonempty; moreover, $\varphi_{ij}(S_i) \subseteq S_j$. Hence the collection $\{S_i\}_{i\in I}$ is an inverse system of nonempty finite sets. Thus

$$\varprojlim_{i\in I} S_i \neq \emptyset$$

(see Sect. 1.1). Let $e \in \varprojlim_{i\in I} S_i$. Then e is an edge of Γ with either $d_0(e) = v$ or $d_1(e) = v$.

(d) This is clear if Γ is finite. Write

$$\Gamma = \varprojlim_{i\in I} \Gamma_i,$$

where each Γ_i is a connected finite quotient graph of Γ (see Proposition 2.1.4(a)). Let Δ_i be the image of Δ in Γ_i under the canonical projection. Then

$$\Delta = \varprojlim_{i\in I} \Delta_i \quad \text{and} \quad \Gamma/\Delta = \varprojlim_{i\in I} \Gamma_i/\Delta_i.$$

Let Λ_i be the image of Λ in Γ_i/Δ_i, and denote by $\tilde{\Lambda}_i$ its inverse image in Γ_i. Since $\tilde{\Lambda} = \varprojlim_{i\in I} \tilde{\Lambda}_i$, $\tilde{\Lambda}$ is connected according to part (b). □

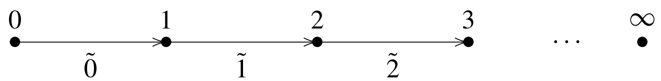
Lemma 2.1.7

(a) *Let D be an abstract subgraph of a profinite graph Γ. Then the topological closure $\bar{D}$ of D in Γ is a profinite graph. If D is connected as an abstract graph* (*see Sect.* A.1 *in Appendix* A), *then $\bar{D}$ is a connected profinite graph.*
(b) *Let $\{\Delta_j \mid j \in J\}$ be a collection of connected profinite subgraphs of a profinite graph Γ. If $\bigcap_{j\in J} \Delta_j \neq \emptyset$, then $\Delta = \overline{\bigcup_{j\in J} \Delta_j}$ is connected.*

Proof To prove (a), let $m \in \bar{D}$. By the continuity of d_i, $d_i(m) \in \overline{V(D)}$ $(i = 1, 2)$, so that $\bar{D}$ is a (profinite) graph with $V(\bar{D}) = \overline{V(D)}$. If $\varphi : \bar{D} \to A$ is a qmorphism of profinite graphs onto a finite graph, then $\varphi(\bar{D}) = \varphi(D) = A$ by continuity. Since D is a connected abstract graph, one easily checks that $\varphi(D)$ is a finite connected graph; hence $\bar{D}$ is a connected profinite graph. This proves (a).

For part (b) note that if $\alpha : \Delta \to A$ is a qmorphism onto a finite graph A, then $\alpha(\Delta_j)$ is a connected finite subgraph of A $(j \in J)$. Since $A = \bigcup_{j\in J} \alpha(\Delta_j)$, and $\bigcap_{j\in J} \alpha(\Delta_j) \neq \emptyset$, it follows that A is a connected abstract graph. □

Example 2.1.8 (A connected profinite graph which is not connected as an abstract graph and with a vertex with no edge beginning or ending at it) Let I be the graph considered in Example 2.1.1(b): $I = \mathbf{N} \,\dot\cup\, \tilde{\mathbf{N}} \,\dot\cup\, \{\infty\}$ is the one-point compactification of a disjoint union of two copies $\mathbf{N}$ and $\tilde{\mathbf{N}} = \{\tilde{n} \mid n \in \mathbf{N}\}$ of the natural numbers; $V(I) = \mathbf{N} \,\dot\cup\, \{\infty\}$, $E(I) = \tilde{\mathbf{N}}$, $d_0(\tilde{n}) = n$, $d_1(\tilde{n}) = n + 1$ for $\tilde{n} \in E(I)$, and $d_i(n) = n$ for $n \in V(I)$ $(i = 1, 2)$.



Then I is a connected profinite graph; to see this consider the connected finite graphs I_n

$$0 \xrightarrow[\tilde{0}]{} 1 \xrightarrow[\tilde{1}]{} 2 \qquad \cdots \qquad n-1 \xrightarrow[\widetilde{n-1}]{} n$$

with vertices $V(I_n) = \{0, 1, 2, \ldots, n\}$ and edges $E(I_n) = \{\tilde{0}, \tilde{1}, \ldots, \widetilde{n-1}\}$ such that $d_0(\tilde{i}) = i$, $d_1(\tilde{i}) = i + 1$ $(i = 0, \ldots, n-1)$ and $d_j(i) = i$ $(i = 0, \ldots, n;\ j = 0, 1)$. If $n \le m$, define $\varphi_{m,n} : I_m \to I_n$ to be the map of graphs that sends the segment $[0, n]$ identically to $[0, n]$, and the segment $[n, m]$ to the vertex n. Then $(I_n, \varphi_{m,n})$ is an inverse system of graphs, and

$$I = \varprojlim_{n \in \mathbf{N}} I_n,$$

where $\infty = (n)_{n \in \mathbf{N}}$. Hence I is a connected profinite graph. We observe that there is no edge e of I which has ∞ as one of its vertices; and so I is not connected as an abstract graph.

Lemma 2.1.9 *Let $\Gamma = \Gamma_1 \mathbin{\dot\cup} \Gamma_2$ be a profinite graph which is the disjoint union of two open profinite subgraphs Γ_1 and Γ_2; then Γ is not connected. In particular, a profinite graph that contains two different vertices and no edges is not connected.*

Proof Collapse Γ_1 to a point v_1 and Γ_2 to a different point v_2 (see Example 2.1.2), to get a disconnected finite quotient graph $\tilde{\Gamma} = \{v_1\} \mathbin{\dot\cup} \{v_2\}$ consisting of two vertices and no edges. □

A maximal connected profinite subgraph of a profinite graph Γ is called a *connected profinite component* of Γ.

Proposition 2.1.10 *Let Γ be a profinite graph.*

(a) *Let $m \in \Gamma$. Then there exists a unique connected profinite component of Γ containing m, which we shall denote by $\Gamma^*(m)$.*
(b) *Any two connected profinite components of Γ are either equal or disjoint.*
(c) *Γ is the union of its connected profinite components.*

Proof Part (c) follows from (a). Part (b) follows from (a) and Lemma 2.1.7(b). To prove (a) observe first that the result is obvious if Γ is finite. By Proposition 2.1.4, Γ can be represented as an inverse limit $\varprojlim_{i \in I} \Gamma_i$ of finite quotient graphs. For $i \in I$, let $\varphi_i : \Gamma \to \Gamma_i$ denote the projection. Since the image of a connected profinite graph is connected, the graphs $\Gamma_i^*(\varphi_i(m))$ form an inverse system. It suffices to show that the profinite subgraph $\varprojlim_{i \in I} \Gamma_i^*(\varphi_i(m))$ of Γ is the connected profinite component of Γ containing m. This profinite subgraph is connected by Proposition 2.1.6(b). If Γ' is a connected profinite subgraph of Γ containing m, then $\Gamma' = \varprojlim_{i \in I} \varphi_i(\Gamma')$. Therefore $\varphi_i(\Gamma') \subseteq \Gamma_i^*(\varphi_i(m))$ for all $i \in I$. Hence

$\Gamma' \subseteq \varprojlim_{i\in I} \Gamma_i^*(\varphi_i(m))$; therefore $\varprojlim_{i\in I} \Gamma_i^*(\varphi_i(m))$ is maximal connected containing m, as desired. The uniqueness of connected profinite components containing m follows from Lemma 2.1.7(b). □

Exercise 2.1.11

(a) Let Δ be a profinite graph. Define the space of connected profinite components of Δ as a quotient space $\Delta/\sim$, where $\sim$ is the equivalence relation defined as follows: $m_1 \sim m_2$ if and only if $\Delta^*(m_1) = \Delta^*(m_2)$. Prove that $\Delta/\sim$ is a profinite space. [Hint: write Δ as an inverse limit of finite quotient graphs.]
(b) Let Δ be a profinite subgraph of a profinite graph Γ. Define the operation of collapsing the connected profinite components of Δ to points as a natural mapping to the quotient space $\Gamma/\sim$, where $\sim$ is the equivalence relation defined as follows: $m_1 \sim m_2$ if $m_1 = m_2$, for $m_1, m_2 \in \Gamma - \Delta$, or $\Delta^*(m_1) = \Delta^*(m_2)$ for $m_1, m_2 \in \Delta$. Prove that $\Gamma/\sim$ is a profinite quotient graph of Γ.

Example 2.1.12 (The Cayley graph) Let G be a profinite group (whose operation is denoted as multiplication and whose identity element is denoted by 1) and let X be a closed subset of G. Put $\tilde{X} = X \cup \{1\}$. Define the *Cayley graph* $\Gamma(G, X)$ of G with respect to the subset X as follows:

$$\Gamma(G, X) = G \times \tilde{X},$$

where $G \times \tilde{X}$ has the product topology. Define the space of vertices of $\Gamma(G, X)$ to be $V(\Gamma(G, X)) = \{(g, 1) \mid g \in G\}$. We identify this space of vertices with G by means of the homeomorphism $(g, 1) \mapsto g$ $(g \in G)$.

Finally, the incidence maps

$$d_0, d_1 : \Gamma(G, X) = G \times \tilde{X} \longrightarrow V\big(\Gamma(G, X)\big) = G$$

are defined by

$$d_0(g, x) = g \quad \text{and} \quad d_1(g, x) = gx, \quad \big(g \in G, x \in X \cup \{1\}\big).$$

Clearly d_0 and d_1 are continuous and they are the identity map when restricted to $V(\Gamma(G, X)) = \{(g, 1) \mid g \in G\} = G$. Therefore the Cayley graph $\Gamma(G, X)$ is a profinite graph.

Note that the space of edges is $E(\Gamma(G, X)) = \Gamma(G, X) - V(\Gamma(G, X)) = G \times (X - \{1\})$:

$$g \xrightarrow{(g,x)} gx,$$

where $x \in X - \{1\}$. It is a closed (and hence clopen) subset of $\Gamma(G, X)$ if and only if 1 is an isolated point of $\tilde{X}$. Observe that if $1 \notin X$, then $V(\Gamma(G, X)) = G$ and $E(\Gamma(G, X)) = G \times X$, and in this case $E(\Gamma(G, X))$ is clopen. If $1 \in X$, then $\tilde{X} = X$. If 1 is in X and it an isolated point of X (for example, if X is finite), then $X - \{1\}$ is also a closed subspace and we have $\Gamma(G, X) = \Gamma(G, X - \{1\})$. Note that the Cayley graph $\Gamma(G, X)$ does not contain loops since the elements of the form $(g, 1)$ are vertices by definition.

Let $\varphi : G \to H$ be a continuous homomorphism of profinite groups and let X be a closed subset of G. Put $Y = \varphi(X)$. Then φ induces a qmorphism of the corresponding Cayley graphs

$$\tilde{\varphi} : \Gamma(G, X) \longrightarrow \Gamma(H, Y).$$

In particular, if U is an open normal subgroup of G and $X_U = \varphi_U(X)$, where $\varphi_U : G \to G/U$ is the canonical epimorphism, then φ_U induces a corresponding epimorphism of Cayley graphs $\widetilde{\varphi_U} : \Gamma(G, X) \to \Gamma(G/U, X_U)$. One easily checks that

$$\Gamma(G, X) = \varprojlim_{U \triangleleft_o G} \Gamma(G/U, X_U)$$

is a decomposition of $\Gamma(G, X)$ as an inverse limit of finite Cayley graphs.

Example 2.1.13 (An infinite connected profinite graph all of whose proper connected profinite subgraphs are finite) Let $\Gamma = \Gamma(\widehat{\mathbf{Z}}, \{1\})$ be the Cayley graph of the free profinite group $\widehat{\mathbf{Z}}$ of rank one with respect the subset $\{1\}$. Then

$$\Gamma = \varprojlim_{n \geq 2} \Gamma\big(\mathbf{Z}/n\mathbf{Z}, \{1\}\big),$$

with canonical maps

$$\varphi_{mn} : \Gamma\big(\mathbf{Z}/m\mathbf{Z}, \{1\}\big) \longrightarrow \Gamma\big(\mathbf{Z}/n\mathbf{Z}, \{1\}\big) \quad (n|m).$$

Let

$$\varphi_n : \Gamma \longrightarrow \Gamma\big(\mathbf{Z}/n\mathbf{Z}, \{1\}\big)$$

denote the projection ($n \in \mathbf{N}$). Assume that Δ is a connected proper profinite subgraph of Γ. Put $\Delta_n = \varphi_n(\Gamma)$. Then Δ_n is a connected subgraph of the finite graph $\Gamma(\mathbf{Z}/n\mathbf{Z}, \{1\})$.

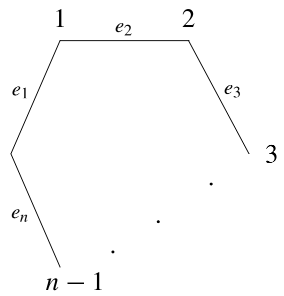
Since $\Delta \neq \Gamma$, there exists some $n_0 \in \mathbf{N}$ such that $\Delta_{n_0} \neq \Gamma(\mathbf{Z}/n_0\mathbf{Z}, \{1\})$. Observe that for every $m \in \mathbf{N}$ with $n_0|m$, the connected components of $\varphi_{mn_0}^{-1}(\Delta_{n_0})$ are isomorphic to Δ_{n_0}. Therefore, $|\Delta_m| = |\Delta_{n_0}|$. Thus Δ is finite.

It is easy to check that if Δ is a proper connected subgraph of Γ with $t+1$ vertices, then there exists a $\gamma \in \widehat{\mathbf{Z}}$ such that the vertices of Δ are $\gamma, \gamma+1, \dots, \gamma+t$ and with edges $(\gamma, 1), (\gamma+1, 1), \dots, (\gamma+t-1, 1)$:

$$\overset{\gamma}{\bullet} \xrightarrow[(\gamma,1)]{\qquad} \overset{\gamma+1}{\bullet} \xrightarrow[(\gamma+1,1)]{\qquad} \overset{\gamma+2}{\bullet} \quad \cdots \quad \overset{\gamma+t-1}{\bullet} \xrightarrow[(\gamma+t-1,1)]{\qquad} \overset{\gamma+t}{\bullet}$$

2.1.14 Circuits. Let $\underline{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_n)$, where $\varepsilon_i = \pm 1$ $(i = 1, \dots, n)$ and $n \geq 1$ is a natural number. Define $\mathrm{Circ}_n(\underline{\varepsilon})$ to be a graph with n vertices (that we take to be the elements of $\mathbf{Z}/n\mathbf{Z}$) and n edges $e_1, \dots, e_n$

$\mathrm{Circ}_n(\underline{\varepsilon})$:



such that $d_0(e_i) = i - 1$ and $d_1(e_i) = i$, if $\varepsilon_i = 1$, and $d_0(e_i) = i$ and $d_1(e_i) = i - 1$, if $\varepsilon_i = -1$. We refer to a graph of the form $\mathrm{Circ}_n(\underline{\varepsilon})$ as a *circuit* of length n or as a n-*circuit*. A circuit of length 1 is a loop. Note that if $n \geq 2$ and $\underline{\varepsilon} = (1, \dots, 1)$, then $\mathrm{Circ}_n(\underline{\varepsilon}) = \Gamma(\mathbf{Z}/n\mathbf{Z}, \{1\})$.

2.2 Groups Acting on Profinite Graphs

Let G be a profinite group and let Γ be a profinite graph. We say that the profinite group G *acts on the profinite graph* Γ on the left, or that Γ is a G-*graph*, if

(i) G acts continuously on the topological space Γ on the left, i.e., there is a continuous map $G \times \Gamma \to \Gamma$, denoted $(g, m) \longmapsto gm$, $g \in G, m \in \Gamma$, such that
$$(gh)m = g(hm) \quad \text{and} \quad 1m = m,$$
for all $g, h \in G, m \in \Gamma$, where 1 is the identity element of G; and
(ii) $d_j(gm) = gd_j(m)$, for all $g \in G, m \in \Gamma$, $j = 0, 1$.

Observe that if G acts on Γ, then for a fixed $g \in G$, the map $\rho_g : \Gamma \to \Gamma$ given by $m \mapsto gm$ $(m \in \Gamma)$ is an automorphism of the graph Γ. Hence (cf. RZ, Remark 5.6.1), G acts on a profinite graph Γ if and only if there exists a continuous homomorphism
$$\rho : G \longrightarrow \mathrm{Aut}(\Gamma),$$
where $\mathrm{Aut}(\Gamma)$ is the group of automorphisms of Γ as a profinite graph, and where the topology on $\mathrm{Aut}(\Gamma)$ is induced by the compact-open topology. The *kernel* of the action of G on Γ is the kernel of ρ, i.e., the closed normal subgroup of G consisting of all the elements $g \in G$ such that $gm = m$, for all $m \in \Gamma$.

One defines actions on the right in a similar manner. We shall consider only left actions in this chapter.

Let G be a profinite group that acts continuously on two profinite graphs Γ and Γ'. A qmorphism of graphs
$$\varphi : \Gamma \longrightarrow \Gamma'$$

is called a *G-map* of graphs if

$$\varphi(gm) = g\varphi(m), \quad \text{for all } m \in \Gamma, g \in G.$$

Assume that a profinite group G acts on a profinite graph Γ and let $m \in \Gamma$. Define

$$G_m = \{g \in G \mid gm = m\}$$

to be the *stabilizer* (or G-stabilizer, if one needs to specify the group G) of the element m. It follows from the continuity of the action and the compactness of G that G_m is a closed subgroup of G. Clearly,

$$G_m \leq G_{d_j(m)}, \quad \text{for every } m \in \Gamma, j = 0, 1.$$

If the stabilizer G_m of every element $m \in \Gamma$ is trivial, i.e., $G_m = 1$, we say that G acts *freely* on Γ. If $m \in \Gamma$, the *G-orbit* of m is the closed subset $Gm = \{gm \mid g \in G\}$.

If a profinite group G acts on a profinite graph Γ, then G acts on the profinite space $V(\Gamma)$ of vertices and G acts on $E(\Gamma)$. The space

$$G\backslash\Gamma = \{Gm \mid m \in \Gamma\}$$

of G-orbits with the quotient topology is a profinite space which admits a natural profinite graph structure as follows:

$$V(G\backslash\Gamma) = G\backslash V(\Gamma), \qquad d_j(Gm) = Gd_j(m), \quad j = 0, 1.$$

We say that $G\backslash\Gamma$ is the *quotient graph of Γ under the action of G*. The corresponding quotient map

$$\Gamma \longrightarrow G\backslash\Gamma$$

is an epimorphism of profinite graphs given by $m \mapsto Gm$ ($m \in \Gamma$, $g \in G$). We observe that it sends edges to edges (it is a morphism).

If $N \triangleleft_c G$, there is an induced action of G/N on $N\backslash\Gamma$ defined by

$$(gN)(Nm) = N(gm), \quad g \in G, \ m \in \Gamma.$$

The following result is straightforward.

Lemma 2.2.1 *Let a profinite group G act on a profinite graph Γ.*

(a) *Let $\mathcal{N}$ be a collection of closed normal subgroups of G filtered from below* (*i.e., the intersection of any two groups in $\mathcal{N}$ contains a group in $\mathcal{N}$*) *and assume that*

$$G = \varprojlim_{N \in \mathcal{N}} G/N.$$

Then the collection of graphs $\{N\backslash\Gamma \mid N \in \mathcal{N}\}$ is an inverse system in a natural way and

$$\Gamma = \varprojlim_{N \in \mathcal{N}} N\backslash\Gamma.$$

(b) *Let $N \triangleleft_c G$. For $m \in \Gamma$, denote by m' the image of m in $N\backslash\Gamma$. Consider the natural action of G/N on $N\backslash\Gamma$ defined above. Then $(G/N)_{m'}$ is the image of G_m under the natural epimorphism $G \to G/N$. In particular, if $G_m \leq N$, for all $m \in \Gamma$, then G/N acts freely on $N\backslash\Gamma$.*

Let G be a profinite group. If $\{\Gamma_i, \varphi_{ij}, I\}$ is an inverse system of profinite G-graphs and G-maps over the directed poset I, then

$$\Gamma = \varprojlim_{i\in I} \Gamma_i$$

is in a natural way a profinite G-graph.

Next we show that every profinite G-graph admits a decomposition as an inverse limit of finite G-graphs.

Proposition 2.2.2 *Let a profinite group G act on a profinite graph Γ.*

(a) *Then there exists a decomposition*

$$\Gamma = \varprojlim_{i\in I} \Gamma_i$$

of Γ as the inverse limit of a system of finite quotient G-graphs Γ_i and G-maps $\varphi_{ij} : \Gamma_i \to \Gamma_j$ $(i \succeq j)$ over a directed poset $(I, \preceq)$.

(b) *If G is finite and acts freely on Γ, then the decomposition of part* (a) *can be chosen so that G acts freely on each Γ_i.*

Proof The proof follows the same pattern as the proof of Proposition 2.1.4; we only indicate the main steps and changes. We prove (a) and (b) at the same time.

Let R be an open equivalence relation on Γ. Assume that G acts continuously on the finite discrete space Γ/R in such a way that the canonical projection $\varphi_R : \Gamma \to \Gamma/R$ is a G-map of G-spaces: this is equivalent to saying that whenever $m, m' \in \Gamma$ and $mR = m'R$, then $(gm)R = (gm')R$, for all $g \in G$ (we term such R a *G-invariant equivalence relation*). Then (see Sect. 1.3) there exists a set $\mathcal{R}$ of G-invariant open equivalence relations on Γ such that $(\mathcal{R}, \preceq)$ is a directed poset, $\{\Gamma/R, \varphi_{RR'}\}$ is an inverse system of finite G-spaces and G-maps over $\mathcal{R}$ and

$$\Gamma = \varprojlim_{R\in\mathcal{R}} \Gamma/R \tag{2.3}$$

as topological G-spaces. Moreover, if G is finite and acts freely on Γ, one can modify the set $\mathcal{R}$ so that the action of G on each Γ/R is free and the decomposition (2.3) still holds.

Let $\mathcal{R}'$ be the subset of $\mathcal{R}$ consisting of those $R \in \mathcal{R}$ such that in addition Γ/R has the structure of a G-graph and $\varphi_R : \Gamma \to \Gamma/R$ is a G-map of G-graphs.

Let $R \in \mathcal{R}$ and apply Construction 2.1.3 to get the maps $\widetilde{\varphi_R} : \Gamma \to \Gamma_{\varphi_R}$ and $\psi_{\varphi_R} : \Gamma_{\varphi_R} \to \Gamma/R$. For $g \in G$ and $m \in \Gamma$, define

$$g\big(\varphi(m), \varphi d_0(m), \varphi d_1(m)\big) = \big(g\varphi(m), g\varphi d_0(m), g\varphi d_1(m)\big).$$

This makes Γ_{φ_R} into a G-graph and one checks that $\widetilde{\varphi_R}$ is a G-map of G-graphs and ψ_{φ_R} is a G-map of G-spaces. Let $\tilde{R}$ be the open equivalence relation on Γ whose equivalence classes are $\{\widetilde{\varphi_R}^{-1}(x) \mid x \in \Gamma_{\varphi_R}\}$, so that $\Gamma_{\varphi_R} = \Gamma/\tilde{R}$. Therefore $\tilde{R} \succeq R$. From this one sees, as in the proof of Proposition 2.1.4, that $\mathcal{R}'$ is a directed poset that is cofinal in $\mathcal{R}$. Observe that if G acts freely on Γ/R, then it acts freely on Γ_{φ_R}. Hence both (a) and (b) follow from the decomposition (2.3) (see Sect. 1.1). □

We remark that part (b) of the above proposition can be sharpened in the following sense. When G is infinite, it obviously cannot act freely on a finite graph; hence, if G acts freely on Γ, it is not possible to obtain a G-decomposition of Γ as in part (a) if in addition one requires that G acts freely on each Γ_i. However, one can obtain a decomposition as in part (a) so that, for each i, a finite quotient G_i of G acts freely on Γ_i, and G is the inverse limit of the G_i. We make this precise in Proposition 3.1.3. The following example shows how to do this in the case of Cayley graphs.

Example 2.2.3 (The Cayley graph as a G-graph) Let G be a profinite group and let X be a closed subset of G. Let $\Gamma(G, X)$ be the Cayley graph of G with respect to X (see Example 2.1.12). Define a left action of G on $\Gamma(G, X)$ by setting

$$g' \cdot (g, x) = \big(g'g, x\big) \quad \forall x \in \tilde{X} = X \cup \{1\},\ g', g \in G.$$

Clearly $gd_i(m) = d_i(gm)$, for all $g \in G$, $m \in \Gamma(G, X)$, $i = 0, 1$. Thus, G acts (continuously and freely) on the Cayley graph $\Gamma(G, X)$.

Now, if $\mathcal{N}$ is the collection of all open normal subgroups of G, we have

$$\Gamma(G, X) = \varprojlim_{N \in \mathcal{N}} \Gamma(G/N, X_N),$$

where X_N is the image of X in G/N. Note that G/N acts freely on $\Gamma(G/N, X_N)$.

The next lemma sometimes provides a useful way of checking whether certain G-graphs are connected.

Lemma 2.2.4

(a) *Let $G = \langle X \rangle$ be an abstract group generated by a subset X. Assume that G acts on an abstract graph Γ. Let Δ be a connected subgraph of Γ such that $\Delta \cap x\Delta \neq \emptyset$, for all $x \in X$. Then*

$$G\Delta = \bigcup_{g \in G} g\Delta$$

is a connected subgraph of Γ.

(b) *Let X be a closed subset of a profinite group G that generates the group topologically, i.e., $G = \overline{\langle X \rangle}$. Assume that G acts on a profinite graph Γ. Let Δ be a connected profinite subgraph of Γ such that $\Delta \cap x\Delta \neq \emptyset$, for all $x \in X$. Then*

$$G\Delta = \bigcup_{g \in G} g\Delta$$

is a connected profinite subgraph of Γ.

(c) *Let G be a profinite group and let X be a closed subset of G. The Cayley graph $\Gamma(G, X)$ is connected if and only if $G = \overline{\langle X \rangle}$.*

Proof (a) Put

$$Y = \left\{ x^{\varepsilon} \;\middle|\; \varepsilon = \pm 1, x \in X \right\},$$

and let Y_n be the set of elements of G that can be written as a product of not more than n elements of Y ($n = 0, 1, 2, \ldots$). Since $G\Delta = \bigcup_{n=0}^{\infty} Y_n \Delta$, and $Y_0 \subseteq Y_1 \subseteq \cdots$, it suffices to prove that $Y_n\Delta$ is a connected graph. We show this by induction on n. If $n = 0$, then $Y_0\Delta = \Delta$. Assume that $Y_n\Delta$ is connected. From our assumption that $x\Delta \cap \Delta \neq \emptyset$, we deduce that $x^{-1}\Delta \cap \Delta \neq \emptyset$, for all $x \in X$. Observe that if w is a word in Y of length $n+1$, then $w = w'x^{\varepsilon}$, for some $w' \in Y_n$ and some $x \in X$; hence $w\Delta \cap w'\Delta \neq \emptyset$; and so, $w\Delta \cup Y_n\Delta$ is connected. It follows that $Y_{n+1}\Delta$ is connected.

(b) By Proposition 2.2.2 there exists a decomposition $\Gamma = \varprojlim \Gamma_i$, where all Γ_i are finite quotient G-graphs of Γ. Hence it suffices to prove the result for Γ finite. In that case the kernel K of the action of G on Γ is an open normal subgroup of G. Therefore, replacing G by its quotient G/K if necessary, we may assume that G is finite; and then the result follows from part (a).

(c) Let $\mathcal{U}$ be the collection of all open normal subgroups of G. Then

$$\Gamma(G, X) = \varprojlim_{U \in \mathcal{U}} \Gamma(G/U, X_U),$$

where X_U is the image of X on G/U under the canonical map $G \to G/U$. Therefore we may assume that G is finite, in which case the result follows from part (a): consider the connected subgraph Δ of $\Gamma(G, X)$ consisting of the vertices 1 and $\{x \mid x \in X\}$ and the collection of edges $\{(1, x) \mid x \in X - \{1\}\}$; then $\Gamma(G, X) = G\Delta$. □

2.3 The Chain Complex of a Graph

We shall use the following notation and terminology. Given a pseudovariety of finite groups $\mathcal{C}$, we say that R is a pro-$\mathcal{C}$ ring if it is an inverse limit of finite rings which are in $\mathcal{C}$ as abelian groups; if $\mathcal{C}$ is the class of all finite rings, we write profinite rather than pro-$\mathcal{C}$. Let X be a profinite space and let R be a pro-$\mathcal{C}$ ring. We denote by $[\![RX]\!]$ the free profinite R-module on the space X. Similarly, $[\![R(X, *)]\!]$ denotes the free profinite R-module on a pointed space $(X, *)$. The complete group algebra $[\![RG]\!]$ is the inverse limit of the finite group algebras

$$[\![RG]\!] = \varprojlim \left[(R/I)(G/U) \right],$$

where I and U range over the open ideals of R and the open normal subgroups of G, respectively.

Let G be a profinite group, and let X be a profinite G-space. Then $[\![RX]\!]$ naturally becomes a profinite $[\![RG]\!]$-module. Similarly, if $(X, *)$ is a pointed profinite

G-space, then the free profinite R-module $[\![R(X,*)]\!]$ is naturally a profinite $[\![RG]\!]$-module.

Let $\varGamma$ be a profinite graph. Define

$$E^*(\varGamma) = \varGamma / V(\varGamma)$$

to be the quotient space of the space $\varGamma$ modulo the subspace of vertices $V(\varGamma)$. We think of $E^*(\varGamma)$ as a pointed space with the image of $V(\varGamma)$ as the distinguished point.

Let R be a profinite ring and consider the free profinite R-modules $[\![R(E^*(\varGamma),*)]\!]$ and $[\![RV(\varGamma)]\!]$ on the pointed profinite space $(E^*(\varGamma),*)$ and on the profinite space $V(\varGamma)$, respectively. Denote by $C(\varGamma, R)$ the chain complex

$$0 \longrightarrow \big[\!\big[R\big(E^*(\varGamma),*\big)\big]\!\big] \xrightarrow{d} \big[\!\big[RV(\varGamma)\big]\!\big] \xrightarrow{\varepsilon} R \longrightarrow 0 \tag{2.4}$$

of free profinite R-modules and continuous R-homomorphisms d and ε determined by $\varepsilon(v) = 1$, for every $v \in V(\varGamma)$, $d(\bar{e}) = d_1(e) - d_0(e)$, where $\bar{e}$ is the image of an edge $e \in E(\varGamma)$ in the quotient space $E^*(\varGamma)$, and $d(*) = 0$. Obviously, $\mathrm{Im}(d) \subseteq \mathrm{Ker}(\varepsilon)$. If we need to emphasize the role of the ring R we sometimes write d^R for the map d.

Note that if $E(\varGamma)$ is closed in $\varGamma$, then $*$ is an isolated point of $E^*(\varGamma)$, and so $[\![R(E^*(\varGamma),*)]\!] = [\![RE(\varGamma)]\!]$; this is the case in many important examples.

The *homology groups* of $\varGamma$ are defined as the homology groups of the chain complex $C(\varGamma, R)$ in the usual way:

$$H_0(\varGamma, R) = \mathrm{Ker}(\varepsilon)/\mathrm{Im}(d), \qquad H_1(\varGamma, R) = \mathrm{Ker}(d).$$

A qmorphism

$$\alpha : \varGamma \longrightarrow \Delta$$

of profinite graphs naturally induces continuous maps

$$\alpha_V : V(\varGamma) \longrightarrow V(\Delta) \quad \text{and} \quad \alpha_{E^*} : \big(E^*(\varGamma),*\big) \longrightarrow \big(E^*(\Delta),*\big),$$

which in turn extend to continuous R-homomorphisms

$$\tilde{\alpha}_V : \big[\!\big[RV(\varGamma)\big]\!\big] \longrightarrow \big[\!\big[RV(\Delta)\big]\!\big] \quad \text{and}$$
$$\tilde{\alpha}_{E^*} : \big[\!\big[R\big(E^*(\varGamma),*\big)\big]\!\big] \longrightarrow \big[\!\big[R\big(E^*(\Delta),*\big)\big]\!\big].$$

Then the following diagram

$$\begin{tikzcd}
0 \arrow[r] & {[\![R(E^*(\varGamma),*)]\!]} \arrow[r, "d"] \arrow[d, "\tilde{\alpha}_{E^*}"'] & {[\![RV(\varGamma)]\!]} \arrow[r, "\varepsilon"] \arrow[d, "\tilde{\alpha}_V"'] & R \arrow[r] \arrow[d, "\mathrm{id}_R"'] & 0 \\
0 \arrow[r] & {[\![R(E^*(\Delta),*)]\!]} \arrow[r, "d"] & {[\![RV(\Delta)]\!]} \arrow[r, "\varepsilon"] & R \arrow[r] & 0
\end{tikzcd}$$

commutes. In other words, the triple $\tilde{\alpha} = (\tilde{\alpha}_{E^*}, \tilde{\alpha}_V, \mathrm{id}_R)$ is a morphism

$$\tilde{\alpha} : C(\varGamma, R) \longrightarrow C(\Delta, R)$$

of complexes. Therefore, if

$$\Gamma = \varprojlim_{i \in I} \Gamma_i$$

is an inverse limit of an inverse system of profinite graphs Γ_i, the corresponding chain complexes $C(\Gamma_i, R)$ form an inverse system and

$$C(\Gamma, R) = \varprojlim_{i \in I} C(\Gamma_i, R).$$

Furthermore, the homomorphism $\tilde{\alpha}$ induces continuous homomorphisms of homology groups

$$\alpha_0^* : H_0(\Gamma, R) \longrightarrow H_0(\Delta, R) \quad \text{and} \quad \alpha_1^* : H_1(\Gamma, R) \longrightarrow H_1(\Delta, R).$$

Of course, α_1^* is just the restriction of $\tilde{\alpha}_{E^*}$ to $\mathrm{Ker}(d)$. The statements in the following lemma are easily verified and we leave them to the reader.

Lemma 2.3.1 *Let R be a profinite ring.*

(a) *Let*

$$\alpha : \Gamma \longrightarrow \Delta$$

be a qmorphism of profinite graphs. If α is surjective, then

$$\alpha_0^* : H_0(\Gamma, R) \longrightarrow H_0(\Delta, R)$$

is surjective. If α is injective, so is

$$\alpha_1^* : H_1(\Gamma, R) \longrightarrow H_1(\Delta, R).$$

(b) *If $\Gamma = \varprojlim \Gamma_i$ is the inverse limit of an inverse system of profinite graphs Γ_i, then*

$$H_0(\Gamma, R) = \varprojlim H_0(\Gamma_i, R) \quad \textit{and} \quad H_1(\Gamma, R) = \varprojlim H_1(\Gamma_i, R).$$

In the next proposition we prove that the connectivity of a profinite graph is equivalent to the triviality of its 0-homology group.

Proposition 2.3.2 *A profinite graph Γ is connected if and only if $H_0(\Gamma, R) = 0$, independently of the choice of the profinite ring R.*

Proof Write Γ as an inverse limit $\Gamma = \varprojlim_{i \in I} \Gamma_i$ of finite quotient graphs Γ_i. By Proposition 2.1.6, Γ is a connected profinite graph if and only if each Γ_i is connected as an abstract graph. On the other hand, by Lemma 2.3.1, $H_0(\Gamma, R) = 0$ if and only if $H_0(\Gamma_i, R) = 0$, for each i. Hence it suffices to prove the theorem for finite Γ. In this case the sequence (2.4) becomes

$$0 \longrightarrow [RE(\Gamma)] \xrightarrow{d} [RV(\Gamma)] \xrightarrow{\varepsilon} R \longrightarrow 0,$$

where if X is a set, $[RX]$ denotes the free R-module on the set X. Observe that $\varepsilon d = 0$, so that $\mathrm{Im}(d) \le \mathrm{Ker}(\varepsilon)$.

Assume first that Γ is connected. Let

$$\varepsilon\left(\sum_{i=1}^{t} n_i v_i\right) = \sum_{i=1}^{t} n_i = 0 \quad \big(v_1, \dots, v_t \in V(\Gamma); n_1, \dots, n_t \in R\big).$$

Fix $v_0 \in V(\Gamma)$. Then $\sum_{i=1}^{t} n_i v_i = \sum_{i=1}^{t} n_i (v_i - v_0)$; hence it suffices to check that for every pair of distinct vertices v, w of Γ, there exists some $c \in [RE(\Gamma)]$ with $d(c) = w - v$. To verify this let $e_1^{\varepsilon_1}, \dots, e_m^{\varepsilon_m}$ be a path from v to w. Define $c = \sum_{i=1}^{s} \varepsilon_i e_i$, where we think of ε_i as an element of R. Then $d(c) = w - v$. Hence the sequence is exact at $[RV(\Gamma)]$, i.e., $H_0(\Gamma, R) = 0$.

Assume now that the sequence is exact at $[RV(\Gamma)]$. Let $v' \in V(\Gamma)$ and let Γ' be the connected component of v' in Γ. Suppose that $\Gamma' \neq \Gamma$, and let Γ'' be the complement of Γ' in Γ; then Γ'' is a subgraph of Γ. Choose $v'' \in V(\Gamma'')$. Clearly $v' - v'' \in \mathrm{Ker}(\varepsilon)$. Then there exists

$$\sum_{i=1}^{s} n_i e_i \in \big[RE(\Gamma)\big] \quad \big(e_i \in E(\Gamma), n_i \in R, i = 1, \dots, s\big)$$

such that $d(\sum_{i=1}^{s} n_i e_i) = v' - v''$. We may assume that v' is a vertex of e_1 and $e_1, \dots, e_t \in \Gamma'$, while $e_{t+1}, \dots, e_s \in \Gamma''$ and v'' is a vertex of e_s. Clearly

$$d\big(\big[RE\big(\Gamma'\big)\big]\big) \le \big[RV\big(\Gamma'\big)\big],$$
$$d\big(\big[RE\big(\Gamma''\big)\big]\big) \le \big[RV\big(\Gamma''\big)\big]$$

and

$$\big[RV(\Gamma)\big] = \big[RV\big(\Gamma'\big)\big] \oplus \big[RV\big(\Gamma''\big)\big].$$

Therefore $d(\sum_{i=1}^{t} n_i e_i) = v'$. However, $v' \notin \mathrm{Ker}(\varepsilon)$, a contradiction. Thus $\Gamma = \Gamma'$, and Γ is connected. □

2.4 π-Trees and $\mathcal{C}$-Trees

Let $\mathcal{C}$ be a pseudovariety of finite groups and consider the set of primes $\pi = \pi(\mathcal{C})$ involved in $\mathcal{C}$ (see Sect. 1.3). Let $\mathbf{Z}_{\hat{\mathcal{C}}}$ denote the pro-$\mathcal{C}$ completion of the group of integers $\mathbf{Z}$. This is the free pro-$\mathcal{C}$ group of rank 1; it also has, in a natural way, a ring structure. One has

$$\mathbf{Z}_{\hat{\mathcal{C}}} = \prod_{p \in \pi} \mathbf{Z}_p / p^{n_p} \mathbf{Z}_p,$$

where

$$n_p = n_p(\mathcal{C}) = \sup\big\{n \bigm| n \in \mathbf{N}, p^n \| C|, C \in \mathcal{C}\big\}.$$

If $n_p = \infty$, then, by convention, we agree that $p^\infty \mathbf{Z}_p = 0$. Note that every abelian pro-$\mathcal{C}$ group is in a unique way a profinite $\mathbf{Z}_{\hat{\mathcal{C}}}$-module.

A profinite graph Γ is said to be a *$\mathcal{C}$-tree* if Γ is connected and $H_1(\Gamma, \mathbf{Z}_{\hat{\mathcal{C}}}) = 0$. Thus Γ is a $\mathcal{C}$-tree if and only if the sequence $C(\Gamma, \mathbf{Z}_{\hat{\mathcal{C}}})$ (see Sect. 2.3)

$$0 \longrightarrow [\![\mathbf{Z}_{\hat{\mathcal{C}}}\big(E^*(\Gamma), *\big)]\!] \xrightarrow{d} [\![\mathbf{Z}_{\hat{\mathcal{C}}} V(\Gamma)]\!] \xrightarrow{\varepsilon} \mathbf{Z}_{\hat{\mathcal{C}}} \longrightarrow 0 \tag{2.5}$$

is exact. Note that if the set of edges $E(\Gamma)$ of Γ is closed, then the sequence (2.5) becomes

$$0 \longrightarrow [\![\mathbf{Z}_{\hat{\mathcal{C}}} E(\Gamma)]\!] \xrightarrow{d} [\![\mathbf{Z}_{\hat{\mathcal{C}}} V(\Gamma)]\!] \xrightarrow{\varepsilon} \mathbf{Z}_{\hat{\mathcal{C}}} \longrightarrow 0.$$

Lemma 2.4.1 *Let $\mathcal{C}$ be a pseudovariety of finite groups. A profinite graph Γ is a $\mathcal{C}$-tree if and only if the sequence $C(\Gamma, \mathbf{F}_p)$*

$$0 \longrightarrow [\![\mathbf{F}_p\big(E^*(\Gamma), *\big)]\!] \xrightarrow{d} [\![\mathbf{F}_p V(\Gamma)]\!] \xrightarrow{\varepsilon} \mathbf{F}_p \longrightarrow 0$$

is exact for every $p \in \pi(\mathcal{C})$, where $\mathbf{F}_p$ is the field with p-elements.

Proof First observe that a proabelian group is the direct product of its p-Sylow subgroups. So, for any profinite space X,

$$[\![\mathbf{Z}_{\hat{\mathcal{C}}} X]\!] = \prod_{p \in \pi(\mathcal{C})} [\![\big(\mathbf{Z}_p / p^{n_p} \mathbf{Z}_p\big) X]\!].$$

Therefore,

$$C(\Gamma, \mathbf{Z}_{\hat{\mathcal{C}}}) = \prod_{p \in \pi(\mathcal{C})} C\big(\Gamma, \mathbf{Z}_p / p^{n_p} \mathbf{Z}_p\big),$$

where $n_p = n_p(\mathcal{C})$. Hence the sequence $C(\Gamma, \mathbf{Z}_{\hat{\mathcal{C}}})$ is exact if and only if the sequence $C(\Gamma, \mathbf{Z}_p / p^{n_p} \mathbf{Z}_p)$ is exact for each $p \in \pi(\mathcal{C})$. Therefore it suffices to prove that $C(\Gamma, \mathbf{Z}_p / p^{n_p} \mathbf{Z}_p)$ is exact if and only if $C(\Gamma, \mathbf{F}_p)$ is exact.

We observe that $C(\Gamma, \mathbf{Z}_p / p^{n_p} \mathbf{Z}_p)$ and $C(\Gamma, \mathbf{F}_p)$ are sequences of free abelian pro-p groups of exponent p^{n_p} and free abelian pro-p groups of exponent p, respectively. Moreover, if X is a profinite space, $[\![\mathbf{F}_p X]\!]$ is the Frattini quotient

$$[\![\big(\mathbf{Z}_p / p^{n_p} \mathbf{Z}_p\big) X]\!] / \Phi\big([\![\big(\mathbf{Z}_p / p^{n_p} \mathbf{Z}_p\big) X]\!]\big)$$

of $[\![(\mathbf{Z}_p / p^{n_p} \mathbf{Z}_p) X]\!]$: this is obvious if X is finite, and in general this can be deduced by a standard inverse limit argument.

Exactness of $C(\Gamma, \mathbf{Z}_p / p^{n_p} \mathbf{Z}_p)$ at $[\![(\mathbf{Z}_p / p^{n_p} \mathbf{Z}_p)(V(\Gamma))]\!]$ is equivalent to exactness of $C(\Gamma, \mathbf{F}_p)$ at $[\![\mathbf{F}_p(V(\Gamma))]\!]$, because any of these statements is equivalent to Γ being connected, according to Proposition 2.3.2. Hence from now on we assume that Γ is connected as a profinite graph, and we must show that injectivity of the map d of $C(\Gamma, \mathbf{Z}_p / p^{n_p} \mathbf{Z}_p)$ is equivalent to injectivity of the map d of $C(\Gamma, \mathbf{F}_p)$.

To prove this we will also work with the chain complex $C(\Gamma, \mathbf{Z}_p)$. Consider the commutative diagram

$$\begin{tikzcd}
[\![\mathbf{Z}_p(E^*(\Gamma),*)]\!] \arrow[r, "d^{\mathbf{Z}_p}"] \arrow[d] & d([\![\mathbf{Z}_p(E^*(\Gamma),*)]\!]) \arrow[d] \\
[\![(\mathbf{Z}_p/p^{n_p}\mathbf{Z}_p)(E^*(\Gamma),*)]\!] \arrow[r, "d'"] \arrow[d] & d([\![(\mathbf{Z}_p/p^{n_p}\mathbf{Z}_p)(E^*(\Gamma),*)]\!]) \arrow[d] \\
[\![\mathbf{F}_p(E^*(\Gamma),*)]\!] \arrow[r, "d^{\mathbf{F}_p}"] & d([\![\mathbf{F}_p(E^*(\Gamma),*)]\!])
\end{tikzcd}$$

where the vertical maps are the natural quotient maps, and the maps $d^{\mathbf{Z}_p}$, d' and $d^{\mathbf{F}_p}$ denote the maps induced by the homomorphisms d of $C(\Gamma, \mathbf{Z}_p)$, $C(\Gamma, \mathbf{Z}_p/p^{n_p}\mathbf{Z}_p)$ and $C(\Gamma, \mathbf{F}_p)$, respectively.

Since the sequence $C(\Gamma, \mathbf{Z}_p)$ is exact at $[\![\mathbf{Z}_p V(\Gamma)]\!]$ and since $\mathbf{Z}_p$ is the free $\mathbf{Z}_p$-module of rank 1, the map ε splits, and we have

$$[\![\mathbf{Z}_p V(\Gamma)]\!] = d\big([\![\mathbf{Z}_p\big(E^*(\Gamma),*\big)]\!]\big) \oplus \mathbf{Z}_p.$$

Similarly, we have

$$[\![\big(\mathbf{Z}_p/p^{n_p}\mathbf{Z}_p\big)V(\Gamma)]\!] = d\big([\![\big(\mathbf{Z}_p/p^{n_p}\mathbf{Z}_p\big)\big(E^*(\Gamma),*\big)]\!]\big) \oplus \mathbf{Z}_p/p^{n_p}\mathbf{Z}_p$$

and

$$[\![\mathbf{F}_p V(\Gamma)]\!] = d\big([\![\mathbf{F}_p\big(E^*(\Gamma),*\big)]\!]\big) \oplus \mathbf{F}_p.$$

From this it follows that the last line of the diagram is obtained from the first or second line by taking quotients modulo the subgroups of p-th powers (the Frattini subgroups); and the second line is obtained from the first by taking quotients modulo the subgroups of p^{n_p}-th powers. It follows that if $d^{\mathbf{Z}_p}$ (respectively, d') is an isomorphism, then so is $d^{\mathbf{F}_p}$. Conversely, assume that $d^{\mathbf{F}_p}$ is an isomorphism. Since $d([\![\mathbf{Z}_p(E^*(\Gamma),*)]\!])$ is a subgroup of $[\![\mathbf{Z}_p V(\Gamma)]\!]$, it is a torsion-free pro-p group, and so a free abelian pro-p group (cf. RZ, Theorem 4.3.3 and Example 3.3.8(c)). Therefore there exists a continuous homomorphism

$$\alpha : d\big([\![\mathbf{Z}_p\big(E^*(\Gamma),*\big)]\!]\big) \longrightarrow [\![\mathbf{Z}_p\big(E^*(\Gamma),*\big)]\!]$$

such that $d^{\mathbf{Z}_p}\alpha$ is the identity map on $d([\![\mathbf{Z}_p(E^*(\Gamma),*)]\!])$; therefore α is injective. On the other hand,

$$\mathrm{Ker}\big(d^{\mathbf{Z}_p}\big) \le \Phi\big([\![\mathbf{Z}_p\big(E^*(\Gamma),*\big)]\!]\big) \quad \text{and}$$
$$\big(\mathrm{Ker}\big(d^{\mathbf{Z}_p}\big)\big) + \mathrm{Im}(\alpha) = [\![\mathbf{Z}_p\big(E^*(\Gamma),*\big)]\!],$$

where $\Phi([\![\mathbf{Z}_p(E^*(\Gamma),*)]\!])$ is the subgroup of p-th powers of $[\![\mathbf{Z}_p(E^*(\Gamma),*)]\!]$, i.e., its Frattini subgroup. So $\mathrm{Im}(\alpha) = [\![\mathbf{Z}_p(E^*(\Gamma),*)]\!]$ (cf. RZ, Corollary 2.8.5). Therefore α is an isomorphism, and hence $d^{\mathbf{Z}_p}$ is an isomorphism. Thus, d' is also an isomorphism. $\square$

The above lemma shows that in fact the concept of a $\mathcal{C}$-tree depends only on the primes involved in the pseudovariety $\mathcal{C}$. This suggests the following definition. Let π be a nonempty set of prime numbers, and denote by $\mathbf{Z}_{\hat{\pi}}$ the profinite group (ring)

$$\mathbf{Z}_{\hat{\pi}} = \prod_{p\in\pi} \mathbf{Z}_p.$$

We say that a profinite graph Γ is a *π-tree* if it is connected as a profinite graph and one has $H_1(\Gamma, \mathbf{Z}_{\hat{\pi}}) = 0$. In other words, Γ is a π-tree if and only if the sequence $C(\Gamma, \mathbf{Z}_{\hat{\pi}})$

$$0 \longrightarrow [\![\mathbf{Z}_{\hat{\pi}}(E^*(\Gamma), *)]\!] \xrightarrow{d} [\![\mathbf{Z}_{\hat{\pi}} V(\Gamma)]\!] \xrightarrow{\varepsilon} \mathbf{Z}_{\hat{\pi}} \longrightarrow 0 \tag{2.6}$$

is exact. If $\pi = \{p\}$ consists of only one prime, we write *p-tree* rather than $\{p\}$-tree. When π is the set of all prime numbers, we normally use the term *profinite tree* rather than π-tree. The following proposition is an immediate consequence of Lemma 2.4.1.

Proposition 2.4.2 *Let $\mathcal{C}$ be a pseudovariety of finite groups and let Γ be a profinite graph. Let $\pi = \pi(\mathcal{C})$. The following conditions are equivalent*:

(a) *Γ is a $\mathcal{C}$-tree*;
(b) *Γ is a π-tree*;
(c) *let R be a quotient ring of $\widehat{\mathbf{Z}}$ such that the order $\#R$ of R as a profinite group involves precisely the primes in the set π. Then the sequence*

$$0 \longrightarrow [\![R(E^*(\Gamma), *)]\!] \xrightarrow{d} [\![RV(\Gamma)]\!] \xrightarrow{\varepsilon} R \longrightarrow 0$$

is exact;
(d) *for a given prime p, let R_p denote one of the following rings*: $\mathbf{Z}_p$, $\mathbf{F}_p$ *or* $\mathbf{Z}_p/p^n\mathbf{Z}_p$, *for some positive integer n. Then, for every $p \in \pi$, the sequence*

$$0 \longrightarrow [\![R_p(E^*(\Gamma), *)]\!] \xrightarrow{d} [\![R_p V(\Gamma)]\!] \xrightarrow{\varepsilon} R_p \longrightarrow 0$$

is exact.

Proposition 2.4.3 *Let π be a nonempty set of prime numbers. Then the following statements hold.*

(a) *Every finite tree is a π-tree.*
(b) *Every connected profinite subgraph of a π-tree is a π-tree.*
(c) *If Δ_1 and Δ_2 are π-subtrees of a π-tree such that $\Delta_1 \cap \Delta_2 \neq \emptyset$, then $\Delta_1 \cup \Delta_2$ is a π-subtree.*
(d) *An inverse limit of π-trees is a π-tree. In particular, an inverse limit of finite trees is a π-tree.*
(e) *If $\emptyset \neq \pi' \subseteq \pi$, then every π-tree is a π'-tree.*

Proof Part (b) follows from Lemma 2.3.1(a). Part (c) follows from (b) and Lemma 2.1.7. The first statement in part (d) is a consequence of Lemma 2.3.1(b);

and the second then follows from (a). Part (e) is a consequence of the definition of a π-tree. To prove (a), let Γ be a finite tree. In this case the sequence (2.6) becomes

$$0 \longrightarrow [\mathbf{Z}_p E(\Gamma)] \xrightarrow{d} [\mathbf{Z}_p V(\Gamma)] \xrightarrow{\varepsilon} \mathbf{Z}_p \longrightarrow 0.$$

Since Γ is connected, this sequence is exact at $[\mathbf{Z}_p V(\Gamma)]$ by Proposition 2.3.2. It remains to see that d is an injection. For this define a map

$$\rho : V(\Gamma) \longrightarrow [\mathbf{Z}_p E(\Gamma)]$$

as follows: fix a vertex $v_0 \in V(\Gamma)$; since Γ is an abstract tree, for each vertex $v \in V(\Gamma)$ there is a unique path $e_1^{\varepsilon_1}, \dots, e_t^{\varepsilon_t}$ from v_0 to v of minimal length; define

$$\rho(v) = \varepsilon_1 e_1 + \cdots + \varepsilon_t e_t \quad \bigl(e_1, \dots, e_t \in E(\Gamma); \varepsilon_i = \pm 1, i = 1, \dots, t\bigr).$$

Since $[\mathbf{Z}_p V(\Gamma)]$ is a free $\mathbf{Z}_p$-module, this map extends to a $\mathbf{Z}_p$-homomorphism (also denoted ρ) $\rho : [\mathbf{Z}_p V(\Gamma)] \to [\mathbf{Z}_p E(\Gamma)]$. Then ρd is the identity map on $[\mathbf{Z}_p E(\Gamma)]$; thus d is an injection. □

Exercise 2.4.4 Let T be a π-tree.

(a) T does not contain circuits.
(b) If $v, w \in V(T)$ and there exists a path p_{vw} from v to w, then there is a unique reduced path from v to w.

Example 2.4.5 (A π-tree which is not an inverse limit of finite trees) It is not always possible to decompose a π-tree as an inverse limit of finite trees. For example, let p be a prime number. The Cayley graph $\Gamma = \Gamma(\mathbf{Z}_p, \{1\})$ is a p-tree (see Theorem 2.5.3 below). Let $\tilde{\Gamma}$ be a finite quotient graph of Γ. Then $\tilde{\Gamma}$ is also a quotient graph of a graph of the form $\Gamma(\mathbf{Z}/p^n\mathbf{Z}, \{1\})$ (see Lemma 2.1.5), which is a circuit. Hence, if $|\tilde{\Gamma}| \geq 2$, then $\tilde{\Gamma}$ is not a tree.

Lemma 2.4.6 *Let Δ be a profinite subgraph of a profinite graph Γ, and let R be a profinite ring. Then*

(a) *$V(\Delta)$ is a closed subspace of $V(\Gamma)$, and $(E^*(\Delta), *)$ is naturally embedded in $(E^*(\Gamma), *)$;*
(b) *$V(\Gamma/\Delta)$ is naturally homeomorphic with $V(\Gamma)/V(\Delta)$, and $E^*(\Gamma/\Delta, *)$ is naturally homeomorphic with $(E^*(\Gamma)/E^*(\Delta), *)$, where, in this last case, the distinguished point $*$ is the image of $E^*(\Delta)$ in $E^*(\Gamma)/E^*(\Delta)$;*
(c)
$$[\![R(E^*(\Gamma/\Delta), *)]\!] \cong [\![R(E^*(\Gamma), *)]\!]/[\![R(E^*(\Delta), *)]\!].$$

Proof Parts (a) and (b) are straightforward. To prove (c) consider the natural continuous map

$$\iota : (E^*(\Gamma/\Delta), *) \longrightarrow [\![R(E^*(\Gamma), *)]\!]/[\![R(E^*(\Delta), *)]\!].$$

We must show that $[\![R(E^*(\Gamma),*)]\!]/[\![R(E^*(\Delta),*)]\!]$ is the free profinite R-module on the space $(E^*(\Gamma/\Delta),*)$ with respect to the map ι (see Sect. 1.7). Let $\varphi : (E^*(\Gamma/\Delta),*) \to A$ be a continuous map of pointed spaces into a profinite R-module A. Then φ induces a continuous map

$$\varphi_1 : \big(E^*(\Gamma),*\big) \longrightarrow A,$$

and this in turn induces a continuous R-homomorphism

$$\overline{\varphi_1} : \big[\!\big[R\big(E^*(\Gamma),*\big)\big]\!\big] \longrightarrow A$$

such that $\overline{\varphi_1}([\![R(E^*(\Delta),*)]\!]) = 0$. Therefore $\overline{\varphi_1}$ induces a continuous R-homomorphism

$$\bar{\varphi} : \big[\!\big[R\big(E^*(\Gamma),*\big)\big]\!\big]/\big[\!\big[R\big(E^*(\Delta),*\big)\big]\!\big] \longrightarrow A$$

such that $\bar{\varphi}\iota = \varphi$. The uniqueness of $\bar{\varphi}$ is clear since $\iota(E^*(\Gamma/\Delta),*)$ generates $[\![R(E^*(\Gamma),*)]\!]/[\![R(E^*(\Delta),*)]\!]$. $\square$

Lemma 2.4.7 *Let Δ be a π-subtree of a connected profinite graph Γ and let*

$$\alpha : \Gamma \longrightarrow \Gamma/\Delta$$

be the corresponding canonical epimorphism of graphs. Then the induced homomorphism

$$\alpha_1^* : H_1(\Gamma, \mathbf{Z}_{\hat{\pi}}) \longrightarrow H_1(\Gamma/\Delta, \mathbf{Z}_{\hat{\pi}})$$

is an isomorphism. In particular, if Γ is a π-tree, then so is Γ/Δ.

Proof We may assume that π consists of just one prime p. Let

$$\beta : \Delta \longrightarrow \Gamma$$

be the natural embedding. Then β and α induce a monomorphism $\tilde{\beta} : C(\Delta, \mathbf{Z}_p) \to C(\Gamma, \mathbf{Z}_p)$ and an epimorphism $\tilde{\alpha} : C(\Gamma, \mathbf{Z}_p) \to C(\Gamma/\Delta, \mathbf{Z}_p)$ of chain complexes, respectively, and the following diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & & & \\
0 \arrow[r] & {[\![\mathbf{Z}_p(E^*(\Delta),*)]\!]} \arrow[r, "d^\Delta"] \arrow[d, "\tilde{\beta}_{E^*}"] & {[\![\mathbf{Z}_p V(\Delta)]\!]} \arrow[r, "\varepsilon^\Delta"] \arrow[d, "\tilde{\beta}_V"] & \mathbf{Z}_p \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
 & {[\![\mathbf{Z}_p(E^*(\Gamma),*)]\!]} \arrow[r, "d^\Gamma"] \arrow[d, "\tilde{\alpha}_{E^*}"] & {[\![\mathbf{Z}_p V(\Gamma)]\!]} \arrow[r, "\varepsilon^\Gamma"] \arrow[d, "\tilde{\alpha}_V"] & \mathbf{Z}_p \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
 & {[\![\mathbf{Z}_p(E^*(\Gamma/\Delta),*)]\!]} \arrow[r, "d^{\Gamma/\Delta}"] \arrow[d] & {[\![\mathbf{Z}_p V(\Gamma/\Delta)]\!]} \arrow[r, "\varepsilon^{\Gamma/\Delta}"] & \mathbf{Z}_p \arrow[r] & 0 \\
 & 0 & & &
\end{tikzcd}$$

commutes. Note that the first row is exact because Δ is a p-tree, the second row is exact because Γ is connected.

By Lemma 2.4.6, $\mathrm{Ker}(\tilde{\alpha}_{E^*}) = \tilde{\beta}_{E^*}([\![\mathbf{Z}_p(E^*(\Delta), *)]\!])$, in other words, the first column of the diagram is an exact sequence. From this it easily follows that α_1^* is an injection. Indeed, let $a \in H_1(\Gamma, \mathbf{Z}_p)$ be such that $\alpha_1^*(a) = 0$; i.e., $a \in [\![\mathbf{Z}_p(E^*(\Gamma), *)]\!]$ with $d^\Gamma(a) = 0$ and $\tilde{\alpha}_{E^*}(a) = 0$. Then there exists a $b \in [\![\mathbf{Z}_p(E^*(\Delta), *)]\!]$ such that $\tilde{\beta}_{E^*}(b) = a$. Now, since d^Δ and $\bar{\beta}_V$ are injections, we deduce from the commutativity of the diagram that $b = 0$. Thus $a = 0$.

Next we observe that $\mathrm{Ker}(\tilde{\alpha}_V) = \tilde{\beta}_V(\mathrm{Ker}(\varepsilon^\Delta))$; indeed, first we notice that this is straightforward if $V(\Gamma)$ is finite; in general we use an inverse limit argument.

Now we can easily deduce that α_1^* is a surjection: if $c \in [\![\mathbf{Z}_p(E^*(\Gamma/\Delta), *)]\!]$ and $d^{\Gamma/\Delta}(c) = 0$, choose $\tilde{c} \in [\![\mathbf{Z}_p(E^*(\Gamma), *)]\!]$ such that $\tilde{\alpha}_{E^*}(\tilde{c}) = c$; then $d^\Gamma(\tilde{c}) \in \mathrm{Ker}(\tilde{\alpha}_V)$, and so there exists a $y \in \mathrm{Ker}(\varepsilon^\Delta)$ with $\tilde{\beta}_V(y) = d^\Gamma(\tilde{c})$; hence there exists a $y' \in [\![\mathbf{Z}_p(E^*(\Delta), *)]\!]$ with $d^\Delta(y') = y$; then $c' = \tilde{c} - \tilde{\beta}_{E^*}(y') \in \mathrm{Ker}(d^\Gamma)$ and $\tilde{\alpha}_{E^*}(c') = c$, as needed. □

Lemma 2.4.8 *Let R be a profinite ring. Then the following statements hold.*

(a) *Let $\{X_i \mid i \in I\}$ be a collection of closed subspaces of a profinite space Y. Set $X = \bigcap_{i \in I} X_i$. Then*

$$[\![RX]\!] = \bigcap_i [\![RX_i]\!].$$

(b) *Let $\{(X_i, *) \mid i \in I\}$ be a collection of closed pointed subspaces of a profinite pointed space $(Y, *)$. Set $(X, *) = \bigcap_{i \in I} (X_i, *)$. Then*

$$\big[\!\!\big[R(X, *)\big]\!\!\big] = \bigcap_i \big[\!\!\big[R(X_i, *)\big]\!\!\big].$$

(c) *Let Y and Z be closed subspaces of the profinite pointed space $(X, *)$ such that $* \in Y$ and $* \notin Z$. Then there are natural isomorphisms*

$$\big[\!\!\big[R(X, *)\big]\!\!\big]/[\![RZ]\!] \cong \big[\!\!\big[R(X/Z, *)\big]\!\!\big] \quad \textit{and}$$
$$\big[\!\!\big[R(X, *)\big]\!\!\big]\big/\big[\!\!\big[R(Y, *)\big]\!\!\big] \cong \big[\!\!\big[R(X/Y, *)\big]\!\!\big].$$

Proof The proofs of (a) and (b) are similar. We only prove (a). Assume first that $I = \{1, 2\}$, i.e., $X = X_1 \cap X_2$. Write Y as the inverse limit

$$Y = \varprojlim_{j \in J} Y_j$$

of its finite quotient spaces. Denote by $\varphi_j : Y \to Y_j$ the projection ($j \in J$), and define $X_{1j} = \varphi_j(X_1)$ and $X_{2j} = \varphi_j(X_2)$.

Since X_{1j} and X_{2j} are finite, we have

$$\big[\!\!\big[R(X_{1j} \cap X_{2j})\big]\!\!\big] = [\![RX_{1j}]\!] \cap [\![RX_{2j}]\!].$$

It is easy to verify that

$$X_1 \cap X_2 = \Big(\varprojlim_{j\in J} X_{1j}\Big) \cap \Big(\varprojlim_{j\in J} X_{2j}\Big) = \varprojlim_{j\in J} (X_{1j} \cap X_{2j}).$$

Hence

$$\begin{aligned}\big[\!\big[R(X_1 \cap X_2)\big]\!\big] &= \Big[\!\Big[R\Big(\varprojlim_{j\in J}(X_{1j}\cap X_{2j})\Big)\Big]\!\Big] = \varprojlim_{j\in J} \big[\!\big[R(X_{1j}\cap X_{2j})\big]\!\big] \\ &= \varprojlim_{j\in J} \big([\![RX_{1j}]\!] \cap [\![RX_{2j}]\!]\big) = \Big(\varprojlim_{j\in J} [\![RX_{1j}]\!]\Big) \cap \Big(\varprojlim_{j\in J} [\![RX_{2j}]\!]\Big) \\ &= [\![RX_1]\!] \cap [\![RX_2]\!]\end{aligned}$$

(for the second and fourth equalities see RZ, Proposition 5.2.2).

Assume now that I is any indexing set. By the case considered above we may assume that the collection $\{X_i \mid i \in I\}$ is filtered from below, i.e., that the intersection of any two sets in the collection contains a set in the collection. So we may think of this collection as an inverse system of sets and

$$X = \bigcap_{i\in I} X_i = \varprojlim_{i\in I} X_i.$$

Also, using again the case above, the collection of profinite R-submodules $\{[\![RX_i]\!] \mid i \in I\}$ of $[\![RY]\!]$ is filtered from below. Therefore,

$$[\![RX]\!] = \Big[\!\Big[R\Big(\varprojlim_{i\in I} X_i\Big)\Big]\!\Big] = \varprojlim_{i\in I} [\![RX_i]\!] = \bigcap_{i\in I} [\![RX_i]\!].$$

(c) We prove the second statement, the first being similar. The quotient map $(X,*) \to (X/Y,*)$ induces a continuous epimorphism of free profinite modules $f : [\![R(X,*)]\!] \to [\![R(X/Y,*)]\!]$. Since $f([\![R(Y,*)]\!]) = 0$, f induces an epimorphism

$$\rho : \big[\!\big[R(X,*)\big]\!\big]\big/\big[\!\big[R(Y,*)\big]\!\big] \longrightarrow \big[\!\big[R(X/Y,*)\big]\!\big].$$

On the other hand, the natural map $(X/Y,*) \to [\![R(X,*)]\!]/[\![R(Y,*)]\!]$ induces a continuous homomorphism

$$\psi : \big[\!\big[R(X/Y,*)\big]\!\big] \longrightarrow \big[\!\big[R(X,*)\big]\!\big]\big/\big[\!\big[R(Y,*)\big]\!\big].$$

Finally, observe that the composition $\psi\rho$ is the identity map on $[\![R(X,*)]\!]/[\![R(Y,*)]\!]$. Thus ρ is an isomorphism. □

Proposition 2.4.9 *Let π be a nonempty set of prime numbers. Suppose that $\{\Delta_i \mid i \in I\}$ is a family of π-subtrees of a π-tree T, and let $\Delta = \bigcap_{i\in I} \Delta_i$. Then Δ is either empty or a π-tree.*

Proof Assume that $\Delta \neq \emptyset$. By Lemma 2.4.8 one has

$$\big[\!\big[\mathbf{Z}_{\hat\pi} V(\Delta)\big]\!\big] = \bigcap_{i\in I} \big[\!\big[\mathbf{Z}_{\hat\pi} V(\Delta_i)\big]\!\big]$$

and

$$[\![\mathbf{Z}_{\hat{\pi}}\big(E^*(\Delta),*\big)]\!] = \bigcap_{i\in I} [\![\mathbf{Z}_{\hat{\pi}}\big(E^*(\Delta_i),*\big)]\!].$$

Consider the exact sequence

$$0 \longrightarrow [\![\mathbf{Z}_{\hat{\pi}}\big(E^*(T),*\big)]\!] \xrightarrow{d} [\![\mathbf{Z}_{\hat{\pi}} V(T)]\!] \xrightarrow{\varepsilon} \mathbf{Z}_{\hat{\pi}} \longrightarrow 0$$

associated with T. Denote by $\varepsilon^{\Delta}, \varepsilon^{\Delta_i}, d^{\Delta}, d^{\Delta_i}$ the restrictions of ε and d to Δ and Δ_i, respectively. Then

$$\mathrm{Ker}\big(\varepsilon^{\Delta}\big) = [\![\mathbf{Z}_{\hat{\pi}} V(\Delta)]\!] \cap \mathrm{Ker}(\varepsilon) = \left(\bigcap_{i\in I} [\![\mathbf{Z}_{\hat{\pi}} V(\Delta_i)]\!]\right) \cap \mathrm{Ker}(\varepsilon) = \bigcap_{i\in I} \mathrm{Ker}\big(\varepsilon^{\Delta_i}\big);$$

moreover,

$$\mathrm{Im}\big(d^{\Delta}\big) = \bigcap_{i\in I} \mathrm{Im}\big(d^{\Delta_i}\big)$$

because d is injective. Since each Δ_i is connected, we have $\mathrm{Ker}(\varepsilon^{\Delta_i}) = \mathrm{Im}(d^{\Delta_i})$, for every i, by Proposition 2.3.2. It follows that $\mathrm{Im}(d^{\Delta}) = \mathrm{Ker}(\varepsilon^{\Delta})$. So, by Proposition 2.3.2, Δ is connected, and therefore a π-tree according to Proposition 2.4.3(b). □

It follows from Proposition 2.4.9 that given a nonempty subset W of a π-tree T, there exists a smallest π-subtree $[W]$ containing W, namely the intersection of all π-subtrees containing W. If W consists of two vertices v and w, we use the notation $[v,w]$ rather than $[\{v,w\}]$ and call it the *chain* connecting v and w. Observe that if $[v,w]$ is finite, then it is just the underlying graph of the unique reduced path from v to w.

Lemma 2.4.10 *A profinite subgraph Δ of a π-tree T is a π-tree if and only if $[v,w] \subseteq \Delta$, for all $v,w \in V(\Delta)$.*

Proof If Δ is a π-tree, then by definition $[v,w] \subseteq \Delta$, for all $v,w \in V(\Delta)$. Conversely, suppose Δ is a profinite subgraph of T and that $[v,w] \subseteq \Delta$, for all $v,w \in V(\Delta)$. To prove that Δ is a π-tree, it suffices to show that Δ is connected (see Proposition 2.4.3(b)). Write T as an inverse limit of finite quotient graphs,

$$T = \varprojlim_{i\in I} T_i,$$

and let $\varphi_i : T \to T_i$ denote the projection ($i \in I$). It suffices to prove that $\varphi_i(\Delta)$ is a connected graph for each $i \in I$. Given vertices $\bar{v}$ and $\bar{w}$ of $\varphi_i(\Delta)$, let $v,w \in V(\Delta)$ with $\varphi_i(v) = \bar{v}$ and $\varphi_i(w) = \bar{w}$. Since $[v,w] \subseteq \Delta$ and $[v,w]$ is a π-tree, we have that $\varphi_i([v,w])$ is a connected subgraph of the finite graph $\varphi_i(\Delta)$ containing $\bar{v}$ and $\bar{w}$. Therefore, $\varphi_i(\Delta)$ is connected. □

Example 2.4.11 (A π-tree that coincides with its infinite chains) Let $\Gamma = \Gamma(\widehat{\mathbf{Z}}, 1)$ be the Cayley graph of the free profinite group $\widehat{\mathbf{Z}}$ of rank 1 with respect to its subset $\{1\}$. This is a π-tree for any nonempty set of prime numbers π (see Theorem 2.5.3 below and Proposition 2.4.3(e)). The proper π-subtrees of Γ are precisely the proper connected profinite subgraphs of Γ, and these are precisely the finite π-subtrees (see Example 2.1.13). Therefore, if v, w are vertices of Γ, then $[v, w] = \Gamma$, unless $[v, w]$ is finite, in which case $[v, w]$ has vertices $\gamma, \gamma + 1, \dots, \gamma + t$, where $\gamma = v$ or $\gamma = w$ and t is a natural number.

Let G be a profinite group that acts on a π-tree T. A π-subtree T' of T is *G-invariant* if whenever $g \in G$ and $m \in T'$, one has $gm \in T'$; and such T' is *minimal* if it does not contain any proper G-invariant π-subtrees. Minimal G-invariant π-subtrees are especially useful when they are unique. In the next proposition we begin the study of minimal G-invariant π-subtrees T' of T. A more systematic study is carried out in Chap. 8.

Proposition 2.4.12 *Let G be a profinite group acting on a π-tree T. Then the following assertions hold.*

(a) *There exists a minimal G-invariant π-subtree D of T.*
(b) *If $|D| > 1$, then D is unique. In particular, if $|G| > 1$ and G acts freely on T or if G is infinite and the stabilizer of some $m \in D$ is finite, then D is the unique minimal G-invariant π-subtree of T.*
(c) *Assume that D is unique. Let $N \triangleleft G$ be such that there exists a unique minimal N-invariant π-subtree L of T. Then $L = D$.*

Proof (a) Consider the collection $\mathcal{T}$ of all G-invariant π-subtrees of T ordered by reverse inclusion. Since $T \in \mathcal{T}$, $\mathcal{T} \neq \emptyset$. Let $\{T_i\}_{i\in I}$ be a linearly ordered chain in $\mathcal{T}$. By the compactness of T, the set $\bigcap T_i$ is nonempty. Then, by Proposition 2.4.9, $\bigcap T_i$ is a G-invariant π-subtree. So $\{T_i\}_{i\in I}$ possesses an upper bound in $\mathcal{T}$. Therefore we can apply Zorn's lemma to conclude that there exists a minimal G-invariant π-subtree.

(b) This will be proved after Corollary 4.1.9.

(c) Let $g \in G$; then N acts on gL and so gL is minimal N-invariant; hence $gL = L$. This means that G acts on L. Therefore $D \subseteq L$; but obviously $L \subseteq D$, since N acts on D; thus $L = D$. $\square$

2.5 Cayley Graphs and $\mathcal{C}$-Trees

A pseudovariety of finite groups $\mathcal{C}_0$ is said to be *closed under extensions with abelian kernel* if whenever

$$1 \longrightarrow A \longrightarrow G \longrightarrow H \longrightarrow 1$$

is an exact sequence of finite groups with $A, H \in \mathcal{C}_0$ and A is abelian, then $G \in \mathcal{C}_0$. By the Kaloujnine–Krassner theorem (cf. Kargapolov and Merzljakov 1979, Theorem 6.2.8) such an extension group G can be embedded in the wreath product A by H; it follows that to check that a pseudovariety of finite groups $\mathcal{C}$ is closed under extensions with abelian kernel, it suffices to verify that any semidirect product of an abelian group in $\mathcal{C}$ by a group in $\mathcal{C}$ is in $\mathcal{C}$.

Next we give an example showing that a pseudovariety which is closed under extensions with abelian kernel is not necessarily extension-closed.

Example 2.5.1 (A pseudovariety closed under extensions with abelian kernel that is not extension-closed) Let $\Delta = A_5$ be the alternating group of degree 5. This is the finite simple nonabelian group with smallest order. Let $\mathcal{C}(\Delta)$ be the collection of all the finite direct products of copies of Δ. Observe that $\mathcal{C}(\Delta)$ is closed under homomorphic images (cf. RZ, Lemma 8.2.4). For a finite group G, denote by $S(G)$ its maximal solvable normal subgroup. Define $\mathcal{V}$ to be the set of all finite groups G such that $G/S(G) \in \mathcal{C}(\Delta)$.

We shall show that $\mathcal{V}$ is a pseudovariety of finite groups that is closed under extensions with abelian kernel, but not extension-closed.

We claim first that $\mathcal{V}$ is a pseudovariety. Clearly $\mathcal{V}$ is closed under finite direct products; moreover, since $\mathcal{C}(\Delta)$ is closed under homomorphic images, so is $\mathcal{V}$. It remains to prove that $\mathcal{V}$ is closed under taking subgroups. Let $G \in \mathcal{V}$ and let H be a proper subgroup of G. We use induction on the order of G to show that $H \in \mathcal{V}$. If G is solvable or $G \cong \Delta$, then H is solvable and the result is clear. Observe that $H/S(H)$ is a quotient of $H/H \cap S(G)$. If $S(G) \neq 1$, the result follows from the induction hypothesis since

$$H/H \cap S(G) \cong HS(G)/S(G) \leq G/S(G) \quad \text{and} \quad \left|G/S(G)\right| < |G|.$$

Thus from now on we may assume that $G \in \mathcal{C}(\Delta)$, i.e., $G = \Delta_1 \times \cdots \times \Delta_n$ $(n \geq 2)$, where each Δ_i is isomorphic to Δ. Since H is a proper subgroup of G, there is some i such that $H_i = H \cap \Delta_i \neq \Delta_i$, $1 \leq i \leq n$. Then H_i is solvable. So $H_i \leq S(H)$ and $S(H/H_i) = S(H)/H_i$. Now, since $H/H_i \leq G/\Delta_i \in \mathcal{V}$, we conclude from the induction hypothesis that

$$H/S(H) = (H/H_i)/S(H/H_i) \in \mathcal{C}(\Delta).$$

This proves the claim.

It follows easily from the definition that $\mathcal{V}$ is closed under extensions with abelian kernel. Let us show now that it is not extension-closed. For this consider the wreath product $R = \Delta \wr C$ of Δ with a group C of order 2; this is a semidirect product of $B = \Delta \times \Delta$ by C; both of these groups are in $\mathcal{V}$; and the action of C on B permutes the two factors Δ. Let $K \triangleleft R$ and assume that K is solvable. We claim that $K = 1$. Note that $K \cap B = 1$, for otherwise K must contain one of the copies of Δ, contradicting the solvability of K. If $K \neq 1$, we have $R = B \times K = B \times C$, contradicting the definition of R. This proves the claim. Therefore $S(R) = 1$. Finally, observe that $R \notin \mathcal{C}(\Delta)$. Thus $R \notin \mathcal{V}$.

If $(X, *)$ is a pointed profinite space, we denote by $F = F_{\mathcal{C}}(X, *)$ the free pro-$\mathcal{C}$ group on the pointed space $(X, *)$. The next two results establish conditions under which the Cayley graph of a free pro-$\mathcal{C}$ group with respect to one of its bases is a $\mathcal{C}$-tree. We begin with a study of the augmentation ideal (see Sect. 1.10) of a free pro-$\mathcal{C}$ group.

Lemma 2.5.2 *Let $\mathcal{C}$ be a pseudovariety of finite groups. Then $\mathcal{C}$ is closed under extensions with abelian kernel if and only if for every pointed profinite space $(Y, *)$, the augmentation ideal $(\!(IF)\!)$ of the complete group algebra $[\![\mathbf{Z}_{\hat{\mathcal{C}}}F]\!]$ of the free pro-$\mathcal{C}$ group $F = F_{\mathcal{C}}(Y, *)$ is a free $[\![\mathbf{Z}_{\hat{\mathcal{C}}}F]\!]$-module on the pointed space $(Y, *)$ with respect to the map $\iota : (Y, *) \to (\!(IF)\!)$ defined by $\iota(y) = y - 1$ $(y \in Y)$.*

Proof The augmentation ideal $(\!(IF)\!)$ is topologically generated by the space $Y - 1 = \{y - 1 \mid y \in Y\}$ as an $[\![\mathbf{Z}_{\hat{\mathcal{C}}}F]\!]$-module (see Sect. 1.10).

Assume first that $\mathcal{C}$ is closed under extensions with abelian kernel. We shall prove that $(\!(IF)\!)$ satisfies the required universal property of a free $[\![\mathbf{Z}_{\hat{\mathcal{C}}}F]\!]$-module with respect to the map ι. We must prove that given a map of pointed spaces $\psi : Y \to M$ to a profinite $[\![\mathbf{Z}_{\hat{\mathcal{C}}}F]\!]$-module M, there exists a unique continuous $[\![\mathbf{Z}_{\hat{\mathcal{C}}}F]\!]$-module homomorphism $\tilde{\psi} : (\!(IF)\!) \to M$ such that $\tilde{\psi}\iota = \psi$.

$$\begin{tikzcd}
y - 1 & (\!(IF)\!) \arrow[r, dotted, "\tilde{\psi}"] & M \\
y \arrow[u, mapsto] & Y \arrow[u, "\iota"] \arrow[ur, "\psi"'] &
\end{tikzcd}$$

Observe that if such a $\tilde{\psi}$ exists, then it is unique since $\iota(Y)$ generates $(\!(IF)\!)$ as a $[\![\mathbf{Z}_{\hat{\mathcal{C}}}F]\!]$-module.

We may assume that M is finite since M is an inverse limit of finite $[\![\mathbf{Z}_{\hat{\mathcal{C}}}F]\!]$-modules (see Sect. 1.7). Note that $M \in \mathcal{C}$ since M is automatically a $\mathbf{Z}_{\hat{\mathcal{C}}}$-module and so an abelian pro-$\mathcal{C}$ group.

Since M is in particular an F-module, we may construct the corresponding semidirect product $M \rtimes F$. We remark that $M \rtimes F$ is a pro-$\mathcal{C}$ group since $\mathcal{C}$ is closed under extensions with abelian kernel. Since F is a free pro-$\mathcal{C}$ group on $(Y, *)$, there exists a unique continuous homomorphism

$$\rho : F \longrightarrow M \rtimes F$$

such that $\rho(y) = (\psi(y), y)$ $(y \in Y)$.

Define now a map

$$\delta : F \longrightarrow M$$

by the equation $(\delta(f), f) = \rho(f)$, for all $f \in F$. Then δ is continuous and it is a derivation, that is,

$$\delta(f_1 f_2) = \delta(f_1) + f_1\delta(f_2), \quad \forall f_1, f_2 \in F$$

(see Sect. 1.10). Now, (see 1.10.7 in Sect. 1.10), there exists an isomorphism

$$\mathrm{Der}(F, M) \cong \mathrm{Hom}_{[\![\mathbf{Z}_{\hat{\mathcal{C}}}F]\!]}\big((\!(IF)\!), M\big),$$

and under this isomorphism δ corresponds to a $[[\mathbf{Z}_{\hat{\mathcal{C}}}F]]$-homomorphism

$$\tilde{\psi} : ((IF)) \longrightarrow M$$

such that $\tilde{\psi}(f-1) = \delta(f)$, for all $f \in F$. Then

$$\tilde{\psi}\iota(y) = \tilde{\psi}(y-1) = \delta(y) = \psi(y), \quad \forall y \in Y,$$

and thus $\tilde{\psi}\iota = \psi$.

Conversely, assume that $((IF))$ is a free $[[\mathbf{Z}_{\hat{\mathcal{C}}}F]]$-module on the pointed space $(Y, *)$ with respect to the map ι, for every profinite pointed space $(Y, *)$, where $F = F(Y, *)$ denotes the free pro-$\mathcal{C}$ group on the pointed profinite space $(Y, *)$. Let $A, H \in \mathcal{C}$, with A abelian. Assume that A is an H-module, and let $G = A \rtimes H$ be the corresponding semidirect product. To prove that $\mathcal{C}$ is closed under extensions with abelian kernel it suffices to show that $G \in \mathcal{C}$, as pointed out above.

Let $\{(a_y, h_y) \mid y \in Y\}$ be a generating set of $G = A \rtimes H$, with $a_y \in A, h_y \in H$, for all $y \in Y$, where $(Y, *)$ is a certain finite pointed indexing set and $a_* = 1, h_* = 1$. Then $H = \langle h_y \mid y \in Y\rangle$. Let $F = F_{\mathcal{C}}(Y, *)$ be the free pro-$\mathcal{C}$ group on the pointed space $(Y, *)$ and let

$$\varphi : F \longrightarrow H$$

be the continuous epimorphism determined by $\varphi(y) = h_y$ $(y \in Y)$. Then the action of H on A induces an action of F on A via φ:

$$f \cdot a = \varphi(f)a, \quad (a \in A, f \in F).$$

Let $\tilde{G} = A \rtimes F$ be the corresponding semidirect product, and let

$$\tilde{\varphi} : \tilde{G} = A \rtimes F \longrightarrow G = A \rtimes H$$

be the epimorphism induced by φ.

Since, by assumption, $((IF))$ is a free $[[\mathbf{Z}_{\hat{\mathcal{C}}}F]]$-module on $(Y, *)$ and A is an $[[\mathbf{Z}_{\hat{\mathcal{C}}}F]]$-module, there exists a continuous $[[\mathbf{Z}_{\hat{\mathcal{C}}}F]]$-homomorphism

$$\tilde{\psi} : ((IF)) \to A$$

such that $\tilde{\psi}(y-1) = a_y$ $(y \in Y)$. Define

$$d : F \longrightarrow A$$

by $d(f) = \tilde{\psi}(f-1)$ $(f \in F)$. Then d is a continuous derivation (see 1.10.7 in Sect. 1.10). Hence the map

$$\rho : F \longrightarrow \tilde{G} = A \rtimes F$$

given by $\rho(f) = (d(f), f)$ $(f \in F)$, is a continuous homomorphism (cf. RZ, Lemma 9.3.6). Define $\alpha : F \to G$ to be the composite $\alpha = \tilde{\varphi}\rho$. Observe that

$$\alpha(y) = (a_y, h_y) \quad (y \in Y);$$

therefore α is an epimorphism, and thus $G \in \mathcal{C}$, as needed. $\square$

Theorem 2.5.3 *Let $\mathcal{C}$ be a pseudovariety of finite groups. Then $\mathcal{C}$ is closed under extensions with abelian kernel if and only if for every profinite pointed space $(Y, *)$, the Cayley graph $\Gamma = \Gamma(F, Y)$ of the free pro-$\mathcal{C}$ group $F = F(Y, *)$ with respect to Y is a $\mathcal{C}$-tree.*

Proof We think of $(Y, *)$ as being embedded in F; in particular $*$ is identified with 1. Since $1 \in Y$, $\Gamma = \Gamma(F, Y) = F \times Y$ and $V(\Gamma) = F \times \{1\}$. Consider the sequence associated with the graph Γ and $\mathbf{Z}_{\hat{\mathcal{C}}}$ as in Eq. (2.4) of Sect. 2.3:

$$0 \longrightarrow \big[\!\big[\mathbf{Z}_{\hat{\mathcal{C}}}\big((F \times Y)/\big(F \times \{1\}\big), *\big)\big]\!\big] \xrightarrow{d} [\![\mathbf{Z}_{\hat{\mathcal{C}}}F]\!] \xrightarrow{\varepsilon} \mathbf{Z}_{\hat{\mathcal{C}}} \longrightarrow 0,$$

where $d(f, y) = fy - f$ $(y \in Y)$ and $\varepsilon(f) = 1$ $(f \in F)$. We have to prove that this sequence is exact for every $(Y, *)$ if and only if $\mathcal{C}$ is closed under extensions with abelian kernel.

By Lemma 2.2.4, Γ is a connected profinite graph since F is topologically generated by Y. Therefore, by Proposition 2.3.2, the above sequence is exact at $[\![\mathbf{Z}_{\hat{\mathcal{C}}}F]\!]$. It remains to prove that d is a monomorphism. Now, $\mathrm{Ker}(\varepsilon)$ is the augmentation ideal $(\!(IF)\!)$ of $[\![\mathbf{Z}_{\hat{\mathcal{C}}}F]\!]$, which is generated as a topological $[\![\mathbf{Z}_{\hat{\mathcal{C}}}F]\!]$-module by the subspace $\{y - 1 \mid y \in Y\}$ (see Sect. 1.10).

On the other hand, $[\![\mathbf{Z}_{\hat{\mathcal{C}}}((F \times Y)/(F \times \{1\}), *)]\!]$ is a free $[\![\mathbf{Z}_{\hat{\mathcal{C}}}F]\!]$-module on the quotient space $F\backslash((F \times Y)/(F \times \{1\}), *)$ (cf. RZ, Proposition 5.7.1). The space $F\backslash((F \times Y)/(F \times \{1\}), *)$ can be identified with the pointed space $(\{(1, y) \mid y \in Y\}, *)$. Since $d(1, y) = 1 - y$ $(y \in Y)$, to show that d is a monomorphism is equivalent to showing that the augmentation ideal $(\!(IF)\!)$ is free on the subspace $(\{1 - y \mid y \in Y\}, *)$, as a profinite $[\![\mathbf{Z}_{\hat{\mathcal{C}}}F]\!]$-module. But, according to Lemma 2.5.2, this is the case if and only if $\mathcal{C}$ is closed under extensions with abelian kernel. $\square$

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