

Chapter 2

Differential Equations

Aims and Objectives

- To review basic methods for solving some differential equations.
- To apply the theory to simple mathematical models.
- To introduce an existence and uniqueness theorem.

On completion of this chapter, the reader should be able to

- solve certain first- and second-order differential equations;
- apply the theory to chemical kinetics and electric circuits;
- interpret the solutions in physical terms;
- understand the existence and uniqueness theorem and its implications.

The basic theory of ordinary differential equations (ODEs) and analytical methods for solving some types of ODEs are reviewed. This chapter is not intended to be a comprehensive study on differential equations, but more an introduction to the theory that will be used in later chapters. Most of the material will be covered in first- and second-year undergraduate mathematics courses. The differential equations are applied to all kinds of models, but this chapter concentrates on chemical kinetics and electric circuits in particular.

This chapter ends with the existence and uniqueness theorem and some analysis.

2.1 Simple Differential Equations and Applications

Definition 1. A differential equation that involves only one independent variable is called an *ordinary differential equation* (ODE). Those involving two or more independent variables are called *partial differential equations* (PDEs). This chapter will be concerned with ODEs only.

The subject of ODEs encompasses analytical, computational, and applicable fields of interest. There are many textbooks written from the elementary to the most advanced, with some focusing on applications and others concentrating on existence theorems and rigorous methods of solution. This chapter is intended to introduce the reader to all three branches of the subject. For more information, the reader should consult the ODE textbooks in the bibliography [2, 7, 10, 15]. To solve ODEs using Mathematica, the reader is directed to [1, 6].

Separable Differential Equations

Consider the differential equation

$$\frac{dx}{dt} = f(t, x) \quad (2.1)$$

and suppose that the function $f(t, x)$ can be factored into a product $f(t, x) = g(t)h(x)$, where $g(t)$ is a function of t and $h(x)$ is a function of x . If f can be factored in this way, then equation (2.1) can be solved by the method of *separation of variables*.

To solve the equation, divide both sides by $h(x)$ to obtain

$$\frac{1}{h(x)} \frac{dx}{dt} = g(t);$$

and integration with respect to t gives

$$\int \frac{1}{h(x)} \frac{dx}{dt} dt = \int g(t) dt.$$

Changing the variables in the integral gives

$$\int \frac{dx}{h(x)} = \int g(t) dt.$$

An analytic solution to (2.1) can be found only if both integrals can be evaluated. The method can be illustrated with some simple examples.

Example 1. Solve the differential equation $\frac{dx}{dt} = -\frac{t}{x}$.

Solution. The differential equation is separable. Separate the variables and integrate both sides with respect to t . Therefore,

$$\int x \frac{dx}{dt} dt = - \int t dt,$$

and so

$$\int x dx = - \int t dt.$$

Integration of both sides yields

$$t^2 + x^2 = r^2,$$

where r^2 is a constant. There are an infinite number of solutions. The *solution curves* are concentric circles of radius r centered at the origin. There are an infinite number of solution curves that would fill the plane if they were all plotted. Three such solution curves are plotted in Fig. 2.1.

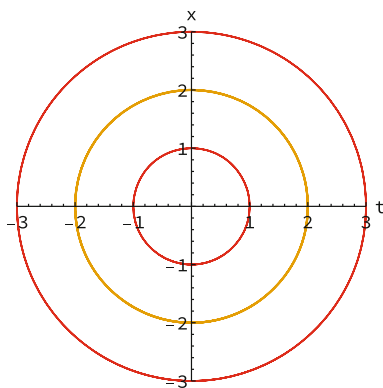


Figure 2.1: Three of an infinite number of solution curves for Example 1.

Example 2. Solve the differential equation $\frac{dx}{dt} = \frac{t}{x^2}$.

Solution. The differential equation is separable. Separate the variables and integrate both sides with respect to t to give

$$\int x^2 dx = \int t dt.$$

Integration of both sides yields

$$\frac{x^3}{3} = \frac{t^2}{2} + C,$$

where C is a constant. Six of an infinite number of solution curves are plotted in Fig. 2.2.

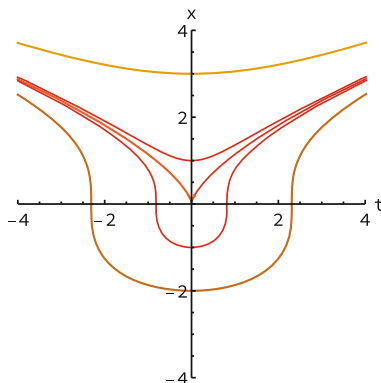


Figure 2.2: Five solution curves for Example 2.

Example 3. The population of a certain species of fish living in a large lake at time t can be modeled using *Verhulst's equation*, otherwise known as the *logistic equation*,

$$\frac{dP}{dt} = P(\beta - \delta P),$$

where $P(t)$ is the population of fish measured in tens of thousands, and β and δ are constants representing the birth and death rates of the fish living in the lake, respectively. Suppose that $\beta = 0.1$, $\delta = 10^{-3}$, and the initial population is 50×10^4 . Solve this *initial value problem* and interpret the results in physical terms.

Solution. Using the methods of separation of variables gives

$$\int \frac{dP}{P(\beta - \delta P)} = \int dt.$$

The solution to the integral on the left may be determined using partial fractions. The general solution is

$$\ln \left| \frac{P}{\beta - \delta P} \right| = \beta t + C_1,$$

or

$$P(t) = \frac{\beta}{\delta + C_2 \beta e^{-\beta t}},$$

computed using Mathematica, where C_1 and C_2 are constants. Substituting the initial conditions, the solution is

$$P(t) = \frac{100}{1 + e^{-0.1t}}.$$

Thus as time increases, the population of fish tends to a value of 100×10^4 . The solution curve is plotted in Fig. 2.3.

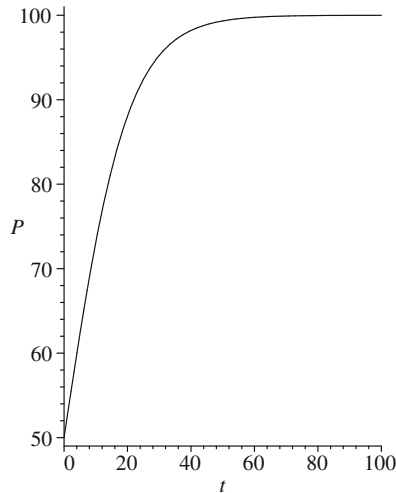


Figure 2.3: Solution curve for the initial value problem in Example 3. Note that the axes would be scaled by 10^4 in applications.

Note the following:

- The quantity $\frac{\beta}{\delta}$ is the ratio of births to deaths and is called the *carrying capacity* of the environment.
- Take care when interpreting the solutions. This and similar continuous models only work for large species populations. The solutions give approximate numbers. Even though time is continuous, the population size is not. For example, you cannot have a fractional living fish, so population sizes have to be rounded out to whole numbers in applications.
- Discrete models can also be applied to population dynamics (see Chap. 13).

Exact Differential Equations

A differential equation of the form

$$M(t, x) + N(t, x) \frac{dx}{dt} = 0 \quad (2.2)$$

is said to be *exact* if there exists a function, say, $F(t, x)$, with continuous second partial derivatives such that

$$\frac{\partial F}{\partial t} = M(t, x), \quad \text{and} \quad \frac{\partial F}{\partial x} = N(t, x).$$

Such a function exists as long as

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t},$$

and then the solution to (2.2) satisfies the equation

$$F(t, x) = C,$$

where C is a constant. Differentiate this equation with respect to t to obtain (2.2).

Example 4. Solve the differential equation

$$\frac{dx}{dt} = \frac{9 - 12t - 5x}{5t + 2x - 4}.$$

Solution In this case, $M(t, x) = -9 + 12t + 5x$ and $N(t, x) = 5t + 2x - 4$. Now

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t} = 5$$

and integration gives the solution $F(t, x) = x^2 + 6t^2 + 5tx - 9t - 4x = C$. There are an infinite number of solution curves, some of which are shown in Fig. 2.4.

Homogeneous Differential Equations

Consider differential equations of the form

$$\frac{dx}{dt} = f\left(\frac{x}{t}\right). \quad (2.3)$$

Substitute $v = \frac{x}{t}$ into (2.3) to obtain

$$\frac{d}{dt}(vt) = f(v).$$

Therefore,

$$v + t \frac{dv}{dt} = f(v),$$

and so

$$\frac{dv}{dt} = \frac{f(v) - v}{t},$$

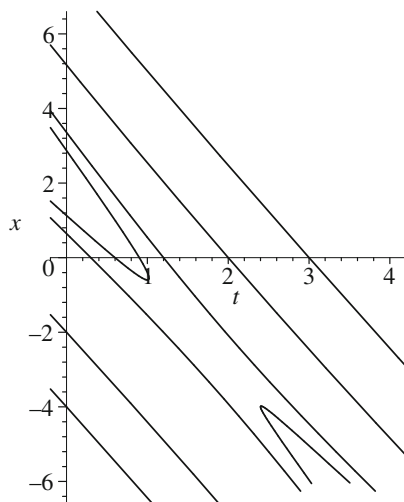


Figure 2.4: Some solution curves for Example 4.

which is separable. A complete solution can be found as long as the equations are integrable, and then, v may be replaced with $\frac{x}{t}$.

Example 5. Solve the differential equation

$$\frac{dx}{dt} = \frac{t - x}{t + x}.$$

Solution. The equation may be rewritten as

$$\frac{dx}{dt} = \frac{1 - \frac{x}{t}}{1 + \frac{x}{t}}. \quad (2.4)$$

Let $v = \frac{x}{t}$. Then, (2.4) becomes

$$\frac{dv}{dt} = \frac{1 - 2v - v^2}{t(1 + v)}.$$

This is a separable differential equation. The general solution is given by

$$x^2 + 2tx - t^2 = C,$$

where C is a constant. Some solution curves are plotted in Fig. 2.5.

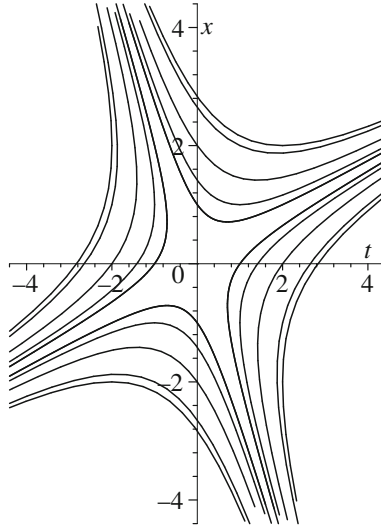


Figure 2.5: Some solution curves for Example 5.

Linear Differential Equations

Consider differential equations of the form

$$\frac{dx}{dt} + P(t)x = Q(t). \quad (2.5)$$

Multiplying through by an integrating factor, say, $J(t)$, (2.5) becomes

$$J \frac{dx}{dt} + JPx = JQ. \quad (2.6)$$

Find J such that (2.6) can be written as

$$\frac{d}{dt}(Jx) = J \frac{dx}{dt} + x \frac{dJ}{dt} = JQ.$$

In order to achieve this, set

$$\frac{dJ}{dt} = JP$$

and integrate to get

$$J(t) = \exp \left(\int P(t) dt \right).$$

Thus, the solution to system (2.5) may be found by solving the differential equation

$$\frac{d}{dt}(Jx) = JQ,$$

as long as the right-hand side is integrable.

Example 6. A chemical company pumps v liters of solution containing mass m grams of solute into a large lake of volume V per day. The inflow and outflow of the water is constant. The concentration of solute in the lake, say, σ , satisfies the differential equation

$$\frac{d\sigma}{dt} + \frac{v}{V}\sigma = \frac{m}{V}. \quad (2.7)$$

Determine the concentration of solute in the lake at time t assuming that $\sigma = 0$ when $t = 0$. What happens to the concentration in the long term?

Solution. This is a linear differential equation, and the integrating factor is given by

$$J = \exp\left(\int \frac{v}{V} dt\right) = e^{\frac{vt}{V}}.$$

Multiply (2.7) by the integrating factor to obtain

$$\frac{d}{dt}\left(e^{\frac{vt}{V}}\sigma\right) = e^{\frac{vt}{V}}\frac{m}{V}.$$

Integration gives

$$\sigma(t) = \frac{m}{v} - ke^{-\frac{vt}{V}},$$

where k is a constant. Substituting the initial conditions, the final solution is

$$\sigma(t) = \frac{m}{v}\left(1 - e^{-\frac{vt}{V}}\right).$$

As $t \rightarrow \infty$, the concentration settles to $\frac{m}{v} gl^{-1}$.

Series Solutions

Another very useful method for determining the solutions to some ODEs is the series solution method. The basic idea is to seek a series solution (assuming that the series converge) of the form

$$x(t) = \sum_{n=0}^{\infty} a_n(t - t_0)^n,$$

about the point t_0 . The method holds for infinitely differentiable functions (that is, functions that can be differentiated as often as desired), and is outlined using two simple examples.

Example 7. Determine a series solution to the initial value problem

$$\frac{dx}{dt} + tx = t^3, \quad (2.8)$$

given that $x(0) = 1$.

Solution. Given that $t_0 = 0$, set $x(t) = \sum_{n=0}^{\infty} a_n t^n$. Substituting into (2.8) gives

$$\sum_{n=1}^{\infty} n a_n t^{n-1} + t \left(\sum_{n=0}^{\infty} a_n t^n \right) = t^3.$$

Combining the terms into a single series

$$a_1 + \sum_{n=1}^{\infty} ((n+1)a_{n+1} + a_{n-1}) t^n = t^3.$$

Equating coefficients gives

$$a_1 = 0, 2a_2 + a_0 = 0, 3a_3 + a_1 = 0, 4a_4 + a_2 = 1, 5a_5 + a_3 = 0, \dots$$

and solving these equations gives $a_{2n+1} = 0$, for $n = 0, 1, 2, \dots$,

$$a_2 = -\frac{a_0}{2}, a_4 = \frac{1 - a_2}{4},$$

and

$$a_{2n} = -\frac{a_{2n-2}}{2n},$$

where $n = 3, 4, 5, \dots$. Based on the assumption that $x(t) = \sum_{n=0}^{\infty} a_n t^n$, substituting $x(0) = 1$ gives $a_0 = 1$. Hence, the series solution to the ODE (2.8) is

$$x(t) = 1 - \frac{1}{2}t^2 + \frac{3}{8}t^4 + \sum_{n=3}^{\infty} (-1)^n \left(\frac{1}{(2n)} \frac{1}{(2n-2)} \cdots \frac{1}{6} \frac{3}{8} \right) t^{2n}.$$

Note that the analytic solution can be found in this case and is equal to

$$x(t) = -2 + t^2 + 3e^{-\frac{t^2}{2}},$$

which is equivalent to the series solution above.

Example 8. Consider the van der Pol equation given by

$$\frac{d^2x}{dt^2} + 2(x^2 - 1) \frac{dx}{dt} + x = 0, \quad (2.9)$$

where $x(0) = 5$ and $\dot{x}(0) = 0$. Use Mathematica to plot a numerical solution against a series solution up to order 6 near to the point $x(0) = 5$.

Solution. Using Mathematica, the series solution is computed to be

$$x(t) = 5 - \frac{5}{2}t^2 + 40t^3 - \frac{11515}{24}t^4 + \frac{9183}{2}t^5 + O(t^6).$$

Figure 2.6 shows the truncated series and numerical solutions for the ODE (2.9) near to $x(0) = 5$. The upper curve is the truncated series approximation that diverges quite quickly away from the numerical solution. Of course, one must also take care that the numerical solution is correct.

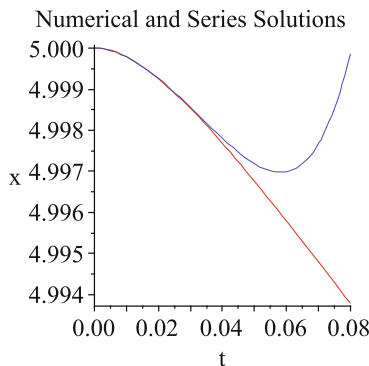


Figure 2.6: [Mathematica] Numerical and truncated series solutions for the van der Pol equation (2.9) near $x(0) = 5$.

2.2 Applications to Chemical Kinetics

Even the simplest chemical reactions can be highly complex and difficult to model. Physical parameters such as temperature, pressure, and mixing are ignored in this text, and differential equations are constructed that are dependent only on the concentrations of the chemicals involved in the reaction. This is potentially a very difficult subject, and some assumptions have to be made to make progress.

The Chemical Law of Mass Action. The rates at which the concentrations of the various chemical species change with time are proportional to their concentrations.

Consider the simple chemical reaction



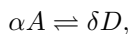
where β and γ are the stoichiometric coefficients, A and B are the reactants, C is the product, and k_1 is the rate constant of the equation. The rate of reaction, say, r , is given by

$$r = \frac{\text{change in concentration}}{\text{change in time}}.$$

For this simple example,

$$r = k_1[A][B] = -\frac{d[A]}{dt} = -\frac{1}{\beta} \frac{d[B]}{dt} = \frac{1}{\gamma} \frac{d[C]}{dt},$$

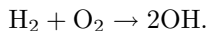
where $[A]$, $[B]$, and $[C]$ represents the concentrations of A , B , and C , respectively. By adding a second chemical equation, a slightly more complex system is produced,



where k_2 is the rate constant of the second equation and α and δ are the stoichiometric coefficients. Two of the possible reaction rate equations for this system now become

$$\frac{d[A]}{dt} = -k_1\beta[A][B] - k_2\alpha[A]^\alpha, \quad \frac{d[D]}{dt} = k_2\delta[A]^\alpha.$$

Consider the following example, where one molecule of hydrogen reacts with one molecule of oxygen to produce two molecules of hydroxyl (OH):



Suppose that the concentration of hydrogen is $[\text{H}_2]$ and the concentration of oxygen is $[\text{O}_2]$. Then from the chemical law of mass action, the rate equation is given by

$$\text{Rate} = k[\text{H}_2][\text{O}_2],$$

where k is called the *rate constant*, and the reaction rate equation is

$$\frac{d[\text{OH}]}{dt} = 2k[\text{H}_2][\text{O}_2].$$

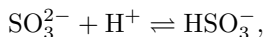
Table 2.1: One of the possible reaction rate equations for each chemical reaction.

Chemical reaction	The reaction rate equation for one species may be expressed as follows:
$A+B \rightarrow C$	$\frac{dc}{dt} = k_f ab = k_f(a_0 - c)(b_0 - c)$
$2A \rightleftharpoons B$	$\frac{db}{dt} = k_f(a_0 - 2b)^2 - k_r b$
$A \rightleftharpoons 2B$	$\frac{db}{dt} = k_f(a_0 - \frac{b}{2}) - k_r b^2$
$A \rightleftharpoons B+C$	$\frac{dc}{dt} = k_f(a_0 - c) - k_r(b_0 + c)(c_0 + c)$
$A+B \rightleftharpoons C$	$\frac{dc}{dt} = k_f(a_0 - c)(b_0 - c) - k_r c$
$A+B \rightleftharpoons C+D$	$\frac{dc}{dt} = k_f(a_0 - c)(b_0 - c) - k_r(c_0 + c)(d_0 + c)$

Unfortunately, it is not possible to write down the reaction rate equations based on the stoichiometric (balanced) chemical equations alone. There may be many mechanisms involved in producing OH from hydrogen and oxygen in the above example. Even simple chemical equations can involve a large number of steps and different rate constants. Suppose in this text that the chemical equations give the *rate-determining steps*.

Suppose that species A, B, C, and D have concentrations $a(t)$, $b(t)$, $c(t)$, and $d(t)$ at time t and initial concentrations a_0 , b_0 , c_0 , and d_0 , respectively. Table 2.1 lists some reversible chemical reactions and one of the corresponding reaction rate equations, where k_f and k_r are the forward and reverse rate constants, respectively.

Example 9. A reaction equation for sulfate and hydrogen ions to form bisulfite ions is given by



where k_f and k_r are the forward and reverse rate constants, respectively. Denote the concentrations by $a = [\text{SO}_3^{2-}]$, $b = [\text{H}^+]$, and $c = [\text{HSO}_3^-]$, and let the initial concentrations be a_0 , b_0 , and c_0 . Assume that there is much more of species H^+ than the other two species, so that its concentration b can be regarded as constant. The reaction rate equation for $c(t)$ is given by

$$\frac{dc}{dt} = k_f(a_0 - c)b - k_r(c_0 + c).$$

Find a general solution for $c(t)$.

Solution. The differential equation is separable and

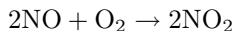
$$\int \frac{dc}{k_f(a_0 - c)b - k_r(c_0 + c)} = \int dt.$$

Integration yields

$$c(t) = \frac{k_f a_0 b - k_r c_0}{k_f b + k_r} - \frac{k_r c_0}{k_f b + k_r} + A e^{(-k_f a_0 - k_r)t},$$

where A is a constant.

Example 10. The chemical equation for the reaction between nitrous oxide and oxygen to form nitrogen dioxide at 25°C ,



obeys the law of mass action. The rate equation is given by

$$\frac{dc}{dt} = k(a_0 - c)^2 \left(b_0 - \frac{c}{2}\right),$$

where $c = [\text{NO}_2]$ is the concentration of nitrogen dioxide, k is the rate constant, a_0 is the initial concentration of NO, and b_0 is the initial concentration of O_2 . Find the concentration of nitrogen dioxide after time t given that $k = 0.00713\text{l}^2\text{M}^{-2}\text{s}^{-1}$, $a_0 = 4\text{Ml}^{-1}$, $b_0 = 1\text{Ml}^{-1}$, and $c(0) = 0\text{Ml}^{-1}$.

Solution. The differential equation is separable and

$$\int \frac{dc}{(4 - c)^2(1 - c/2)} = \int k dt.$$

Integrating using partial fractions gives

$$kt = \frac{1}{c - 4} + \frac{1}{2} \ln |c - 4| - \frac{1}{2} \ln |c - 2| + \frac{1}{4} - \frac{1}{2} \ln 2.$$

It is not possible to obtain $c(t)$ explicitly, so numerical methods are employed using Mathematica. The concentration of nitrogen dioxide levels off at two moles per liter as time increases, as depicted in Fig. 2.7.

Chemical reactions displaying periodic behavior will be dealt with in Chapter 8. There may be a wide range of timescales involved in chemical reactions, and this can lead to *stiff* systems. Loosely speaking, a stiff system of differential equations is one in which the velocity or magnitude of the vector field changes rapidly in phase space.

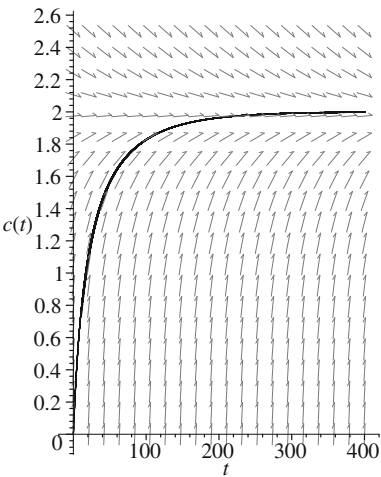


Figure 2.7: [Mathematica] The Concentration of NO₂ in moles per liter against time in seconds.

2.3 Applications to Electric Circuits

For many years, differential equations have been applied to model simple electrical and electronic circuits. If an oscilloscope is connected to the circuit, then the results from the analysis can be seen to match very well with what happens physically. As a simple introduction to electric circuits, linear systems will be considered and the basic definitions and theory will be introduced. The section ends with an introduction to the nonlinear circuit element known as the memristor.

Current and Voltage

The *current* I flowing through a conductor is proportional to the number of positive charge carriers that pass a given point per second. The unit of current is the *ampere* A . A *coulomb* is defined to be the amount of charge that flows through a cross section of wire in 1 second when a current of $1A$ is flowing, so 1 amp is 1 coulomb per second. As the current passes through a circuit element, the charge carriers exchange energy with the circuit elements, and there is a *voltage drop* or *potential difference* measured in joules per coulomb, or *volts* V .

Consider simple electric circuits consisting of voltage sources, resistors, inductors, and capacitors, or *RLC* circuits. A series *RLC* circuit is shown schematically in Fig. 2.8. The voltage drop across a resistor and the current flowing through it are related by Ohm’s Law.

Ohm’s Law. The voltage drop V across a resistor is proportional to the current I flowing through it:

$$V = IR,$$

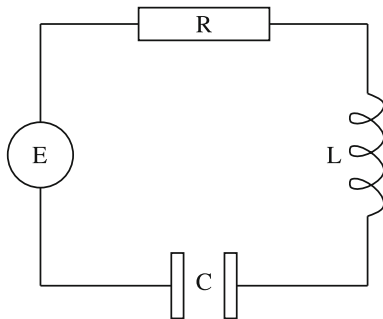


Figure 2.8: Schematic of a simple RLC series circuit.

where R is the *resistance* of the resistor measured in ohms (Ω).

A changing electric current can create a changing magnetic field that induces a voltage drop across a circuit element, such as a coil.

Faraday's Law. The voltage drop across an inductor is proportional to the rate of change of the current:

$$V = L \frac{dI}{dt},$$

where L is the *inductance* of the inductor measured in henries (H).

A capacitor consists of two plates insulated by some medium. When connected to a voltage source, charges of opposite sign build up on the two plates, and the total charge on the capacitor is given by

$$q(t) = q_0 + \int_{t_0}^t I(s) ds,$$

where q_0 is the initial charge.

Coulomb's Law. The voltage drop across a capacitor is proportional to the charge on the capacitor:

$$V(t) = \frac{1}{C} q(t) = \frac{1}{C} \left(q_0 + \int_{t_0}^t I(s) ds \right),$$

where C is the *capacitance* of the capacitor measured in farads (F).

The physical laws governing electric circuits were derived by G.R. Kirchhoff in 1859.

Kirchhoff's Current Law. The algebraic sum of the currents flowing into any junction of an electric circuit must be zero.

Kirchhoff's Voltage Law. The algebraic sum of the voltage drops around any closed loop in an electric circuit must be zero.

Applying Kirchhoff's voltage law to the RLC circuit gives

$$V_L + V_R + V_C = E(t),$$

where V_R , V_L , and V_C are the voltage drops across R , L , and C , respectively, and $E(t)$ is the voltage source, or applied electromotive force (EMF). Substituting for the voltages across the circuit components gives

$$L \frac{dI}{dt} + RI + \frac{1}{C}q = E(t).$$

Since the current is the instantaneous rate of change in charge, $I = \frac{dq}{dt}$, this equation becomes

$$L \frac{d^2q}{dt^2} + R \frac{dq}{dt} + \frac{1}{C}q = E(t). \quad (2.10)$$

This differential equation is called a *linear second-order differential equation*. It is linear because there are no powers of the derivatives, and second order since the order of the highest occurring derivative is two. This equation can be solved by the method of *Laplace transforms* [12]; there are other methods available, and readers should use whichever method they feel most comfortable with. The method of Laplace transforms can be broken down into four distinct steps when finding the solution of a differential equation:

- rewrite equation (2.10) in terms of Laplace transforms;
- insert any given initial conditions;
- rearrange the equation to give the transform of the solution;
- find the inverse transform.

The method is illustrated in the following examples.

Example 11. Consider a series resistor-inductor electrical circuit. Kirchhoff's voltage law gives

$$L \frac{dI}{dt} + RI = E.$$

Given that $L = 10H$; $R = 2\Omega$, and $E = 50 \sin(t)V$, find an expression for the current in the circuit if $I(0) = 0$.

Solution. Take Laplace transforms of both sides. Then,

$$10(s\bar{I} - I(0)) + 2\bar{I} = \frac{50}{s^2 + 1}.$$

Inserting the initial condition and rearranging,

$$\bar{I}(5s + 1) = \frac{25}{s^2 + 1},$$

and splitting into partial fractions,

$$\bar{I} = \frac{25}{26} \frac{1}{s^2 + 1} - \frac{125}{26} \frac{s}{s^2 + 1} - \frac{125}{126} \frac{1}{(s - 1/5)}.$$

Take inverse Laplace transforms to give

$$I(t) = \frac{25}{26} \sin(t) - \frac{125}{26} \cos(t) - \frac{125}{126} e^{-\frac{t}{5}}.$$

The periodic expression $\frac{25}{26} \sin(t) - \frac{125}{26} \cos(t)$ is called the *steady state*, and the term $\frac{125}{126} e^{-\frac{t}{5}}$ is called the *transient*. Note that the transient decays to zero as $t \rightarrow \infty$.

Example 12. Differentiate equation (2.10) with respect to time and substitute for $\frac{dq}{dt}$ to obtain

$$L \frac{d^2 I}{dt^2} + R \frac{dI}{dt} + \frac{1}{C} I = \frac{dE}{dt}.$$

The second-order differential equation for a certain RLC circuit is given by

$$\frac{d^2 I}{dt^2} + 5 \frac{dI}{dt} + 6I = 10 \sin(t).$$

Solve this differential equation given that $I(0) = \dot{I}(0) = 0$ (a *passive circuit*).

Solution. Take Laplace transforms of both sides:

$$(s^2 \bar{I} - sI(0) - \dot{I}(0)) + 5(s\bar{I} - I(0)) + 6\bar{I} = \frac{10}{s^2 + 1}.$$

Substitute the initial conditions to obtain

$$\bar{I}(s^2 + 5s + 6) = \frac{10}{s^2 + 1}.$$

Splitting into partial fractions gives

$$\bar{I} = \frac{2}{s + 2} - \frac{1}{s + 3} + \frac{1}{s^2 + 1} - \frac{s}{s^2 + 1}.$$

Take inverse transforms to get

$$I(t) = 2e^{-2t} - e^{-3t} + \sin(t) - \cos(t).$$

The Memristor. The examples discussed thus far have concerned electric circuits with linear elements; however, nonlinear electric circuits are now coming to the fore. It is now widely acknowledged that Professor Leon Chua is the father of nonlinear circuit theory. Chua's famous nonlinear electric circuit is discussed in Chapter 8, and the circuit is easy to construct even in school physics laboratories. It has long been believed that there are only three fundamental passive circuit elements such as the capacitor, the inductor, and the resistor. In 1971, Chua [3] used mathematics to prove the existence of a fourth fundamental nonlinear element which acts like a resistor with memory, and he called the new device the memristor. The three well-known circuit elements are described by the equations

$$\frac{1}{C} = \frac{dv}{dq}, L = \frac{d\phi}{di}, R = \frac{dv}{di},$$

where $\frac{1}{C}$ is the inverse capacitance, L is inductance, R is incremental resistance, v is voltage, i is current, q is charge, and ϕ is flux. In addition, the current and voltage are described by the following physical laws

$$i = \frac{dq}{dt}, v = \frac{d\phi}{dt}.$$

This gives five relationships on three elements and leaves a gap in the harmonic symmetry of Chua's aesthetics. Chua discovered the missing functional relationship between charge and flux which is given by

$$M = \frac{d\phi}{dq},$$

where M is the memristance. Fig. 2.9 displays the relationships between the four fundamental elements.

In 1976, Chua and Kang [4] discovered that a memristor displays a pinched hysteresis and suggested that this effect could be used as a test to determine if a device could be truly categorized as a memristor. A pinched hysteresis loop is demonstrated in Chapter 21, and the Mathematica program for plotting the loop is listed within the Mathematica commands section of that chapter.

In 2008, a team at HP Laboratories [13] announced that they had evidence that many nanoscale electronic devices which involve the motion of charged atomic or molecular species act as memristors. Their analysis was based on results from a thin film of titanium dioxide, and they are currently building devices for computer logic, nanoelectronic memories, and neuromorphic computer architectures. A long-term project of HP Labs Research has been the development of *The Machine*, which was supposed to reinvent the fundamental architecture of computing. Among the principal components to be used were the memristor and silicon photonics using optical communications; unfortunately, in June 2015, HP Labs announced that memristors were to be removed from The

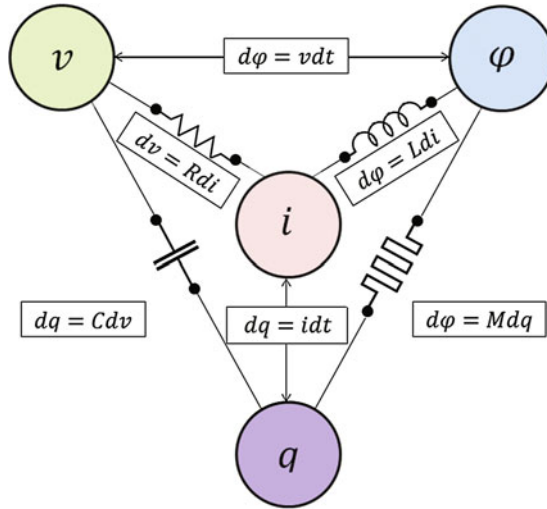


Figure 2.9: The memristor: the missing link discovered.

Machine’s road map. Some researchers believe that Strukov’s memristor modeling equations [13] do not simulate the devices physics very well but believe that Chang’s and Yakopcic’s models [8] provide a good compromise.

It is now understood that man-made memristive devices have been around for over two hundred years. In 2012, Prodromakis et al. [9] published a paper entitled “Two centuries of memristors”. Indeed, it is now known that the first demonstration of a memristor device took place at the Royal Institution in 1808. Sir Humphrey Davy produced a 1000 V carbon arc discharge, and modern technology has demonstrated a pinched hysteresis effect in this system.

Incredibly, natural memristors have been around for hundreds of millions of years, and there are memristors in plants and early life forms. Chua [5] shows that sodium and potassium ion channel memristors are the key to generating action potentials in the Hodgkin-Huxley equations (see Chapter 21), and he explains some unresolved anomalies with the original equations. In terms of neurobiology, the tutorial shows that synapses are locally passive memristors and that neurons act as locally active memristors. Chua also shows that the circuits used to model the Josephson junction effect should include memristor elements to explain the workings of these devices accurately. The author and Borresen believe it is possible to make superfast low-power computers using Josephson junctions acting as neurons connected together with memristors acting as axons and synapses. More detail is provided in Chapter 21.

2.4 Existence and Uniqueness Theorem

Definition 2. A function $\mathbf{f}(\mathbf{x})$ with $\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is said to satisfy a *Lipschitz condition* in a domain $D \subset \mathbb{R}^n$ if there exists a constant, say, L , such that

$$\|\mathbf{f}(\mathbf{x}_1) - \mathbf{f}(\mathbf{x}_2)\| \leq L \|\mathbf{x}_1 - \mathbf{x}_2\|,$$

where $\mathbf{x}_1, \mathbf{x}_2 \in D$.

If the function \mathbf{f} satisfies the Lipschitz condition, then it is said to be *Lipschitz continuous*. Note that Lipschitz continuity in \mathbf{x} implies continuity in \mathbf{x} , but the converse is not always true.

Existence and Uniqueness Theorem. Suppose that \mathbf{f} is continuously Lipschitz; then for an initial point $\mathbf{x}_0 \in D$, the autonomous differential equation

$$\frac{d\mathbf{x}}{dt} = \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \quad (2.11)$$

has a unique solution, say, $\phi_t(\mathbf{x}_0)$, that is defined on the maximal interval of existence.

Note that (2.11) is called autonomous as long as \mathbf{f} is independent of t . The proof of this theorem can be found in most textbooks that specialize in the theory of ODEs. As far as the reader is concerned, this theorem implies that as long as \mathbf{f} is continuously differentiable, i.e., $\mathbf{f} \in C^1(D)$, then two distinct solutions cannot intersect in finite time.

The following simple examples involving first-order ODEs illustrate the theorem quite well.

Example 13. Solve the following linear differential equations, and state the maximal interval of existence for each solution:

- (a) $\dot{x} = x, x(0) = 1;$
- (b) $\dot{x} = x^2, x(0) = 1;$
- (c) $\dot{x} = 3x^{\frac{2}{3}}, x(0) = 0.$

Solutions.

- (a) The solution to this elementary differential equation is $x(t) = e^t$, which is unique and defined for all t . The maximal interval of existence in this case is $-\infty < t < \infty$. Note that $f(x) = x$ is continuously differentiable.
- (b) The solution is given by

$$x(t) = \frac{1}{1-t},$$

which is not defined for $t = 1$. Therefore, there is a unique solution on the maximal interval of existence given by $-\infty < t < 1$.

- (c) The function $f(x) = 3x^{\frac{2}{3}}$ is not continuously differentiable and does not satisfy the Lipschitz condition at $x = 0$; $\frac{\partial f}{\partial x} = 2x^{-\frac{1}{3}}$ is not continuous at $x = 0$. Integration gives

$$\int \frac{1}{3}x^{-\frac{2}{3}} dx = \int dt,$$

with general solution $x(t) = t^3 + C$. The solution to the initial value problem is therefore $x(t) = t^3$. The point $x = 0$ is zero when $\dot{x} = 0$. This means that a solution starting at this point should stay there for all t . Thus, there are two solutions starting at $x_0 = 0$, namely $\phi_1(t) = t^3$ and $\phi_2(t) = 0$. In fact, there are infinitely many solutions starting at $x_0 = 0$. In this case, there exist solutions but they are not unique.

Note that the solution would be unique on the maximal interval of existence $0 < t < \infty$ if the initial condition were $x(1) = 1$.

Consider autonomous differential equations of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad (2.12)$$

where $\mathbf{x} \in \mathbb{R}^n$.

Definition 3. A *critical point* (equilibrium point, fixed point, stationary point) is a point that satisfies the equation $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) = 0$. If a solution starts at this point, it remains there forever.

Definition 4. A critical point, say, \mathbf{x}_0 , of the differential equation (2.12) is called *stable* if given $\epsilon > 0$, there is a $\delta > 0$, such that for all $t \geq t_0$, $\|\mathbf{x}(t) - \mathbf{x}_0(t)\| < \epsilon$, whenever $\|\mathbf{x}(t_0) - \mathbf{x}_0(t_0)\| < \delta$, where $\mathbf{x}(t)$ is a solution of (2.12).

A critical point that is not stable is called an *unstable* critical point.

Example 14. Find and classify the critical points for the following one-dimensional differential equations.

- (a) $\dot{x} = x$;
- (b) $\dot{x} = -x$;
- (c) $\dot{x} = x^2 - 1$.

Solutions.

- (a) There is one critical point at $x_0 = 0$. If $x < 0$, then $\dot{x} < 0$, and if $x > 0$, then $\dot{x} > 0$. Therefore, x_0 is an unstable critical point. Solutions starting either side of x_0 are repelled away from it.
- (b) There is one critical point at $x_0 = 0$. If $x < 0$, then $\dot{x} > 0$, and if $x > 0$, then $\dot{x} < 0$. Solutions starting either side of x_0 are attracted toward it. The critical point is stable.

- (c) There are two critical points, one at $x_1 = -1$ and the other at $x_2 = 1$. If $x > 1$, then $\dot{x} > 0$; if $-1 < x < 1$, then $\dot{x} < 0$; and if $x < -1$, then $\dot{x} > 0$. Therefore, solutions starting near to x_1 but not on it are attracted toward this point, and x_1 is a stable critical point. Solutions starting near x_2 but not on it move away from this point, and x_2 is an unstable critical point.

By linearizing near a critical point, one can obtain a quantitative measure of stability as demonstrated below. Consider one-dimensional systems here; higher-dimensional systems will be investigated in other chapters.

Linear Stability Analysis

Let x^* be a critical point of $\dot{x} = f(x)$, $x \in \mathfrak{R}$. Consider a *small perturbation*, say, $\xi(t)$, away from the critical point at x^* to give $x(t) = x^* + \xi(t)$. A simple analysis is now applied to determine whether the perturbation grows or decays as time evolves. Now

$$\dot{\xi} = \dot{x} = f(x) = f(x^* + \xi),$$

and after a Taylor series expansion,

$$\dot{\xi} = f(x^*) + \xi f'(x^*) + \frac{\xi^2}{2} f''(x^*) + \cdots.$$

In order to apply a linear stability analysis, the nonlinear terms are ignored. Hence,

$$\dot{\xi} = \xi f'(x^*),$$

since $f(x^*) = 0$. Therefore, the perturbation $\xi(t)$ grows exponentially if $f'(x^*) > 0$ and decays exponentially if $f'(x^*) < 0$. If $f'(x^*) = 0$, then higher-order derivatives must be considered to determine the stability of the critical point.

A linear stability analysis is used extensively throughout the realms of nonlinear dynamics and will appear in other chapters of this book.

Example 15. Use a linear stability analysis to determine the stability of the critical points for the following differential equations:

- (a) $\dot{x} = \sin(x)$;
- (b) $\dot{x} = x^2$;
- (c) $\dot{x} = e^{-x} - 1$.

Solutions.

- (a) There are critical points at $x_n = n\pi$, where n is an integer. When n is even, $f'(x_n) = 1 > 0$, and these critical points are unstable. When n is odd, $f'(x_n) = -1 < 0$, and these critical points are stable.
- (b) There is one critical point at $x_0 = 0$ and $f'(x) = 2x$ in this case. Now $f'(0) = 0$ and $f''(0) = 2 > 0$. Therefore, x_0 is attracting when $x < 0$ and repelling when $x > 0$. The critical point is called *semistable*.

- (c) There is one critical point at $x_0 = 0$. Now $f'(0) = -1 < 0$, and therefore, the critical point at the origin is stable.

The theory of autonomous systems of ODEs in two dimensions will be discussed in the next chapter.

2.5 Mathematica Commands in Text Format

For more information on solving differential equations using Mathematica, the reader should look for DSolve and NDSolve under the Help tab.

(* See Example 1: Solving simple ODEs. *)

```
In[1]:=DSolve[x'[t]==-t/x[t],x[t],t]
```

(* See Example 8 and Fig. 2.6: Series solutions for van der Pol. *)

```
In[2]:=DEqn=x''[t]+2*(x[t]^2-1)*x'[t]+x[t];
```

```
In[3]:=Ser=Series[DEqn,{t,0,6}];
```

```
In[4]:=Serone=Ser/.{x[0]->5,x'[0]->0};
```

```
In[5]:=Eqns=LogicalExpand[Serone==0];
```

```
In[6]:=Coeffs=Solve[Eqns];
```

```
In[7]:=Serx=Series[x[t],{t,0,5}];
```

```
In[8]:=ApproxSol=Normal[Serx]/.{x[0]->5,x'[0]->0}/.Coeffs[[1]]
```

```
In[9]:=Plot[ApproxSol,{t,0,0.08}]
```

(* See Example 10 and Fig. 2.7: Chemical kinetics. *)

```
In[10]:=k=.00713;a0=4;b0=1;sol=NDSolve[{c'[t]==k (a0-c[t]^2)(b0-c[t]/2),
c[0]==0},c,{t,0,700}]
```

```
In[11]:=Plot[Evaluate[c[t]/.sol],{t,0,700}]
```

(* See Example 12: A second-order ODE. *)

```
In[12]:=DSolve[{i''[t]+5i'[t]+6i[t]==10Sin[t],i[0]==0,i'[0]==0},i[t],t]
```

(* Solving an IVP. *)

```
In[13]:=DSolve[{x'[t]==2 Sqrt[x[t]],x[t0]==x0},x[t],t]
```

```
In[14]:=Simplify[%]
```

(* See Exercise 7: Solving a system of 3 ODEs *)

```
In[15]:=DSolve[{x'[t]==-a x[t],y'[t]==a x[t]-b y[t],z'[t]==b y[t],x[0]==M,
y[0]==0,z[0]==0},{x[t],y[t],z[t]},t];
```

```
In[16]:=Simplify[%]
```

2.6 Exercises

1. Sketch some solution curves for the following differential equations

(a) $\frac{dy}{dx} = -\frac{y}{x};$

(b) $\frac{dy}{dx} = \frac{2y}{x};$

(c) $\frac{dy}{dx} = \frac{y}{2x};$

(d) $\frac{dy}{dx} = \frac{y^2}{x};$

(e) $\frac{dy}{dx} = -\frac{xy}{x^2+y^2};$

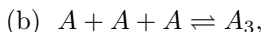
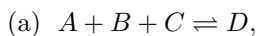
(f) $\frac{dy}{dx} = \frac{y}{x^2}.$

2. Fossils are often dated using the differential equation

$$\frac{dA}{dt} = -\alpha A,$$

where A is the amount of radioactive substance remaining, α is a constant, and t is measured in years. Assuming that $\alpha = 1.5 \times 10^{-7}$, determine the age of a fossil containing radioactive substance A if only 30% of the substance remains.

3. Write down the chemical reaction rate equations for the reversible reaction equations



given that the forward rate constant is k_f and the reverse rate constant is k_r , in each case. Assume that the chemical equations are the rate-determining steps.

4. (a) Consider a series resistor-inductor circuit with $L = 2\text{ H}$, $R = 10\ \Omega$ and an applied EMF of $E = 100\sin(t)$. Use an integrating factor to solve the differential equation, and find the current in the circuit after 0.2 seconds given that $I(0) = 0$.
- (b) The differential equation used to model a series resistor-capacitor circuit is given by

$$R\frac{dQ}{dt} + \frac{Q}{C} = E,$$

where Q is the charge across the capacitor. If a variable resistance $R = 1/(5+t)\ \Omega$ and a capacitance $C = 0.5\text{ F}$ are connected in series

with an applied EMF, $E = 100\text{ V}$, find the charge on the capacitor given that $Q(0) = 0$.

5. (a) A forensic scientist is called to the scene of a murder. The temperature of the corpse is found to be 75°F , and one hour later, the temperature has dropped to 70°F . If the temperature of the room in which the body was discovered is a constant 68°F , how long before the first temperature reading was taken did the murder occur? Assume that the body obeys Newton's Law of Cooling,

$$\frac{dT}{dt} = \beta(T - T_R),$$

where T is the temperature of the corpse, β is a constant, and T_R is room temperature.

- (b) The differential equation used to model the concentration of glucose in the blood, say, $g(t)$, when it is being fed intravenously into the body, is given by

$$\frac{dg}{dt} + kg = \frac{G}{100V},$$

where k is a constant, G is the rate at which glucose is admitted, and V is the volume of blood in the body. Solve the differential equation and discuss the results.

- (c) Single fiber muscle can be modeled using simple differential equations [11]. Download our preprint paper on "Hysteresis in muscle" from ResearchGate, and use Mathematica to reproduce the results of the Hill model given in that paper.

6. Show that the series solution of the Airy equation

$$\frac{d^2x}{dt^2} - tx = 0,$$

where $x(0) = a_0$ and $\dot{x}(0) = a_1$, used in physics to model the diffraction of light, is given by

$$\begin{aligned} x(t) = & a_0 \left(1 + \sum_1^{\infty} \left(\frac{t^{3k}}{(2.3)(5.6) \cdots ((3k-1)(3k))} \right) \right) \\ & + a_1 \left(t + \sum_1^{\infty} \left(\frac{t^{3k+1}}{(3.4)(6.7) \cdots ((3k)(3k+1))} \right) \right). \end{aligned}$$

7. A chemical substance A changes into substance B at a rate α times the amount of A present. Substance B changes into C at a rate β times the amount of B present. If initially only substance A is present and its amount is M , show that the amount of C present at time t is

$$M + M \left(\frac{\beta e^{-\alpha t} - \alpha e^{-\beta t}}{\alpha - \beta} \right).$$

8. Two tanks A and B , each of volume V , are filled with water at time $t = 0$. For $t > 0$, volume v of solution containing mass m of solute flows into tank A per second; mixture flows from tank A to tank B at the same rate; and mixture flows away from tank B at the same rate. The differential equations used to model this system are given by

$$\frac{d\sigma_A}{dt} + \frac{v}{V}\sigma_A = \frac{m}{V}, \quad \frac{d\sigma_B}{dt} + \frac{v}{V}\sigma_B = \frac{v}{V}\sigma_A,$$

where $\sigma_{A,B}$ are the concentrations of solute in tanks A and B , respectively. Show that the mass of solute in tank B is given by

$$\frac{mV}{v} \left(1 - e^{-vt/V} \right) - mte^{-vt/V}.$$

9. In an epidemic the rate at which healthy people become infected is a times their number, the rates of recovery and death are, respectively, b and c times the number of infected people. If initially there are N healthy people and no sick people, find the number of deaths up to time t . Is this a realistic model? What other factors should be taken into account?
10. (a) Determine the maximal interval of existence for each of the following initial value problems:
- (i) $\dot{x} = x^4, x(0) = 1$;
 - (ii) $\dot{x} = \frac{x^2-1}{2}, x(0) = 2$;
 - (iii) $\dot{x} = x(x-2), x(0) = 3$.
- (b) For what values of t_0 and x_0 does the initial value problem

$$\dot{x} = 2\sqrt{x}, \quad x(t_0) = x_0,$$

have a unique solution?

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